CHAPTER 7

Quadratic Equation

SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS

(a) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(b) The expression $b^2 - 4\,ac = D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then;

(i) $\alpha + \beta = -b/a$

(ii) $\alpha\beta = c/a$

(iii) $|\alpha - \beta| = \sqrt{D}/|a|$

(d) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ *i.e.*

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ *i.e.* x^2 –(sum of roots)x + product of roots = 0.

Nature of Roots

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal).

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + i\,q$ is one root of a quadratic equation, then the other root must be the conjugate $p - i\,q$ & vice versa.

$(p, q \in R \;\&\; i = \sqrt{-1})$.

(*b*) Consider the quadratic equation $ax^2 + bx + c = 0$

where $a, b, c \in Q$ & $a \neq 0$ then;

(*i*) If D is a perfect square, then roots are rational.

(*ii*) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$. (if a, b, c are rational)

Common Roots of Two Quadratic Equations

(*a*) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is

$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

(*b*) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

Roots Under Particular Cases

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(*a*) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(*b*) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(*c*) If $a = c \Rightarrow$ roots are reciprocal to each other

(*d*) If $\left.\begin{matrix} a > 0\ c < 0 \\ a < 0\ c > 0 \end{matrix}\right\} \Rightarrow$ roots are of opposite signs.

(*e*) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(*f*) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive.

(g) If sign of a = sign of $b \neq$ sign of $c \Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of $c \neq$ sign of $a \Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSION

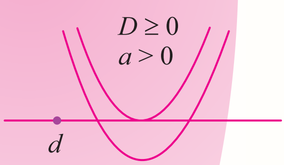
Maximum & Minimum Values of expression $y = ax^2 + bx + c$ is $\frac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

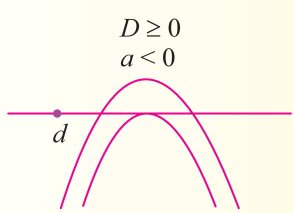
$$y \in \left[\frac{-D}{4a}, \infty\right) \text{ if } a > 0 \quad \& \quad y \in \left(-\infty, \frac{-D}{4a}\right] \text{ if } a < 0.$$

LOCATION OF ROOTS

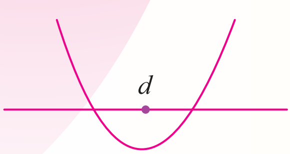
Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R, a \neq 0$

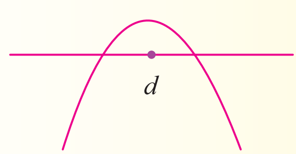
(a) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are $\boldsymbol{D \geq 0; a.f(d) > 0}$ **&** $\boldsymbol{(-b/2a) > d}$**.**





(b) Conditions for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is $\boldsymbol{a.f(d) < 0}$**.**

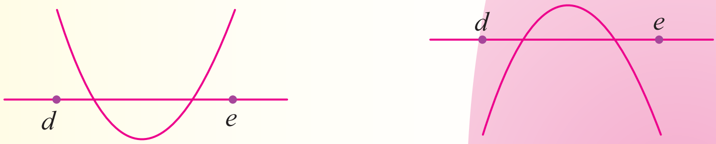




(*c*) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d,e) *i.e.*, $d < x < e$ is $f(d).\,f(e) < 0$.

(*d*) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$D \geq 0$; $a.f(d) > 0$ & $af(e) > 0$; $d < (-b/2a) < e$



GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES

$f(x, y) = ax^2 + 2\,hxy + by^2 + 2gx + 2\,fy + c$ may be resolved into two linear factors if;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ OR } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

THEORY OF EQUATIONS

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots\alpha_n$ are the roots of the equation;

$f(x) = a_0\,x^n + a_1\,x^{n-1} + a_2\,x^{n-2} + \ldots\ldots + a_{n-1}\,x + a_n = 0$ where $a_0, a_1, \ldots\ldots a_n$ are constants $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0}, \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0},$$

$$\alpha_1\alpha_2\alpha_3\ldots\ldots\ldots\alpha_n = (-1)^n \frac{a_n}{a_0}$$

NOTES

(*i*) Every odd degree equation has at least one real root whose sign is opposite to that of its last term, when coefficient of highest degree term is (+)ve {If not then make it (+)ve}.

Ex. $x^3 - x^2 + x - 1 = 0$

(*ii*) Even degree polynomial whose last term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(*iii*) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.

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