Chapter 04

APPLICATIONS OF DERIVATIVES

1. DERIVATIVE AS RATE OF CHANGE

In various fields of applied mathematics one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.

An economist may want to study how the investment changes with respect to variations in interest rates.

A physician may want to know, how small changes in dosage can affect the body's response to a drug.

A physicist may want to know the rate of change of distance with respect to time.

All questions of the above type can be interpreted and represented using derivatives.

Definition :

The **average rate of change** of a function $f(x)$ with respect to x over an interval $[a, a + h]$ is defined as $\dfrac{f(a+h) - f(a)}{h}$.

Definition :

The **instantaneous rate of change** of f with respect to x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$, provided the limit exists.

NOTES :

To use the word 'instantaneous', x may not be representing time. We usually use the word 'rate of change' to mean 'instantaneous rate of change'.

2. EQUATIONS OF TANGENT & NORMAL

(I) The value of the derivative at P (x_1, y_1) gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \text{Slope of tangent at}$$

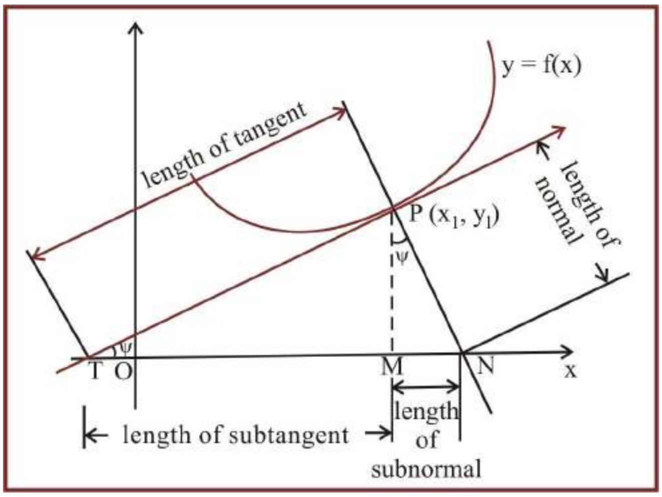
P $(x_1, y_1) = m$ (say).

(II) Equation of tangent at (x_1, y_1) is ;

$$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \times (x - x_1)$$

(III) Equation of normal at (x_1, y_1) is ;

$$(y - y_1) = \left(\frac{-1}{\frac{dy}{dx}}\right)_{(x_1, y_1)} \times (x - x_1)$$



NOTES :

1. The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.
2. If the tangent at any point P on the curve is parallel to X-axis then $dy/dx = 0$ at the point P.
3. If the tangent at any point on the curve is parallel to Y-axis, then $dy/dx = \infty$ or $dx/dy = 0$.
4. If the tangent at any point on the curve is equally inclined to both the axes then $dy/dx = \pm 1$.
5. If the tangent at any point makes equal intercept on the coordinate axes then $dy/dx = \pm 1$.
6. Tangent to a curve at the point P (x_1, y_1) can be drawn even though dy/dx at P does not exist. e.g. $x = 0$ is a tangent to $y = x^{2/3}$ at (0, 0).
7. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$.

(IV) (a) Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$

(d) Length of Subnormal (MN) = $y_1 f'(x_1)$

(V) Differential :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, y = tan x then dy = $\sec^2 x$ dx.

In general dy = $f'(x)$ dx.

NOTES :

$d(c) = 0$ where 'c' is a constant.

$d(u + v - w) = du + dv - dw$

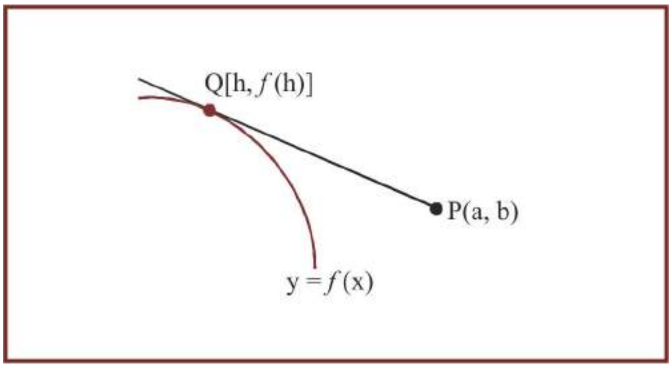
$d(uv) = udv + vdu$

* The relation $dy = f'(x)\,dx$ can be written as $\dfrac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

3. TANGENT FROM AN EXTERNAL POINT

Given a point P (a, b) which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve $y = f(x)$, passing through (a, b) can be found by solving for the point of contact Q.

And equation of tangent is $y - b = \dfrac{f(h) - b}{h - a}(x - a)$

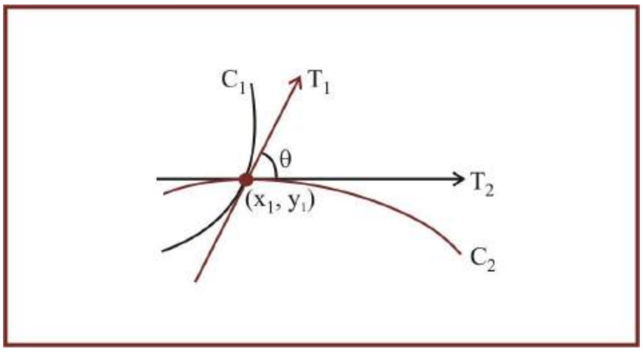


4. ANGLE BETWEEN THE CURVES

Angle between two intersecting curves is defined as the acute angle between their tangents or the normals at the point of intersection of two curves.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1).



NOTES :

(i) The angle is defined between two curves if the curves are intersecting. This can be ensured by finding their point of intersection or by graphically.

(ii) If the curves intersect at more than one point then angle between curves is found out with respect to the point of intersection.

(iii) Two curves are said to be **orthogonal** if angle between them at each point of intersection is right angle i.e. $m_1 m_2 = -1$.

5. SHORTEST DISTANCE BETWEEN TWO CURVES

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

6. ERRORS AND APPROXIMATIONS

(a) Errors

Let $y = f(x)$

From definition of derivative, $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$

$\dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$ approximately

or $\delta y = \left(\dfrac{dy}{dx}\right) . \delta x$ approximately

Definition :

(i) δx is known as absolute error in x.

(ii) $\dfrac{\delta x}{x}$ is known as relative error in x.

(iii) $\frac{\delta x}{x} \times 100$ is known as percentage error in x.

NOTES :

δx and δy are known as differentials.

(b) Approximations

From definition of derivative,

$\therefore$ Derivative of $f(x)$ at $(x = a) = f'(a)$

or $$f'(a) = \lim_{\delta x \to 0} \frac{f(a+\delta x) - f(a)}{\delta x}$$

or $$\frac{f(a+\delta x) - f(a)}{\delta x} \to f'(a) \quad \text{(approximately)}$$

$$f(a+\delta x) = f(a) + \delta x f'(a) \quad \text{(approximately)}$$

7. DEFINITIONS

1. A function $f(x)$ is called an **Increasing Function** at a point $x = a$ if in a sufficiently small neighbourhood around $x = a$ we have

$$f(a+h) > f(a)$$
$$f(a-h) < f(a)$$

Similarly **Decreasing Function** if

$$f(a+h) < f(a)$$
$$f(a-h) > f(a)$$

Above statements hold true irrespective of whether f is non derivable or even discontinuous at $x = a$

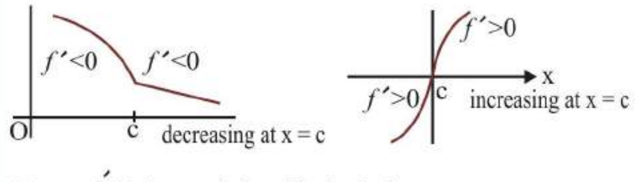
2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.

3. A function which in a given interval is increasing or decreasing is called **"Monotonic"** in that interval.

4. Tests for increasing and decreasing of a function at a point : If the derivative $f'(x)$ is positive at a point $x = a$, then the function $f(x)$ at this point is increasing. If it is negative, then the function is decreasing.

NOTES :

Even if $f'(a)$ is not defined, f can still be increasing or decreasing. (Look at the cases below).



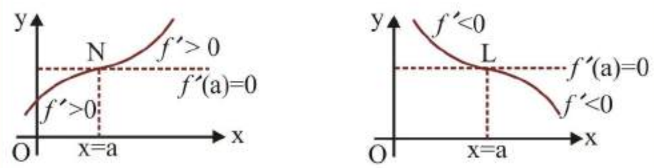
Note : $f'(c)$ is not defined in both the cases.

NOTES :

If $f'(a) = 0$, then for $x = a$ the function may be still increasing or it may be decreasing as shown. It has to be identified by a separate rule.

e.g. $f(x) = x^3$ is increasing at every point.

Note that, $dy/dx = 3x^2$.



NOTES :

1. If a function is invertible it has to be either increasing or decreasing.
2. If a function is continuous, the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.
3. If f is increasing in [a, b] and is continuous then $f(b)$ is the greatest and $f(a)$ is the least value of f in [a, b]. Similarly if f is decreasing in [a, b] then $f(a)$ is the greatest value and $f(b)$ is the least value.

5. (a) ROLLE'S Theorem :

Let $f(x)$ be a function of x subject to the following conditions :

(i) $f(x)$ is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = 0$.

(b) LMVT Theorem :

Let $f(x)$ be a function of x subject to the following conditions :

(i) $f(x)$ is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

Then there exists at least one point x = c such that $a < c < b$ where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically, the slope of the secant line joining the curve at x = a & x = b is equal to the slope of the tangent line drawn to the curve at x = c.

Note the following : Rolles theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

NOTES :

Physical Interpretation of LMVT :

Now $[f(b) - f(a)]$ is the change in the function f as x changes from a to b so that $\dfrac{f(b) - f(a)}{b - a}$ is the average rate of change of the function over the interval [a, b]. Also $f'(c)$ is the actual rate of change of the function for x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.

This interpretation of the theorem justifies the name "Mean Value" for the theorem.

(c) Application of rolles theorem for isolating the real roots of an equation $f(x) = 0$

Suppose a & b are two real numbers such that ;

(i) $f(x)$ & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(ii) $f(a)$ & $f(b)$ have opposite signs.

(iii) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one real root of the equation $f(x) = 0$ between a & b.

8. HOW MAXIMA & MINIMA ARE CLASSIFIED

1. Maxima & Minima

A function $f(x)$ is said to have a local maximum at $x = a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of x = a. Symbolically

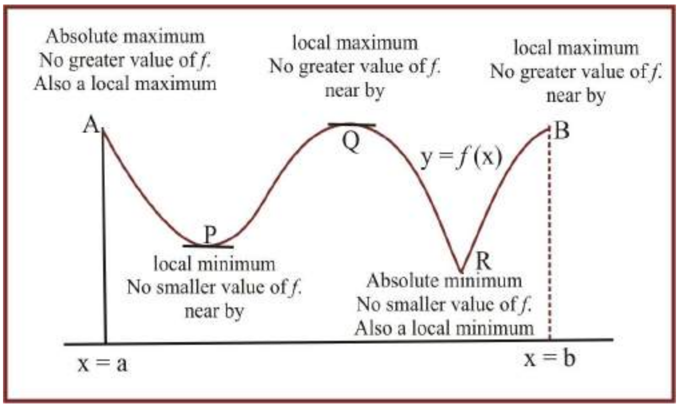
$$\left.\begin{array}{l} f(a) > f(a+h) \\ f(a) > f(a-h) \end{array}\right] \Rightarrow x=a \text{ gives maxima}$$

for a sufficiently small positive h.

Similarly, a function $f(x)$ is said to have a local minimum value at $x = b$ if $f(b)$ is least than every other value assumed by $f(x)$ in the immediate neighbourhood at x = b. Symbolically if

$$\left.\begin{array}{l} f(b) < f(b+h) \\ f(b) < f(b-h) \end{array}\right] \Rightarrow x=b \text{ gives minima for a sufficiently}$$

small positive h.



NOTES :

(i) The local maximum & local minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) The term 'extremum' is used both for maxima or a minima.

(iii) A local maximum (local minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) A function can have several local maximum & local minimum values & a local minimum value may even be greater than a local maximum value.

(v) Maxima & minima of a continuous function occur alternately & between two consecutive maxima there is a minima & vice versa.

2. A necessary condition for maxima & minima

If $f(x)$ is a maxima or minima at $x = c$ & if $f'(c)$ exists then $f'(c) = 0$.

NOTES :

(i) The set of values of x for which $f'(x) = 0$ are often called as stationary points. The rate of change of function is zero at a stationary point.

(ii) In case $f'(c)$ does not exist $f(c)$ may be a maxima or a minima & in this case left hand and right hand derivatives are of opposite signs.

(iii) The greatest (global maxima) and the least (global minima) values of a function f in an interval [a, b] are $f(a)$ or $f(b)$ or are given by the values of x which are critical points.

(iv) **Critical points** are those where :

(i) $\frac{dy}{dx} = 0$, if it exists; (ii) or it fails to exist

3. Sufficient condition for extreme values

First Derivative Test

$$\left.\begin{matrix} f'(c-h) > 0 \\ f'(c+h) < 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local maxima,}$$

where h is a sufficiently small positive quantity

$$\text{Similarly } \left.\begin{matrix} f'(c-h) < 0 \\ f'(c+h) > 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local minima,}$$

where h is a sufficiently small positive quantity

Note: $f'(c)$ in both the cases may or may not exist. If it exists, then $f'(c) = 0$.

NOTES :

If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c, then $f(x)$ is either strictly increasing or decreasing throughout this neighbourhood implying that $f(c)$ is not an extreme value of f.

4. Use of second order derivative in ascertaining the maxima or minima

(a) $f(c)$ is a minima of the function f, if $f'(c) = 0$ & $f''(c) > 0$.

(b) $f(c)$ is a maxima of the function f, if $f'(c) = 0$ & $f''(c) < 0$.

NOTES :

If $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaining the maxima or minima.

5. Summary-working rule

First : When possible, draw a figure to illustrate them problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second : Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

Third : If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

Fourth : Test each values of x for which $f'(x) = 0$ to determine whether it provides a maxima or minima or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0$

$\Rightarrow$ y is minima.

If d^2y/dx^2 is negative when $dy/dx = 0$

$\Rightarrow$ y is maxima.

If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) $$\text{If } \frac{dy}{dx} \text{ is } \left.\begin{matrix} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{matrix}\right] \Rightarrow \text{a maxima occurs at } x = x_0.$$

But if dy/dx changes sign from negative to zero to positive as x advances through x_0, there is a minima. If dy/dx does not change sign, neither a maxima nor a minima. Such points are called **INFLECTION POINTS**.

Fifth : If the function $y = f(x)$ is defined for only a limited range of values $a \le x \le b$ then examine $x = a$ & $x = b$ for possible extreme values.

Sixth : If the derivative fails to exist at some point, examine this point as possible maxima or minima.

(In general, check at all Critical Points).

NOTES :

- If the sum of two positive numbers x and y is constant than their product is maximum if they are equal, i.e. $x + y = c$, $x > 0$, $y > 0$, then

$$xy = \frac{1}{4}\left[(x+y)^2 - (x-y)^2\right]$$

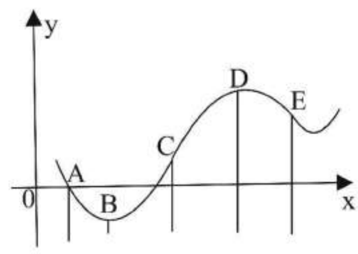
- If the product of two positive numbers is constant then their sum is least if they are equal.
  i.e. $(x+y)^2 = (x-y)^2 + 4xy$

9. USEFUL FORMULAE OF MENSURATION TO REMEMBER

- Volume of a cuboid = $l$bh.
- Surface area of a cuboid = 2 (lb + bh + hl).
- Volume of a prism = area of the base × height.
- Lateral surface of a prism = perimeter of the base × height.
- Total surface of a prism = lateral surface + 2 area of the base
  (Note that lateral surfaces of a prism are all rectangles).
- Volume of a pyramid $= \frac{1}{3}$ area of the base × height.
- Curved surface of a pyramid $= \frac{1}{2}$ (perimeter of the base) × slant height.
  (Note that slant surfaces of a pyramid are triangles).
- Volume of a cone $= \frac{1}{3}\pi r^2 h$.
- Curved surface of a cylinder $= 2\pi rh$.
- Total surface of a cylinder $= 2\pi rh + 2\pi r^2$.
- Volume of a sphere $= \frac{4}{3}\pi r^3$.
- Surface area of a sphere $= 4\pi r^2$.
- Area of a circular sector $= \frac{1}{2}r^2\theta$, where θ is in radians.

10. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINTS OF INFLECTION

The sign of the 2nd order derivative determines the concavity of the curve. Such point such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if :



(i) $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave upwards

(ii) $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave downwards.

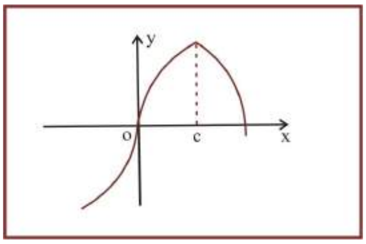
At the point of inflection we find that $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2y}{dx^2}$ changes sign.

Inflection points can also occur if $\frac{d^2y}{dx^2}$ fails to exist (but changes its sign). For example, consider the graph of the function defined as,

$$f(x) = \begin{bmatrix} x^{3/5} & \text{for } x \in (-\infty, 1) \\ 2 - x^2 & \text{for } x \in (1, \infty) \end{bmatrix}$$

NOTES :

The graph below exhibits two critical points one is a point of local maximum (x = c) & the other a point of inflection (x = 0). This implies that not every Critical Point is a point of extrema.



SOLVED EXAMPLES

Example – 1

If the function $f(x) = 2x^3 – 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) 3

Ans. (b)

Sol. For maximum and minima $f'(x)=0$

$\Rightarrow 6x^2-18ax+12a^2=0$

$\Rightarrow x = a, 2a$

Also, $f''(x) = 12x - 18x$

$f''(a)<0 \Rightarrow \max at 'a'$

$f''(2a) > 0 \Rightarrow \min at '2a'$

So, $p = a$ and $q = 2a$

Given $p^2 = q$

$\Rightarrow a^2=2a \Rightarrow a^2-2a = 0$

$\Rightarrow a(a-2)=0 \Rightarrow a = 0, a = 2$

Example – 2

The real number x when added to its inverse gives the minimum value of the sum at x equal to

(a) 1 (b) – 1

(c) – 2 (d) 2

Ans. (a)

Sol. $f(x) = x+\frac{1}{2}$

$f'(x) = 1-\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$

Now $f'(x) = 0$

$\Rightarrow x = \pm 1$

$\because f''(1) > 0$

$\Rightarrow x = 1$ is point of minima.

Example – 3

A function $y = f(x)$ has a second order derivative $f'' = 6(x–1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x – 5$, then the function is

(a) $(x – 1)^2$ (b) $(x – 1)^3$

(c) $(x + 1)^3$ (d) $(x + 1)^2$

Ans. (b)

Sol. Given $f''(x)=6(x-1)$

$\Rightarrow f'(x) = \frac{6(x-1)^2}{2}+c$

$\Rightarrow 3 = 3 + c$ $\left[\because \text{tangent at } x = 2 \text{ is } y = 3x+5 \right.$

$\Rightarrow c = 0$ $\left. \Rightarrow f'(2) = 3 \right.$

so $f'(x) = 3(x-1)^2$

$\Rightarrow f(x)=(x-1)^3+c_1$ as curve passes through (2,1)

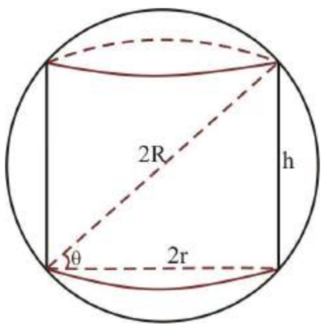
$\Rightarrow 1=(2-1)^3+c_1$

$\Rightarrow c_1 = 0$ hence $f(x)=(x-1)^3$

Example – 4

Find the maximum surface area of a cylinder that can be inscribed in a given sphere of radius R.

Sol.



Let r be the radius and *h* be the height of cylinder. Consider the right triangle shown in the figure.

$2r = 2R\cos\theta$ and $\qquad h = 2R\sin\theta$

Surface area of the cylinder $= 2\pi rh + 2\pi r^2$

$\Rightarrow \quad S(\theta) = 4\pi R^2 \sin\theta\cos\theta + 2\pi R^2\cos^2\theta$

$\Rightarrow \quad S(\theta) = 2\pi R^2\sin 2\theta + 2\pi R^2\cos^2\theta$

$\Rightarrow \quad S'(\theta) = 4\pi R^2\cos 2\theta - 2\pi R^2\sin 2\theta$

$S'(\theta) = 0 \quad \Rightarrow \quad 2\cos 2\theta - \sin 2\theta = 0$

$\Rightarrow \quad \tan 2\theta = 2 \quad \Rightarrow \quad \theta = \theta_0 = 1/2\tan^{-1}2$

$S''(\theta_0) = -8\pi R^2\sin 2\theta - 4\pi R^2\cos 2\theta$

$$S''(\theta) = -8\pi R^2\left(\frac{2}{\sqrt{5}}\right) - 4\pi R^2\left(\frac{1}{\sqrt{5}}\right) < 0$$

Hence surface area is maximum for $\theta = \theta_0 = 1/2\tan^{-1}2$

$S_{max} = 2\pi R^2\sin 2\theta_0 + 2\pi R^2\cos^2\theta_0$

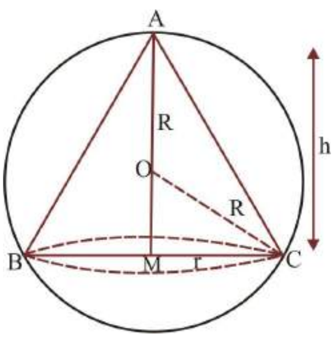
$$\Rightarrow \quad S_{max} = 2\pi R^2\left(\frac{2}{\sqrt{5}}\right) + 2\pi R^2\left(\frac{1+1/\sqrt{5}}{2}\right)$$

$$\Rightarrow \quad S_{max} = \pi R^2(1+\sqrt{5})$$

Example – 5

Find the semi-vertical angle of the cone of maximum curved surface area that can be inscribed in a given sphere of radius R.

Sol.



Let h be the height of cone and r be the radius of the cone. Consider the right ΔOMC where O is the centre of sphere and AM is perpendicular to the base BC of cone.

$OM = h - R$, $OC = R$, $MC = r$

$R^2 = (h-R)^2 + r^2 \qquad \ldots(i)$

and $\quad r^2 + h^2 = l^2 \qquad \ldots(ii)$

where l is the slant height of cone.

Curve surface area $= C = \pi r l$

Using (i) and (ii), express C in terms of h only.

$$C = \pi r\sqrt{r^2+h^2} \Rightarrow C = \pi\sqrt{2hR - h^2}\,\sqrt{2hR}$$

We will maximise C^2.

Let $C^2 = f(h) = 2\pi^2 hR(2hR - h^2) = 2\pi^2R(2h^2R - h^3)$

$\Rightarrow \quad f'(h) = 2\pi^2R(4hR - 3h^2)$

$f'(h) = 0 \Rightarrow 4hR - 3h^2 = 0 \Rightarrow h(4R - 3h) = 0$

$\Rightarrow \quad h = 4R/3.$

$f''(h) = 2\pi^2R(4R - 6h)$

$$f''\left(\frac{4R}{3}\right) = 2\pi R^2(4R - 8R) < 0$$

Hence curved surface area is maximum for $h = \dfrac{4R}{3}$

Using (i), we get :

$$r^2 = 2hR - h^2 = \frac{8R^2}{9} \Rightarrow r = \frac{2\sqrt{2}}{3}R$$

Semi–vertical angle $= \theta = \tan^{-1} r/h = \tan^{-1} 1/\sqrt{2}$.

