Continuity and Differentiability

No. of Questions	Maximum Marks	Time
30	120	1 Hour

GENERAL INSTRUCTIONS

- This test contains 30 MCQ's. For each question only one option is correct. Darken the correct circle/ bubble in the Response Grid provided on each page.
- You have to evaluate your Response Grids yourself with the help of solutions provided at the end of this book.
- Each correct answer will get you 4 marks and 1 mark shall be deduced for each incorrect answer. No mark will be given/ deducted if no bubble is filled. Keep a timer in front of you and stop immediately at the end of 60 min.
- The sheet follows a particular syllabus. Do not attempt the sheet before you have completed your preparation for that syllabus.
- After completing the sheet check your answers with the solution booklet and complete the Result Grid. Finally spend time to analyse your performance and revise the areas which emerge out as weak in your evaluation.

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1. Let $f : R \to R$ be a function such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}, \ f(0)=0 \ \text{and} \ f'(0)=3. \ \text{Then}$$

   (a) $f(x)$ is a quadratic function
   (b) $f(x)$ is continuous but not differentiable
   (c) $f(x)$ is differentiable in R
   (d) $f(x)$ is bounded in R

2. If the function $f(x) = \begin{cases} (\cos x)^{1/x}, x \neq 0 \\ k, x = 0 \end{cases}$ is continuous at $x=0$,

   then the value of k is
   (a) 1
   (b) −1
   (c) 0
   (d) e

3. If $x^2 + y^2 = a^2$ and $k = \dfrac{1}{a}$, then k is equal to

   (a) $\dfrac{|y''|}{\sqrt{1+y'}}$
   (b) $\dfrac{|y''|}{\sqrt{\left(1+y'^2\right)^3}}$
   (c) $\dfrac{2y''}{\sqrt{1+y'^2}}$
   (d) $\dfrac{y''}{2\sqrt{\left(1+y'^2\right)^3}}$

4. Let a function $f : R \to R$ satisfy the equation $f(x+y) = f(x) + f(y)$ for all x, y, If the function $f(x)$ is continuous at x = 0, then
   (a) $f(x) = 0$ for all x
   (b) $f(x)$ is continuous for all positive real x
   (c) $f(x)$ is continuous for all x
   (d) None of these

RESPONSE GRID	1. ⓐⓑⓒⓓ	2. ⓐⓑⓒⓓ	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ

5. Differential coefficient of $\tan^{-1}\dfrac{2x}{1-x^2}$ with respect to $\sin^{-1}\dfrac{2x}{1+x^2}$ will be

(a) 1   (b) -1

(c) $-1/2$ (d) x

6. The values of a, b and c which make the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & x < 0 \\ c & , x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , x > 0 \end{cases}$$

continuous at x = 0 are

(a) $a = \dfrac{-3}{2}$, $c = \dfrac{1}{2}$, $b = 0$ (b) $a = \dfrac{3}{2}$, $c = \dfrac{1}{2}$, $b \ne 0$

(c) $a = \dfrac{-3}{2}$, $c = \dfrac{1}{2}$, $b \ne 0$ (d) None of these

7. Let f(x), g(x) be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = [f(x)]^2 + [g(x)]^2$. If $h(0) = 5$, then $h(10) =$
(a) 10   (b) 5
(c) 15   (d) None of these

8. The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer function less than or equal to y), is discontinuous at :
(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

9. If $f(x) = \lim\limits_{n\to\infty} \dfrac{\tan \pi x^2 + (x+1)^n \sin x}{x^2 + (x+1)^n}$, then

(a) f is continuous at x = 0
(b) f is differentiable at x = 0
(c) f is continuous but not differentiable at x = 0
(d) None of these

10. The value of p for which the function

$$f(x) = \begin{cases} \dfrac{(4^x - 1)^3}{\sin\dfrac{x}{p} \log\left[1 + \dfrac{x^2}{3}\right]}, & x \ne 0 \\ 12(\log 4)^3, & x = 0 \end{cases}$$

may be continuous at x = 0, is
(a) 1   (b) 2
(c) 3   (d) None of these

11. In the mean value theorem $\dfrac{f(b) - f(a)}{b - a} = f'(c)$, if $a = 0$, $b = 1/2$ and $f(x) = x(x-1)(x-2)$, the value of c is –

