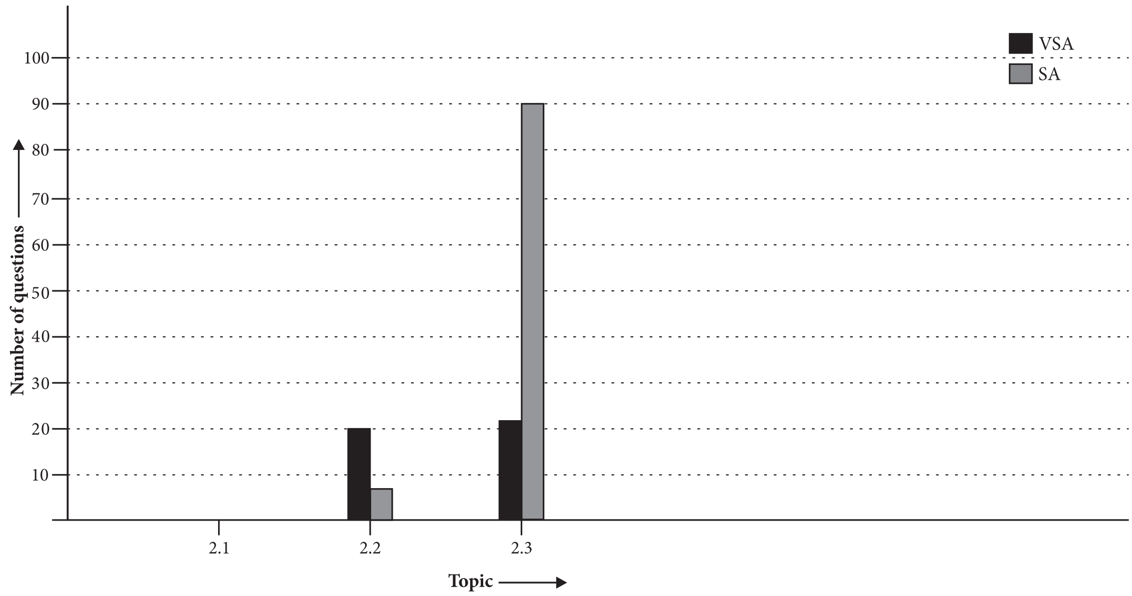
02 Inverse Trigonometric Functions

Topicwise Analysis of Last 10 Years' CBSE Board Questions



- Maximum weightage is of *Properties of Inverse Trigonometric Functions*
- Maximum VSA and SA type questions were asked from *Properties of Inverse Trigonometric Functions*
- No VBQ & LA type questions were asked till now

QUICK RECAP

INVERSE TRIGONOMETRIC FUNCTIONS

- Trigonometric functions are not one-one and onto over their natural domains and ranges *i.e.*, R(real numbers). But some restrictions on domains and ranges of trigonometric function ensures the existence of their inverses.
  Let $y = f(x) = \cos x$, then its inverse is $x = \cos^{-1} y$
- The domains and ranges (principal value branches) of inverse trigonometric functions are as follows :

Functions	Domain	Range
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	R	$(0, \pi)$
$y = \text{cosec}^{-1} x$	$R-(-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
$y = \sec^{-1} x$	$R-(-1, 1)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$

- The value of the inverse trigonometric functions which lies in its principal value branch is called the principal value of inverse trigonometric function.