Example – 6

If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$:

(a) $f'(c) = 2g'(c)$ $\qquad$ (b) $2f'(c) = g'(c)$
(c) $2f'(c) = 3g'(c)$ $\qquad$ (d) $f'(c) = g'(c)$

Ans. (a)

Sol. By LMVT

$$f'(c) = \frac{f(1) - f(0)}{1-0} = \frac{6-2}{1} = 4$$

$$g'(c) = \frac{g(1) - g(0)}{1-0} = \frac{2-0}{1} \Rightarrow 2$$

$$\Rightarrow f'(c) = 2g'(c)$$

Example – 7

If $2a + 3b + 6c = 0$ ($a, b, c, \in R$), then the quadratic equation $ax^2 + bx + c = 0$ has

(a) at least one root in (0, 1) $\quad$ (b) at least one root in [2, 3]
(c) at least one root in [4, 5] $\quad$ (d) none of the above

Ans. (a)

Sol. Let us consider $f(x)=\dfrac{ax^3}{3}+\dfrac{bx^2}{2}+cx$

$\therefore f(0)=0$ and $f(1)=\dfrac{a}{3}+\dfrac{b}{2}+c$

$=\dfrac{2a+3b+6c}{6}=0$(given).

As $f(0)=f(1)=0$ and $f(x)$ is continuous and differentiable also in $[0,1]$.

$\therefore$ By Rolle's theorem $f(x)=0$

$\Rightarrow ax^2+bx+c=0$ has at least one root in the interval (0, 1).

Example – 8

Find the approximate value of $(0.007)^{1/3}$.

Sol. Let $f(x)=(x)^{1/3}$

Now, $f(x+\delta x)-f(x)=f'(x).\delta x=\dfrac{\delta x}{3x^{2/3}}$

we may write, $0.007=0.008-0.001$

Taking $x=0.008$ and $\delta x=-0.001$, we have

$$f(0.007)-f(0.008)=-\frac{0.001}{3(0.008)^{2/3}}$$

or $\quad f(0.007)-(0.008)^{1/3}=-\dfrac{0.001}{3(0.2)^2}$ or

$$f(0.007)=0.2-\frac{0.001}{3(0.04)}=0.2-\frac{1}{120}=\frac{23}{120}$$

Hence $(0.007)^{1/3}=\dfrac{23}{120}$.

Example – 9

Discuss concavity and convexity and find points of inflexion of $y=x^2e^{-x}$.

Sol. Let $f(x)=x^2e^{-x}$.

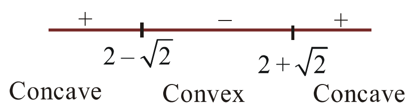
Differentiate w.r.t.x to get :

$$f'(x)=e^{-x}(2x)+(-e^{-x})x^2$$
$$=xe^{-x}[2-x]$$

Differentiate again w.r.t. x to get :

$$f''(x)=(2-2x)e^{-x}+(2x-x^2)(-e^{-x})$$
$$=e^{-x}(2-2x-2x+x^2)$$
$$=e^{-x}(x^2-4x+2)$$
$$=e^{-x}(x-(2-\sqrt{2}))(x-(2+\sqrt{2}))$$

See the figure and observe how the sign of f'' (x) changes.



Sign of $f''(x)$ is changing at $x=2\pm\sqrt{2}$.

Therefore points of inflextion of $f(x)$ are $x=2\pm\sqrt{2}$.

$$f''(x)\geq 0\ \forall\ x\in[-\infty,2-\sqrt{2}]\cup[2+\sqrt{2},\infty]$$

Therefore $f(x)$ is "Concave upward"

$$\forall\ x\in(-\infty,2-\sqrt{2}]\cup[2+\sqrt{2},\infty)$$

Similarly we can observe

$$f''(x)\leq 0\ \forall\ x\in[2-\sqrt{2},2+\sqrt{2}]$$

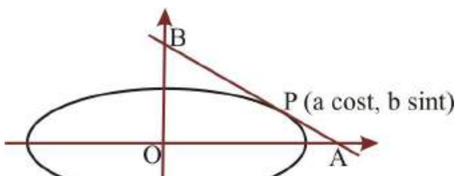
Therefore $f(x)$ is "Convex downwards"

$$\forall\ x\in[2-\sqrt{2},2+\sqrt{2}]$$

Example –10

Prove that the minimum intercept made by axes on the tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is a + b. Also find the ratio in which the point of contact divides this intercept.

Sol.



Intercept made by the axes on the tangent is the length of the portion of the tangent intercepted between the axes.

Consider a point P on the ellipse whose coordinates are

x = a cost, y = b sint (where t is the parameter)

$$\frac{dx}{dt}=-a\sin t$$
$$\frac{dy}{dt}=b\cos t$$

The equation of the tangent is :

$$y - b\sin t = \frac{b\cos t}{-a\sin t}(x - a\cos t)$$

$\Rightarrow \quad \frac{x}{a}\cos t + \frac{y}{b}\sin t = 1$

$\Rightarrow \quad OA = \frac{a}{\cos t},\ OB = \frac{b}{\sin t}$

Length of intercept $= l = AB = \sqrt{\frac{a^2}{\cos^2 t} + \frac{b^2}{\sin^2 t}}$

We will minimise l^2.

Let $l^2 = f(t) = a^2\sec^2 t + \text{cosec}^2 t$

$\Rightarrow \quad f'(t) = 2a^2\sec^2 t\tan t - 2b^2\text{cosec}^2 t\cot t$

$f'(t) = 0 \quad \Rightarrow \quad a^2\sin^4 t = b^2\cos^4 t$

$\Rightarrow \quad t = \tan^{-1}\sqrt{b/a}$

$f''(t) = 2a^2(\sec^4 t + 2\tan^2 t\sec^2 t)$
$+ 2b^2(\text{cosec}^4 t + 2\,\text{cosec}^2 t\cot^2 t)$, which is positive.

Hence $f(t)$ is minimum for $\tan t = \sqrt{\frac{b}{a}}$.

$\Rightarrow \quad l_{min} = \sqrt{a^2(1 + b/a) + b^2(1 + a/b)}$

$\Rightarrow \quad l_{min} = a + b$

$$PA^2 = \left(a\cos t - \frac{a}{\cos t}\right)^2 + b^2\sin^2 t$$

$$= \frac{a^2\sin^4 t}{\cos^2 t} + b^2\sin^2 t$$

$$= (a^2\tan^2 t + b^2)\sin^2 t$$

$$= (ab + b^2)\frac{b}{a+b} = b^2 \Rightarrow PA = b$$

Hence $\frac{PA}{PB} = \frac{b}{a} \Rightarrow$ P divides AB in the ratio b : a

Example – 11

Find the equation of tangent to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at (x_0, y_0). Hence prove that the length of the portion of tangent intercepted between the axes is constant.

Sol. Method 1 :

$x^{2/3} + y^{2/3} = a^{2/3}$ Differentiating wrt x,

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_0, y_0)} = -\left(\frac{x_0}{y_0}\right)^{-\frac{1}{3}}$$

$$\Rightarrow \left.\frac{dy}{dx}\right]_{(x_0, y_0)} = -\left(\frac{y_0}{x_0}\right)^{\frac{1}{3}}$$

$\Rightarrow$ equation is $y - y_0 = -\left(\frac{y_0}{x_0}\right)^{\frac{1}{3}}(x - x_0)$

$\Rightarrow \quad x_0^{1/3}y - y_0x_0^{1/3} = -xy_0^{1/3} + x_0y_0^{1/3}$

$\Rightarrow \quad x\,y_0^{1/3} + yx_0^{1/3} = x_0y_0^{1/3} + y_0x_0^{1/3}$

$\Rightarrow \quad \frac{x\,y_0^{1/3}}{x_0^{1/3}\,y_0^{1/3}} + \frac{y\,x_0^{1/3}}{x_0^{1/3}\,y_0^{1/3}} = x_0^{2/3} + y_0^{2/3}$

$\Rightarrow$ equation of tangent is : $\frac{x}{x_0^{1/3}} + \frac{y}{y_0^{1/3}} = a^{2/3}$

Length intercepted between the axes :

length $= \sqrt{(\text{x intercept})^2 + (\text{y intercept})^2}$

x intercept $= x_0^{1/3}a^{2/3}$

y intercept $= = y_0^{1/3}a^{2/3}$

$$= \sqrt{\left(x_0^{1/3}\,a^{2/3}\right)^2 + \left(y_0^{1/3}\,a^{2/3}\right)^2}$$

$$= \sqrt{x_0^{2/3}a^{4/3} + y_0^{2/3}a^{4/3}}$$

$$= a^{2/3}\sqrt{x_0^{2/3} + y_0^{2/3}}$$

$$= a^{2/3}\sqrt{a^{2/3}}$$

= a i.e. constant.

Method 2 :

Express the equation in parametric form

$x = a\sin^3 t, \quad y = a\cos^3 t$

$$\frac{dx}{dt} = 3a\sin^2 t\cos t,\ \frac{dy}{dt} = -3a\cos^2 t\sin t$$

Equation of tangent is :

$$(y - a\cos^3 t) = \frac{-3a\cos^2 t\sin t}{3a\sin^2 t\cos t}(x - a\sin^3 t)$$

$\Rightarrow \quad y\sin t - a\sin t\cos^3 t = -x\cos t + a\sin^3 t\cos t$

$\Rightarrow \quad x\cos t + y\sin t = a\sin t\cos t$

$\Rightarrow \quad \frac{x}{\sin t} + \frac{y}{\cos t} = a$

in terms of (x_0, y_0) equation is :

$$\frac{x}{(x_0/a)^{1/3}} + \frac{y}{(y_0/a)^{1/3}} = a$$

Length of tangent intercepted between axes

$$=\sqrt{(x_{int})^2+(y_{int})^2}$$

$$=\sqrt{a^2\sin^2 t+a^2\cos^2 t}=a \text{ which is constant}$$

NOTES :

1. The parametric form is very useful in these type of problems.
2. Equation of tangent can also be obtained by substituting b = a and m = 2/3 in the result $\frac{x}{a}\left(\frac{x_0}{a}\right)^{m-1}+\frac{y}{b}\left(\frac{y_0}{b}\right)^{m-1}=1$.

Example – 12

For the curve $xy = c^2$, prove that

(i) the intercept between the axes on the tangent at any point is bisected at the point of contact.

(ii) the tangent at any point makes with the co-ordinate axes a triangle of constant area.

Sol. Let the equation of the curve in parametric form be $x = ct$, $y = c/t$

$$\frac{dx}{dt}=c$$

$$\frac{dy}{dt}=\frac{-c}{t^2}$$

Let the point of contact be (ct, c/t)

Equation of tangent is :

$$y-c/t=\frac{-c/t^2}{c}(x-ct)$$

$\Rightarrow$ $t^2y - ct = -x + ct$

$\Rightarrow$ $x + t^2y = 2\,ct$ ....(i)

(i) Let the tangent cut the x and y axes at A and B respectively.

Writing the equations as : $\frac{x}{2ct}+\frac{y}{2c/t}=1$

$\Rightarrow$ $x_{intercept} = 2ct$, $y_{intercept} = 2\,c/t$

$\Rightarrow$ $A \equiv (2ct, 0)$ and $B\equiv\left(0,\frac{2c}{t}\right)$

mid point of $AB\equiv\left(\frac{2ct+0}{2},\frac{0+2c/t}{2}\right)\equiv(ct, c/t)$

Hence, the point of contact bisects AB.

(ii) If O is the origin,

Area of triangle Δ OAB = 1/2 (OA) (OB)

$$=\frac{1}{2}(2ct)\frac{(2c)}{t}$$

$= 2\,c^2$

i.e. constant for all tangents because it is independent of t.

Example – 13

Find critical points of $f(x) = x^{2/3}(2x - 1)$.

Sol. $f(x) = 2x^{5/3} - x^{2/3}$

Differentiate w.r.t. x to get,

$$f'(x)=\frac{10}{3}x^{2/3}-\frac{2}{3}x^{-1/3}=\frac{2}{3}\frac{(5x-1)}{x^{1/3}}.$$

For critical points,

$f'(x) = 0$ or $f'(x)$ is not defined.

Put $f'(x) = 0$ to get $x=\frac{1}{5}$.

$f'(x)$ is not defined when denominator = 0.

$\Rightarrow$ $x^{1/3}=0$ $\Rightarrow$ $x=0$

Now we can say that x = 0 and $x=\frac{1}{5}$ are critical points as

$f(x)$ exists at both x = 0 and $x=\frac{1}{5}$.

$\Rightarrow$ Critical points of $f(x)$ are $x = 0$, $x=\frac{1}{5}$.

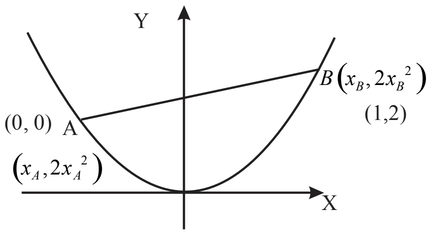
Example – 14

The ends A and B of a rod of length $\sqrt{5}$ are sliding along the curve $y = 2x^2$. Let x_A and x_B be the x-coordinate of the ends. At the moment when A is at (0, 0) and B is at (1, 2), find the value of the derivative $\frac{dx_B}{dx_A}$.

Sol. We have $y = 2x^2$

$(AB)^2=(x_B-x_A)^2+(2x_B^2-2x_A^2)^2=5\left(As\,|AB|=\sqrt{5}\right)$

or $(x_B-x_A)^2+4(x_B^2-x_A^2)^2=5$



Differentiating w.r.t. x_A and denoting $\frac{dx_B}{dx_A} = D$

$2(x_B - x_A)(D-1) + 8(x_B^2 - x_A^2)(2x_B D - 2x_A) = 0$

Put $x_A = 0, x_B = 1$

$2(1-0)(D-1) + 8(1-0)(2D-0) = 0$

$2D - 2 + 16D = 0 \Rightarrow D = 1/9 \Rightarrow \frac{dx_B}{dx_A} = \frac{1}{9}$

Example – 15

The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is

(a) $y=0$ (b) $y=1$

(c) $y=2$ (d) $y=3$

Ans. (d)

Sol. Tangent is parallel to x-axis

$$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0 \Rightarrow x = 2 \Rightarrow y = 3$$

Example – 16

For $0 < x \le \frac{\pi}{2}$, show that $x - \frac{x^3}{6} < \sin x < x$.

Sol. Let $f(x) = \sin x - x$

$f'(x) = \cos x - 1 = -(1 - \cos x) = -2\sin^2 x/2 < 0$

$\therefore$ $f(x)$ is a decreasing function

for $x > 0$

$\therefore$ $f(x) < f(0) \Rightarrow \sin x - x < 0$ ($\because f(0) = 0$)

$\Rightarrow$ $\sin x < x$ ......(1)

Now let $g(x) = x - \frac{x^3}{6} - \sin x$ $\therefore$ $g'(x) = 1 - \frac{x^2}{2} - \cos x$

To find sign of $g'(x)$ we consider $\phi(x) = 1 - \frac{x^2}{2} - \cos x$

$\therefore$ $\phi'(x) = -x + \sin x < 0$ [From (1)]

$\therefore$ $\phi(x)$ is a decreasing function $\Rightarrow g'(x) < 0$

$\Rightarrow$ $g(x)$ is a decreasing function $\because x > 0$

$\Rightarrow$ $g(x) < g(0)$

$\Rightarrow$ $x - \frac{x^3}{6} - \sin x < 0$ ($\because g(0) = 0$)

$\Rightarrow$ $x - \frac{x^3}{6} < \sin x$ ......(2)

Combining (1) and (2) we get $x - \frac{x^3}{6} < \sin x < x$.

Example – 17

Show that $x/(1+x) < \log(1+x) < x$ for $x > 0$.

Sol. Let $f(x) = \log(1+x) - \frac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \frac{(1+x) - x}{(1+x)^2}$$

$$f'(x) = \frac{x}{(1+x)^2} > 0 \ for\ x > 0$$

$\Rightarrow$ $f(x)$ is increasing.

Hence $x > 0 \Rightarrow f(x) > f(0)$ by the definition of the increasing function.

$$\Rightarrow \log(1+x) - \frac{x}{1+x} > \log(1+0) - \frac{0}{1+0}$$

$$\Rightarrow \log(1+x) - \frac{x}{1+x} > 0$$

$$\Rightarrow \log(1+x) > \frac{x}{1+x} \quad \text{... (i)}$$

Now, let $g(x) = x - \log(1+x)$

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \text{ for } x > 0$$

$\Rightarrow$ $g(x)$ is increasing.

Hence $x > 0 \Rightarrow g(x) > g(0)$

$\Rightarrow$ $x - \log(1+x) > 0 - \log(1+0)$

$\Rightarrow \quad x - \log(1+x) > 0$

$\Rightarrow \quad x > \log(1+x) \quad \text{... (ii)}$

Combining (i) and (ii), we get :

$$\frac{x}{1+x} < \log(1+x) < x$$

Example – 18

Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is

(a) $\pi/2$ (b) $\pi/3$

(c) $\pi/6$ (d) $\pi/4$

Ans. (a)

Sol. Given equation $y = x^2 - 5x + 6$,given point (2,0),(3,0)

$$\therefore \frac{dy}{dx} = 2x - 5$$

$$\text{say } m_1 = \left(\frac{dy}{dx}\right)_{\substack{x=2\\y=0}} = 4 - 5 = -1$$

$$\text{and } m_2 = \left(\frac{dy}{dx}\right)_{\substack{x=3\\y=0}} = 6 - 5 = 1$$

since $m_1 m_2 = -1$

$\Rightarrow$ tangents are at right angle i.e . $\frac{\pi}{2}$

Example – 19

Determine the absolute extrema for the following function and interval.

$$g(t) = 2t^3 + 3t^2 - 12t + 4 \text{ on } [0, 2]$$

Sol. Differentiate w.r.t. t

$$g'(t) = 6t^2 + 6t - 12 = 6(t+2)(t-1)$$

Note that this problem is almost identical to the first problem. The only difference is the interval that we were working on.

The first step is to again find the critical points. From the first example we know these are $t = -2$ and $t = 1$. At this point it's important to recall that we only want the critical points that actually fall in the interval in question. This means that we only want $t = 1$ since $t = -2$ falls outside the interval so reject it.

Now for absolute maxima

We have,

$$\text{Max } \{g(1), g(0), g(2)\}$$

i.e., $\text{Max } \{-3, 4, 8\}$

On comparing all these values we get $g(t)$ has absolute max. as 8 at $t = 2$ and similarly absolute minimum of $g(t)$ is -3 at $t = 1$.

Example – 20

Let f be differentiable for all x.

If $f(1) = -2$ and $f'(x) \geq 2$ for $x \in [1, 6]$, then

(a) $f(6) < 8$ (b) $f(6) \geq 8$

(c) $f(6) = 5$ (d) $f(6) < 5$

Ans. (b)

Sol. Using LMVT,

$$\frac{f(6) - f(1)}{6-1} = f'(c) \textit{ for some } c \in (1,6)$$

$$\Rightarrow \frac{f(6) - (-2)}{5} \geq 2$$

$$\Rightarrow f(6) \geq 8$$

Example – 21

Find points of local maximum and local minimum of $f(x) = x^{2/3}(2x - 1)$.

Sol. Let $f(x) = 2x^{5/3} - x^{2/3}$

Differentiate w.r.t. x to get :

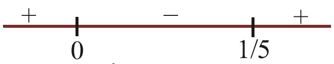
$$f'(x) = 2\left(\frac{5}{3}\right)x^{2/3} - \frac{2}{3}x^{-1/3} = \frac{2}{3}\frac{(5x-1)}{x^{1/3}}$$

By taking $f'(x) = 0$ or $f'(x)$ is not defined.

Critical points of $f(x)$ are $x = \frac{1}{5}$ and $x = 0$.

Using the following figure, we can determine how sign of $f'(x)$ is changing at $x = 0$ and $x = \frac{1}{5}$.

from figure,



Sign of $f'(x)$ in various intervals

$x = 0$ is point of local maximum as sign of $f'(x)$ changes from positive to negative and $x = \frac{1}{5}$ is a point of local minimum as sign of $f'(x)$ is changing from negative to positive.

Example – 22

Find the local maximum and local minimum values of the function $y = x^x$.

Sol. Let $f(x) = y = x^x$

$\Rightarrow \log y = x \log x$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log x$$

$$\Rightarrow \frac{dy}{dx} = x^x (1 + \log x)$$

$f'(x) = 0 \quad \Rightarrow x^x (1 + \log x) = 0$

$\Rightarrow \log x = -1 \quad \Rightarrow x = e^{-1} = 1/e.$

Method 1 : (First Derivative Test)

$f'(x) = x^x(1 + \log x)$

$f'(x) = x^x \log x$

$x < 1/e \quad \Rightarrow ex < 1$

$\Rightarrow \quad f'(x) < 0$

$x > 1/e \quad \Rightarrow ex > 1$

$\Rightarrow \quad f'(x) > 0$

The sign of $f'(x)$ changes from – ve to + ve around $x = 1/e$.

In other words, $f(x)$ changes from decreasing to increasing at $x = 1/e$.

Hence $x = 1/e$ is a point of local minimum.

Local minimum value $= (1/e)^{1/e} = e^{-1/e}$.

Method II : (Second Derivative Test)

$$f''(x) = (1 + \log x)\frac{d}{dx}x^x + x^x\left(\frac{1}{x}\right)$$

$$= x^x(1 + \log x)^2 + x^{x-1}$$

$f''(1/e) = 0 + (e)^{(e-1)/e} > 0.$

Hence $x = 1/e$ is a point of local minimum.

Local minimum value is $(1/e)^{1/e} = e^{-1/e}$.

Example – 23

The function $g(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at

(a) $x = 2$ (b) $x = -2$

(c) $x = 0$ (d) $x = 1$

Ans. (a)

Sol. Let $g(x) = \frac{x}{2} + \frac{2}{x}$

$\therefore g'(x) = \frac{1}{2} - \frac{2}{x^2}$

for maxima and minima $g'(x) = 0 \Rightarrow x = \pm 2$

Again $g''(x) = \frac{4}{x^3} > 0$ for $x = 2$

< 0 for $x = -2 \quad \therefore x = 2$ is point of minima

Example – 24

Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds?

(a) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(b) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(c) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(d) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

Ans. (b)

Sol. Let $f(x) = x^3 - px + q$

Now $f'(x) = 0$, i.e. $3x^2 - p = 0$

$$\Rightarrow x = -\sqrt{\frac{p}{3}}, \sqrt{\frac{p}{3}}$$

Also, $f''(x) = 6x \Rightarrow f''\left(-\sqrt{\frac{p}{3}}\right) < 0 \; and \; f''\left(\sqrt{\frac{p}{3}}\right) > 0$

Thus maxima at $-\sqrt{\frac{p}{3}}$ and minima at $\sqrt{\frac{p}{3}}$

Example – 25

Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$

(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P
(d) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

Ans. (b)

Sol. $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$P'(x) = 4x^3 + 3ax^2 + 2bx + c$

$P'(0) = 0 \Rightarrow c = 0$

Now, $P'(x) = x\,(4x^2 + 3ax + 2b)$

As $P'(x) = 0$ has no real roots except

$x = 0$, we have

Discriminant of $4x^2 + 3ax + 2b$ is less than zero.
i.e., $(3a)^2 - (4)(4)(2b) < 0$

then $4x^2 + 3ax + 2b > 0 \; \forall x \in R$

(If $a > 0, b^2 - 4ac < 0$ then $ax^2 + bx + c > 0 \; \forall x \in R$)

So $P'(x) < 0$ if $x \in [-1, 0]$ i.e., decreasing

and $P'(x) > 0$ if $x \in [0, 1]$ i.e., increasing

Max. of $P(x) = P(1)$

But minimum of $P(x)$ doesn't occur at $x = -1$, i.e., $P(-1)$ is not the minimum.