(a) $1 - \dfrac{\sqrt{15}}{6}$ (b) $1 + \sqrt{15}$

(c) $1 - \dfrac{\sqrt{21}}{6}$ (d) $1 + \sqrt{21}$

12. If $y = (1 + 1/x)^x$, then $\dfrac{2\sqrt{y_2(2) + 1/8}}{(\log 3/2 - 1/3)}$ is equal to –

(a) 3   (b) 4
(c) 1   (d) 2

13. If $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} (\log a)^n$, then at x = 0, f(x)

(a) has no limit
(b) is discontinuous
(c) is continuous but not differentiable
(d) is differentiable

14. Let $f(x) = g(x) \cdot \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$, where g is a continuous function then $\lim\limits_{x\to 0} f(x)$ does not exist if

(a) g(x) is any constant function
(b) g(x) = x
(c) $g(x) = x^2$
(d) g(x) = x h(x), where h(x) is a polynomial

15. If $f(x) = \cot^{-1}\left(\dfrac{x^x - x^{-x}}{2}\right)$, then $f'(1)$ is equal to

(a) -1 (b) 1
(c) $\log 2$ (d) $-\log 2$

16. Let $f: R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is NOT differentiable is
(a) $\{-1, 1\}$ (b) $\{-1, 0\}$
(c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

17. The function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \dfrac{\pi}{4}$. The value of $f\left(\dfrac{\pi}{4}\right)$ so that f is continuous at $x = \dfrac{\pi}{4}$ is

(a) $\sqrt{e}$ (b) $\dfrac{1}{\sqrt{e}}$

(c) 2 (d) None of these

18. If g is the inverse function of f and $f'(x) = \sin x$, then $g'(x)$ is

(a) $\csc\{g(x)\}$ (b) $\sin\{g(x)\}$

(c) $-\dfrac{1}{\sin\{g(x)\}}$ (d) $\cos\{g(x)\}$

19. Let $f(x) = \begin{cases} \dfrac{1 - \sin^3 x}{3\cos^2 x}, & x < \dfrac{\pi}{2} \\ p, & x = \dfrac{\pi}{2} \\ \dfrac{q(1 - \sin x)}{(\pi - 2x)^2}, & x > \dfrac{\pi}{2} \end{cases}$

If $f(x)$ is continuous at $x = \dfrac{\pi}{2}$, $(p, q) =$

(a) $(1, 4)$ (b) $\left(\dfrac{1}{2}, 2\right)$

(c) $\left(\dfrac{1}{2}, 4\right)$ (d) None of these

20. Which of the following functions is differentiable at $x = 0$?
(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$
(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

21. Let $f(x) = \dfrac{(e^x - 1)^2}{\sin\left(\dfrac{x}{a}\right)\log\left(1 + \dfrac{x}{4}\right)}$ for $x \neq 0$, and $f(0) = 12$. If f is continuous at $x = 0$, then the value of a is equal to
(a) 1 (b) -1
(c) 2 (d) 3

22. If the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots + a_1 x = 0$

$a_1 \neq 0, n \geq 2$, has a positive root $x = \alpha$, then the equation

$n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots\ldots + a_1 = 0$ has a positive root, which is
(a) greater than α
(b) smaller than α
(c) greater than or equal to α
(d) equal to α

23. If $f(x) = \begin{cases} \dfrac{\sin[x]}{[x] + 1}, & \text{for } x > 0 \\ \dfrac{\cos\dfrac{\pi}{2}[x]}{[x]}, & \text{for } x < 0 \\ k, & \text{at } x = 0 \end{cases}$; where $[x]$ denotes the greatest integer less than or equal to x, then in order that f be continuous at $x = 0$, the value of k is
(a) equal to 0 (b) equal to 1
(c) equal to -1 (d) indeterminate

24. If $f(x) = \dfrac{\tan[x]\pi}{[1 + |\log(\sin^2 x + 1)|]}$, where $[.]$ denotes the greatest integer function and $|.|$ stands for the modulus of the function, then $f(x)$ is
(a) discontinuous $\forall x \in I$
(b) continuous $\forall x$
(c) non differentiable $\forall x \in I$
(d) a periodic function with fundamental period 1.