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

- $\sin^{-1}(\sin x) = x,\ \forall -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
  $\cos^{-1}(\cos x) = x,\ \forall\ 0 \le x \le \pi$
  $\tan^{-1}(\tan x) = x,\ \forall -\frac{\pi}{2} < x < \frac{\pi}{2}$

- $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x,\ \forall\ x \ge 1 \text{ or } x \le -1$
  $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x,\ \forall\ x \ge 1 \text{ or } x \le -1$
  $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x,\ \forall\ x > 0$
  $= -\pi + \cot^{-1} x, \forall\ x < 0$

- $\sin(\sin^{-1}x) = x,\ \forall -1 \le x \le 1$
  $\cos(\cos^{-1}x) = x,\ \forall -1 \le x \le 1$
  $\tan(\tan^{-1}x) = x,\ \forall\ x \in R$

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2},\ \forall -1 \le x \le 1$
  $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ \forall\ x \in R$
  $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2},\ \forall\ x \le -1 \text{ or } x \ge 1$

- $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2},\ \forall\ 0 \le x \le 1$
  $\sin^{-1} x = -\cos^{-1}\sqrt{1-x^2},\ \forall -1 \le x < 0$
  $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2},\ \forall\ 0 \le x \le 1$
  $\cos^{-1} x = \pi - \sin^{-1}\sqrt{1-x^2},\ \forall -1 \le x < 0$

- $\sin^{-1}(-x) = -\sin^{-1}x,\ \forall -1 \le x \le 1$
  $\cos^{-1}(-x) = \pi - \cos^{-1}x,\ \forall -1 \le x \le 1$
  $\tan^{-1}(-x) = -\tan^{-1}x,\ \forall\ x \in R$
  $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ \forall\ x \in R$
  $\sec^{-1}(-x) = \pi - \sec^{-1}x,\ \forall\ |x| \ge 1$
  $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x,\ \forall\ |x| \ge 1$

- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right),\ \forall\ xy < 1$
  $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right),$
  $-1 \le x, y \le 1,\ x^2 + y^2 \le 1$
  $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right),$
  $-1 \le x, y \le 1,\ x + y \ge 0$
  $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right),\ xy > -1$
  $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\},$
  $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$
  $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\},$
  $-1 \le x, y \le 1$ and $x \le y$.

- $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right),\ \forall -1 \le x \le 1$
  $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ \forall\ x \ge 0$
  $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right),\ \forall |x| < 1$
  $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right),\ \forall -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$
  $2\cos^{-1} x = \sin^{-1}(2x\sqrt{1-x^2}),\ \forall \frac{1}{\sqrt{2}} \le x \le 1$
  $2\cos^{-1}x = \cos^{-1}(2x^2 - 1),\ \forall\ 0 \le x \le 1$

- $3\sin^{-1}x = \sin^{-1}(3x - 4x^3),\ \forall -\frac{1}{2} \le x \le \frac{1}{2}$
  $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x),\ \forall \frac{1}{2} \le x \le 1$
  $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ \forall -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Previous Years' CBSE Board Questions

2.2 Basic Concepts

VSA (1 mark)

1. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$. *(Foreign 2014)*

2. Write the principal value of $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$. *(AI 2014C)*

3. Find the value of the following :
$\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$ *(AI 2014C)*

4. Write the principal value of
$\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$. *(Delhi 2013)*

5. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$. *(AI 2013)*

6. Write the principal value of
$\left[\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\left(-\frac{1}{2}\right)\right]$. *(Delhi 2013C)*

7. Write the principal value of
$[\tan^{-1}(-\sqrt{3})+\tan^{-1}(1)]$. *(AI 2013C)*

8. Write the principal value of
$\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right)$ *(Delhi 2012)*

9. Using principal values, write the value of
$\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$. *(AI 2012C)*

10. Evaluate : $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right]$. *(Delhi 2011, 2008)*

11. Write the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$. *(Delhi 2011C)*

12. Using principal values, write the value of
$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. *(AI 2011C, Delhi 2010)*

13. Find the principal value of
$\sin^{-1}\left(\frac{-1}{2}\right)+\cos^{-1}\left(\frac{-1}{2}\right)$. *(Delhi 2010)*

14. Find the principal value of $\sec^{-1}(-2)$. *(AI 2010)*

15. Using the principal values, evaluate the following : $\tan^{-1}1+\sin^{-1}\left(\frac{-1}{2}\right)$. *(Delhi 2009 C)*