Example – 26

For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then, f has

(a) local minimum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local minimum at 2π
(d) local maximum at π and 2π

Ans. (c)

Sol. $f'(x) = \sqrt{x} \sin x, f'(x) = 0$

$\Rightarrow x = 0$ or $\sin x = 0$

$$\Rightarrow x = 2\pi, \pi \left(\because x \in \left(0, \frac{5\pi}{2}\right)\right)$$

$$f''(x) = \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x = \frac{1}{2\sqrt{x}}(2x \cos x + \sin x)$$

$f''(\pi) < 0$ and $f''(2\pi) > 0$

$\Rightarrow$ Local maxima at $x = \pi$ and local minima at $x = 2\pi$

Example – 27

Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is

(a) 1 (b) 0

(c) $-\frac{1}{2}$ (d) -1

Ans. (d)

Sol. $\lim_{x \to 1+} f(x) = 1$

As $f(-1) = k + 2$

As f has a local minimum at $x = -1$

$f(-1^+) \ge f(-1) \le f(-1^-) \Rightarrow 1 \ge k+2$

$\Rightarrow k + 2 \le 1. \quad \therefore k \le -1$

Thus $k = -1$ is a possible value.

Example – 28

Let a, b $\in$ R be such that the function f given by $f(x) = \log|x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement I f has local maximum at $x = -1$ and $x = 2$.

Statement II $a = \frac{1}{2}$ and $b = \frac{-1}{4}$.

(a) Statement I is false, Statement II is true.

(b) Statement I is true, Statement II is true;

Statement II is a correct explanation for Statement I.

(c) Statement I is true, Statement II is true;

Statement II is not a correct explanation for Statement I.

(d) Statement I is true, Statement II is false.

Ans. (c)

Sol. Given $f(x) = \text{In}|x| + bx^2 + ax$

$$\therefore f'(x) = \frac{1}{x} + 2bx + a$$

at $x = -1$, $f'(-1) = -1 - 2b + a = 0$

$\Rightarrow a - 2b = 1$ ...(i)

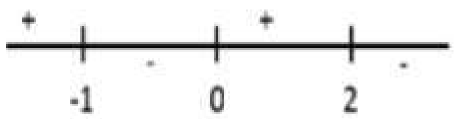
at $x = 2$, $f'(-2) = \frac{1}{2} + 4b + a = 0$

$\Rightarrow a + 4b = -\frac{1}{2}$ ...(ii)

Solving (i) and (ii) we get,

$$a = \frac{1}{2},\ b = -\frac{1}{4}.$$

$$\Rightarrow f'(x) = \frac{1}{x} - \frac{x}{2} + \frac{1}{2} = \frac{2 - x^2 + x}{2x} = \frac{-(x+1)(x-2)}{2x}$$



$\Rightarrow$ maxima as $x = -1.2$

Hence both statement are true but statement II. is not correct explanation of statement I.

Example – 29

The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that

(a) it makes angle $\frac{\pi}{2} + \theta$ with x–axis

(b) it passes through the origin

(c) it is a constant distance from the origin

(d) it passes through $\left(a\frac{\pi}{2}, -a\right)$

Ans. (a,c)

Sol. $\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{d\theta}{dx} = \tan\theta =$ Slope of tangent

$\therefore$ Slope of normal to the curve

$$= -\cot\theta \left(= \tan\left(\frac{\pi}{2} + \theta\right)\right)$$

Now, equation of normal to the curve

$$[y - a(\sin\theta - \theta\cos\theta)] = -\frac{\cos\theta}{\sin\theta}(x - a(\cos\theta + \theta\sin\theta))$$

$$\Rightarrow x\cos\theta + y\sin\theta = a(1)$$

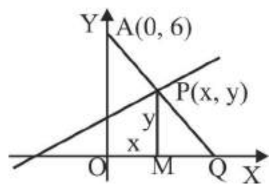
Now, distance from $(0,0)$ to $x\cos\theta + y\sin\theta = a$ is

$$\text{distance } (d) = \left|\frac{(0 + 0 - a)}{1}\right|$$

$\therefore$ distance is constant = |a|.

Example – 30

A point P (x, y) moves along the line whose equation is $x - 2y + 4 = 0$ in such a way that y increases at the rate of 3 units/sec. The point A (0, 6) is joined to P and the segment AP is prolonged to meet the x-axis in a point Q. Find how fast the distance from the origin to Q is changing when P reaches the point (4, 4).



Sol. The rate of change of y is given and it is desired to find the rate of change of OQ, which we denote by z. If MP is perpendicular to the x-axis, MP = y and OM = x.

The triangles OAQ and MPQ are similar, hence

$$\frac{z}{6} = \frac{z - x}{y} \Rightarrow yz = 6z - 6x \Rightarrow z = \frac{6x}{6 - y}$$

Substituting the value of x from the equation of the given line, we have

$$z = \frac{12\,(y-2)}{6-y}$$

$$\frac{dz}{dt} = \frac{48}{(6-y)^2}\frac{dy}{dt}$$

Setting y = 4 and $\frac{dy}{dt} = 3$, we obtain $\frac{dz}{dt} = 36$ that is, z is increasing at the rate of 36 units/sec.

Example – 31

The maximum distance from origin of a point on the curve

$$x = a\sin t - b\sin\left(\frac{at}{b}\right)$$

$y = a\cos t - b\cos\left(\frac{at}{b}\right)$, both a, b > 0, is

(a) a – b (b) a + b

(c) $\sqrt{a^2+b^2}$ (d) $\sqrt{a^2-b^2}$

Ans. (b)

Sol. Let A(0, 0) and B(x, y)

$$\therefore AB^2 = x^2 + y^2$$

$$\Rightarrow AB=\sqrt{a^2(\sin^2 t+\cos^2 t)+b^2\left(\sin^2\left(\frac{at}{b}\right)+\cos^2\left(\frac{at}{b}\right)\right)-2ab\cos\left(t-\frac{at}{b}\right)}$$

$$=\sqrt{a^2+b^2-2ab\cos\alpha} \quad = \sqrt{a^2+b^2+2ab}$$

($\because$ expression will take max value when as $\cos\alpha = -1$)

= (a + b)

Example – 32

The greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on [0, 1] is

(a) 1 (b) 2

(c) 3 (d) $\frac{1}{3}$

Ans. (b)

Sol. We have

$$f(x)=(x+1)^{\frac{1}{3}}-(x-1)^{\frac{1}{3}}$$

$$\therefore f'(x)=\left[\frac{1}{(x+1)^{2/3}}-\frac{1}{(x-1)^{2/3}}\right]\cdot\frac{1}{3}=\frac{(x-1)^{2/3}-(x+1)^{2/3}}{3(x^2-1)^{2/3}}$$

for critical points : $f'(x) = 0$ or not defined.

Clearly, $f'(x)$ does not exist at $x = \pm 1$

Now, $f'(x)=0 \Rightarrow (x-1)^{2/3}=(x+1)^{2/3} \Rightarrow x=0$

Clearly, so x = 0, + 1 are critical point in [0, 1].

f(0) = 2 and $f(1) = 2^{1/3}$

Hence greatest value = 2

Example – 33

If x is real, the maximum value of $\frac{3x^2+9x+17}{3x^2+9x+7}$ is

(a) $\frac{1}{4}$ (b) 41

(c) 1 (d) $\frac{17}{7}$

Ans. (b)

Sol. For the range of the expression

$$\frac{3x^2+9x+17}{3x^2+9x+7} = y = \frac{ax^2+bc+c}{px^2+qx+r},$$

[find the solution of the inequality $Ay^2 + By + K \geq 0$

Where $A = q^2 - 4pr = -3$, $B = 4ar + 4pc - 2bq = 126$

$K = b^2 - 4ac = -123$

i.e., solve $3y^2 - 126 + y - 123 \geq 0$

$\Rightarrow 3y^2 - 126y + 123 \leq 0 \Rightarrow y^2 - 42y + 41 \leq 0$

$\Rightarrow (y-1)(y-42) \leq 0 \Rightarrow 1 \leq y \leq 42$

$\Rightarrow$ Maximum value of y is 42

Example – 34

If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of (p + q) is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\sqrt{2}$ (d) 2

Ans. (c)

Sol. I[st] solution :

Let p = cos θ , q = sin θ where $0 \leq \theta \leq \frac{\pi}{2}$

p + q = cos θ + sin θ

$\Rightarrow$ maximum value of (p + q) = $\sqrt{2}$

II[nd] solution :

By using $A.M \geq G.M, \frac{p^2+q^2}{2} \geq pq \Rightarrow pq \leq \frac{1}{2}$

$(p+q)^2 = p^2+q^2+2pq \Rightarrow (p+q) \leq \sqrt{2}$

Example – 35

Let f be a function defined by $f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Statement I $x = 0$ is point of minima of f.

Statement II $f'(0) = 0$

(a) Statement I is false, Statement II is true.

(b) Statement I is true, Statement II is true; Statement II is correct explanation for Statement I.

(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(d) Statement I is true, Statement II is false.

Ans. (c)

Sol. $f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

In right neighbourhood of '0'

$\tan x > x \Rightarrow \frac{\tan x}{x} > 1$

In left neighbourhood of '0'

$\tan x < x \Rightarrow \frac{\tan x}{x} > 1 (\because \tan x < 0)$

at $x = 0, f(x) = 1$

$\Rightarrow x = 0$ is point of minima

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{\tan h}{h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\tan h - h}{h^2} = 0$$

hence $f'(0) = 0$

$\Rightarrow$ statement I is true and statement II is true.

Example – 36

If $f(\theta) = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$ and A and B are respectively the maximum and the minimum values of $f(\theta)$, then (A, B) is equal to:

(a) (3, -1) (b) $\left(4, 2-\sqrt{2}\right)$

(c) $\left(2+\sqrt{2}, 2-\sqrt{2}\right)$ (d) $\left(2+\sqrt{2}, -1\right)$

Ans. (c)

Sol. $f(\theta) = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$

$\Rightarrow (1+\sin\theta\cos\theta) - \cos\theta.(-\sin\theta - \cos\theta) + \left(-\sin^2\theta + 1\right)$

$\Rightarrow f(\theta) = 2 + \sin 2\theta + \cos 2\theta$

$\Rightarrow f(\theta)_{\min} = 2 - \sqrt{2}$

$\Rightarrow f(\theta)_{\max} = 2 + \sqrt{2}$

Example – 37

Find the interval in which

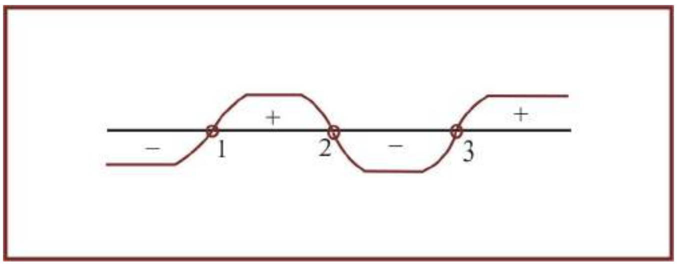
$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$ is increasing.

Sol. Given $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$

$\therefore f'(x) = 4x^3 - 24x^2 + 44x - 24$

$= 4(x^3 - 6x^2 + 11x - 6)$

$= 4(x-1)(x-2)(x-3)$



For increasing function $f'(x) > 0$

or $4(x-1)(x-2)(x-3) > 0$

or $(x-1)(x-2)(x-3) > 0$

$\therefore x \in (1, 2) \cup (3, \infty)$

Example – 38

Find the interval in which $f(x) = x - 2\sin x,\ 0 \le x \le 2\pi$ is increasing

Sol. Given $f(x) = x - 2\sin x$

$\therefore f'(x) = 1 - 2\cos x$

$f'(x) > 0$ or $1 - 2\cos x > 0 \qquad \therefore \cos x < \frac{1}{2}$

or $\quad -\cos x > -\frac{1}{2}$

or $\quad \cos(\pi + x) > \cos\frac{2\pi}{3}$

or $\quad 2n\pi - \frac{2\pi}{3} < \pi + x < 2n\pi + \frac{2\pi}{3},\ n \in I$

or $\quad 2n\pi - \frac{5\pi}{3} < x < 2n\pi - \frac{\pi}{3}$

For $\quad n = 1,\ \frac{\pi}{3} < x < \frac{5\pi}{3}$ which is true $(\because 0 \le x \le 2\pi)$

Hence, $x \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$

Example – 39

Find the intervals of monotonicity of the function $f(x) = \frac{|x-1|}{x^2}$.

Sol. The given function $f(x)$ can be written as :

$$f(x) = \frac{|x-1|}{x^2} = \begin{cases} \frac{1-x}{x^2} & ; \quad x < 1, x \ne 0 \\ \frac{x-1}{x^2} & ; \quad x \ge 1 \end{cases}$$

Consider x < 1

$$f'(x) = \frac{-2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3}$$

For increasing, $f'(x) > 0 \qquad \Rightarrow \frac{x-2}{x^3} > 0$

$\Rightarrow \quad x(x-2) > 0 \qquad$ [as x^2 is positive]

$\Rightarrow \quad x \in (-\infty, 0) \cup (2, \infty)$.

Combining with $x < 1$, we get $f(x)$ is increasing in $x < 0$ and decreasing in $x \in (0, 1)$ ... (i)

Consider x ≥ 1

$$f'(x) = \frac{-1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$$

$$\begin{array}{ccccccc} + & & - & & + & & - \\ \hline & 0 & & 1 & & 2 & \end{array}$$

For increasing $f'(x) > 0$

$\Rightarrow \quad (2-x) > 0 \qquad$ [as x^3 is positive]

$\Rightarrow \quad (x-2) < 0$.

$\Rightarrow \quad x < 2$.

Combining with $x > 1$, $f(x)$ is increasing in $x \in (1, 2)$ and decreasing in $x \in (2, \infty)$ ... (ii)

Combining (i) and (ii), we get :

$f(x)$ is strictly increasing on $x \in (-\infty, 0) \cup (1, 2)$ and strictly decreasing on $x \in (0, 1) \cup (2, \infty)$.

Example – 40

The function $f(x) = \log(x-2)^2 - x^2 + 4x + 1$ increases on the interval

(a) (1, 2) $\qquad$ (b) (2, 3)

(c) (5/2, 3) $\qquad$ (d) (2, 4)

Ans. (b,c)

Sol. $\quad f(x) = 2\log(x-2) - x^2 + 4x + 1$

$$\Rightarrow \quad f'(x) = \frac{2}{x-2} - 2x + 4$$

$$\Rightarrow \quad f'(x) = 2\left[\frac{1-(x-2)^2}{x-2}\right] = -2\frac{(x-1)(x-3)}{x-2}$$

$$\Rightarrow \quad f'(x) = -\frac{2(x-1)(x-3)(x-2)}{(x-2)^2}$$

$\therefore \quad f'(x) > 0 \Rightarrow -2(x-1)(x-3)(x-2) > 0$

$\Rightarrow \quad (x-1)(x-2)(x-3) < 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (2, 3)$.

$$\begin{array}{ccccccccc} & - & & + & & - & & + & \\ \hline -\infty & & 1 & & 2 & & 3 & & \infty \end{array}$$

Example – 41

A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched ?

	Interval	**Function**
(a)	$(-\infty,-4)$	x^3+6x^2+6
(b)	$\left(-\infty,\frac{1}{3}\right]$	$3x^2-2x+1$
(c)	$[2,\infty)$	$2x^3-3x^2-12x+6$
(d)	$(-\infty,\infty)$	x^3-3x^2+3x+3

Ans. (b)

Sol. For function to be increasing, $f'(x)>0$

(a) $f'(x)=3x(x+4)\Rightarrow$ increasing in $(-\infty,-4)\cup(0,\infty)$

(b) $f'(x)=2(3x-1)\Rightarrow$ decreasing in $\left(-\infty,\frac{1}{3}\right)$

(c) $f'(x)=6(x+1)(x-2)\Rightarrow$ increasing in $(-\infty,-1)\cup(2,\infty)$

(d) $f'(x)=3(x-1)^2\Rightarrow$ increasing in $(-\infty,\infty)$

so (b) match is incorrect.

Example – 42

The function $f(x)=\tan^{-1}(\sin x+\cos x)$ is an increasing function in

(a) $\left(0,\frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2},\frac{\pi}{4}\right)$

Ans. (d)

Sol. $f'(x)=\dfrac{1}{1+(\sin x+\cos x)^2}\cdot(\cos x-\sin x)$

$f'(x)=\dfrac{\cos x-\sin x}{2+\sin 2x}$

If $f'(x)>0$ then $f'(x)$ is increasing function

For $-\frac{\pi}{2}<x<\frac{\pi}{4}$, $\cos x>\sin x$

Hence $y=f'(x)$ is increasing in $\left(-\frac{\pi}{2},\frac{\pi}{4}\right)$

Example – 43

The function $f(x)=\cot^{-1}x+x$ increases in the interval

(a) $(1,\infty)$ (b) $(-1,\infty)$

(c) $(-\infty,\infty)$ (d) $(0,\infty)$

Ans. (c)

Sol. $f(x)=\cot^{-1}x+x$

$f'(x)=\dfrac{-1}{1+x^2}+1=\dfrac{x^2}{1+x^2}>0\ \forall x\in R$

Example – 44

A spherical balloon is filled with 4500π cu m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu m/ min, then the rate (in m/min) at which the radius of the balloon decreases 49 min after the leakage began is

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$

(c) $\frac{2}{9}$ (d) $\frac{9}{2}$

Ans. (c)

Sol. $\frac{dv}{dt}=-72\pi m^3/\min, v_0=4500\pi$

$v=\frac{4}{3}\pi r^3\ \therefore\ \frac{dv}{dt}=\frac{4}{3}\pi\times 3r^2\times\frac{dr}{dt}$

After 49 min, $v=v_0+49.\frac{dv}{dt}=4500\pi-49\times 72\pi$

$=4500\pi-3528\pi=972\pi$

$\Rightarrow 972\pi=\frac{4}{3}\pi r^3\Rightarrow r^3=243\times 3=36\Rightarrow r=9$

$\therefore 72\pi=4\pi\times 81\times\frac{dr}{dt}=-\frac{18}{81}=-\frac{2}{9}$

Thus, radius decreases at a rate of $\frac{2}{9}$ m/min

Example – 45

A point on the parabola $y^2=18x$ at which the ordinate increases at twice the rate of the abscissa, is

(a) $(2,4)$ (b) $(2,-4)$

(c) $\left(-\frac{9}{8},\frac{9}{2}\right)$ (d) $\left(\frac{9}{8},\frac{9}{2}\right)$

Ans. (d)

Sol. $y^2 = 18x \Rightarrow 2y\dfrac{dy}{dx} = 18 \Rightarrow \dfrac{dy}{dx} = \dfrac{9}{y}$

Given $\dfrac{dy}{dx} = 2 \Rightarrow \dfrac{9}{y} = 2 \Rightarrow y = \dfrac{9}{2} \Rightarrow x = \dfrac{9}{8}$

Example – 46

If the volume of a spherical ball is increasing at the rate of 4π cc/sec, then the rate of increase of its radius (in cm/sec), when the volume is 288 πcc, is:

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{9}$

(c) $\dfrac{1}{36}$ (d) $\dfrac{1}{24}$

Ans. (c)

Sol. $\dfrac{dV}{dt} = 4\pi cc/sec$

we know $V = \dfrac{4}{3}\pi r^3$

$\dfrac{dV}{dt} = \dfrac{4}{3}\pi 3r^2 \dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt}$

when $V = 288\pi cc$

$\Rightarrow r^3 = 216$

$\Rightarrow r = 6$

$\dfrac{dV}{dt} = 4\pi r^2 . \dfrac{dr}{dt}$

$\Rightarrow 4\pi = 4\pi \times 36 \times \dfrac{dr}{dt}$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{1}{36}$

Example – 47

The period T of a simple pendulum is

$$T = 2\pi\sqrt{\frac{(\ell)}{g}}$$

Find the maximum error in T due to possible errors upto 1% in l and 2.5% in g.

Sol. Since $T = 2\pi\sqrt{\dfrac{(\ell)}{g}} = 2\pi\left(\dfrac{\ell}{g}\right)^{1/2}$

Taking logarithm on both sides, we get

$$\ln T = \ln\ 2\pi + \frac{1}{2}\ \ln l - \frac{1}{2}\ \ln g$$

Differentiating both sides, we get

$$\frac{dT}{T} = 0 + \frac{1}{2}.\frac{dl}{l} - \frac{1}{2}.\frac{dg}{g}$$

or $\left(\dfrac{dT}{T}\times 100\right) = \dfrac{1}{2}\left(\dfrac{dl}{l}\times 100\right) - \dfrac{1}{2}\left(\dfrac{dg}{g}\times 100\right)$

$\left(\dfrac{dT}{T}\times 100\right) = \dfrac{1}{2}\ (1 \pm 2.5)\ \left(\because \dfrac{dl}{l}\times 100 = 1 \text{ and } \dfrac{dg}{g}\times 100 = 2.5\right)$

$\therefore$ Maximum error in T = 1.75%.

Example – 48

If the Rolle's theorem holds for the function $f(x) = 2x^3 + ax^2 + bx$ in the interval [-1, 1] for the point $c = \dfrac{1}{2}$, then the value of 2a + b is

(a) 1 (b) -1

(c) 2 (d) -2

Ans. (b)

Sol. $f(x) = 2x^3 + ax + bx$

Given Rolle's theorem is applicable

$\Rightarrow f(-1) = f(1)$

$\Rightarrow -2 + a - b = 2 + a + b$

$\Rightarrow b = -2$

$f'(x) = 6x^2 + 2ax + b$

$= 6x^2 + 2ax - 2$

$f'\left(\dfrac{1}{2}\right) = 0$

$\Rightarrow a = \dfrac{1}{2}$

$\Rightarrow 2a + b = -1$

Example – 49

Find the equation of the tangent to $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ at the point (x_0, y_0).

Sol. $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ Differentiating wrt x,

$\Rightarrow \quad \frac{mx^{m-1}}{a^m}+\frac{my^{m-1}}{b^m}\frac{dy}{dx}=0$

$\Rightarrow \quad \frac{dy}{dx}=-\frac{b^m}{a^m}\left(\frac{x}{y}\right)^{m-1}$

$\Rightarrow$ at the given point (x_0, y_0), slope of tangent is

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)}=-\left(\frac{b}{a}\right)^m\left(\frac{x_0}{y_0}\right)^{m-1}$$

$\Rightarrow$ the equation of tangent is

$$y-y_0=-\left(\frac{b}{a}\right)^m\left(\frac{x_0}{y_0}\right)^{m-1}(x-x_0)$$

$$a^m yy_0^{m-1}-a^m y_0^m=-b^m x\, x_0^{m-1}+b^m x_0^m$$

$$a^m yy_0^{m-1}+b^m x\, x_0^{m-1}=a^m y_0^m+b^m x_0^m$$

using the equation of given curve, the right side can be replaced by $a^m b^m$.