25. If $\sqrt{1-x^{2n}} + \sqrt{1-y^{2n}} = a(x^n - y^n)$, then $\sqrt{\dfrac{1-x^{2n}}{1-y^{2n}}}\dfrac{dy}{dx}$ is equal to

(a) 1
(b) x/y
(c) $\dfrac{x^{n-1}}{y^{n-1}}$
(d) None of these

26. The equation $e^{x-8} + 2x - 17 = 0$ has
(a) two real roots
(b) one real root
(c) eight real roots
(d) four real roots

27. If $f''(x) = -f(x)$ and $g(x) = f'(x)$ and

$$F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2 \text{ and given that}$$

$F(5) = 5$, then $F(10)$ is equal to –
(a) 5
(b) 10
(c) 0
(d) 15

28. Choose the correct statements –
(a) If $f'(a^+)$ and $f'(a^-)$ exist finitely at a point, then f is continuous at $x = a$.
(b) The function $f(x) = 3 \tan 5x - 7$ is differentiable at all points in its domain.

(c) The existence of $\lim\limits_{x \to c}(f(x) + g(x))$ does not imply of existence of $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$.
(d) All of these

29. **Statement-1 :** If g (x) is a differentiable function g (1) ≠ 0, g (–1) ≠ 0 and Rolles theorem is not applicable to $f(x) = \dfrac{x^2 - 1}{g(x)}$ in $[-1,1]$, then g (x) has atleast one root in (–1, 1).

Statement-2 : If $f(a) = f(b)$, then Rolles theorem is applicable for $x \in (a, b)$.

(a) Statement- 1 is True, Statement-2 is True, Statement-2 is a correct explanation for Statement -1
(b) Statement -1 is True, Statement -2 is True ; Statement-2 is NOT a correct explanation for Statement - 1
(c) Statement -1 is False, Statement -2 is True
(d) Statement - 1 is True, Statement- 2 is False

30. **Statement-1 :** $f(x) = |x| \sin x$, is differentiable at $x = 0$.

Statement-2 : If f (x) is not differentiable and g (x) is differentiable at $x = a$, then $f(x) . g(x)$ can still be differentiable at $x = a$.

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement -1
(b) Statement -1 is True, Statement -2 is True ; Statement-2 is NOT a correct explanation for Statement - 1
(c) Statement -1 is False, Statement -2 is True
(d) Statement - 1 is True, Statement- 2 is False

RESPONSE GRID	25. ⓐⓑⓒⓓ	26. ⓐⓑⓒⓓ	27. ⓐⓑⓒⓓ	28. ⓐⓑⓒⓓ	29. ⓐⓑⓒⓓ
	30. ⓐⓑⓒⓓ				

MATHEMATICS CHAPTERWISE SPEED TEST-79

Total Questions	30	Total Marks	120
Attempted		Correct	
Incorrect		Net Score	
Cut-off Score	41	Qualifying Score	60
Success Gap = Net Score – Qualifying Score			
Net Score = (Correct × 4) – (Incorrect × 1)			

Speed Test-79

1. (c) We have

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}, \quad f(0)=0 \text{ and } f'(0)=3$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{f(3x)+f(3h)}{3}-\dfrac{f(3x)+f(0)}{3}}{h} = \lim_{h\to 0}\frac{f(3h)-f(0)}{3h}=3$$