16. Find the principal value of $\tan^{-1}(-1)$. *(Delhi 2008 C)*

17. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. *(AI 2008 C)*

18. Find the value of the following.
$\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$. *(AI 2007)*

SA (4 marks)

19. Prove that
$\cos^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{56}{33}$. *(AI 2013C)*

20. Prove that :
$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$ *(AI 2012, Delhi 2010C, 2009 C)*

21. Prove that :
$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$. *(AI 2012, Delhi 2010)*

22. Prove that :
$\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)=\pi$. *(Foreign 2008)*

2.3 Properties of Inverse Trigonometric Functions

VSA (1 mark)

23. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x. *(Delhi 2014)*

24. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$, $xy < 1$, then write the value of $x + y + xy$. *(AI 2014, Delhi 2012C)*

25. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$. *(Delhi 2013)*

26. Write the principal value of $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)$. *(AI 2013)*

27. Evaluate $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ *(AI 2013C, Delhi 2009)*

28. Find the principal value of $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$. *(AI 2012)*

29. Find the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$. *(Delhi 2011)*

30. Write the principal value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$. *(Delhi 2011, AI 2009)*

31. Using principal values, evaluate the following.
$\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$. *(AI 2011, 2009C, 2008)*

32. Find the principal value of $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$. *(AI 2010)*

33. If $\sin^{-1}\left(\frac{1}{3}\right)+\cos^{-1}(x)=\frac{\pi}{2}$, then find x. *(Delhi 2010C)*

34. If $\sin^{-1}(x)+\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}$, then find x. *(Delhi 2010C)*

35. Using principal value, find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$. *(AI 2010C)*

36. If $\tan^{-1}\left(\sqrt{3}\right)+\cot^{-1}\left(x\right)=\frac{\pi}{2}$, then find x. *(AI 2010C)*

37. Show that, $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x$. *(Foreign 2008)*

38. Write into the simplest form:
$\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$ *(Delhi 2007)*

SA (4 marks)

39. Solve for x : $2\tan^{-1}(\cos x) = \tan^{-1}(2\text{cosec } x)$ *(Delhi 2016, 2014C, Foreign 2015, AI 2009)*

40. Prove that :
$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$
(Delhi 2016, 2008, 2008C, AI 2010, 2009C, 2008)

41. Solve the equation for x :
$\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$ *(AI 2016)*

42. If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, prove that
$\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$ *(AI 2016)*

43. Solve for x :
$\tan^{-1}\left(\frac{x-2}{x-1}\right)+\tan^{-1}\left(\frac{x+2}{x+1}\right)=\frac{\pi}{4}$ *(Foreign 2016)*

44. Prove that :
$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2}, x\in\left(0,\frac{\pi}{4}\right)$
(Foreign 2016, Delhi 2014, 2014C, 2011,AI 2009)

45. If $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1}x)$, then find x. *(Delhi 2015)*

46. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, then find x. *(Delhi 2015)*

47. Prove the following:
$\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)=0,$
$(0 < xy, yz, zx < 1)$ *(AI 2015)*

48. Solve for x :
$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}.$
(AI 2015, Foreign 2008)

49. If $\tan^{-1}\left(\frac{1}{1+1\cdot 2}\right)+\tan^{-1}\left(\frac{1}{1+2\cdot 3}\right)$
$+\ldots+\tan^{-1}\left(\frac{1}{1+n\cdot(n+1)}\right)=\tan^{-1}\theta,$
then find the value of θ. *(Foreign 2015)*

50. Solve for x : $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}.$
(Delhi 2015C, 2013C, 2009, 2008, AI 2012C, 2009C)

51. Prove that: $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$
(Delhi 2015C)

52. Prove that
$2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\sin^{-1}\left(\frac{31}{25\sqrt{2}}\right).$
(AI 2015C)

53. Solve for x :
$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x,\ x>0.$
(AI 2015C, 2014C, 2010C, 2009, Delhi 2008C, Foreign 2008)