$\therefore \quad a^m yy_0^{m-1}+b^m x\, x_0^{m-1}=a^m b^m$

$\Rightarrow$ the equation of tangent is

$$\frac{x}{a}\left(\frac{x_0}{a}\right)^{m-1}+\frac{y}{b}\left(\frac{y_0}{b}\right)^{m-1}=1$$

Example – 50

Find the equation of the tangent to $x^3 = ay^2$ at the point A (at^2, at^3). Find also the Point where this tangent meets the curve again.

Sol. Equation of tangent to : $x = at^2$, $y = at^3$ is

$$\frac{dx}{dt}=2at, \frac{dy}{dt}=3at^2$$

$$y-at^3=\frac{3\,at^2}{2\,at}\left(x-at^2\right)$$

$\Rightarrow \quad 2y-2at^3=3tx-3at^3$

i.e. $\quad 3tx-2y-at^3=0$

Let B $\left(at_1^2, at_1^3\right)$ be the point where it again meets the curve.

$\Rightarrow$ slope of tangent at A = slope of AB

$$\frac{3at^2}{2at}=\frac{a\left(t^3-t_1^3\right)}{a\left(t^2-t_1^2\right)}$$

$\Rightarrow \quad \frac{3t}{2}=\frac{t^2+t_1^2+t\,t_1}{t+t_1}$

$\Rightarrow \quad 3t^2+3\,tt_1=2t^2+2t_1^2+2\,t\,t_1$

$\Rightarrow \quad 2t_1^2-t\,t_1-t^2=0$

$\Rightarrow \quad (t_1-t)(2t_1+t)=0$

$\Rightarrow \quad t_1=t$ or $t_1=-t/2$

The relevant value is $t_1=-t/2$

Hence the meeting point B is

$$=\left[a\left(\frac{-t}{2}\right)^2, a\left(\frac{-t}{2}\right)^3\right]=\left[\frac{at^2}{4}, \frac{-at^3}{8}\right]$$

Example – 51

The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point

(a) (a, 0) (b) (0, a)

(c) (0, 0) (d) (a, a)

Ans. (a)

Sol. $\frac{dx}{d\theta}=-a\sin\theta$ and $\frac{dy}{d\theta}=a\cos\theta \Rightarrow \frac{dy}{dx}=-\cot\theta$

$\therefore$ slope of normal at $\theta=\tan\theta$

$\therefore$ the equation of normal at θ is

$y-a\sin\theta=\tan\theta(x-a-a\cos\theta)$

$\Rightarrow x\sin\theta-y\cos\theta=a\sin\theta$

$\Rightarrow y=(x-a)\tan\theta$

which always pasess through (a,0)

Example –52

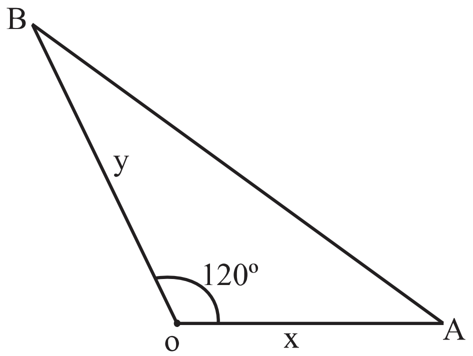
Two ships A and B are sailing straight away from a fixed point O along routes such that $\angle AOB$ is always 120°. At a certain instance, OA = 8 km, OB = 6 km and the ship A is sailing at the rate of 20 km/hr while the ship B sailing at the rate of 30 km/hr. Then the distance between A and B is changing at the rate (in km/hr):

(a) $\frac{260}{\sqrt{37}}$ (b) $\frac{260}{37}$

(c) $\frac{80}{\sqrt{37}}$ (d) $\frac{80}{37}$

Ans. (a)

Sol.



Let OA = x and OB = y

$$\frac{dx}{dt} = 20\ km/hr,\ \frac{dy}{dt} = 30\ km/hr.$$

When OA = 8, OB = 6

Applying cosine formula in ΔAOB.

$$\cos 120° = \frac{x^2 + y^2 - (AB)^2}{2xy} \quad -\frac{1}{8} = \frac{64 + 36 - (AB)^2}{2 \times 8 \times 6}$$

$\Rightarrow$ -48 = 64 + 36 – $(AB)^2$

$\Rightarrow AB = 2\sqrt{37}$

Again applying cosine formula in ΔAOB

When OA = x and OB = y

$$\Rightarrow -\frac{1}{2} = \frac{x^2 + y^2 - (AB)^2}{2xy}$$

$$\Rightarrow (AB)^2 = x^2 + y^2 + xy$$

AB = distance between A and B = Z (let)

$z^2 = x^2 + y^2 + xy$

differentiate w.r.t. “t”

$$2z.\frac{dz}{dt} = 2x.\frac{dx}{dt} + 2y.\frac{dy}{dt} + x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\Rightarrow 2 \times 2\sqrt{37}\ \frac{dz}{dt} = 16 \times 20 + 12 \times 30 + 240 + 120$$

$$\Rightarrow 4\sqrt{37}\ \frac{dz}{dt} = 1040$$

$$\Rightarrow \frac{dz}{dt} = \frac{260}{\sqrt{37}}\ km/hr$$

Example – 53

If 2a + 3b + 6c = 0, a, b, c $\in$ R then show that the equation $ax^2 + bx + c = 0$ has at least one root between 0 and 1.

Sol. Given 2a + 3b + 6c = 0

or $\frac{a}{3} + \frac{b}{2} + c = 0$ .... (i)

Let $f'(x) = ax^2 + bx + c$

On integrating both sides, we get

$$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + k$$

Now, $f(1) = \frac{a}{3} + \frac{b}{2} + c + k$ [From (i)]

$= 0 + k = k$

and $f(0) = 0 + 0 + 0 + k = k$

Since $f(x)$ is a polynomial of three degree, it is continuous and differentiable and $f(0) = f(1)$, then by Rolle’s theorem $f'(x) = 0$ i.e., $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

Example – 54

If $f(x) = (x – 1)(x – 2)(x – 3)$ and a = 0, b = 4., find ‘c’ using Lagrange’s mean value theorem.

Sol. We have $f(x) = (x – 1)(x – 2)(x – 3) = x^3 – 6x^2 + 11x – 6$

$\therefore$ $f(a) = f(0) = (0 – 1)(0 – 2)(x – 3) = – 6$

and $f(b) = f(4) = (4 – 1)(4 – 2)(4 – 3) = 6$

$$\therefore \quad \frac{f(b) - f(a)}{b - a} = \frac{6 - (-6)}{4 - 0} = \frac{12}{4} = 3 \quad .... (1)$$

Also $f'(x) = 3x^2 - 12x + 11$

gives $f'(c) = 3c^2 - 12c + 11$

From LMVT, $\frac{f(b) - f(a)}{b - a} = f'(c)$ .... (2)

$\Rightarrow$ $3 = 3c^2 – 12c + 11$ {From (1) and (2)}

$\Rightarrow$ $3c^2 – 12c + 8 = 0$

$$\therefore \quad c = \frac{12 \pm \sqrt{144 - 96}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$$

As both of these values of c lie in the open interval (0, 4). Hence both of these are required values of c.

Example – 55

A value of c for which conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3], is

(a) $\log_3 e$ (b) $\log_e 3$

(c) $2 \log_3 e$ (d) $\frac{1}{2}\log_e 3$

Ans. (c)

Sol. By LMVT

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3\text{-}1}$$

$$f'(c) = \frac{\log_e 3 - \log_e 1}{2} = \frac{1}{2}\log_e 3$$

$$\Rightarrow \frac{1}{c} = \frac{1}{2}\log_e 3 = \frac{1}{2\log_3 e} \;\therefore\; c = 2\log_3 e$$

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

Derivative as rate Measure

1. Gas is being pumped into a spherical balloon at the rate of 30 ft^3/min. Then, the rate at which the radius increases when it reaches the value 15 ft, is

(a) $\frac{1}{30\pi}$ ft/min (b) $\frac{1}{15\pi}$ ft/min

(c) $\frac{1}{20}$ ft/min (d) $\frac{1}{15}$ ft/min

2. The position of a point in time 't' is given by $x = a + bt - ct^2$, $y = at + bt^2$. Its acceleration at time 't' is

(a) $b - c$ (b) $b + c$

(c) $2b - 2c$ (d) $2\sqrt{b^2 + c^2}$

3. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is

(a) $\frac{1}{18\pi}$ cm/min (b) $\frac{1}{36\pi}$ cm/min

(c) $\frac{5}{6\pi}$ cm/min (d) $\frac{1}{54\pi}$ cm/min

4. The rate of change of the surface area of a sphere of radius r, when the radius is increasing at the rate of 2 cm/s is proportional to

(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$

(c) r (d) r^2

5. For what values of x is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x ?

(a) $-3, -\frac{1}{3}$ (b) $-3, \frac{1}{3}$

(c) $3, -\frac{1}{3}$ (d) $3, \frac{1}{3}$

6. If a particle moving along a line follows the law $s = \sqrt{1+t}$, then the acceleration is proportional to

(a) square of the velocity

(b) cube of the displacement

(c) cube of the velocity

(d) square of the displacement

7. If a particle is moving such that the velocity acquired is proportional to the square root of the distance covered, then its acceleration is

(a) a constant (b) $\propto s^2$

(c) $\propto \frac{1}{s^2}$ (d) $\propto \frac{1}{s}$

Errors and Approximations

8. If $y = x^n$, then the ratio of relative errors in y and x is

(a) 1 : 1 (b) 2 : 1

(c) 1 : n (d) n : 1

9. If the ratio of base radius and height of a cone is 1 : 2 and percentage error in radius is λ %, then the error in its volume is

(a) λ % (b) 2λ%

(c) 3λ% (d) none of these

10. The height of a cylinder is equal to the radius. If an error of α % is made in the height, then percentage error in its volume is

(a) α % (b) 2α%

(c) 3α% (d) none of these

Equation of Tangents and Normals

11. For the curve $y = 3 \sin\theta \cos\theta$, $x = e^{\theta}\sin\theta, 0 \le \theta \le \pi$, the tangent is parallel to x-axis when θ is:

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

12. The curve $y - e^{xy} + x = 0$ has a vertical tangent at

(a) (1, 1) (b) (0, 1)

(c) (1, 0) (d) no point

13. If the line $ax + by + c = 0$ is a tangent to the curve $xy = 4$, then the possible answer is

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$

(c) $a < 0, b > 0$ (d) none of these

14. The tangent to the curve $5x^2 + y^2 = 1$ at $\left(\frac{1}{3}, -\frac{2}{3}\right)$ passes through the point

(a) (0, 0) (b) (1, –1)

(c) (–1, 1) (d) none of these

15. The equation of the tangent to the curve $y = \sqrt{9 - 2x^2}$ at the point where the ordinate and the abscissa are equal, is

(a) $2x + y - 3\sqrt{3} = 0$ (b) $2x + y + 3\sqrt{3} = 0$

(c) $2x - y - 3\sqrt{3} = 0$ (d) none of these

16. The tangent to the curve $x^2 + y^2 = 25$ is parallel to the line $3x - 4y = 7$ at the point

(a) (–3, –4) (b) (3, –4)

(c) (3, 4) (d) none of these

17. If the tangent at each point of the curve $y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$ makes an acute angle with the positive direction of x-axis, then

(a) $a \geq 1$ (b) $-1 \leq a \leq 1$

(c) $a \leq -1$ (d) none of these

18. The equation of the tangent to the curve $(1 + x^2)\, y = 2 - x$, where it crosses the x-axis, is

(a) $x + 5y = 2$ (b) $x - 5y = 2$

(c) $5x - y = 2$ (d) $5x + y - 2 = 0$

19. The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\, dt,\ x \in R$, which are parallel to the line $y = 2x$, are equal to

(a) ± 1 (b) ± 2

(c) ± 3 (d) ± 4

Length of tangent, normal, subtangent and subnormal

20. The length of subtangent to the curve $x^2 y^2 = a^4$ at the point (–a, a) is

(a) 3a (b) 2a

(c) a (d) 4a

21. For the parabola $y^2 = 4ax$, the ratio of the sub-tangent to the abscissa is

(a) 1 : 1 (b) 2 : 1

(c) 1 : 2 (d) 3 : 1

22. The length of subtangent to the curve $x^2y^2 = a^4$ at the point (–a, a) is

(a) 3a (b) 2a

(c) a (d) 4a

23. The product of the lengths of subtangent and subnormal at any point of a curve is

(a) square of the abscissae (b) square of the ordinate

(c) constant (d) None of these

Angle of intersection between the curves

24. The curves $x^3 + p\,xy^2 = -2$ and $3x^2y - y^3 = 2$ are orthogonal for

(a) $p = 3$ (b) $p = -3$

(c) no value of p (d) $p = \pm 3$

25. The two tangents to the curve $ax^2 + 2hxy + by^2 = 1,\ a > 0$ at the points where it crosses x-axis, are

(a) parallel (b) perpendicular

(c) inclined at an angle $\frac{\pi}{4}$ (d) none of these

26. The lines $y = -\frac{3}{2}x$ and $y = -\frac{2}{5}x$ intersect the curve $3x^2 + 4xy + 5y^2 - 4 = 0$ at the points P and Q respectively. The tangents drawn to the curve at P and Q

(a) intersect each other at angle of 45°

(b) are parallel to each other

(c) are perpendicular to each other

(d) none of these

27. The angle between the curves $y = \sin x$ and $y = \cos x$ is

(a) $\tan^{-1}(2\sqrt{2})$ (b) $\tan^{-1}(3\sqrt{2})$

(c) $\tan^{-1}(3\sqrt{3})$ (d) $\tan^{-1}(5\sqrt{2})$

28. The angle between the tangents to the curve $y^2 = 2ax$ at the points where $x = \frac{a}{2}$, is

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

29. The angle between the tangents at those points on the curve $y = (x + 1)(x - 3)$ where it meets x-axis, is

(a) $\tan^{-1}\left(\frac{15}{8}\right)$ (b) $\tan^{-1}\left(\frac{8}{15}\right)$

(c) $\frac{\pi}{4}$ (d) none of these

30. The angle at which the curves $y = \sin x$ and $y = \cos x$ intersect in $[0, \pi]$, is

(a) $\tan^{-1} 2\sqrt{2}$ (b) $\tan^{-1}\sqrt{2}$

(c) $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (d) none of these

31. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$

(a) cut at right angles (b) touch each other

(c) cut at an angle $\frac{\pi}{3}$ (d) cut at an angle $\frac{\pi}{4}$

32. The two curves $y = 3^x$ and $y = 5^x$ intersect at an angle

(a) $\tan^{-1}\left(\frac{\log 5 - \log 3}{1 + \log 3.\log 5}\right)$ (b) $\tan^{-1}\left(\frac{\log 3 + \log 5}{1 - \log 3.\log 5}\right)$

(c) $\tan^{-1}\left(\frac{\log 3 + \log 5}{1 + \log 3.\log 5}\right)$ (d) none of these

33. The angle of intersection of the curve $y = x^2$ & $6y = 7 - x^3$ at $(1, 1)$ is

(a) $\pi/5$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

Increasing and Decreasing Functions

34. The function $f(x) = 2x^2 - \log|x|$ monotonically decreases for

(a) $x \in (-\infty, -1/2] \cup (0, 1/2]$

(b) $x \in (-\infty, 1/2]$

(c) $x \in [-1/2, 0) \cup [1/2, \infty)$

(d) none of these

35. The interval in which the function x^3 increases less rapidly than $6x^2 + 15x + 5$ is :

(a) $(-\infty, -1)$ (b) $(-5, 1)$

(c) $(-1, 5)$ (d) $(5, \infty)$

36. The function $y = \frac{2x^2 - 1}{x^4}$ is

(a) a decreasing function for all $x \in R - \{0\}$

(b) a increasing function for all $x \in R - \{0\}$

(c) increasing for $x > 0$

(d) none of these

37. The function $f(x) = \frac{\sin x}{x}$ is decreasing in the interval

(a) $\left(-\frac{\pi}{2}, 0\right)$ (b) $\left(0, \frac{\pi}{2}\right)$

(c) $(0, \pi)$ (d) none of these

38. If $f(x) = \frac{1}{x+1} - \log(1 + x)$, $x > 0$, then f is

(a) an increasing function

(b) a decreasing function

(c) both increasing and decreasing function

(d) None of the above

39. Let $f(x) = \int_1^x e^x (x - 1)(x - 2)\,dx$. Then, f decreases in the interval

(a) $(-\infty, 2)$ (b) $(-2, -1)$

(c) $(1, 2)$ (d) $(2, \infty)$

40. If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a strictly decreasing function of x in the largest possible interval $[-2, -2/3]$ then

(a) $\lambda = 4$ (b) $\lambda = 2$

(c) $\lambda = -1$ (d) λ has no real value

41. The length of the longest interval, in which the function $3 \sin x - 4 \sin^3 x$ is increasing, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) π

42. The function $f(x) = x + \cos x$ is

(a) always increasing

(b) always decreasing

(c) increasing for certain range of x

(d) None of the above

43. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have ?

(a) 5 (b) 7

(c) 1 (d) 3

Maxima and minima

44. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$ has

(a) no maxima and minima

(b) one maxima and one minima

(c) two maxima

(d) two minima

45. The greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on [0, 1] is :

(a) 1 (b) 2

(c) 3 (d) $2^{1/3}$

46. The function $f(x) = x^2(x-2)^2$

(a) decreases on $(0, 1) \cup (2, \infty)$

(b) increase on $(-\infty, 0) \cup (1, 2)$

(c) has a local maximum value 0

(d) has a local maximum value 1

47. The maximum value of the function $y = x(x-1)^2, 0 \le x \le 2$ is

(a) 0 (b) 4/27

(c) –4 (d) none of these

48. The point in the interval $[0, 2\pi]$, where $f(x) = e^x \sin x$ has maximum slope, is

(a) 0 (b) $\frac{\pi}{2}$

(c) 2π (d) $\frac{3\pi}{2}$

49. The minimum value of x^x is attained (where x is positive real number) when x is equal to :

(a) e (b) e^{-1}

(c) 1 (d) e^2

50. The maximum value of $x^3 - 3x$ in the intveral [0, 2], is

(a) –2 (b) 0

(c) 2 (d) None of these

51. If $A > 0, B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{3}$

(c) 3 (d) $\sqrt{3}$

52. The maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12

(c) 16 (d) 32

53. The function $f(x) = \int_1^x \{2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\} dt$ attains its local maximum value at x =

(a) 1 (b) 2

(c) 3 (d) 4

54. The maximum area of the rectangle that can be inscribed in a circle of radius r, is

(a) πr^2 (b) r^2

(c) $\pi r^2/4$ (d) $2r^2$

55. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is

(a) $\frac{3}{2}x^2$ (b) $\sqrt{\frac{x^3}{8}}$

(c) $\frac{1}{2}x^2$ (d) πx^2

56. The greatest and the least value of the function,

$f(x) = \sqrt{1-2x+x^2} - \sqrt{1+2x+x^2}, x \in (-\infty, \infty)$ are

(a) 2, –2 (b) 2, –1

(c) 2, 0 (d) none

57. The function $f(x) = 2x^3 - 15x^2 + 36x + 4$ has local maxima at

(a) $x = 2$ (b) $x = 4$

(c) $x = 0$ (d) $x = 3$

58. The maximum value of xy subject to $x + y = 8$, is

(a) 8 (b) 16

(c) 20 (d) 24

59. Let $f(x) = (1+b^2)x^2 + 2bx + 1$ and m (b) the minimum value of $f(x)$ for a given b. As b varies, the range of m (b) is

(a) [0, 1] (b) (0, 1/2]

(c) $\left[\frac{1}{2}, 1\right]$ (d) (0, 1]

60. $f(x) = 1 + [\cos x]\, x$, in $0 \le x \le \frac{\pi}{2}$

(a) has a minimum value 0

(b) has a maximum value 2

(c) is continuous in $\left[0, \frac{\pi}{2}\right]$

(d) is not differentiable at $x = \frac{\pi}{2}$

61. The minimum value of $2^{(x^2-3)^3+27}$, is

(a) 2^{27} (b) 2

(c) 1 (d) 4

62. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(a) ab (b) 2 ab

(c) a/b (d) $\sqrt{ab}$

63. The difference between the greatest and least values of the function, $f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$ is :

(a) 4/3 (b) 1

(c) 9/4 (d) 1/6

64. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a Δ OPQ, where O is the origin, if the area of the Δ OPQ is least, then the slope of the line PQ is

(a) $-\frac{1}{4}$ (b) -4

(c) -2 (d) $-\frac{1}{2}$

Numerical Value Type Questions

65. The radius of the base of a cone is increasing at the rate of 3 cm/minute and the altitude is decreasing at the rate of 4 cm/minute. The rate of change of lateral surface when the radius = 7 cm and altitude = 24 cm, in cm^2/min is:

66. A ladder 10 metres long rests with one end against a vertical wall, the other end on the floor. The lower end moves away from the wall at the rate of 2 metres/minute. The rate at which the upper end falls when its base is 6 metres away from the wall, in M/min is :

67. If the distance 's' metres travelled by a particle in t seconds is given by $s = t^3 - 3t^2$, then the velocity of the particle when the acceleration is zero in m/s is

68. An object is moving in the clockwise direction around the unit circle $x^2 + y^2 = 1$. As it passes through the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, its y-coordinate is decreasing at the rate of 3 unit per second. The rate at which the x-coordinate changes at this point is (in unit per second)

69. If $V = \frac{4}{3}\pi r^3$, at what rate in cubic units is V increasing when $r = 10$ and $\frac{dr}{dt} = 0.01$?

70. Side of an equilateral triangle expands at the rate of 2 cm/s. The rate of increase of its area when each side is 10 cm, in cm^2/sec is:

71. The radius of a sphere is changing at the rate of 0.1 cm/s. The rate of change of its surface area when the radius is 200 cm, in cm^2/sec is:

72. The surface area of a sphere when its volume is increasing at the same rate as its radius, in sq. unit is :

73. The surface area of a cube is increasing at the rate of 2 cm^2/s. When its edge is 90 cm, the volume is increasing at the rate of (in cm^3/sec)

74. The sides of an equilateral triangle are increasing at the rate of 2 cm/s. The rate at which the area increases, when the side is 10 cm, in cm^2/s is:

75. The distance moved by the particle in time t is given by $x = t^3 - 12t^2 + 6t + 8$. At the instant when its acceleration is zero, the velocity is

76. The circumference of a circle is measured as 28 cm with an error of 0.01 cm. The percentage error in the area is

77. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is

78. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3)$ is equal to

79. Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

80. If $f(x)$ is differentiable in the interval [2, 5], where $f(2) = \frac{1}{5}$ and $f(5) = \frac{1}{2}$, then there exists a number c, $2 < c < 5$ for which $f'(c)$ is equal to

EXERCISE - 2 : PREVIOUS YEAR JEE MAINS QUESTIONS

1. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1): **(2015)**

(a) meets the curve again in the third quadrant.

(b) meets the curve again in the fourth quadrant.

(c) does not meet the curve again.

(d) meets the curve again in the second quadrant.

2. Let f(x) be a polynomial of degree four having extreme value at x = 1 and x = 2. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right] = 3$, then f(2) is equal to: **(2015)**

(a) 0 (b) 4

(c) –8 (d) –4

3. If Rolle's theorem holds for the function $f(x) = 2x^3 + bx^2 + cx$, $x \in [-1, 1]$, at the point $x = \frac{1}{2}$, then 2b + c equals : **(2015/Online Set–1)**

(a) 1 (b) 2

(c) –1 (d) –3

4. Let k and K be the minimum and the maximum values of the function $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$ in [0, 1] respectively, then the ordered pair (k, K) is equal to: **(2015/Online Set–2)**

(a) $(2^{-0.4}, 1)$ (b) $(2^{-0.4}, 2^{0.6})$

(c) $(2^{-0.6}, 1)$ (d) $(1, 2^{0.6})$

5. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side =x unit and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then : **(2016)**

(a) $(4 - \pi)x = \pi r$ (b) $x = 2r$

(c) $2x = r$ (d) $2x = (\pi + 4)r$

6. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right)$.