∴ f(x) = 3x + c, ∵ f(0) = 0 ⇒ c = 0

∴ f(x) = 3x

2. (a) $\lim\limits_{x\to 0}(\cos x)^{1/x} = k \Rightarrow \lim\limits_{x\to 0}\frac{1}{x}\log(\cos x) = \log k$

$$\Rightarrow \lim_{x\to 0}\frac{1}{x}\lim_{x\to 0}\log\cos x = \log k$$

$$\Rightarrow \lim_{x\to 0}\frac{1}{x}\times 0 = \log_e k \Rightarrow k = 1.$$

3. (b) $x^2+y^2=a^2 \Rightarrow 2x+2yy'=0 \Rightarrow y'=-x/y$

$$\Rightarrow yy'+x=0$$

$$\Rightarrow yy''+y'^2+1=0 \Rightarrow y=-\left(\frac{1+y'^2}{y''}\right) \quad \text{....(i)}$$

$$\therefore k=\frac{1}{a}=\left|\frac{1}{\sqrt{x^2+y^2}}\right|=\left|\frac{1}{y\sqrt{1+\frac{x^2}{y^2}}}\right|=\left|\frac{1}{y\sqrt{1+y_1^2}}\right| \quad \left[\because y'=-\frac{x}{y}\right]$$

$$=\left|\frac{-y''}{(1+y'^2)\sqrt{1+y'^2}}\right|=\frac{|y''|}{(1+y'^2)^{3/2}}$$

4. (c) Since f(x) is continuous at x = 0 ∴ $\lim\limits_{x\to 0}f(x)=f(0)$

Take any point x = a, then at x = a

$$\lim_{x\to a}f(x)=\lim_{h\to 0}f(a+h)$$

$$=\lim_{h\to 0}[f(a)+f(h)] \quad [\because f(x+y)=f(x)+f(y)]$$

$$=f(a)+\lim_{h\to 0}f(h)=f(a)+f(0)=f(a+0)=f(a)$$

∴ f(x) is continuous at x = a. Since x = a is any arbitrary point, therefore f(x) is continuous for all x.

5. (a) Let $u=\tan^{-1}\frac{2x}{1-x^2}$ ...... (i)

and $v=\sin^{-1}\frac{2x}{1+x^2}$ ...... (ii)

In equation (i) put, x = tan θ

$$\therefore u=\tan^{-1}\left[\frac{2\tan\theta}{1-}\right]^{-1}$$

$$\Rightarrow u=2\theta \Rightarrow \frac{du}{d\theta}=2 \text{ ...... (a)}$$

In equation (ii), put x = tan θ

$$\therefore v=\sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]=\sin^{-1}(\sin 2\theta)$$

$$\Rightarrow v=2\theta \Rightarrow \frac{dv}{d\theta}=2 \text{ .... (b)}$$

From equations (a) and (b),

$$\frac{du}{dv}=\frac{du}{d\theta}\times\frac{d\theta}{dv}=2\times\frac{1}{2}=1$$

∴ Required differential coefficient will be 1.

6. (c) In the definition of the function, b ≠ 0, then f(x) will be undefined in x > 0.

∵ f(x) is continuous at x = 0, ∴ LHL = RHL = f(0)

$$\Rightarrow \lim_{\substack{x\to 0\\x<0}}\frac{\sin(a+1)x+\sin x}{x}=\lim_{\substack{x\to 0\\x<0}}\frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}=c$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right)=\lim_{x\to 0}\frac{\sqrt{1+bx}-1}{bx}=c$$

$$\Rightarrow (a+1)+1=\lim_{x\to 0}\frac{(1+bx)-1}{bx(\sqrt{1+bx}+1)}=c$$

$$\Rightarrow a+2=\lim_{x\to 0}\frac{1}{\sqrt{1+bx}+1}=c$$

$$\Rightarrow a+2=\frac{1}{2}=c \quad \therefore a=-\frac{3}{2}, c=\frac{1}{2}, b\neq 0$$

7. (b) Since f'(x) = g(x), f'(x) = g'(x)

Put f'(x) = – f(x). Hence g'(x) = –f(x)

we have h'(x) = 2f(x) f'(x) + 2g(x) g'(x)

= 2[f(x) g(x) + g(x) [–f(x)]] = 2 [f(x) g(x) – f(x) g(x)] = 0

∴ h(x) = C, a constant

∴ h(0) = C i.e. C = 5

h(x) = 5 for all x. Hence h (10) = 5.