54. Prove that
$2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$
(Delhi 2014)

55. Prove that :
$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\frac{-1}{\sqrt{2}}\le x\le 1$
(AI 2014, 2011, 2010C)

56. If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$; find the value of x. *(AI 2014)*

57. Solve for x : $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right).$
(Foreign 2014, AI 2013)

58. Prove that: $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18=\cot^{-1}3.$
(Foreign 2014)

59. Prove that :
$\cos^{-1}(x)+\cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right\}=\frac{\pi}{3}.$ *(AI 2014C)*

60. Solve for x : $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}.$
(AI 2014C, Delhi 2009 C)

61. Prove that : $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{36}{85}$
(AI 2014C, Delhi 2012, 2010 C)

62. Find the value of the following :
$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right],$
$|x|<1$, $y>0$ and $xy<1$ *(Delhi 2013)*

63. Prove that,
$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$
(Delhi 2013, 2012C, 2008C, AI 2011)

64. Show that:
$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}.$ *(AI 2013)*

65. Write the value of the following :
$\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right)$ *(Delhi 2013C)*

66. Prove that: $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$
(Delhi 2013C)

67. Solve for x :
$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ *(AI 2013C)*

68. Prove that
$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2},\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
(Delhi 2012)

69. Prove the following:
$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}$ *(AI 2012)*

70. Solve for x :
$\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}.$
(Delhi 2012C, 2009C, 2008, AI 2010, 2008)

71. Prove that :
$$\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}$$
(Delhi 2012C, AI 2009 C)

72. Find the value of
$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$$
(Delhi 2011)

73. Prove that :
$$2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{31}{17}\right).$$
(AI 2011, Delhi 2009)

74. Prove that:
$$2\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$$
(Delhi 2011C)

75. Solve for x:
$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3},\ -1<x<1$$
(Delhi 2011C)

76. Prove that :
$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\tan^{-1}\frac{4}{3}$$
(AI 2011C)

77. Solve for x : $\cos(2\sin^{-1}x)=\frac{1}{9},\ x>0$
(AI 2011C)

78. Prove that :
$$\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x\in(0,1)$$
(Delhi 2010)

79. Prove that : $\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)=\pi.$
(Delhi 2010)

80. Prove that :
$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]=\sqrt{\frac{1+x^2}{2+x^2}}$$
(AI 2010)

81. Prove that :
$$\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$
(AI 2010)

82. Solve for x :
$$\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4};\ 0<x<\sqrt{6}$$
(Delhi 2010 C)

83. Solve for x :
$$\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right);x>0$$
(Delhi 2010 C)

84. Prove that : $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$ *(AI 2010C)*

85. Solve for x :
$$\cos^{-1}x+\sin^{-1}\left(\frac{x}{2}\right)=\frac{\pi}{6}$$
(AI 2010C)

86. Prove that :
$$\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}.$$
(Delhi 2009 C)

87. Prove that : $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{4}{7}.$
(Foreign 2008)

88. Solve for x :
$$\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x,\ 0<x<1.$$
(Delhi 2008 C)

89. Write into the simplest form:
$$\cot^{-1}\left(\sqrt{1+x^2}-x\right).$$
(Delhi 2007)

90. Prove that :
$$\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(2\frac{\sqrt{2}}{3}\right).$$
(AI 2007)

Detailed Solutions

1. Given $\cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

$=\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)=\frac{2\pi}{3}+2\times\frac{\pi}{6}=\pi$

[$\because$ Range of $\cos^{-1}$ is $[0,\pi]$ & of $\sin^{-1}$ is $[-\pi/2,\pi/2]$]

2. Here, $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]=\tan^{-1}(-1)=-\frac{\pi}{4}$.

This is the required principal value as it should lie in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

3. $\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$

$=\cot\left(\frac{\pi}{2}-2\cot^{-1}\left(\cot\frac{\pi}{6}\right)\right)=\cot\left(\frac{\pi}{2}-2.\frac{\pi}{6}\right)$