A normal to y = f (x) at $x = \frac{\pi}{6}$ also passes through the point : **(2016)**

(a) $\left(0, \frac{2\pi}{3}\right)$ (b) $\left(\frac{\pi}{6}, 0\right)$

(c) $\left(\frac{\pi}{4}, 0\right)$ (d) (0, 0)

7. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in R$, meets the curve again at a point Q, then the coordinates of Q are : **(2016/Online Set–1)**

(a) $(t^2 + 3, -t^3 - 1)$ (b) $(4t^2 + 3, -8t^3 - 1)$

(c) $(t^2 + 3, t^3 - 1)$ (d) $(16t^2 + 3, -64t^3 - 1)$

8. The minimum distance of a point on the curve $y = x^2 - 4$ from the origin is : **(2016/Online Set–1)**

(a) $\frac{\sqrt{19}}{2}$ (b) $\sqrt{\frac{15}{2}}$

(c) $\frac{\sqrt{15}}{2}$ (d) $\sqrt{\frac{19}{2}}$

9. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point: **(2017)**

(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{2}, -\frac{1}{3}\right)$ (d) $\left(\frac{1}{2}, \frac{1}{3}\right)$

10. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is: **(2017)**

(a) 12.5 (b) 10

(c) 25 (d) 30

11. The tangent at the point (2,–2) to the curve, $x^2y^2 - 2x = 4(1 - y)$ does not pass through the point : **(2017/Online Set–1)**

(a) $\left(4, \frac{1}{3}\right)$ (b) (8, 5)

(c) (–4, –9) (d) (–2, –7)

12. The function f defined by $f(x) = x^3 - 3x^2 + 5x + 7$, is **(2017/Online Set–2)**

(a) increasing in R.

(b) decreasing in R.

(c) decreasing in $(0, \infty)$ and increasing in $(-\infty, 0)$.

(d) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$.

13. If the curves $y^2 = 6x, 9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is **(2018)**

(a) $\frac{9}{2}$ (b) 6

(c) $\frac{7}{2}$ (d) 4

14. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$,

$x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is : **(2018)**

(a) $2\sqrt{2}$ (b) 3

(c) -3 (d) $-2\sqrt{2}$

15. If a right circular cone, having maximum volume, is inscribed in a sphere of radius 3 cm, then the curved surface area (in cm²) of this cone is : **(2018/Online Set–1)**

(a) $6\sqrt{2}\pi$ (b) $6\sqrt{3}\pi$

(c) $8\sqrt{2}\pi$ (d) $8\sqrt{3}\pi$

16. Let f(x) be a polynomial of degree 4 having extreme values at x = 1 and x = 2. If $\lim_{x \to 0}\left(\frac{f(x)}{x^2} + 1\right) = 3$ then f(-1) is equal to: **(2018/Online Set–2)**

(a) $\frac{9}{2}$ (b) $\frac{5}{2}$

(c) $\frac{3}{2}$ (d) $\frac{1}{2}$

17. Let M and m be respectively the absolute maximum and the absolute minimum values of the function, $f(x) = 2x^3 - 9x^2 + 12x + 5$ in the interval [0, 3]. Then M-m is equal to : **(2018/Online Set–3)**

(a) 5 (b) 9

(c) 4 (d) 1

18. The shortest distance between the line y = x and the curve $y^2 = x - 2$ is: **(2019-04-08/Shift-1)**

(a) 2 (b) $\frac{7}{8}$

(c) $\frac{7}{4\sqrt{2}}$ (d) $\frac{11}{4\sqrt{2}}$

19. If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25, x \in R$ then **(2019-04-08/Shift-1)**

(a) $S_1 = \{-2\}; S_2 = \{0, 1\}$ (b) $S_1 = \{-2, 0\}; S_2 = \{1\}$

(c) $S_1 = \{-2, 1\}; S_2 = \{0\}$ (d) $S_1 = \{-1\}; S_2 = \{0, 2\}$

20. Let $f : [0, 2] \to R$ be a twice differentiable function such that $f''(x) > 0$, for all $x \in (0, 2)$. If $\phi(x) = f(x) + f(2 - x)$, then $\phi(x)$ is : **(2019-04-08/Shift-1)**

(a) increasing on (0, 1) and decreasing on (1, 2).

(b) decreasing on (0, 2)

(c) decreasing on (0, 1) and increasing on (1, 2).

(d) increasing on (0, 2)

21. The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 is : **(2019-04-08/Shift-2)**

(a) $\sqrt{6}$ (b) $\frac{2}{3}\sqrt{3}$

(c) $2\sqrt{3}$ (d) $\sqrt{3}$

22. If the tangent to the curve, $y = x^3 + ax - b$ at the point (1, -5) is perpendicular to the line, $-x + y + 4 = 0$, then which one of the following points lies on the curve? **(2019-04-09/Shift-1)**

(a) (-2, 1) (b) (-2, 2)

(c) (2, -1) (d) (2, -2)

23. Let S be the set of all values of x for which the tangent to the curve $y = f(x) = x^3 - x^2 - 2x$ at (x, y) is parallel to the line segment joining the points $(1, f(1))$ and $(-1, f(-1))$ then S is equal to: **(2019-04-09/Shift-1)**

(a) $\left\{\frac{1}{3}, 1\right\}$ (b) $\left\{-\frac{1}{3}, -1\right\}$

(c) $\left\{\frac{1}{3}, -1\right\}$ (d) $\left\{-\frac{1}{3}, 1\right\}$

24. If $f(x)$ is a non-zero polynomial of degree four, having local extreme points at $x = -1, 0, 1$ then the set $S = \{x \in R : f(x) = f(0)\}$ contains exactly k real values, then k is **(2019-04-09/Shift-1)**

25. A water tank has the shape of an inverted right circular cone, whose semi-vertical angle is $\tan^{-1}\left(\frac{1}{2}\right)$. Water is poured into it at a constant rate of 5 cubic meter per minute. Then the rate (in m/min.), at which the level of water is rising at the instant when the depth of water in the tank is 10m; is: **(2019-04-09/Shift-2)**

(a) $\frac{1}{15\pi}$ (b) $\frac{1}{10\pi}$

(c) $\frac{2}{\pi}$ (d) $\frac{1}{5\pi}$

26. Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $\forall x \in R$. Then the set of all $x \in R$, where the function $h(x) = (fog)(x)$ is increasing, is: **(2019-04-10/Shift-1)**

(a) $\left[-1, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$ (b) $\left[0, \frac{1}{2}\right] \cup [1, \infty)$

(c) $[0, \infty)$ (d) $\left[-\frac{1}{2}, 0\right] \cup [1, \infty)$

27. If the tangent to the curve $y = \frac{x}{x^2 - 3}, x \in R\left(x \neq \pm\sqrt{3}\right)$, at a point $(\alpha, \beta) \neq (0, 0)$ on it is parallel to the line $2x + 6y - 11 = 0$ then: **(2019-04-10/Shift-2)**

(a) $|6\alpha + 2\beta| = 19$ (b) $|6\alpha + 2\beta| = 9$

(c) $|2\alpha + 6\beta| = 19$ (d) $|2\alpha + 6\beta| = 9$

28. A spherical iron ball of radius 10 cm is coated with a layer of ice of uniform thickness that melts at a rate of $50\text{cm}^3/\text{min}$ When the thickness of the ice is 5 cm, then the rate at which the thickness (in cm/min) of the ice decreases, is: **(2019-04-10/Shift-2)**

(a) $\frac{1}{18\pi}$ (b) $\frac{1}{36\pi}$

(c) $\frac{5}{6\pi}$ (d) $\frac{1}{9\pi}$

29. Let $a_1, a_2, a_3, \ldots$ be an A.P. with $a_6 = 2$ then the common difference of this A.P., which maximises the product $a_1.a_4.a_5$ is: **(2019-04-10/Shift-2)**

(a) $\frac{3}{2}$ (b) $\frac{8}{5}$

(c) $\frac{6}{5}$ (d) $\frac{2}{3}$

30. A 2 m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate 25 cm/sec., then the rate (in cm / sec.) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is: **(2019-04-12/Shift-1)**

(a) $25\sqrt{3}$ (b) $\frac{25}{\sqrt{3}}$

(c) $\frac{25}{3}$ (d) 25

31. The maximum volume (in cu.m) of the right circular cone having slant height 3 m is: (2019-01-09/Shift-1)

(a) 6π (b) $3\sqrt{3}\pi$

(c) $\frac{4}{3}\pi$ (d) $2\sqrt{3}\pi$

32. If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan\theta|$ is equal to: (2019-01-09/Shift-1)

(a) $\frac{4}{9}$ (b) $\frac{8}{15}$

(c) $\frac{7}{17}$ (d) $\frac{8}{17}$

33. The shortest distance between the point $\left(\frac{3}{2},0\right)$ and the curve $y=\sqrt{x},(x>0)$, is: (2019-01-10/Shift-1)

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{3}{2}$ (d) $\frac{5}{4}$

34. The tangent to the curve, $y = xe^{x^2}$ passing through the point (1, e) also passes through the point: (2019-01-10/Shift-2)

(a) $(2, 3e)$ (b) $\left(\frac{4}{3},2e\right)$

(c) $\left(\frac{5}{3},2e\right)$ (d) $(3, 6e)$

35. A helicopter is flying along the curve given by $y-x^{3/2}=7,\ (x\geq 0)$. A soldier positioned at the point $\left(\frac{1}{2},7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is: (2019-01-10/Shift-2)

(a) $\frac{\sqrt{5}}{6}$ (b) $\frac{1}{3}\sqrt{\frac{7}{3}}$

(c) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (d) $\frac{1}{2}$

36. The maximum value of the function

$f(x)=3x^3-18x^2+27x-40$ on the set

$S=\{x\in R: x^2+30\leq 11x\}$ is : (2019-01-11/Shift-1)

37. Let $f(x)=\frac{x}{\sqrt{a^2+x^2}}-\frac{d-x}{\sqrt{b^2+(d-x)^2}},\ x\in\mathbb{R}$, where a, b and d are non-zero real constants. Then: (2019-01-11/Shift-2)

(a) f is an increasing function of x

(b) f is a decreasing function of x

(c) f' is not a continuous function of x

(d) f is neither increasing nor decreasing function of x

38. If the function f given by

$f(x)=x^3-3(a-2)x^2+3ax+7,\ f(0)=7$ for some $a\in R$ is increasing in $(0,1]$ and decreasing in $[1,5)$, then a root of the equation, $\frac{f(x)-14}{(x-1)^2}=0\ (x\neq 1)$ (2019-01-12/Shift-2)

(a) -7 (b) 5

(c) 7 (d) 6

39. Let P (h, k) be a point on the curve $y = x^2 + 7x + 2$, nearest to the line, $y = 3x - 3$. Then the equation of the normal to the curve at P is : (2020-09-02/Shift-1)

(a) $x + 3y - 62 = 0$ (b) $x - 3y - 11 = 0$

(c) $x - 3y + 22 = 0$ (d) $x + 3y + 26 = 0$

40. If p(x) be a polynomial of degree three that has a local maximum value 8 at $x = 1$ and a local minimum value 4 at $x = 2$; then p (0) is equal to : (2020-09-02/Shift-1)

(a) 12 (b) – 12

(c) –24 (d) 6

41. If the tangent to the curve $y = x + \sin y$ at a point *(a, b)* is parallel to the line joining $\left(0,\frac{3}{2}\right)$ and $\left(\frac{1}{2},2\right)$, then : (2020-09-02/Shift-1)

(a) $b=\frac{\pi}{2}+a$ (b) $|a+b|=1$

(c) $|b-a|=1$ (d) $b=a$

42. The equation of the normal to the curve

$y=(1+x)^{2y}+\cos^2(\sin^{-1}x)$ at $x=0$ is : **(2020-09-02/Shift-2)**

(a) $y+4x=2$ (b) $2y+x=4$

(c) $x+4y=8$ (d) $y=4x+2$

43. Let $f:(-1,\infty)\to R$ be defined by $f(0)=1$ and $f(x)=\frac{1}{x}\log_e(1+x), x\neq 0$ .Then the functionf : **(2020-09-02/Shift-2)**

(a) increases in $(-1,\infty)$

(b) decreases in $(-1, 0)$ and increases in $(0,\infty)$

(c) increases in $(-1, 0)$ and decreases in $(0,\infty)$

(d) decreases in $(-1,\infty)$.

44. The function, $f(x)=(3x-7)x^{2/3}, x\in R$ is increasing for all x lying in : **(2020-09-03/Shift-1)**

(a) $\left(-\infty,-\frac{14}{15}\right)\cup(0,\infty)$

(b) $\left(-\infty,\frac{14}{15}\right)$

(c) $(-\infty,0)\cup\left(\frac{14}{15},\infty\right)$

(d) $(-\infty,0)\cup\left(\frac{3}{7},\infty\right)$

45. Suppose $f(x)$ is a polynomial of degree four, having critical points at $-1, 0, 1$. If $T=\{x\in R\mid f(x)=f(0)\}$, then the sum of squares of all the element of T is : **(2020-09-03/Shift-2)**

(a) 6 (b) 2

(c) 8 (d) 4

46. If the surface area of a cube is increasing at a rate of 3.6 cm^2/sec, retaining its shape; then the rate of change of its volume (in cm^3/sec), when the length of a side of the cube is 10cm, is : **(2020-09-03/Shift-2)**

(a) 9 (b) 10

(c) 18 (d) 20

47. The area (in sq. units) of the largest rectangle ABCD whose vertices A and B lie on the x-axis and vertices C and D lie on the parabola, $y=x^2-1$ below the x-axis, is: **(2020-09-04/Shift-2)**

(a) $\frac{2}{3\sqrt{3}}$ (b) $\frac{4}{3}$

(c) $\frac{1}{3\sqrt{3}}$ (d) $\frac{4}{3\sqrt{3}}$

48. If $x=1$ is a critical point of the function $f(x)=(3x^2+ax-2-a)e^x$, then: **(2020-09-05/Shift-2)**

(a) $x=1$ is a local minima and $x=-\frac{2}{3}$ is a local maxima of f.

(b) $x=1$ is a local maxima and $x=-\frac{2}{3}$ is a local minima of f.

(c) $x=1$ and $x=-\frac{2}{3}$ are local minima of f.

(d) $x=1$ and $x=-\frac{2}{3}$ are local maxima of f.

49. Which of the following points lies on the tangent to the curve $x^4e^y+2\sqrt{y+1}=3$ at the point (1,0)? **(2020-09-05/Shift-2)**

(a) (2,6) (b) (2,2)

(c) (–2,6) (d) (–2,4)

50. The position of a moving car at time t is given by $f(t)=at^2+bt+c, t>0$, where a, b and c are real numbers greater than 1. Then the average speed of the car over the time interval $[t_1,t_2]$ is attained at the point: **(2020-09-06/Shift-1)**

(a) $[t_1+t_2]/2$ (b) $2a(t_1+t_2)+b$

(c) $(t_2-t_1)/2$ (d) $a(t_2-t_1)+b$

51. Let AD and BC be two vertical poles at A and B respectively on a horizontal ground. If AD = 8m, BC = 11m and AB = 10m; then the distance (in meters) of a point M on AB from the point A such that MD^2+MC^2 is minimum is _____. **(2020-09-06/Shift-1)**

52. The set of all real values of λ for which the function

$f(x)=(1-\cos^2 x).(\lambda+\sin x), x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

has exactly one maxima and exactly one minima, is:

(2020-09-06/Shift-2)

(a) $\left(-\frac{3}{2},\frac{3}{2}\right)-\{0\}$ (b) $\left(-\frac{1}{2},\frac{1}{2}\right)-\{0\}$

(c) $\left(-\frac{3}{2},\frac{3}{2}\right)$ (d) $\left(-\frac{1}{2},\frac{1}{2}\right)$

53. If the tangent to the curve, $y=f(x)=x\log_e x,(x>0)$ at a point (c,f(c)) is parallel to the line-segment joining the points (1, 0) and (e, e), then c is equal to :

(2020-09-06/Shift-2)

(a) $e^{\left(\frac{1}{1-e}\right)}$ (b) $\frac{e-1}{e}$

(c) $\frac{1}{e-1}$ (d) $e^{\left(\frac{1}{e-1}\right)}$

54. For all twice differentiable functions f : R → R, with $f(0)=f(1)=f'(0)=0$, **(2020-09-06/Shift-2)**

(a) $f''(x)=0$, at every point $x\in(0,1)$

(b) $f''(x)\neq 0$, at every point $x\in(0,1)$

(c) $f''(x)=0$, for some $x\in(0,1)$

(d) $f''(0)=0$

55. Let the function, f: [-7, 0] → R be continuous on [-7,0] and differentiable on (-7,0). If f (-7) =-3 and $f'(x)\le 2$ for all $x\in(-7, 0)$, then for all such functions f, $f(-1)+f(0)$ lies in the interval: **(7-1-2020/Shift-1)**

(a) [-6, 20] (b) (-∞, 20]

(c) (-∞,11] (d) [-3,11]

56. Let $f(x)$ be a polynomial of degree 5 such that $x=\pm 1$ are its critical points. If $\lim_{x\to 0}\left(2+\frac{f(x)}{x^3}\right)=4$, then which one of the following is not true? **(7-1-2020/Shift-2)**

(a) $f(1)-4f(-1)=4$

(b) $x = 1$ is a point of maxima and $x = -1$ is a point of minimum of f.

(c) f is an odd function.

(d) $x = 1$ is a point of minima and $x = -1$ is a point of maxima of f.

57. The value of c in Lagrange's mean value theorem for the function $f(x)=x^3-4x^2+8x+11$, where $x\in[0,1]$ is :

(7-1-2020/Shift-2)

(a) $\frac{4-\sqrt{7}}{3}$ (b) $\frac{2}{3}$

(c) $\frac{\sqrt{7}-2}{3}$ (d) $\frac{4-\sqrt{5}}{3}$

58. If c is a point at which Rolle's theorem holds for the function, $f(x)=\log_e\left(\frac{x^2+\alpha}{7x}\right)$ in the interval [3, 4], where $\alpha\in R$ then $f''(c)$ is equal to:

(8-1-2020/Shift-1)

(a) $-\frac{1}{24}$ (b) $\frac{-1}{12}$

(c) $\frac{\sqrt{3}}{7}$ (d) $\frac{1}{12}$

59. Let $f(x)=x\cos^{-1}(\sin(-|x|)), x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then which of the following is true? **(8-1-2020/Shift-1)**

(a) $f'(0)=-\frac{\pi}{2}$

(b) f' is decreasing in $\left(-\frac{\pi}{2},0\right)$ and increasing in $\left(0,\frac{\pi}{2}\right)$

(c) f is not differentiable at x = 0

(d) f' is increasing in $\left(-\frac{\pi}{2},0\right)$ and decreasing in $\left(0,\frac{\pi}{2}\right)$

60. Let the normal at a P on the curve $y^2-3x^2+y+10=0$ intersect the y-axis at $\left(0,\frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then |m| is equal to _____.

(8-1-2020/Shift-1)

61. The length of the perpendicular from the origin, on the normal to the curve, $x^2+2xy-3y^2=0$ at the point (2,2) is: **(8-1-2020/Shift-2)**

(a) 2 (b) $2\sqrt{2}$

(c) $4\sqrt{2}$ (d) $\sqrt{2}$

62. Let $f(x)$ be a polynomial of degree 3 such that $f(-1)=10$, $f(1)=-6$, $f(x)$ has a critical point at $x=-1$ and $f'(x)$ has a critical point at $x=1$. Then the local minima at $x=$______ **(8-1-2020/Shift-2)**

63. A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness that melts at the rate of 50 cm³/min. When the thickness of ice is 5 cm, then the rate (in cm/min.) at which the thickness of ice decreases, is: **(9-1-2020/Shift-1)**

(a) $\frac{5}{6\pi}$ (b) $\frac{1}{54\pi}$

(c) $\frac{1}{36\pi}$ (d) $\frac{1}{19\pi}$

64. Let f be any function continuous on [a.b] and twice differentiable on (a,b). If for all $x\in(a,b), f'(x)>0$ and $f''(x)<0$, then for any $c\in(a,b), \frac{f(c)-f(a)}{f(b)-f(c)}$ is greater than: **(9-1-2020/Shift-1)**

(a) $\frac{b-c}{c-a}$ (b) 1

(c) $\frac{c-a}{b-c}$ (d) $\frac{b+a}{b-a}$

65. Let a function $f:[0,5]\to R$, be continuous, $f(1)=3$ and F be defined as: $F(x)=\int_1^x t^2 g(t)\,dt$, where $g(t)=\int_1^t f(u)\,du$. Then for the function F, the point $x=1$ is **(9-1-2020/Shift-2)**

(a) a point of inflection (b) a point of local maxima

(c) a point of local minima (d) not a critical point

66. The function $f(x)=\frac{4x^3-3x^2}{6}-2\sin x+(2x-1)\cos x$ **(24-02-2021/Shift-1)**

(a) Decreases in $\left(-\infty,\frac{1}{2}\right]$ (b) Increases in $\left(-\infty,\frac{1}{2}\right]$

(c) Increases in $\left[\frac{1}{2},\infty\right)$ (d) Decreases in $\left[\frac{1}{2},\infty\right)$

67. The minimum value of α for which the equation $\frac{4}{\sin x}+\frac{1}{1-\sin x}=\alpha$ has at least one solution in $\left(0,\frac{\pi}{2}\right)$ is ________. **(24-02-2021/Shift-1)**

68. Let $f:R\to R$ be defined as

$$f(x)=\begin{cases}-55x, & \text{if } x<-5\\ 2x^3-3x^2-120x, & \text{if } -5\le x\le 4\\ 2x^3-3x^2-36x-336, & \text{if } x>4,\end{cases}$$

Let $A=\{x\in R: f \text{ is increasing}\}$. Then A is equal to **(24-02-2021/Shift-2)**

(a) $(-\infty,-5)\cup(4,\infty)$ (b) $(-5,\infty)$

(c) $(-5,-4)\cup(4,\infty)$ (d) $(-\infty,-5)\cup(-4,\infty)$

69. If P Is a point on the parabola $y=x^2+4$ which is closest to the straight line $y=4x-1$, then the co-ordinates of P are **(24-02-2021/Shift-2)**

(a) (3,13) (b) (2,8)

(c) (-2,8) (d) (1,5)

70. If the curve $y=ax^2+bx+c, x\in R$, passes through the point (1, 2) and the tangent line to this curve at origin is $y=x$, then the possible values of a, b, c are : **(24-02-2021/Shift-2)**

(a) $a=1, b=1, c=0$ (b) $a=\frac{1}{2}, b=\frac{1}{2}, c=1$

(c) $a=-1, b=1, c=1$ (d) $a=1, b=0, c=1$

71. If the curves, $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{c}+\frac{y^2}{d}=1$ intersect each other at an angle of 90°, then which of the following relations is TRUE? **(25-02-2021/Shift-1)**

(a) $ab=\frac{c+d}{a+b}$ (b) $a-c=b+d$

(c) $a+b=c+d$ (d) $a-b=c-d$

72. If Rolle's theorem holds for the function $f(x)=x^3-ax^2+bx-4,\ x\in[1,2]$ with $f'\left(\frac{4}{3}\right)=0$, then ordered pair (a,b) is equal to: **(25-02-2021/Shift-1)**

(a) (5, 8) (b) (–5, 8)

(c) (5, –8) (d) (–5, –8)

73. Let $f(x)$ be a polynomial of degree 6 in x, in which the coefficient of x^6 is unity and it has extrema at $x=-1$ and $x=1$. If $\lim\limits_{x\to 0}\frac{f(x)}{x^3}=1$, then $5\cdot f(2)$ is equal to _____.