8. (d) $f(x)=[x]^2-[x^2]=(-1)^2-(0)^2=0, -1<x<0 \Rightarrow 0<x^2<1$

$=0-0=0, \ 0\le x<1$ and $=1-1=0, \ 1\le x<\sqrt{3}$

and $=1-3=-2, \ \sqrt{3}\le x<\sqrt{4}$

∴ From above it is clear that the function is discontinuous at

9. (d) $f(0) = \dfrac{0 + 1 \times 0}{0 + 1} = 0$

$\displaystyle \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \lim_{n \to \infty} \frac{\tan \pi x^2 + (x+1)^n \sin x}{x^2 + (x+1)^n}$

$= \displaystyle \lim_{x \to 0^-} \frac{\tan \pi x^2}{x^2} \quad (\text{If } x \to 0^-, x+1 < 1)$

$= \pi \hspace{3cm} \therefore \ \text{LHL} \neq f(0)$

$\therefore$ f(x) is not continuous at x = 0 hence not differentiable also.

10. (d) For $f(x)$ to be continuous at $x = 0$, we should have

$\displaystyle \lim_{\infty \to 0} f(x) = f(0) = 12 (\log 4)^3$

$\displaystyle \lim_{\infty \to 0} f(x) = \lim_{x \to 0} \left(\frac{4^x - 1}{x} \right)^3 \times \frac{\left(\dfrac{x}{p} \right)}{\left(\sin \dfrac{x}{p} \right)} \cdot \frac{p x^2}{\log \left(1 + \dfrac{1}{3} x^2 \right)}$

$= (\log 4)^3 \cdot 1 \cdot p \cdot \displaystyle \lim_{x \to 0} \left(\frac{x^2}{\dfrac{1}{3} x^2 - \dfrac{1}{18} x^4 + \dots} \right)$

$= 3p \, (\log 4)^3 \cdot$ Hence $p = 4$.

11. (c) $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$\Rightarrow 3c^2 - 6c + 2 = \dfrac{3/8 - 0}{1/2 - 0} = \dfrac{3}{4}$

$\Rightarrow c = 1 \pm \dfrac{\sqrt{21}}{6} \Rightarrow c = 1 + \dfrac{\sqrt{21}}{6} \notin \left(0, \dfrac{1}{2} \right) \Rightarrow c = 1 - \dfrac{\sqrt{21}}{6}$

12. (a) Let $y = \left(1 + \dfrac{1}{x} \right)^x$

Taking logarithm of both sides, we get

$\log y = x \left[\log \left(1 + \dfrac{1}{x} \right) \right]$

$\Rightarrow \dfrac{1}{y} y_1(x) = \dfrac{x^2}{x+1} \left(-\dfrac{1}{x^2} \right) + \log \left(1 + \dfrac{1}{x} \right)$

$= -\dfrac{1}{x+1} + \log \left(1 + \dfrac{1}{x} \right) \hspace{1cm} \dots\dots (1)$

Since, $y(2) = (1 + 1/2)^2 = 9/4$

so, $y_1(2) = (9/4) \left(-\dfrac{1}{3} + \log \dfrac{3}{2} \right)$

Again differentiate eq (1) w.r.t (x), we get

$\dfrac{y(x) y_2(x) - [y_1(x)]^2}{(y(x))^2} = \dfrac{1}{(1+x)^2} - \dfrac{1}{x(x+1)}$

By putting $x = 2$, we get

$\dfrac{y(2) y_2(2) - (y_1}{(y(2))^2} \quad$ ——

Now, put value of $y(2)$ and $y_1(2)$

$\Rightarrow y_2(2) = \left(\dfrac{9}{4} \right) \left(-\dfrac{1}{3} + \log \dfrac{3}{2} \right)^2 - \dfrac{1}{8}$