$=\cot\left(\frac{\pi}{2}-\frac{\pi}{3}\right)=\cot\frac{\pi}{6}=\sqrt{3}$

4. $\tan^{-1}(1)+\cos^{-1}\left(\frac{-1}{2}\right)$

$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)=\frac{\pi}{4}+\frac{2\pi}{3}=\frac{11\pi}{12}$

5. $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$=\tan^{-1}\left[2\sin\left(2.\frac{\pi}{6}\right)\right]=\tan^{-1}\left[2\sin\frac{\pi}{3}\right]$

$=\tan^{-1}\left[2.\frac{\sqrt{3}}{2}\right]=\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}$.

6. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$

$=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{\pi}{6}+\frac{2\pi}{3}=\frac{5\pi}{6}$

7. $\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}(1)$

$=\tan^{-1}\left(-\tan\frac{\pi}{3}\right)+\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

$=\tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)+\frac{\pi}{4}=-\frac{\pi}{3}+\frac{\pi}{4}=-\frac{\pi}{12}$.

8. $\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right)$

$=\cos^{-1}\left(\cos\frac{\pi}{3}\right)-2\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

9. Principal value of

$\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+2.\frac{\pi}{6}=\frac{2\pi}{3}$.

10. $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]=\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right)\right]$

$=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\frac{\pi}{2}=1$

11. Let $\sin^{-1}\left(\frac{-1}{2}\right)=\theta$

Then, $\sin\theta=\frac{-1}{2}=\sin\left(-\frac{\pi}{6}\right)$, where $\frac{-\pi}{6}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

12. The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$=\sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)=-\frac{\pi}{3}$, where $\frac{-\pi}{3}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

13. $\sin^{-1}\left(\frac{-1}{2}\right)+\cos^{-1}\left(\frac{-1}{2}\right)$

$=\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)+\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)=\frac{-\pi}{6}+\frac{2\pi}{3}=\frac{\pi}{2}$

14. Let $\sec^{-1}(-2)=y$. Then, $\sec y=-2$.

$\sec y=-2=-\sec\left(\frac{\pi}{3}\right)=\sec\left(\pi-\frac{\pi}{3}\right)=\sec\left(\frac{2\pi}{3}\right)$

We know that the range of principal value branch of $\sec^{-1}$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ and $\sec\left(\frac{2\pi}{3}\right)=-2$

Hence, principal value of $\sec^{-1}(-2)=\frac{2\pi}{3}$

15. $\tan^{-1}(1)+\sin^{-1}\left(\frac{-1}{2}\right)$

$=\tan^{-1}(1)+\sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right)=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}$

$\therefore$ Required principal value is $\frac{\pi}{12}$

16. Let $\tan^{-1}(-1)=x \Rightarrow -1=\tan x$

We know that the range of principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Then, $-1=\tan\left(-\frac{\pi}{4}\right)$, where $-\frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Hence, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

17. Principal value of

$\cos^{-1}\frac{\sqrt{3}}{2}=\cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right]=\frac{\pi}{6}$, where $\frac{\pi}{6}\in[0,\pi]$

18. Here, $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\left[\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)+\sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right)\right]$

$=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}=\frac{3\pi}{4}$

19. Let $x=\cos^{-1}\left(\frac{4}{5}\right)$ and $y=\cos^{-1}\left(\frac{12}{13}\right)$

$\Rightarrow \cos x=\frac{4}{5}$ and $\cos y=\frac{12}{13}$

Now, $\sin x=\sqrt{1-\cos^2 x}$ and $\sin y=\sqrt{1-\cos^2 y}$

$\Rightarrow \sin x=\sqrt{1-\frac{16}{25}}$ and $\sin y=\sqrt{1-\frac{144}{169}}$

