1.COMPLEX NUMBERS

1. DEFINITION : Complex numbers are definited as expressions of the form a + ib where a, b ∈ R & $i = \sqrt{-1}$. It is denoted by z i.e. z = a + ib. 'a' is called as real part of z (Re z) and 'b' is called as imaginary part of z (Im z). www.MathsBySuhag.com , www.TekoClasses.com

EVERY COMPLEX NUMBER CAN BE REGARDED AS

Purely real	Purely imaginary	Imaginary
if b = 0	if a = 0	if b ≠ 0

Note :

(a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complete Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is both purely real as well as purely imaginary but not imaginary.

(c) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(d) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.www. Maths By Suhag .com

2. CONJUGATE COMPLEX :

If z = a + ib then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

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(i) $z + \bar{z} = 2\,\mathrm{Re}(z)$ **(ii)** $z - \bar{z} = 2i\,\mathrm{Im}(z)$ **(iii)** $z\bar{z} = a^2 + b^2$ which is real

(iv) If z lies in the 1st quadrant then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.

3. ALGEBRAIC OPERATIONS :

The algebraic operations on complex numbers are similiar to those on real numbers treating *i* as a polynomial. Inequalities in complex numbers are not defined. There is no validity if we say that complex number is positive or negative.

e.g. z > 0, 4 + 2i < 2 + 4 i are meaningless .

However in real numbers if $a^2 + b^2 = 0$ then a = 0 = b but in complex numbers,

$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.www.MathsBySuhag.com , www.TekoClasses.com

4. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts coincide.

5. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS:

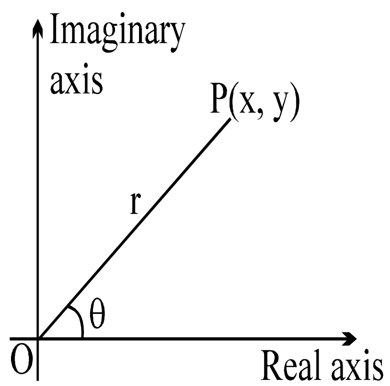
(a) Cartesian Form (Geometric Representation) :

Every complex number z = x + i y can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).

length OP is called modulus of the complex number denoted by $|z|$ & θ is called the argument or amplitude

eg . $|z| = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\frac{y}{x}$ (angle made by OP with positive x–axis)



NOTE :(i) $|z|$ is always non negative . Unlike real numbers $|z| = \begin{bmatrix} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{bmatrix}$ is **not correct**

(ii) Argument of a complex number is a many valued function . If θ is the argument of a complex number then 2 nπ + θ ; n ∈ I will also be the argument of that complex number. Any two arguments of a complex number differ by 2nπ.

(iii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.

(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) By specifying the modulus & argument a complex number is defined completely. For the complex number 0 + 0 i the argument is not defined and this is the only complex number which is given by its modulus.www.MathsBySuhag.com , www.TekoClasses.com

(vi) There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

(b) Trignometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note: cos θ + i sin θ is also written as CiS θ.

Also $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \frac{e^{ix} - e^{-ix}}{2}$ are known as Euler's identities.

(c) Exponential Representation :

$z = re^{i\theta}$; $|z| = r$; $\arg z = \theta$; $\bar{z} = re^{-i\theta}$

6. IMPORTANT PROPERTIES OF CONJUGATE / MODULI / AMPLITUDE :

If $z, z_1, z_2 \in C$ then ;

(a) $z + \bar{z} = 2\,\mathrm{Re}(z)$; $z - \bar{z} = 2i\,\mathrm{Im}(z)$; $\overline{(\bar{z})} = z$; $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$;

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; $\overline{z_1 z_2} = \bar{z}_1 \,.\, \bar{z}_2$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$; $z_2 \ne 0$

(b) $|z| \ge 0$; $|z| \ge \mathrm{Re}(z)$; $|z| \ge \mathrm{Im}(z)$; $|z| = |\bar{z}| = |-z|$; $z\bar{z} = |z|^2$;

$|z_1 z_2| = |z_1| \,.\, |z_2|$; $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \ne 0$, $|z^n| = |z|^n$;

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$www.MathsBySuhag.com , www.TekoClasses.com

$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$ **[TRIANGLE INEQUALITY]**

(c) **(i)** $\mathrm{amp}(z_1 \,.\, z_2) = \mathrm{amp}\, z_1 + \mathrm{amp}\, z_2 + 2k\pi$. $k \in I$

(ii) $\mathrm{amp}\left(\frac{z_1}{z_2}\right) = \mathrm{amp}\, z_1 - \mathrm{amp}\, z_2 + 2k\pi$; $k \in I$

(iii) $\mathrm{amp}(z^n) = n\,\mathrm{amp}(z) + 2k\pi$.

where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(7) VECTORIAL REPRESENTATION OF A COMPLEX :