$4 \left(y_2(2) + \dfrac{1}{8} \right) = 9 \left(\log \dfrac{3}{2} - \dfrac{1}{3} \right)^2$

$\Rightarrow$ Required expression = 3

13. (d) We have, $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} (\log a)^n = \sum_{n=0}^{\infty} \frac{(x \log a)^n}{n!}$

$= e^{x \log a} = e^{\log a^x} = a^x$

$Lf'(0) = \displaystyle\lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{a^{-h} - 1}{-h} = \log_e a$

$Rf'(0) = \displaystyle\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{a^h - 1}{h} = \log_e a$

Since $Lf'(0) = Rf'(0)$, $\quad \therefore \ $ f(x) is differentiable at x = 0

Since every differentiable function is continuous, therefore, f(x) is continuous at x = 0.

14. (a) $\displaystyle\lim_{x \to 0^+} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = \lim_{x \to 0^+} \frac{1 - e^{-2/x}}{1 + e^{-2/x}} = 1$

and $\displaystyle\lim_{x \to 0^-} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = \lim_{x \to 0^-} \frac{e^{2/x} - 1}{e^{2/x} + 1} = -1.$

Hence $\displaystyle\lim_{x \to 0} f(x)$ exists if $\displaystyle\lim_{x \to 0} g(x) = 0$.

If $g(x) = a \neq 0$ (constant) then

$\displaystyle\lim_{x \to 0+} f(x) = a$ and $\displaystyle\lim_{x \to 0-} f(x) = -a.$

Thus $\displaystyle\lim_{x \to 0} f(x)$ doesn't exist in this case.

$\therefore \displaystyle\lim_{x \to 0} f(x)$ exists in case of (b), (c) and (d) each.

15. (a) Let $f(x) = \cot^{-1} \left(\dfrac{x^x - x^{-x}}{2} \right)$

Take out x^{-x} common

$f(x) = \cot^{-1} \left(\dfrac{x^{2x} - 1}{2 x^x} \right)$

Put $x^x = \tan \theta$

$\therefore \ f(x) = \cot^{-1} \left\{ \dfrac{\tan^2 \theta - 1}{2 \tan \theta} \right\} = \cot^{-1}(-\cot 2\theta)$

$= \pi - \cot^{-1} (\cot 2\theta) \quad \left[\because \tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta} \right]$

$\Rightarrow f(x) = \pi - 2\theta = \pi - 2 \tan^{-1} (x^x)$

Differentiate w.r.t. x, we get

$$f'(x) = -\frac{2}{1+x^{2x}} \cdot x^x (1+\log x)$$

$\therefore$ At $x = 1$

$$f'(1) = \frac{-2}{1+1}(1+0) = -1.$$

16. (d) $f(x) = \max. \{x, x^3\}$

$$= \begin{cases} x\ ; & x < -1 \\ x^3; & -1 \le x \le 0 \\ x\ ; & 0 \le x \le 1 \\ x^3; & x \ge 1 \end{cases}$$

$\therefore$ $f'(x) = \begin{cases} 1\ ; & x < -1 \\ 3x^2; & -1 \le x \le 0 \\ 1\ ; & 0 \le x \le 1 \\ 3x^2; & x \ge 1 \end{cases}$

Clearly f is not differentiable at $-1, 0$ and 1.

17. (b) f is continuous at $x = \pi/4$, if $\lim\limits_{x \to \pi/4} f(x) = f(\pi/4)$.