(25-02-2021/Shift-1)

74. The shortest distance between the line $x-y=1$ and the curve $x^2=2y$ is : **(25-02-2021/Shift-2)**

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) 0

75. If the curves $x=y^4$ and $xy=k$ cut at right angles, then $(4k)^6$ is equal to: **(25-02-2021/Shift-2)**

76. The maximum slope of the curve

$y=\frac{1}{2}x^4-5x^3+18x^2-19x$ occurs at the point:

(26-02-2021/Shift-1)

(a) $\left(3,\frac{21}{2}\right)$ (b) $(0,0)$

(c) $(2,9)$ (d) $(2,2)$

77. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is **(26-02-2021/Shift-2)**

(a) A right angle triangle having two of its sides of length 2r and r.

(b) An equilateral triangle of height $\frac{2r}{3}$.

(c) An isosceles triangle with base equal to 2r.

(d) An equilateral triangle having each of its side of length $\sqrt{3}r$.

78. Let a be an integer such that all the real roots of the polynomial $2x^5+5x^4+10x^3+10x^2+10x+10$ lie in the interval $(a,\ a+1)$. **(26-02-2021/Shift-2)**

Then, $|a|$ is equal to _________.

79. The range of $a\in R$ for which the function

$f(x)=(4a-3)(x+\log_e 5)+2(a-7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)$,

$x\neq 2n\pi,\ n\in N$ has critical points is:

(16-03-2021/Shift-1)

(a) $(-\infty,-1]$ (b) $(-3,1)$

(c) $\left[-\frac{4}{3},2\right]$ (d) $[1,\infty)$

80. Let f be a real valued function, defined on R - {-1, 1} and given by $f(x)=3\log_e\left|\frac{x-1}{x+1}\right|-\frac{2}{x-1}$

Then in which of the following intervals, function f(x) is increasing? **(16-03-2021/Shift-2)**

(a) $(-\infty,\infty)-\{-1,1\}$

(b) $\{-\infty,-1\}\cup\left(\left[\frac{1}{2},\infty\right)-\{1\}\right)$

(c) $\left(-1,\frac{1}{2}\right]$

(d) $\left(-\infty,\frac{1}{2}\right]-\{-1\}$

81. Consider the function $f: R \to R$ defined by

$$f(x)=\begin{cases}\left(2-\sin\left(\frac{1}{x}\right)\right)|x|, & x\neq 0\\ 0, & x=0\end{cases}.$$ Then f is :

(17-03-2021/Shift-2)

(a) monotonic on $(-\infty, 0)\cup(0, \infty)$

(b) not monotonic on $(-\infty, 0)$ and $(0, \infty)$

(c) monotonic on $(-\infty, 0)$ only

(d) monotonic on $(0, \infty)$ only

82. Let $f:[-1, 1]\to R$ be defined as $f(x)=ax^2+bx+c$ for all $x\in[-1, 1]$, where $f''(x)$ is $\frac{1}{2}$. If $f(x)\le\alpha$, $x\in[-1, 1]$, then the least value of α is equal to ....

(17-03-2021/Shift-2)

83. Let a tangent be drawn to the ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ where $\theta\in\left(0, \frac{\pi}{2}\right)$. Then the value of θ such that the sum of intercepts on axes made by this tangent is minimum is equal to: **(18-03-2021/Shift-2)**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

84. Let 'a' be a real number such that the function $f(x)=ax^2+6x-15, x\in R$ is increasing in $\left(-\infty, \frac{3}{4}\right)$ and decreasing in $\left(\frac{3}{4}, \infty\right)$. Then the function $g(x)=ax^2-6x+15, x\in R$ has a: **(20-07-2021/Shift-1)**

(a) local minimum at $x=-\frac{3}{4}$

(b) local maximum at $x=\frac{3}{4}$

(c) local minimum at . .

(d) local maximum at $x=-\frac{3}{4}$

85. The sum of all the local minimum values of the twice differentiable function $f: R\to R$ defined by

$$f(x)=x^3-3x^2-\frac{3f''(2)}{2}x+f''(1)$$ is?

(20-07-2021/Shift-2)

(a) –22 (b) 0

(c) –27 (d) 5

86. Let $f: R\to R$ be defined as

$$f(x)=\begin{cases}-\frac{4}{3}x^3+2x^2+3x, & x>0\\ 3xe^x, & x\le 0\end{cases}$$

Then f is is increasing function in the interval.

(22-07-2021/Shift-2)

(a) $\left(-1, \frac{3}{2}\right)$ (b) $\left(\frac{-1}{2}, 2\right)$

(c) $(0, 2)$ (d) $(-3, -1)$

87. Let

$$f(x)=3\sin^4 x+10\sin^3 x+6\sin^2 x-3, x\in\left[-\frac{\pi}{6}, \frac{\pi}{2}\right].$$

Then, f is? **(25-07-2021/Shift-1)**

(a) Increasing in $\left(-\frac{\pi}{6}, 0\right)$

(b) Decreasing in $\left(0, \frac{\pi}{2}\right)$

(c) Decreasing in $\left(-\frac{\pi}{6}, 0\right)$

(d) Increasing in $\left(-\frac{\pi}{6}, \frac{\pi}{2}\right)$

88. The number of real roots of the equation $e^{6x}-e^{4x}-2e^{3x}-12e^{2x}+e^{x}+1=0$ is ?

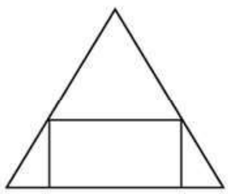
(25-07-2021/Shift-1)

(a) 1 (b) 6

(c) 4 (d) 2

89. If a rectangle is inscribed in an equilateral triangle of side length $2\sqrt{2}$ as shown in the figure, then the square of the largest area of such a rectangle is -

(25-07-2021/Shift-2)



90. Let $f:(a,b)\to R$ be twice differentiable function such that $f(x)=\int_a^x g(t)dt$ for a differentiable function $g(x)$.

If $f(x)=0$ has exactly five distinct roots in (a, b), then $g(x)g'(x)=0$ has at least: **(27-07-2021/Shift-2)**

(a) seven roots in (a, b) (b) five roots in (a, b)

(c) three roots in (a, b) (d) twelve roots in (a, b)

91. A wire of length 36 m is cut into two pieces, one of the pieces is bent to form a square and the other is bent to form a circle. If the sum of the areas of the two figures is minimum, and the circumference of the circle is K (meter), then $\left(\frac{4}{\pi}+1\right)k$ is equal to _______.

(26-08-2021/Shift-1)

92. The local maximum value of the function $f(x)=\left(\frac{2}{x}\right)^{x^2}$, $x>0$ is : **(26-08-2021/Shift-2)**

(a) $(e)^{\frac{2}{e}}$ (b) $\left(\frac{4}{\sqrt{e}}\right)^{\frac{e}{4}}$

(c) $\left(2\sqrt{e}\right)^{\frac{1}{e}}$ (d) 1

93. A wire of length 20 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a regular hexagon. Then the length of the side (in meters) of the hexagon, so that the combined area of the square and the hexagon is minimum, is: **(27-08-2021/Shift-1)**

(a) $\frac{10}{3+2\sqrt{3}}$ (b) $\frac{5}{3+\sqrt{3}}$

(c) $\frac{10}{2+3\sqrt{3}}$ (d) $\frac{5}{2+\sqrt{3}}$

94. The number of distinct real roots of the equation is

$3x^4+4x^3-12x^2+4=0$ _______.

(27-08-2021/Shift-1)

95. A box open from top is made from a rectangular sheet of dimension x from each of the four corners and folding up the flaps. If the volume of the box is maximum, then x is equal to : **(27-08-2021/Shift-2)**

(a) $\frac{a+b-\sqrt{a^2+b^2-ab}}{6}$

(b) $\frac{a+b+\sqrt{a^2+b^2-ab}}{6}$

(c) $\frac{a+b-\sqrt{a^2+b^2-ab}}{12}$

(d) $\frac{a+b-\sqrt{a^2+b^2+ab}}{6}$

96. The number of real roots of the equation $e^{4x}+2e^{3x}-e^{x}-6=0$ is? **(31-08-2021/Shift-1)**

(a) 0 (b) 1

(c) 4 (d) 2

97. If 'R' is the least value of 'a' such that the function $f(x)=x^2+ax+1$ is increasing on $[1,2]$ and 'S" is the greatest value of 'a' such that the function $f(x)=x^2+ax+1$ is decreasing on $[1,2]$, then the value $|R-S|$ is _______ ? **(31-08-2021/Shift-1)**

98. Let f be any continuous function on $[0,2]$ and twice differentiable on $(0,2)$. If $f(0)=0, f(1)=1$ and $f(2)=2$, then: **(31-08-2021/Shift-2)**

(a) $f''(x)>0$ for all $x\in(0,2)$

(b) $f'(x)=0$ for some $x\in[0,2]$

(c) $f''(x)=0$ for all $x\in(0,2)$

(d) $f''(x)=0$ for some $x\in(0,2)$

99. An angle of intersection of the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x^2+y^2=ab, a<b$ is **(31-08-2021/Shift-2)**

(a) $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$

(b) $\tan^{-1}\left(\frac{a+b}{\sqrt{ab}}\right)$

(c) $\tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$

(d) $\tan^{-1}\left(2\sqrt{ab}\right)$

100. Let $f(x)$ be a cubic polynomial with $f(1)=-10, f(-1)=6$, and has a local minima at $x=-1$. Then $f(3)$ is equal to ___ **(31-08-2021/Shift-2)**

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

Objective Questions I [Only one correct option]

1. The two curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ at the point (1,2)

(a) intersect orthogonally (b) intersect at an angle $\frac{\pi}{3}$

(c) touch each other (d) none of these

2. The function 'f' is defined by $f(x) = x^p (1-x)^q$ for all $x \in R$, where p,q are positive integers, has a local maximum value, for x equal to :

(a) $\frac{pq}{p+q}$ (b) 1

(c) 0 (d) $\frac{p}{p+q}$

3. The triangle formed by the tangent to the parabola $y = x^2$ at the point with abscissa x_1, the y-axis and the straight line $y = x_1^2$ has the greatest area where $x_1 \in [1, 3]$. Then x_1 equals:

(a) 3 (b) 2

(c) 1 (d) none

4. Let f be a differentiable function with $f(2) = 3$ and $f'(2) = 5$, and let g be the function defined by $g(x) = x f(x)$. y-intercept of the tangent line to the graph of 'g' at point with abscissa 2, is

(a) 20 (b) 8

(c) – 20 (d) – 18

5. If $px^2 + qx + r = 0$, $p, q, r \in R$ has no real zero and the line $y + 2 = 0$ is tangent to $f(x) = px^2 + qx + r$ then

(a) $p + q + r > 0$ (b) $p - q + r > 0$

(c) $r < 0$ (d) None of these

6. If $P(x) = a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$ be a polynomial in $x \in R$ with $0 < a_1 < a_2 < ... < a_n$, then P(x) has

(a) no point of minima

(b) only one point of minima

(c) only two points of minima

(d) none of these

7. Consider the following statements S and R :

S : Both sin x and cos x are decreasing functions in the interval $\left(\frac{\pi}{2}, \pi\right)$

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b). Which of the following is true ?

(a) Both S and R are wrong.

(b) Both S and R are correct, but R is not the correct explanation for S.

(c) S is correct and R is the correct explanation for S.

(d) S is correct and R is wrong.

8. If the function f (x) increases in the interval (a, b) then the function $\phi(x) = [f(x)]^2$.

(a) Increases in (a, b)

(b) decreases in (a, b)

(c) we cannot say that $\phi(x)$ increases or decreases in (a, b)

(d) none of these

9. If at any point on a curve the sub-tangent and sub-normal are equal, then the length of the normal is equal to

(a) $\sqrt{2}$ ordinate (b) ordinate

(c) $\sqrt{2 \text{ ordinate}}$ (d) none of these

10. A curve passes through the point (2, 0) and the slope of the tangent at any point (x, y) is $x^2 - 2x$ for all values of x then $3y_{\text{local max}}$ is equal to

(a) 4 (b) 3

(c) 1 (d) 2

11. The radius of a right circular cylinder increases at a constant rate. Its altitude is a linear function of the radius and increases three times as fast as radius. When the radius is 1 cm the altitude is 6 cm. When the radius is 6 cm, the volume is increasing at the rate of 1 cu cm/sec. When the radius is 36 cm, the volume is increasing at a rate of n cu cm/sec. The value of 'n' is equal to

(a) 12 (b) 22

(c) 30 (d) 33

12. Slope of tangent to the curve

$y = 2e^x \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)$, where $0 \le x \le 2\pi$ is minimum at x =

(a) 0 (b) π

(c) 2π (d) none of these

13. Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$ the set of values of b for which $f(x)$ have greatest value at x = 1 is given by :

(a) $1 \le b \le 2$

(b) $b = \{1, 2\}$

(c) $b \in (-\infty, -1)$

(d) $\left[-\sqrt{130}, -\sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{130}\right]$

14. A curve is represented parametrically by the equation $x = t + e^{at}$ and $y = -t + e^{at}$ when $t \in R$ and $a > 0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are

(a) (1, 0) (b) (1/e, 0)

(c) (e, 0) (d) (2e, 0)

15. If $ax + \frac{b}{x} \ge c$ for all positive x, where a, b, c > 0, then

(a) $ab < \frac{c^2}{4}$ (b) $ab \ge \frac{c^2}{4}$

(c) $ab \ge \frac{c}{4}$ (d) none of these

16. If $f(x)$ is a differentiable function and $\phi(x)$ is twice differentiable function and α and β are roots of the equation $f(x) = 0$ and $\phi'(x) = 0$ respectively, then which of the following statement is true ? $(\alpha < \beta)$.

(a) there exists exactly one root of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ and (α, β)

(b) there exists at least one root of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ and (α, β)

(c) there exists odd number of roots of the equation $\phi'(x).f'(x) + \phi''(x).f(x) = 0$ and (α, β)

(d) None of these

17. The sub-normal at any point of the curve $x^2y^2 = a^2(x^2 - a^2)$ varies as

(a) $(\text{abscissa})^{-3}$ (b) $(\text{abscissa})^3$

(c) $(\text{ordinate})^{-3}$ (d) none of these

18. The sub-tangent at any point of the curve $x^m y^n = a^{m+n}$ varies as

(a) $(\text{abscissa})^2$ (b) $(\text{abscissa})^3$

(c) abscissa (d) ordinate

19. The length of the perpendicular from the origin to the normal of curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is

(a) a (b) a/2

(c) a/3 (d) none of these

20. If t, n, t', n' are the lengths of tangent, normal, subtangent & subnormal at a point $P(x_1, y_1)$ on any curve $y = f(x)$ then

(a) $t^2 + n^2 = t'n'$ (b) $\frac{1}{t^2} + \frac{1}{n^2} = \frac{1}{t'n'}$

(c) $t'n' = tn$ (d) $nt' = n't$

21. Find the shortest distance between $xy = 9$ and $x^2+y^2 = 1$.

(a) $3\sqrt{2}+1$ (b) 2

(c) 4 (d) $3\sqrt{2}-1$

22. The largest area of a rectangle which has one side on the x-axis and the two vertices on the curve $y = e^{-x^2}$ is

(a) $\sqrt{2}\,e^{-1/2}$ (b) $2\,e^{-1/2}$

(c) $e^{-1/2}$ (d) none

23. If $(x - a)^{2n}(x - b)^{2m+1}$, where m and n are positive integers and a > b, is the derivative of a function *f*, then

(a) x = a gives neither a maximum nor a minimum

(b) x = a gives a maximum

(c) x = b gives neither a maximum nor a minimum

(d) none of these

24. Let (h, k) be a fixed point, where h > 0, k > 0. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. The minimum area of the Δ OPQ, O being the origin, is

(a) $2kh$ (b) kh

(c) $4kh$ (d) none of these

25. The set of all values of the parameters a for which the points of local minimum of the function $y = 1 + a^2 x - x^3$ satisfy the inequality $\frac{x^2+x+2}{x^2+5x+6} \le 0$ is

(a) an empty set

(b) $\left(-3\sqrt{3}, -2\sqrt{3}\right)$

(c) $\left(2\sqrt{3}, 3\sqrt{3}\right)$

(d) $\left(-3\sqrt{3}, -2\sqrt{3}\right) \cup \left(2\sqrt{3}, 3\sqrt{3}\right)$

26. The volume of the largest possible right circular cylinder that can be inscribed in a sphere of radius $= \sqrt{3}$ is:

(a) $\frac{4}{3}\sqrt{3}\pi$ (b) $\frac{8}{3}\sqrt{3}\pi$

(c) 4π (d) 2π

27. Tangent of acute angle between the curves $y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$ at their points of intersection is

(a) $\frac{5\sqrt{3}}{2}$ (b) $\frac{3\sqrt{5}}{2}$

(c) $\frac{5\sqrt{3}}{4}$ (d) $\frac{3\sqrt{5}}{4}$

28. A tangent to the curve $y = 1 - x^2$ is drawn so that the abscissa x_0 of the point of tangency belongs to the interval [0, 1]. The tangent at x_0 meets the x-axis and y-axis at A & B respectively. The minimum area of the triangle OAB, where O is the origin is

(a) $\frac{2\sqrt{3}}{9}$ (b) $\frac{4\sqrt{3}}{9}$

(c) $\frac{2\sqrt{2}}{9}$ (d) none

29. If the polynomial equation

$a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0 = 0$, n positive integer, has two different real roots α and β, then between α and β, the equation

$n a_n x^{n-1} + (n-1)\, a_{n-1}\, x^{n-2} + ... + a_1 = 0$ has

(a) exactly one root (b) atmost one root

(c) atleast one root (d) no root

30. If $f(x) = \begin{vmatrix} \sin x & \sin a & \sin b \\ \cos x & \cos a & \cos b \\ \tan x & \tan a & \tan b \end{vmatrix}$, where $0 < a < b < \frac{\pi}{2}$,

then the equation $f'(x) = 0$ has, in the interval (a, b)

(a) atleast one root (b) atmost one root

(c) no root (d) none of these

31. If $f(x) = \frac{x^2}{2 - 2\cos x}$; $g(x) = \frac{x^2}{6x - 6\sin x}$ where $0 < x < 1$,

then :

(a) both 'f' and 'g' are increasing functions

(b) 'f' is decreasing and 'g' is increasing function

(c) 'f' is increasing and 'g' is decreasing function

(d) both 'f' and 'g' are decreasing function

32. For $x \in \left(0, \tan^{-1}\sqrt{\frac{5}{2}}\right)$, the function

$$f(x) = \cot^{-1}\left(\frac{\sqrt{2}\sin x + \sqrt{5}\cos x}{\sqrt{7}}\right)$$

(a) increases in $\left(0, \tan^{-1}\sqrt{\frac{5}{2}}\right)$

(b) decreases in $\left(0, \tan^{-1}\sqrt{\frac{5}{2}}\right)$

(c) increases in $\left(0, \tan^{-1}\sqrt{\frac{2}{5}}\right)$ and decreases in $\left(\tan^{-1}\sqrt{\frac{2}{5}}, \tan^{-1}\sqrt{\frac{5}{2}}\right)$

(d) increases in $\left(\tan^{-1}\sqrt{\frac{2}{5}}, \tan^{-1}\sqrt{\frac{5}{2}}\right)$ and decreases in $\left(0, \tan^{-1}\sqrt{\frac{2}{5}}\right)$

33. If $f(x)=a^{\{a^{|x|}\operatorname{sgn}x\}}$; $g(x)=a^{[a^{|x|}\operatorname{sgn}x]}$ for a > 1 and $x\in R$, where { } & [ ] denote the fractional part and integral part functions respectively, then which of the following statements hold good for the function h (x), where $(\ln a)\,h(x)=(\ln f(x)+\ln g(x))$.

(a) 'h' is even and increasing

(b) 'h' is odd and decreasing

(b) 'h' is even and decreasing

(d) 'h' is odd and increasing

34. The sum of tangent and sub-tangent at any point of the curve $y=a\log(x^2-a^2)$ varies as

(a) abscissa

(b) product of the coordinates

(c) ordinate

(d) none of these

35. For the curve $x^{m+n}=a^{m-n}y^{2n}$, where a is a positive constant and m, n are positive integers

(a) $(\text{sub-tangent})^m \propto (\text{sub-normal})^n$

(b) $(\text{sub-normal})^m \propto (\text{sub-tangent})^n$

(c) the ratio of subtangent and subnormal is constant

(d) none of the above

36. $|\sin 2x|-|x|-a=0$ does not have solution if a lies in

(a) $\left(\frac{3\sqrt{3}-\pi}{6},\infty\right)$ (b) $\left(\frac{3\sqrt{3}+\pi}{6},\infty\right)$

(c) $(1,\infty)$ (d) None of these

37. Let $f(x)=\begin{cases}-x^2, & \text{for } x<0\\ x^2+8, & \text{for } x\ge 0\end{cases}$. Then the x-intercept of the line that is tangent to both portions of the graph of $y=f(x)$ is

(a) zero (b) –1

(c) –3 (d) –4

38. The least area of a circle circumscribing any right triangle of area S is :

(a) πS (b) $2\pi S$

(c) $\sqrt{2}\,\pi S$ (d) $4\pi S$

39. The minimum value of $a\tan^2 x+b\cot^2 x$ equals the maximum value of $a\sin^2\theta+b\cos^2\theta$ where a > b > 0, when

(a) a = b (b) a = 2b

(c) a = 3b (d) a = 4b

40. A function f such that $f'(a)=f''(a)=\ldots f^{2n}(a)=0$ and f has a local maximum value b at x = a, if $f(x)$ is

(a) $(x-a)^{2n+2}$ (b) $b-1-(x+1-a)^{2n+1}$

(c) $b-(x-a)^{2n+2}$ (d) $(x-a)^{2n+2}-b$.

41. A truck is to be driven 300 km on a highway at a constant speed of x kmph. Speed rules of the highway required that $30\le x\le 60$. The fuel costs Rs. 10 per litre and is consumed at the rate of $2+\frac{x^2}{600}$ liters per hour. The wages of the driver are Rs. 200 per hour. The most economical speed to drive the truck, in kmph, is

(a) 30 (b) 60

(c) $30\sqrt{3.3}$ (d) $20\sqrt{3.3}$

42. The curve $y=\frac{2x}{1+x^2}$ has

(a) exactly three points of inflection separated by a point of maximum and a point of minimum

(b) exactly two points of inflection with a point of maximum lying between them

(c) exactly two points of inflection with a point of minimum lying between them

(d) exactly three points of inflection separated by two points of maximum

43. Let $f(x)=\begin{cases}\left|x^3+x^2+3x+\sin x\right|\left(3+\sin 1/x\right), & x\ne 0\\ 0, & x=0\end{cases}$ then number of points (where $f(x)$ attains its minimum value) is

(a) 1 (b) 2

(c) 3 (d) infinite many

44. The number of points with integral coordinates where tangent exists in the curve $y=\sin^{-1}2x\sqrt{1-x^2}$ is

(a) 0 (b) 1

(c) 3 (d) None

Objective Questions II [One or more than one correct option]

45. The abscissa of a point on the curve $xy = (a + x)^2$, the tangent at which cuts off equal intercepts on the coordinate axes is

(a) $-a/\sqrt{2}$ (b) $\sqrt{2}\,a$

(c) $\sqrt{2}\,a/2$ (d) $-\sqrt{2}\,a$

46. If f is an even function then

(a) f^2 increases on (a, b)

(b) f cannot be monotonic

(c) f^2 need not increases on (a, b)

(d) f has inverse

47. The function $y = \dfrac{2x-1}{x-2}$ $(x \neq 2)$ with codomain $= R - \{2\}$

(a) is its own inverse

(b) decreases at all values of x in the domain

(c) has a graph entirely above x–axis

(d) is bound for all x.