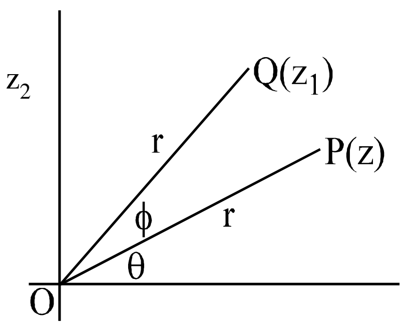
Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

NOTE :(i) If $\vec{OP} = z = re^{i\theta}$ then $\vec{OQ} = z_1 = re^{i(\theta + \phi)} = z \,.\, e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}e^{i\phi}$

(ii) If A, B, C & D are four points representing the complex numbers z_1, z_2, z_3 & z_4 then

AB || CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

AB ⊥ CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary]



(iii) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then **(a)** $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$ **(b)** $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

8. DEMOIVRE'S THEOREM :

Statement : cos nθ + i sin nθ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$ ¥ n ∈ Q. The theorem is very useful in determining the roots of any complex quantity

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

9. CUBE ROOT OF UNITY :(i)The cube roots of unity are $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$.

(ii) If w is one of the imaginary cube roots of unity then $1 + w + w^2 = 0$. In general $1 + w^r + w^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are :

$\cos 0 + i \sin 0$; $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the verties of an equilateral triangle.www.MathsBySuhag.com , www.TekoClasses.com

(v) The following factorisation should be remembered :
(a, b, c $\in$ R & ω is the cube root of unity)
$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;
$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;
$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

10. n^{th} ROOTS OF UNITY :www.MathsBySuhag.com , www.TekoClasses.com
If $1, \alpha_1, \alpha_2, \alpha_3 ..... \alpha_{n-1}$ are the n, n^{th} root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$ &

(ii) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n
$= n$ if p is an integral multiple of n

(iii) $(1 - \alpha_1)(1 - \alpha_2) ...... (1 - \alpha_{n-1}) = n$ &
$(1 + \alpha_1)(1 + \alpha_2) .... (1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1 . \alpha_1 . \alpha_2 . \alpha_3 .... \alpha_{n-1} = 1$ or -1 according as n is odd or even.

11. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{n+1}{2}\right)\theta$.

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\frac{n+1}{2}\right)\theta$.

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

12. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

(A) If z_1 & z_2 are two complex numbers then the complex number $z = \frac{nz_1 + mz_2}{m+n}$ divides the joins of z_1 & z_2 in the ratio m : n.

Note:(i) If a , b , c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(ii) If the vertices A, B, C of a Δ represent the complex nos. z_1, z_2, z_3 respectively, then :

(a) Centroid of the $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$: **(b)** Orthocentre of the $\Delta ABC =$

$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \textbf{ OR } \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

(c) Incentre of the $\Delta ABC = (az_1 + bz_2 + cz_3) \div (a + b + c)$.

(d) Circumcentre of the ΔABC = :
$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(B) $amp(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the x– axis.

(C) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(D) The equation of a line joining z_1 & z_2 is given by ;
$z = z_1 + t(z_1 - z_2)$ where t is a perameter.www.MathsBySuhag.com , www.TekoClasses.com

(E) $z = z_1(1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to oz_1.

(F) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}$ = 0. This is also the condition for three complex numbers to be collinear.

(G) Complex equation of a straight line through two given points z_1 & z_2 can be written as $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$, which on manipulating takes the form as
$\bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

(H) The equation of circle having centre z_0 & radius ρ is : $|z - z_0| = \rho$ or
$z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$, r is real centre $-\alpha$ & radius $\sqrt{\alpha\bar{\alpha} - r}$. Circle will be real if $\alpha\bar{\alpha} - r \geq 0$.

(I) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is :

(i) $\arg\frac{z - z_2}{z - z_1} = \pm\frac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

(J) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is, the number $\frac{z_3 - z_1}{z_3 - z_2} . \frac{z_4 - z_2}{z_4 - z_1}$ is real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real $\Rightarrow \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \frac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$

13.(a) Reflection points for a straight line : Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if ; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

(b) Inverse points w.r.t. a circle :www.MathsBySuhag.com , www.TekoClasses.com
Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if :
(i) the point O, P, Q are collinear and on the same side of O. **(ii)** $OP . OQ = \rho^2$.
Note that the two points z_1 & z_2 will be the inverse points w.r.t. the circle
$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$.

14. PTOLEMY'S THEOREM :www.MathsBySuhag.com , www.TekoClasses.com
It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides.
i.e. $|z_1 - z_3|\ |z_2 - z_4| = |z_1 - z_2|\ |z_3 - z_4| + |z_1 - z_4|\ |z_2 - z_3|$.

15. LOGARITHM OF A COMPLEX QUANTITY :

(i) $\text{Log}_e(\alpha + i\beta) = \frac{1}{2}\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$ where $n \in I$.

(ii) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi + \frac{\pi}{2}\right)}$, $n \in I$.