Now, $L = \lim\limits_{x \to \pi/4} (\sin 2x)^{\tan^2 2x}$

$\Rightarrow \log L = \lim\limits_{x \to \pi/4} \tan^2 2x \log \sin 2x$

$= \lim\limits_{x \to \pi/4} \dfrac{\log \sin 2x}{\cot^2 2x} \left(\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x \to \pi/4} \dfrac{2 \cot 2x}{-2 \cot 2x \cos ec^2 2x.2} = -\dfrac{1}{2}$

or $L = e^{-1/2}$ $\therefore$ $f(\pi/4) = e^{-1/2} = 1/\sqrt{e}$

18. (a) Given $f^{-1}(x) = g(x)$

$\Rightarrow x = f[g(x)]$

Diff. both side w.r.t (x)

$\Rightarrow 1 = f'[g(x)].g'(x) \Rightarrow g'(x) = \dfrac{1}{f'(g(x))}$

Given, $f'(x) = \sin x$

$\therefore f'(g(x)) = \sin[g(x)]$

$\Rightarrow \dfrac{1}{f'(g(x))} = \cosec[g(x)]$

Hence, $g'(x) = \cosec[g(x)]$

19. (c) $f[(\pi/2)^-] = \lim\limits_{h \to 0} \dfrac{1-\sin^3[(\pi/2)-h]}{3\cos^2[(\pi/2)-h]}$

$= \lim\limits_{h \to 0} \dfrac{1-\cos^3 h}{3\sin^2 h} = \dfrac{1}{2}$

$$f[(\pi/2)^+] = \lim\limits_{h \to 0} \dfrac{q[1-\sin\{(\pi/2)+h\}]}{[\pi - 2\{(\pi/2)+h\}]^2} = \lim\limits_{h \to 0} \dfrac{q(1-\cosh)}{4h^2} = \dfrac{q}{8}$$

$\therefore p = \dfrac{1}{2} = \dfrac{q}{8} \Rightarrow p = \dfrac{1}{2}, q = 4.$

20. (d) $|x|$ is non-differentiable function at $x = 0$ as L.H.D $= -1$ and R.H. D $= 1$

$\therefore |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

But $\cos|h|$ is differentiable

$\therefore$ Any combination of two such functions will be non-differentiable. Hence option (a) and (b) are ruled out.

Now, consider $\sin|x| + |x|$

$L' = \lim\limits_{h \to 0} \dfrac{\sin|-h| + |-h|}{-h}$

$= \lim\limits_{h \to 0} \dfrac{\sin h}{-h} - 1 = -1 - 1 = -2$

$R' = \lim\limits_{h \to 0} \dfrac{\sin|h| + |h|}{h}$

$= \lim\limits_{h \to 0} \dfrac{\sin h}{h} + 1 = 1 + 1 = 2$

Consider $\sin|x| - |x|$

$L' = \lim\limits_{h \to 0} \dfrac{\sin|-h| - |-h|}{-h}$

$= \lim\limits_{h \to 0} \dfrac{\sin h}{-h} + 1 = 0$

$R' = \lim\limits_{h \to 0} \dfrac{\sin|h| - |h|}{h}$

$= \lim\limits_{h \to 0} \dfrac{\sin h}{h} - 1 = 0$

Hence, $\sin|x| - |x|$ is differentiable at $x = 0$.

21. (d) $\underset{x \to 0}{\text{Lt}} \dfrac{(e^x - 1)^2}{\sin\left(\dfrac{x}{a}\right) \log\left(1 + \dfrac{x}{4}\right)}$

$= \underset{x \to 0}{\text{Lt}} \dfrac{\dfrac{(e^x - 1)^2}{x} \cdot x^2}{\dfrac{x}{a} \cdot \dfrac{\sin\left(\dfrac{x}{a}\right)}{\left(\dfrac{x}{a}\right)} \cdot \dfrac{\log\left(1 + \dfrac{x}{4}\right)}{\dfrac{x}{4}} \cdot \dfrac{x}{4}}$

$\Rightarrow 4a = 12 \Rightarrow a = 3$

22. (b) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x = 0$

The other given equation,

$na_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots + a_1 = 0 = f'(x)$

Given $a_1 \neq 0 \Rightarrow f(0) = 0$

Again $f(x)$ has root α, $\Rightarrow f(\alpha) = 0$

$\therefore \ f(0) = f(\alpha)$

$\therefore$ By Rolle's theorem,

$f'(x) = 0$ has root between $(0, \alpha)$

Hence $f'(x)$ has a positive root smaller than α.