48. Let $g'(x) > 0$ and $f'(x) < 0$, $\forall\ x \in R$, then

(a) $g(f(x+1)) > g(f(x-1))$

(b) $f(g(x-1)) > f(g(x+1))$

(c) $g(f(x+1)) < g(f(x-1))$

(d) $g(g(x+1)) < g(g(x-1))$

49. If $f(x) = x^3 - x^2 + 100x + 1001$, then

(a) $f(2000) > f(2001)$

(b) $f\left(\dfrac{1}{1999}\right) > f\left(\dfrac{1}{2000}\right)$

(c) $f(x+1) > f(x-1)$

(d) $f(3x-5) > f(3x)$

50. An extremum of the function,

$$f(x) = \frac{2-x}{\pi}\cos\pi(x+3) + \frac{1}{\pi^2}\sin\pi(x+3)\ 0 < x < 4 \text{ occurs}$$

at :

(a) $x = 1$ (b) $x = 2$

(c) $x = 3$ (d) $x = \pi$

51. The length of the perpendicular from the origin to the normal of curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at a point θ is 'a', if $\theta =$

(a) $\pi/4$ (b) $\pi/3$

(c) $\pi/2$ (d) $\pi/6$

52. The points on the curve $y = x\sqrt{1-x^2}$, $-1 \leq x \leq 1$ at which the tangent line is vertical are

(a) $(-1, 0)$ (b) $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{2}\right)$

(c) $(1, 0)$ (d) $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

53. Let the parabolas $y = x(c - x)$ and $y = -x^2 - ax + b$ touch each other at the point (1, 0), then

(a) $a + b + c = 0$ (b) $a + b = 2$

(c) $b + c = 1$ (d) $a - c = -2$

54. The value of parameter a so that the line $(3-a)x + ay + (a^2 - 1) = 0$ is normal to the curve $xy = 1$, may lie in the interval

(a) $(-\infty, 0)$ (b) $(1, 3)$

(c) $(0, 3)$ (d) $(3, \infty)$

55. Which of the following pair (s) of curves is/are orthogonal.

(a) $y^2 = 4ax$; $y = e^{-x/2a}$

(b) $y^2 = 4ax$; $x^2 = 4ay$ at (0, 0)

(c) $xy = a^2$; $x^2 - y^2 = b^2$

(d) $y = ax$; $x^2 + y^2 = c^2$

56. If $\phi(x) = f(x) + f(2a - x)$ and $f''(x) > 0$, $a > 0$, $0 \leq x \leq 2a$ then

(a) $\phi(x)$ increases in [a, 2a]

(b) $\phi(x)$ increases in [0, a]

(c) $\phi(x)$ decreases in [0, a]

(d) $\phi(x)$ decreases in [a, 2a]

57. Let $f(x) = x^{m/n}$ for $x \in R$ where m and n are integers, m even and n odd and $0 < m < n$. Then

(a) $f(x)$ decreases on $(-\infty, 0]$

(b) $f(x)$ increases on $[0, \infty)$

(c) $f(x)$ increases on $(-\infty, 0]$

(d) $f(x)$ decreases on $[0, \infty)$

58. For function $f(x)=\frac{\ell n\, x}{x}$, which of the following statements are true.

(a) $f(x)$ has horizontal tangent at $x = e$

(b) $f(x)$ cuts the x–axis only at one point

(c) $f(x)$ is many – one function

(d) $f(x)$ has one vertical tangent

59. If $f(x)=\frac{x}{1+x\tan x}$, $x\in\left(0,\frac{\pi}{2}\right)$, then

(a) $f(x)$ has exactly one point of minimum

(b) $f(x)$ has exactly one point of maximum

(c) $f(x)$ is increasing in $\left(0,\frac{\pi}{2}\right)$

(d) maximum occurs at x_0 where $x_0=\cos x_0$

60. Let $f(x)=(x-1)^4(x-2)^n$, $n\in N$. then $f(x)$ has

(a) local minimum at $x = 2$ if n is even

(b) local minimum at $x = 1$ if n is odd

(c) local maximum at $x = 1$ if n is odd

(d) local minimum at $x = 1$ if n is even

61. The angle between the tangent at any point P and the line joining P to the origin, where P is a point on the curve $ln(x^2+y^2)=c\tan^{-1}\frac{y}{x}$, c is a constant, is

(a) independent of x and y

(b) dependent on c

(c) independent of c but dependent on x

(d) none of these

62. The point on the curve $xy^2 = 1$, which is nearest to the origin is

(a) $(2^{1/3}, 2^{1/6})$
(b) $(2^{-1/3}, 2^{1/6})$
(c) $(2^{-1/3}, -2^{1/6})$
(d) $(-2^{-1/3}, 2^{1/6})$

63. Let $g(x) = -\frac{f(-1)}{2}x^2(x-1) - f(0)(x^2-1) + \frac{f(1)}{2}x^2(x+1) - f'(0)\,x(x-1)(x+1)$ where f is a thrice differentiable function. Then the correct statements are

(a) there exists $x\in(-1, 0)$ such that $f'(x) = g'(x)$

(b) there exists $x\in(0, 1)$ such that $f''(x) = g''(x)$

(c) there exists $x\in(-1, 1)$ such that $f'''(x) = g'''(x)$

(d) there exists $x\in(-1,1)$ such that $f'''(x)=3f(1)-3f(-1)-6f'(0)$

64. If $f:[-1, 1]\to R$ is a continuously differentiable function such that $f(1)>f(-1)$ and $|f'(y)|\le 1$ for all $y\in[-1, 1]$ then

(a) there exists an $x\in[-1, 1]$ such that $f'(x)>0$

(b) there exists an $x\in[-1, 1]$ such that $f'(x)<0$

(c) $f(1)\le f(-1)+2$

(d) $f(-1)\,.\,f(1)<0$

65. In a triangle ABC

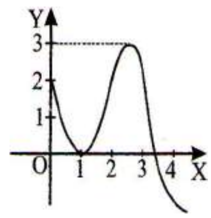
(a) $\sin A\sin B\sin C\le\frac{3\sqrt{3}}{8}$

(b) $\sin^2A+\sin^2B+\sin^2C\le\frac{9}{4}$

(c) $\sin A\sin B\sin C$ is always positive

(d) $\sin^2A+\sin^2B = 1+\cos C$

66. The diagram shows the graph of the derivative of a function $f(x)$ for $0\le x\le 4$ with $f(0)=0$. Which of the following could be correct statements for $y=f(x)$?



(a) Tangent line to $y = f(x)$ at $x = 0$ makes an angle of $\sec^{-1}\sqrt{5}$ with the x-axis.

(b) f is strictly increasing in $(0, 3)$

(c) $x = 1$ is both an inflection point as well as point of local extremum.

(d) Number of critical point on $y=f(x)$ is two.

Numerical Value Type Questions

67. If A is the area of the triangle formed by positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $\left(1, \sqrt{3}\right)$ then $A/\sqrt{3}$ is equal to

68. A cylinderical vessel of volume $25\frac{1}{7}$ cu metres, open at the top is to be manufactured from a sheet of metal. (The value of π is taken as 22/7). If r and h are the radius and height of the vessel so that amount of metal is used in the least possible then rh is equal to

69. Let α be the angle in radians between $\frac{x^2}{36}+\frac{y^2}{4}=1$ and the circle $x^2 + y^2 = 12$ at their points of intersection. If $\alpha = \tan^{-1}\frac{k}{2\sqrt{3}}$, then find the value of k^2.

70. If α is an integer satisfying $|\alpha| \le 5 - |[x]|$, where x is a real number for which $2x \tan^{-1} x$ is greater than or equal to $ln(1 + x^2)$, then find the number of maximum possible values of α. (where [.] represents the greatest integer function)

71. The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. If A is the maximum area of the triangle QSR then $3\sqrt{3}$ A is equal to _____.

72. If $f(x)$ is a twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, where $a < b < c < d < e$, find the minimum number of zeroes of $g(x) = (f'(x))^2 + f''(x)f(x)$ in the interval [a, e].

73. If the length of the interval of 'a' such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution is k then find 4k.

74. If k is a positive integer, such that

(i) $\cos^2 x \sin x > -\frac{7}{k}$, for all x

(ii) $\cos^2 x \sin x < -\frac{7}{k+1}$ for some x, then k must be equal to

Assertion & Reason

(A) **If ASSERTION is true, REASON is true, REASON is a correct explanation for ASSERTION.**

(B) **If ASSERTION is true, REASON is true, REASON is not a correct explanation for ASSERTION.**

(C) **If ASSERTION is true, REASON is false.**

(D) **If ASSERTION is false, REASON is true.**

75. **Assertion :** Let $f(x) = 5 - 4(x - 2)^{2/3}$, then at x = 2 the function $f(x)$ attains neither least value nor greatest value.

Reason : x = 2 is the only critical point of $f(x)$.

(a) A (b) B

(c) C (d) D

76. **Assertion :** for any triangle ABC

$$\sin\left(\frac{A+B+C}{3}\right) \ge \frac{\sin A + \sin B + \sin C}{3}$$

Reason : $y = \sin x$ is concave downward for $x \in (0, \pi]$.

(a) A (b) B

(c) C (d) D

77. **Assertion :** The minimum distance of the fixed point $(0, y_0)$, where $0 \le y_0 \le \frac{1}{2}$, from the curve $y = x^2$ is y_0.

Reason : Maxima and minima of a function is always a root of the equation $f'(x) = 0$.

(a) A (b) B

(c) C (d) D

78. **Assertion :** The equation $3x^2 + 4ax + b = 0$ has at least one root in (0, 1), if $3 + 4a = 0$.

Reason : $f(x) = 3x^2 + 4ax + b$ is continuous and differentiable in the interval (0, 1).

(a) A (b) B

(c) C (d) D

79. **Assertion :** Let $f : [0, \infty) \to [0, \infty)$ and $g : [0, \infty) \to [0, \infty)$ be non-increasing and non-decreasing functions respectively and $h(x) = g(f(x))$. If f and g are differentiable for all points in their respective domains and $h(0) = 0$ then $h(x)$ is constant function.

Reason : $g(x) \in [0, \infty) \Rightarrow h(x) \ge 0$ and $h'(x) \le 0$.

(a) A (b) B

(c) C (d) D

80. **Assertion :** The ratio of length of tangent to length of normal is directly proportional to the ordinate of the point of tangency at the curve $y^2 = 4ax$.

Reason : Length of normal & tangent to a curve

$y = f(x)$ is $\left|y\sqrt{1+m^2}\right|$ and $\left|\frac{y\sqrt{1+m^2}}{m}\right|$, where $m = \frac{dy}{dx}$.

(a) A (b) B
(c) C (d) D

81. **Assertion :** Among all the rectangles of given perimeter, the square has the largest area. Also among all the rectangles of given area, the square has the least perimeter.

Reason : For $x > 0$, $y > 0$, if $x + y =$ const, then xy will be maximum for $y = x$ and if $xy =$ const, then $x + y$ will be minimum for $y = x$.

(a) A (b) B
(c) C (d) D

82. **Assertion :** If $g(x)$ is a differentiable function $g(1) \neq 0$, $g(-1) \neq 0$ and Rolles theorem is not applicable to $f(x) = \frac{x^2-1}{g(x)}$ in $[-1, 1]$, then $g(x)$ has atleast one root in $(-1, 1)$

Reason : If $f(a) = f(b)$, then Rolles theorem is applicable for $x \in (a, b)$

(a) A (b) B
(c) C (d) D

83. **Assertion :** The tangent at $x = 1$ to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = -2$.

Reason : When a equation of a tangent solved with the curve, repeated roots are obtained at point of tangency.

(a) A (b) B
(c) C (d) D

84. **Assertion :** Tangent drawn at the point (0, 1) to the curve $y = x^3 - 3x + 1$ meets the curve thrice at one point only.

Reason : Tangent drawn at the point (1, –1) to the curve $y = x^3 - 3x + 1$ meets the curve at 1 point only.

(a) A (b) B
(c) C (d) D

85. **Assertion :** Shortest distance between

$$|x| + |y| = 2 \ \& \ x^2 + y^2 = 16 \text{ is } 4 - \sqrt{2}$$

Reason : Shortest distance between the two non intersecting differentiable curves lies along the common normal.

(a) A (b) B
(c) C (d) D

86. **Assertion :** If $f(x)$ is increasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

Reason : If $f(x)$ is decreasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

(a) A (b) B
(c) C (d) D

87. **Assertion :** The largest term in the sequence

$$a_n = \frac{n^2}{n^3+200}, n \in N \text{ is } \frac{(400)^{2/3}}{600}.$$

Reason : $f(x) = \frac{x^2}{x^3+200}$, $x > 0$, then at $x = (400)^{1/3}$, $f(x)$ is maximum.

(a) A (b) B
(c) C (d) D

Match the Following

Each question has two columns. Four options are given representing matching of elements from Column-I and Column-II. Only one of these four options corresponds to a correct matching.For each question, choose the option corresponding to the correct matching.

88.

	Column–I		Column–II
(A)	Circular plate is expanded by heat from radius 5 cm to 5.06 cm. Approximate increase in area is	**(P)**	4
(B)	If an edge of a cube increases by 1% then percentage increase in volume is	**(Q)**	0.6π
(C)	If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is twice the rate of decrease of x, then x is equal to (rate of decrease is non-zero)	**(R)**	3
(D)	Rate of increase in area of equilateral triangle of side 15cm, when each side is increasing at the rate of 0.1 cm/sec; is	**(S)**	$3\sqrt{3}/4$

The correct matching is :

(a) (A–Q; B–R; C–P; D–S)
(b) (A–R; B–P; C–Q; D–S)
(c) (A–S; B–Q; C–P; D–S)
(d) (A–P; B–Q; C–R; D–S)

89. **Column–I** **Column–II**

(A) If portion of the tangent at any point on the curve $x = at^3$, $y=at^4$ between the axes is divided by the abscissa of the point of contact in the ratio m : n externally, then $|n + m|$ is equal to (m and n are coprime) **(P)** 0

(B) The area of triangle formed by normal at the point (1, 0) on the curve $x = e^{\sin y}$ with axes is **(Q)** 1/2

(C) If the angle between curves $x^2y=1$ and $y = e^{2(1-x)}$ at the point (1, 1) is θ then $\tan\theta$ is equal to **(R)** 7

(D) The length of sub-tangent at any point on the curve $y = be^{x/3}$ is equal to **(S)** 3

The correct matching is :

(a) (A–R; B–Q; C–P; D–S)

(b) (A–Q; B–R; C–P; D–S)

(c) (A–P; B–Q; C–R; D–S)

(d) (A–S; B–P; C–Q; D–S)

90. **Column - I** **Column - II**

(A) The dimensions of the rectangle of perimeter 36 cm, which sweeps out the largest volume when revolved about one of its sides, are **(P)** 6

(B) Let A (–1, 2) and B (2, 3) be two fixed points, A point P lying on y = x such that perimeter of triangle PAB is minimum, then sum of the abscissa and ordinate of point P, is **(Q)** 12

(C) If x_1 and x_2 are abscissae of two points on the curve $f(x) = x - x^2$ in the interval [0, 1], then maximum value of expression $(x_1+x_2) - (x_1^2 + x_2^2)$ is **(R)** 4

(D) The number of non-zero integral values of 'a' for which the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ is concave upward along the entire real line is **(S)** 1/2

(T) 2

The correct matching is :

(a) (A–R; B–P; C–S; D–Q)

(b) (A–S; B–R; C–P; D–Q)

(c) (A–P,Q; B–R; C–S; D–R)

(d) (A–Q; B–S; C–P; D–R)

91. **Column - I** **Column - II**

(A) The equation $x\,\log x = 3 - x$ has at least one root in **(P)** (0, 1)

(B) If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in **(Q)** (1, 3)

(C) If $c = \sqrt{3}$ & $f(x) = x + \frac{1}{x}$ then interval of x in which LMVT is applicable for $f(x)$, is **(R)** (0, 3)

(D) If $c = \frac{1}{2}$ & $f(x) = 2x - x^2$, then interval of x in which LMVT is applicable for $f(x)$, is **(S)** (–1, 1)

The correct matching is :

(a) (A–P; B–R; C–Q; D–P)

(b) (A–R; B–S; C–Q; D–P)

(c) (A–Q; B–S; C–R; D–P)

(d) (A–R; B–S; C–P; D–P)

92.

	Column - I		Column - II
(A)	If x is real, then the greatest and least value of the expression $\frac{x+2}{2x^2+3x+6}$ is	**(P)**	3
(B)	If $a + b = 1$; $a > 0$, $b > 0$, then the minimum value of $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ is	**(Q)**	$\frac{1}{3}$
(C)	The maximum value attained by $y = 10 - \lvert x-10\rvert, -1 \le x \le 3$, is	**(R)**	5
(D)	If $P(t^2, 2t), t \in [0, 2]$ is an arbitrary point on parabola $y^2=4x$. Q is foot of perpendicular from focus S on the tangent at P, then maximum area of triangle PQS is	**(S)**	$-\frac{1}{13}$

The correct matching is :

(a) (A–S; B–P; C–P; D–R)

(b) (A–Q; B–S; C–P; D–R)

(c) (A–R; B–Q; C–P; D–S)

(d) (A–S; B–R; C–P; D–Q)

Paragraph Type Questions

Using the following, solve Q.93 to Q. 95

Passage

If $y = \int_{u(x)}^{v(x)} f(t)\,dt$, let us define $\frac{dy}{dx}$ in a different manner as $\frac{dy}{dx} = v'(x) f^2(v(x)) - u'(x) f^2(u(x))$ and the equation of the tangent at (a,b) as

$$y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x-a)$$

93. If $y = \int_{x}^{x^2} t^2\,dt$, then equation of tangent at x = 1 is

(a) $y = x + 1$ (b) $x + y = 1$

(c) $y = x - 1$ (d) $y = x$

94. If $F(x) = \int_1^x e^{t^2/2}(1-t^2)\,dt$, then $\frac{d}{dx}F(x)$ at $x = 1$ is

(a) 0 (b) 1

(c) 2 (d) –1

95. If $y = \int_{x^3}^{x^4} \ell nt\,dt$, then $\lim_{x\to 0^+} \frac{dy}{dx}$ is

(a) 0 (b) 1

(c) 2 (d) –1

Using the following passage, solve Q.96 to Q.98

Passage

Consider a function $f(x) = \left(\alpha - \frac{1}{\alpha} - x\right)(4 - 3x^2)$ where 'α' is a positive parameter

96. Number of points of extrema of $f(x)$ for a given value of α is

(a) 0 (b) 1

(c) 2 (d) 3

97. Absolute difference between local maximum and local minimum values of $f(x)$ in terms of α is

(a) $\frac{4}{9}\left(\alpha+\frac{1}{\alpha}\right)^3$ (b) $\frac{2}{9}\left(\alpha+\frac{1}{\alpha}\right)^3$

(c) $\left(\alpha+\frac{1}{\alpha}\right)^3$ (d) independent of α

98. Least possible value of the absolute difference between local maximum and local minimum values of $f(x)$ is

(a) $\frac{32}{9}$ (b) $\frac{16}{9}$

(c) $\frac{8}{9}$ (d) $\frac{1}{9}$

Using the following passage, solve Q.99 to Q.101

Passage

Consider the function $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$ where $0 \le x \le 1$.

99. The interval in which $f(x)$ is increasing is

(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right)$ (d) $\left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{2}{3}, 1\right)$

100. The interval in which $f(x)$ is decreasing is

(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{2}\right)$

(c) $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right)$ (d) $\left(0, \frac{1}{2}\right) \cup \left(\frac{2}{3}, 1\right)$

101. Let RMVT is applicable for $f(x)$ on (a, b) then $a + b + c$ is (where c is point such that $f'(c) = 0$)

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{3}{2}$

Using the following passage, solve Q.102 to Q.104

Passage

Let $y = a\sqrt{x} + bx$ be curve, $(2x - y) + \lambda(2x + y - 4) = 0$ be family of lines.

102. If curve has slope $-\frac{1}{2}$ at $(9, 0)$ then a tangent belonging to family of lines is

(a) $x + 2y - 5 = 0$ (b) $x - 2y + 3 = 0$

(c) $3x - y - 1 = 0$ (d) $3x + y - 5 = 0$

103. A line of the family cutting positive intercepts on axes and forming triangle with coordinate axes, then minimum length of the line segment between axes is

(a) $(2^{2/3} - 1)^{3/2}$ (b) $(2^{2/3} + 1)^{3/2}$

(c) $7^{3/2}$ (d) 27

104. Two perpendicular chords of curve $y^2 - 4x - 4y + 4 = 0$ belonging to family of lines form diagonals of a quadrilateral. Minimum area of quadrilateral is

(a) 16 (b) 32

(c) 64 (d) 50

Using the following passage, solve Q.105 to Q.107

Passage

If $y = f(x)$ is a curve and if there exists two points $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$ on it such that $f'(x_1) = -\frac{1}{f'(x_2)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ then the tangent at x_1 is normal at x_2 for that curve.

105. Number of such lines on the curve $y = \sin x$ is

(a) 1 (b) 0

(c) 2 (d) infinite

106. Number of such lines on the curve $y = |\ln x|$ is

(a) 1 (b) 2

(c) 0 (d) infinite

107. Number of such line on the curve $y^2 = x^3$ is

(a) 1 (b) 2

(c) 3 (d) 0

Using the following passage, solve Q.108 to Q.110

Passage

Let $f'(\sin x) < 0$ and $f''(\sin x) > 0\ \forall\ x \in \left(0, \frac{\pi}{2}\right)$

Now consider a function $g(x) = f(\sin x) + f(\cos x)$

108. $g(x)$ decreases if x belongs to

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ (d) none of these

109. $g(x)$ increase if x belongs to

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{8}, \frac{\pi}{3}\right)$ (d) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

110. The set of critical points of $g(x)$ is

(a) $\left\{\frac{\pi}{8}, \frac{\pi}{6}\right\}$ (b) $\left\{\frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{3}\right\}$

(c) $\left\{\frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{4}\right\}$ (d) none of these

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

Objective Questions I [Only one correct option]

1. For all $x \in (0, 1)$ **(2000)**

(a) $e^x < 1 + x$ (b) $\log_e(1+x) < x$

(c) $\sin x > x$ (d) $\log_e x > x$

2. Let $f(x) = \int e^x (x-1)(x-2)dx$. Then f decreases in the interval **(2000)**

(a) $(-\infty, -2)$ (b) $(-2, -1)$

(c) $(1, 2)$ (d) $(2, \infty)$

3. Let $f(x) = \begin{cases} |x|, & \text{for } 0<|x|\le 2 \\ 1, & \text{for } x=0 \end{cases}$ Then, at $x = 0$, f has **(2000)**

(a) a local maximum (b) no local maximum

(c) a local minimum (d) no extremum

4. If the normal to the curve, $y = f(x)$ at the point $(3, 4)$ makes an angle $3\pi/4$ with the positive x–axis, then $f'(3)$ is equal to **(2000)**

(a) -1 (b) $-3/4$

(c) $4/3$ (d) 1

5. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is **(2001)**

(a) increasing in $\left[-\frac{1}{2}, 1\right]$ (b) decreasing in R

(c) increasing in R (d) decreasing in $\left[-\frac{1}{2}, 1\right]$

6. The maximum value of $(\cos\alpha_1).(\cos\alpha_2)\ldots(\cos\alpha_n)$, under the restrictions $0 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le \frac{\pi}{2}$ and $(\cot\alpha_1).(\cot\alpha_2)\ldots(\cot\alpha_n) = 1$ is **(2001)**

(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$

(c) $\frac{1}{2n}$ (d) 1

7. The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is **(2002)**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) π

8. The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is (are) **(2002)**

(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\sqrt{\frac{11}{3}}, 0\right)$

(c) $(0, 0)$ (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

9. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is **(2002)**

(a) $3y = 9x + 2$ (b) $y = 2x + 1$

(c) $2y = x + 8$ (d) $y = x + 2$

10. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that min $f(x) >$ max $g(x)$, then the relation between b and c, is – **(2003)**

(a) no real value of b & c (b) $0 < c < b\sqrt{2}$

(c) $|c| < |b|\sqrt{2}$ (d) $|c| > |b|\sqrt{2}$

11. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$ **(2004)**

(a) $f(x)$ is strictly increasing function

(b) $f(x)$ has a local maxima

(c) $f(x)$ is strictly decreasing function

(d) $f(x)$ is bounded.