$\Rightarrow \sin x=\frac{3}{5}$ and $\sin y=\frac{5}{13}$

We know that, $\cos(x+y)=\cos x\cos y-\sin x\sin y$

$=\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}$

$\Rightarrow \cos(x+y)=\frac{48}{65}-\frac{15}{65}=\frac{33}{65}$

$\Rightarrow x+y=\cos^{-1}\left(\frac{33}{65}\right)$

$\therefore \cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$

Now, $\cos^{-1}x=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$\therefore \cos^{-1}\left(\frac{33}{65}\right)=\tan^{-1}\left(\frac{\sqrt{1-\left(\frac{33}{65}\right)^2}}{\frac{33}{65}}\right)=\tan^{-1}\left(\frac{56}{35}\right)$

$\therefore \cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\tan^{-1}\left(\frac{56}{35}\right)$

20. *Refer to answer 19.*

21. Let $x=\cos^{-1}\left(\frac{12}{13}\right)$ and $y=\sin^{-1}\left(\frac{3}{5}\right)$

or $\cos x=\frac{12}{13}$ and $\sin y=\frac{3}{5}$

Now, $\sin x=\sqrt{1-\cos^2 x}$ and $\cos y=\sqrt{1-\sin^2 y}$

$\Rightarrow \sin x=\sqrt{1-\frac{144}{169}}$ and $\cos y=\sqrt{1-\frac{9}{25}}$

$\Rightarrow \sin x=\frac{5}{13}$ and $\cos y=\frac{4}{5}$

We know that,

$\sin(x+y)=\sin x\cos y+\cos x\sin y$

$=\frac{5}{13}\times\frac{4}{5}+\frac{12}{13}\times\frac{3}{5}=\frac{20}{65}+\frac{36}{65}=\frac{56}{65}$

$\Rightarrow x+y=\sin^{-1}\left(\frac{56}{65}\right)$

or, $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$

22. Let $\sin^{-1}\left(\frac{12}{13}\right)=x, \cos^{-1}\left(\frac{4}{5}\right)=y,\ \tan^{-1}\left(\frac{63}{16}\right)=z$

Then, $\sin x=\frac{12}{13}$, $\cos y=\frac{4}{5}$, $\tan z=\frac{63}{16}$

Therefore, $\cos x=\frac{5}{13}$, $\sin y=\frac{3}{5}$,

$\tan x=\frac{12}{5}$ and $\tan y=\frac{3}{4}$

We have, $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$$= \frac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{12}{5} \times \frac{3}{4}} = -\frac{63}{16}$$

Hence, $\tan(x + y) = -\tan z$

i.e., $\tan(x + y) = \tan(-z)$ or $\tan(x + y) = \tan(\pi - z)$

Therefore, $x + y = -z$ or $x + y = \pi - z$

Since, x, y and z are positive, $x + y \neq -z$

Hence, $x + y + z = \pi$

or, $\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right) = \pi.$

23. $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$

or $\sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}1$

$\Rightarrow \sin^{-1}\frac{1}{5} + \frac{\pi}{2} - \sin^{-1}x = \frac{\pi}{2}$

$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow \sin^{-1}\frac{1}{5} = \sin^{-1}x \Rightarrow x = \sin\left(\sin^{-1}\frac{1}{5}\right) = \frac{1}{5}$

24. Given: $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$ $(xy < 1)$

$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{4} \Rightarrow \frac{x+y}{1-xy} = \tan\frac{\pi}{4} = 1$

$\Rightarrow x + y = 1 - xy \Rightarrow x + y + xy = 1$

25. Since $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, for $|x| < 1$

So, $2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{2 \times \frac{1}{5}}{1 - \left(\frac{1}{5}\right)^2}\right)$

$= \tan^{-1}\left(\frac{\frac{2}{5}}{\frac{24}{25}}\right) = \tan^{-1}\left(\frac{5}{12}\right)$

$\therefore \tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{12}$

26. $\tan^{-1}\left(\sqrt{3}\right) - \cot^{-1}\left(-\sqrt{3}\right)$

$= \tan^{-1}\left(\sqrt{3}\right) + \cot^{-1}[\sqrt{3}] - \pi = \frac{\pi}{2} - \pi = \frac{-\pi}{2}$

$\therefore$ Required principal value is $-\frac{\pi}{2}$

27. We know that, $\sin^{-1}(\sin x) = x$

Therefore, $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{3\pi}{5}$

But $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value of $\sin^{-1}x$.