23. (a) If f is continuous at $x = 0$, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow f(0) = \lim_{x\to 0^-} f(x)$$

$$k = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[0-h]}{[0-h]}$$

$$k = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h]}{[-h]} = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h-1]}{[-h-1]}$$

$$k = \lim_{h\to 0} \frac{\cos\left(-\frac{\pi}{2}\right)}{-1}; k = 0$$

24. (b) The denominator of the given function is always defined
Also, $\tan [x]\pi = \tan n \pi = 0$ [[x] = integer, say n]
$\therefore f(x) = 0 \ \forall \ x$
$\therefore f(x)$ is continuous and differentiable for all x.

25. (c) Put $x^n = \cos\alpha$, $y^n = \cos\beta$

$$\Rightarrow a = \frac{\sin\alpha + \sin\beta}{\cos\alpha - \cos\beta} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= -\cot\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow 2\cot^{-1}(-a) = \alpha - \beta$$
$$\Rightarrow \cos^{-1}(x^n) - \cos^{-1}(y^n) = 2\cot^{-1}(-a)$$

$$\Rightarrow \frac{y^{n-1}}{\sqrt{1-y^{2n}}}\frac{dy}{dx} = \frac{x^{n-1}}{\sqrt{1-x^{2n}}} \Rightarrow \sqrt{\frac{1-x^{2n}}{1-y^{2n}}}\frac{dy}{dx} = \frac{x^{n-1}}{y^{n-1}}$$

26. (b) Clearly $x = 8$ satisfies the given equation. Assume that $f(x) = e^{x-8} + 2x - 17 = 0$ has a real root α other than $x = 8$. We may suppose that $\alpha > 8$ (the case for $\alpha < 8$ is exactly similar). Applying Rolle's theorem on $[8, \alpha]$, we get $\beta \in (8, \alpha)$, such that $f'(\beta) = 0$.

But $f'(\beta) = e^{\beta - 8} + 2$, so that $e^{\beta-8} = -2$ which is not possible, Hence there is no real root other than 8.

27. (a) $F'(x) = \left[f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right) \right]$

Here, $g(x) = f'(x)$
and $g'(x) = f''(x) = -f(x)$

so $F'(x) = f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) = 0$

$\Rightarrow F(x)$ is constant function
so $F(10) = 5$

28. (d) (a) $\displaystyle\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ exist finitely

$$\therefore \ \lim_{h\to 0^+} f(a+h) - f(a)$$

$$= \lim_{h\to 0^+}\left(\frac{f(a+h)-f(a)}{h}\right) h = 0$$

$$\Rightarrow \lim_{h\to 0^+} f(a+h) = f(a)$$

Similarly, $\displaystyle\lim_{h\to 0^-} f(a+h) = f(a)$

$\therefore f$ is continuous at x = a

(b) Function is not differentiable at $5x = (2n+1)\dfrac{\pi}{2}$ only, which are not in domain

(c) Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = -\dfrac{1}{x^2}$,

$\displaystyle\lim_{x\to 0} f(x) + g(x)$ exists whatever $\displaystyle\lim_{x\to 0} f(x)$ and $\displaystyle\lim_{x\to 0} g(x)$ does not exist.

29. (d) Statement 1 : As $f(-1) = f(1)$ and Rolles theorem is not applicable, then it implies $f(x)$ is either discontinuous or $f'(x)$ does not exist at atleast one point in $(-1, 1)$
$\Rightarrow g(x) = 0$ at atleast one value of x in $(-1, 1)$.
Statement 2 is false. Consider the example in statement-1.

30. (a) $f(x) = |x| \sin x$

L.H.D. $= \displaystyle\lim_{h\to 0} \frac{|0-h|\sin(0-h) - 0}{h}$

$= \displaystyle\lim_{h\to 0} \frac{-h\sin h}{h} = 0$

R.H.D. $= \displaystyle\lim_{h\to 0} \frac{|0+h|\sin(0+h) - 0}{h}$

$f(x)$ is differentiable at $x = 0$