12. If $f(x)$ is differentiable and strictly increasing function, then the value of $\lim_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is **(2004)**

(a) 1 (b) 0

(c) -1 (d) 2

13. Tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the mid point of the intercept made by the tangents between the coordinate axes is **(2004)**

(a) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ (b) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

(c) $\frac{x^2}{2}+\frac{y^2}{4}=1$ (d) $\frac{x^2}{4}+\frac{y^2}{2}=1$

14. The angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is **(2004)**

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

15. The second degree polynomial $f(x)$, satisfying $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$ for all $x \in (0, 1)$: **(2005)**

(a) $f(x) = \phi$

(b) $f(x) = ax + (1 - a)x^2;\ \forall\ a \in (0, \infty)$

(c) $f(x) = ax + (1 - a)x^2;\ a \in (0, 2)$

(d) No such polynomial

16. The tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at **(2005)**

(a) (6, 7) (b) (–6, 7)

(c) (6, –7) (d) (–6, –7)

17. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ **(2007)**

(a) on the left of x = c (b) on the right of x = c

(c) at no point (d) at all points

18. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by

$g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is **(2008)**

(a) even and is strictly increasing in $(0, \infty)$

(b) odd and is strictly decreasing in $(-\infty, \infty)$

(c) odd and is strictly increasing in $(-\infty, \infty)$

(d) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

19. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{\frac{2}{3}}, & -1 < x < 2 \end{cases}$ is **(2008)**

(a) 0 (b) 1

(c) 2 (d) 3

20. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then **(2010)**

(a) a = b and c ≠ b (b) a = c and a ≠ b

(c) a ≠ b and c ≠ b (d) a = b = c

21. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is **(2013)**

(a) 6 (b) 4

(c) 2 (d) 0

22. Consider all rectangles lying in the region

$$\left\{(x, y) \in R \times R : 0 \le x \le \frac{\pi}{2} \text{ and } 0 \le y \le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is **(2020)**

(a) $\frac{3\pi}{2}$ (b) π

(c) $\frac{\pi}{2\sqrt{3}}$ (d) $\frac{\pi\sqrt{3}}{2}$

Objective Questions II [One or more than one correct option]

23. If $f(x)$ is cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local minimum at $x = 0$, then **(2006)**

(a) the distance between (–1, 2) and $(a, f(a))$ where x = a is the point of local minima, is $2\sqrt{5}$.

(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$

(c) $f(x)$ has local minima at x = 1

(d) the value of $f(0) = 5$

24. If $f(x)=\begin{cases} e^x, & 0\le x\le 1 \\ 2-e^{x-1}, & 1<x\le 2 \\ x-e, & 2<x\le 3 \end{cases}$

and $g(x)=\int_0^x f(t)\,dt,\ x\in[1,3]$, then **(2006)**

(a) g (x) has local maxima at $x=1+log_e 2$ and local minima at x = e

(b) $f(x)$ has local maxima at x = 1 and local minima at x = 2

(c) g (x) has no local minima

(d) $f(x)$ has no local maxima

25. For the function $f(x)=x\cos\frac{1}{x},\ x\ge 1$. **(2009)**

(a) for at least one x in the interval $[1,\infty)$, $f(x+2)-f(x)<2$

(b) $\lim\limits_{x\to\infty} f'(x)=1$

(c) for all x in the interval $[1,\infty)$, $f(x+2)-f(x)>2$

(d) $f'(x)$ is strictly decreasing in the interval $[1,\infty)$

26. Let f be a real-valued function defined on the interval $(0,\infty)$, by $f(x)=ln\,x+\int\limits_0^x\sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true ? **(2010)**

(a) $f''(x)$ exists for all $x\in(0,\infty)$

(b) $f'(x)$ exists for all $x\in(0,\infty)$ and f' is continuous on $(0,\infty)$, but not differentiable on $(0,\infty)$

(c) there exists $\alpha>1$ such that $|f'(x)|<|f(x)|$ for all $x\in(\alpha,\infty)$

(d) there exists $\beta>0$ such that $|f(x)|+|f'(x)|\le\beta$ from all $x\in(0,\infty)$

27. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are **(2013)**

(a) 24 (b) 32

(c) 45 (d) 60

28. The function $f(x)=2|x|+|x+2|-||x+2|-2|x||$ has a local minimum or a local maximum at x is equal to **(2013)**

(a) -2 (b) $\frac{-2}{3}$

(c) 2 (d) $\frac{2}{3}$

29. Let $a\in R$ and let $f:R\to R$ be given by $f(x)=x^5-5x+a$.

Then **(2014)**

(a) $f(x)$ has three real roots if $a>4$

(b) $f(x)$ has only one real root if $a>4$

(c) $f(x)$ has three real roots if $a<-4$

(d) $f(x)$ has three real roots if $-4<a<4$

30. Let $f:R\to(0,\infty)$ and $g:R\to R$, be twice differentiable functions such that f'' and g'' are continuous functions on . Suppose $f'(2)=g(2)=0$, $f''(2)\ne 0$ and $g'(2)\ne 0$. If then **(2016)**

(a) f has a local minimum at x = 2

(b) f has a local maximum at x = 2

(c) $f''(2)=f(2)$

(d) $f(x)-f''(x)=0$ for at least one $x\in$

31. Let f: $R\to R$ be given by

$$f(x)=\begin{cases} x^5+5x^4+10x^3+10x^2+3x+1 & x<0 \\ x^2-x+1 & 0\le x<1 \\ (2/3)x^3-4x^2+7x-(8/3) & 1\le x<3 \\ (x-2)ln(x-2)-x+(10/3) & x\ge 3 \end{cases}$$

Then which of the following options is/are correct? **(2019)**

(a) f' is not differentiable at x=1

(b) f is increasing on $(-\infty,0)$

(c) f is onto

(d) f' has a local maximum at x=1

32. Let $f: R \to R$ be given by $f(x) = (x-1)(x-2)(x-5)$. Define $f(x) = \int_0^x f(t)\, dt,\ x > 0$. Then which of the following options is/are correct? **(2019)**

(a) $f(x)$ has a local maximum at $x = 2$

(b) $f(x)$ has a local minimum at $x = 1$

(c) $f(x)$ has two local maxima and one local minimum in $(0, \infty)$

(d) $f(x) \neq 0$, for all $x \in (0, 5)$

33. Let $f(x) = \dfrac{\sin \pi x}{x^2}, x > 0$.

Let $x_1 < x_2 < x_3 \ldots < x_n < \ldots$ be all points of local maximum of f and $y_1 < y_2 < y_3 < \ldots < y_n < \ldots$ be all the points of local minimum of f Then which of the following options is/are correct? **(2019)**

(a) $|x_n - y_n| > 1$ for every n

(b) $x_1 < y_1$

(c) $x_{n+1} - x_n > 2$ for every n

(d) $x_n \in \left(2n, 2n + \dfrac{1}{2}\right)$ for every n

34. Let $f : R \to R$ be defined by $f(x) = \dfrac{x^2 - 3x - 6}{x^2 + 2x + 4}$

Then which of the following statements is (are) TRUE? **(2021)**

(a) f is decreasing in the interval (-2, -1)

(b) f is increasing in the interval (1, 2)

(c) f is onto

(d) Range of f is $\left[-\dfrac{3}{2}, 2\right]$

Numerical Value Type Questions

35. A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin. **(2002)**

36. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum. **(2003)**

37. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point P (6, 8) to the circle and the chord of contact is maximum. **(2003)**

38. If $f(x)$ is twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, where $a < b < c < d < e$, then the minimum number of zeroes of $g(x) = \{f'(x)\}^2 + f''(x) \cdot f(x)$ in the interval [a, e] is ? **(2006)**

39. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is .... **(2009)**

40. The maximum value of the expression

$\dfrac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is...... **(2010)**

41. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010(x - 2009)(x - 2010)^2(x - 2011)^3(x - 2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is ... **(2010)**

42. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is .... **(2011)**

43. Let p (x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p (1) = 6 and p (3) = 2, then p′ (0) is **(2012)**

44. Let $f: R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **(2012)**

45. A vertical line passing through the point (h, 0) intersects the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the ΔPQR, $\Delta_1 = \max\limits_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2 \le h \le 1} \Delta(h)$, then $\dfrac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2$ is equal to **(2013)**

46. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is **(2014)**

47. For a polynomial g (x) with real coefficient, let m_g denote the number of distinct real roots of g (x). Suppose S is the set of polynomials with real coefficient defined by

$S = \{(x^2 - 1)^2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3) : a_0, a_1, a_2, a_3 \in R\}$.

For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is .... **(2020)**

48. Let the function $f:(0,\pi)\to R$ be defined by

$$f(\theta)=(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^4$$

Suppose the function g has a local minimum at θ precisely when $\theta\in\{\lambda_1\pi,.....,\lambda_r\pi\}$

where $0<\lambda_1<......<\lambda_r<1$. Then the value of $\lambda_1+.....+\lambda_r$ is .... **(2020)**

Assertion & Reason

49. Consider the folloiwng statement S and R :

S : Both sin x & cos x are decreasing functions in the interval ($\pi/2$, π).

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).

Which of the following is true ? **(2000)**

(a) both S and R are wrong

(b) both S and R are correct, but R is not the correct explanation for S.

(c) S is correct and R is the correct explanation for S

(d) S is correct and R is wrong.

Match the Following

Each question has two columns. Four options are given representing matching of elements from Column-I and Column-II. Only one of these four options corresponds to a correct matching.For each question, choose the option corresponding to the correct matching.

50. Let the functions defined in Column I have domain ($-\pi/2$, $\pi/2$)

Column I	Column II
(A) x + sin x	**(p)** increasing
(B) sec x	**(q)** decreasing
	(r) neither increasing nor decreasing

(2008)

Paragraph Type Questions

Using the following passage, solve Q.51 to Q.53

Passage

Consider the function f : $(-\infty, \infty)\to(-\infty, \infty)$ defined by

$$f(x)=\frac{x^2-ax+1}{x^2+ax+1};\ 0<a<2.$$ **(2008)**

51. Which of the following is true ?

(a) $(2+a)^2 f''(1)+(2-a)^2 f''(-1)=0$

(b) $(2-a)^2 f''(1)-(2+a)^2 f''(-1)=0$

(c) $f'(1)f'(-1)=(2-a)^2$

(d) $f'(1)f'(-1)=-(2+a)^2$

52. Which of the following is true ?

(a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x=1$.

(b) $f(x)$ is increasing on $(-1, 1)$ and has a local maximum at $x=1$.

(c) $f(x)$ is increasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x=1$.

(d) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x=1$.

53. Let $g(x)=\int_0^{e^x}\frac{f'(t)}{1+t^2}dt$. Which of the following is true ?

(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$

(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$

(d) $g'(x)$ does not change sign $(-\infty, \infty)$

Using the following passage, solve Q.54 to Q.56

Passage

Consider the polynomial $f(x)=1+2x+3x^2+4x^3$. Let s be the sum of all distinct real roots of f (x) and let $t=|s|$ **(2010)**

54. The real numbers s lies in the interval

(a) $\left(-\frac{1}{4},0\right)$ (b) $\left(-11,-\frac{3}{4}\right)$

(c) $\left(-\frac{3}{4},-\frac{1}{2}\right)$ (d) $\left(0,\frac{1}{4}\right)$

55. The area bounded by the curve $y=f(x)$ and the lines $x=0$, $y=0$ and $x=t$, lies in the interval

(a) $\left(\frac{3}{4},3\right)$ (b) $\left(\frac{21}{64},\frac{11}{16}\right)$

(c) (9, 10) (d) $\left(0,\frac{21}{64}\right)$

56. The function $f'(x)$ is

(a) increasing in $\left(-t,-\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4},t\right)$

(b) decreasing in $\left(-t,-\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4},t\right)$

(c) increasing in $(-t, t)$

(d) decreasing in $(-t, t)$

Using the following passage, solve Q.57 and Q.58

Passage

Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in R$ and let

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t \right) f(t)\, dt \text{ for all } x \in (1, \infty)$$

57. Which of the following is true ? **(2012)**

(a) g is increasing on $(1, \infty)$

(b) g is decreasing on $(1, \infty)$

(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$

(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

58. Consider the statements

P : There exists some $x \in R$ such that

$f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in R$ such that

$2f(x) + 1 = 2x(1 + x)$

Then, **(2012)**

(a) Both P and Q are true (b) P is true and Q is false

(c) P is false and Q is true (d) Both P and Q are false

Using the following passage, solve Q.59 and Q.60

Passage

Let $f : [0, 1] \to R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x, x \in [0, 1]$.

59. Which of the following is true for $0 < x < 1$? **(2013)**

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

60. If the function $e^{-x} f(x)$ assumes its minimum in the interval $[0, 1]$ at $x = \frac{1}{4}$, which of the following is true ? **(2013)**

(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$

(b) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$

(d) $f'(x) < f(x), \frac{3}{4} < x < 1$

Using the following passage, solve Q.61 to Q.63

Passage

Let $f(x) = x + \log_e x - x \log_e x,\ x \in (0, \infty)$.

Column 1 contains information about zeros of f(x), f'(x) and f''(x).

Column 2 contains information about the limiting behavior of f(x), f'(x) and f''(x) at infinity.

Column 3 contains information about increasing/decreasing nature of f(x) and f'(x).

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in $(0, 1)$
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in $(0, 1)$
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

(2017)

61. Which of the following options is the only CORRECT combination ?

(a) (I) (ii) (R) (b) (IV) (i) (S)

(c) (III) (iv) (P) (d) (II) (iii) (S)

62. Which of the following options is the only CORRECT combination ?

(a) (I) (i) (P) (b) (II) (ii) (Q)

(c) (III) (iii) (R) (d) (IV) (iv) (S)

63. Which of the following options is the only INCORRECT combination ?

(a) (II) (iii) (P) (b) (I) (iii) (P)

(c) (III) (i) (R) (d) (II) (iv) (Q)

Using the following passage, solve Q.64 and Q.65

Passage

Let $f_1:(0,\infty)\to R$ and $f_2:(0,\infty)\to R$ be defined by

$$f_1(x)=\int_0^x\prod_{j=1}^{21}(t-j)^j\,dt,\ x>0$$

and $f_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450, x>0,$

Where, for any positive integer n and real numbers $a_1, a_2, \ldots\ldots, a_n, \prod_{i=1}^{n}a_i$ denotes the product of $a_1, a_2, \ldots\ldots, a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i=1, 2$, in the interval $(0, \infty)$

(2021)

64. The value of $2m_1+3n_1+m_1n_1$ is-------.

65. The value of $6m_2+4n_2+8m_2n_2$ is ---------.

Text

66. Let $-1\le p\le 1$. Show that the equation $4x^3-3x-p=0$ has a unique root in the interval $\left[\frac{1}{2},1\right]$ and identify it. **(2001)**

67. If $P(1)=0$ and $\frac{dP(x)}{dx}>P(x)$ for all $x\ge 1$, then prove that $P(x)>0$ for all $x>1$. **(2003)**

68. Using the relation $2(1-\cos x)<x^2$, $x\ne 0$ or otherwise, prove that $\sin(\tan x)\ge x,\ \forall\ x\in\left[0,\frac{\pi}{4}\right]$. **(2003)**

69. Prove that $\sin x+2x\ge\frac{3x\,.\,(x+1)}{\pi}\ \forall\ x\in\left[0,\frac{\pi}{2}\right]$.

(Justify the inequality, if any used). **(2004)**

70. If $|f(x_1)-f(x_2)|\le(x_1-x_2)^2$, for all $x_1, x_2\in R$. Find the equation of tangent to the curve $y=f(x)$ at the point $(1, 2)$. **(2005)**

Answer Key

CHAPTER -4 | APPLICATION OF DERIVATIVE

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

1. (a)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (d)
6. (c)	**7.** (a)	**8.** (d)	**9.** (c)	**10.** (c)
11. (c)	**12.** (c)	**13.** (a)	**14.** (d)	**15.** (a)
16. (b)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (c)
21. (b)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (a)
26. (c)	**27.** (a)	**28.** (d)	**29.** (b)	**30.** (a)
31. (a)	**32.** (a)	**33.** (d)	**34.** (a)	**35.** (c)
36. (d)	**37.** (b)	**38.** (b)	**39.** (c)	**40.** (a)
41. (a)	**42.** (a)	**43.** (c)	**44.** (b)	**45.** (b)
46. (d)	**47.** (d)	**48.** (c)	**49.** (b)	**50.** (c)
51. (b)	**52.** (b)	**53.** (a)	**54.** (d)	**55.** (c)
56. (a)	**57.** (a)	**58.** (b)	**59.** (d)	**60.** (b)
61. (c)	**62.** (b)	**63.** (c)	**64.** (c)	
65. (169.65)		**66.** (1.5)	**67.** (-3)	**68.** (5.2)
69. (12.57)	**70.** (17.32)	**71.** (502.65)		**72.** (1)
73. (45)	**74.** (17.32)	**75.** (-42)	**76.** (0.07)	
77. (-3)	**78.** (1)	**79.** (2.12)	**80.** (0.1)	

EXERCISE - 2 : PREVIOUS YEAR JEE MAIN QUESTIONS

1. (b)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (b)
6. (a)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (c)
11. (d)	**12.** (a)	**13.** (a)	**14.** (a)	**15.** (d)
16. (a)	**17.** (b)	**18.** (c)	**19.** (c)	**20.** (c)
21. (c)	**22.** (d)	**23.** (d)	**24.** 3.00	**25.** (d)
26. (b)	**27.** (a)	**28.** (a)	**29.** (b)	**30.** (b)
31. (d)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (c)
36. 122.00	**37.** (a)	**38.** (c)	**39.** (d)	**40.** (b)
41. (c)	**42.** (c)	**43.** (d)	**44.** (c)	**45.** (d)
46. (a)	**47.** (d)	**48.** (a)	**49.** (c)	**50.** (a)
51. 5	**52.** (a)	**53.** (d)	**54.** (c)	**55.** (b)
56. (d)	**57.** (a)	**58.** (d)	**59.** (b)	**60.** 4.00
61. (b)	**62.** (3.00)	**63.** (d)	**64.** (c)	**65.** (c)
66. (c)	**67.** (9.00)	**68.** (c)	**69.** (b)	**70.** (a)
71. (d)	**72.** (a)	**73.** (144.00)		**74.** (a)
75. (4.00)	**76.** (d)	**77.** (d)	**78.** (2.00)	**79.** (c)
80. (b)	**81.** (b)	**82.** (5.00)	**83.** (b)	**84.** (d)
85. (c)	**86.** (a)	**87.** (c)	**88.** (d)	**89.** (3.0)
90. (a)	**91.** (36.00)	**92.** (a)	**93.** (a)	**94.** (4.0)
95. (a)	**96.** (b)	**97.** (2.00)	**98.** (d)	**99.** (c)
100. (22.00)				

ANSWER KEY

CHAPTER -4 | APPLICATION OF DERIVATIVE

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

1. (c)	**2.** (d)	**3.** (a)	**4.** (c)	**5.** (c)
6. (b)	**7.** (d)	**8.** (c)	**9.** (a)	**10.** (a)
11. (d)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (b)
16. (b)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (b)
21. (d)	**22.** (a)	**23.** (a)	**24.** (a)	**25.** (d)
26. (c)	**27.** (c)	**28.** (b)	**29.** (c)	**30.** (a)
31. (c)	**32.** (d)	**33.** (d)	**34.** (b)	**35.** (a)
36. (a)	**37.** (b)	**38.** (a)	**39.** (d)	**40.** (c)
41. (b)	**42.** (a)	**43.** (a)	**44.** (c)	**45.** (a,c)
46. (b,c)	**47.** (a,b)	**48.** (b,c)	**49.** (b,c)	**50.** (b,d)
51. (a,b,c,d)		**52.** (a,c)	**53.** (a,c,d)	**54.** (a,d)
55. (a,b,c,d)		**56.** (a,c)	**57.** (a,b)	
58. (a,b,c)	**59.** (a,c)	**60.** (a,c,d)	**61.** (a,b)	**62.** (b,c)
63. (a,b,c,d)		**64.** (a,c)	**65.** (a,b,c)	
66. (a,b,d)	**67.** (2)	**68.** (4)	**69.** (16)	**70.** (11)
71. (4)	**72.** (6)	**73.** (25)	**74.** (18)	**75.** (d)
76. (a)	**77.** (c)	**78.** (d)	**79.** (a)	**80.** (a)
81. (a)	**82.** (c)	**83.** (d)	**84.** (c)	**85.** (d)
86. (d)	**87.** (d)	**88.** (a)	**89.** (a)	**90.** (c)
91. (a)	**92.** (a)	**93.** (c)	**94.** (a)	**95.** (a)
96. (d)	**97.** (a)	**98.** (a)	**99.** (d)	**100.** (c)
101. (d)	**102.** (b)	**103.** (b)	**104.** (b)	**105.** (b)
106. (c)	**107.** (b)	**108.** (b)	**109.** (b)	**110.** (d)

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

1. (b)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (a)
6. (a)	**7.** (a)	**8.** (d)	**9.** (d)	**10.** (d)
11. (a)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (c)
16. (d)	**17.** (a)	**18.** (c)	**19.** (c)	**20.** (d)
21. (c)	**22.** (c)	**23.** (b,c)	**24.** (a,b)	
25. (b,c,d)	**26.** (b,c)	**27.** (a,c)	**28.** (a,b)	**29.** (b,d)
30. (a,d)	**31.** (a,c,d)	**32.** (a,b,d)	**33.** (a,c,d)	**34.** (a,b)
35. (18)	**36.** (2,1)	**37.** (5 unit)		**38.** (6)
39. (7)	**40.** (2)	**41.** (1)	**42.** (2)	**43.** (2)
44. (5)	**45.** (9)	**46.** (8)	**47.** (5.00)	
48. (0.50)	**49.** (d)	**50.** (A−p; B−r)		
51. (a)	**52.** (a)	**53.** (b)	**54.** (c)	**55.** (a)
56. (b)	**57.** (b)	**58.** (c)	**59.** (d)	**60.** (c)
61. (d)	**62.** (b)	**63.** (c)	**64.** (57.00)	
65. (6.00)	**66.** $\cos\left(\frac{1}{3}\cos^{-1}p\right)$		**70.** $y-2=0$	