So, $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{5}\right)\right]$

$= \sin^{-1}\left[\sin\left(\frac{2\pi}{5}\right)\right] = \frac{2\pi}{5}$ and $\frac{2\pi}{5} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}$

28. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \tan^{-1}\sqrt{3} - \sec^{-1}\left(\sec\frac{2\pi}{3}\right)$

$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}.$

$\therefore$ Principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = -\frac{\pi}{3}$

29. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq \frac{3\pi}{4}$ as the principal value branch of $\tan^{-1}\theta$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right)$

$= \tan^{-1}\left[-\tan\left(\frac{\pi}{4}\right)\right]$

$= \tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$ $[\because -\tan\theta = \tan(-\theta)]$

$= -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$

30. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$ as principal value branch of $\cos^{-1}\theta$ is $[0, \pi]$

So, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left[\cos\left(\pi+\frac{\pi}{6}\right)\right]$

$=\cos^{-1}\left(-\cos\frac{\pi}{6}\right)=\cos^{-1}\left(\cos\left(\pi-\frac{\pi}{6}\right)\right)=\frac{5\pi}{6}$

where $\frac{5\pi}{6}\in[0,\pi]$

Hence, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\frac{5\pi}{6}$

31. We know that the range of principal value branch of $\cos^{-1}\theta$ is $[0, \pi]$ and $\sin^{-1}\theta$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Then, $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$=\frac{2\pi}{3}+\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)$

$=\frac{2\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{2\pi}{3}+\frac{\pi}{3}=\pi$

32. We know that, $\sin^{-1}(\sin x) = x$

Therefore, $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)=\frac{4\pi}{5}$

But $\frac{4\pi}{5}\notin\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

So, $\sin\left(\frac{4\pi}{5}\right)=\sin\left(\pi-\frac{\pi}{5}\right)=\sin\frac{\pi}{5}$ and $\frac{\pi}{5}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Therefore, $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)=\frac{\pi}{5}$.

33. $\sin^{-1}\left(\frac{1}{3}\right)+\cos^{-1}(x)=\frac{\pi}{2}$

$\Rightarrow\ \sin^{-1}\frac{1}{3}=\frac{\pi}{2}-\cos^{-1}x\ \Rightarrow\ \sin^{-1}\frac{1}{3}=\sin^{-1}x$

$\left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

$\Rightarrow x=\frac{1}{3}$

34. $\sin^{-1}x+\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}$

$\Rightarrow\ \sin^{-1}x+\frac{\pi}{3}=\frac{\pi}{2}\Rightarrow\sin^{-1}x=\frac{\pi}{2}-\frac{\pi}{3}$

$\Rightarrow\ \sin^{-1}x=\frac{\pi}{6}\Rightarrow x=\sin\frac{\pi}{6}\Rightarrow x=\frac{1}{2}$

35. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)\neq\frac{13\pi}{6}$ as the range of principal value branch of $\cos^{-1}\theta$ is $[0, \pi]$

So, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{6}\right)\right)$

$=\cos^{-1}\left(\cos\frac{\pi}{6}\right)=\frac{\pi}{6}$

$\therefore\ \cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\frac{\pi}{6}$

36. $\tan^{-1}(\sqrt{3})+\cot^{-1}(x)=\frac{\pi}{2}$

$\Rightarrow\ \frac{\pi}{3}+\cot^{-1}x=\frac{\pi}{2}\Rightarrow\cot^{-1}x=\frac{\pi}{6}$

$\Rightarrow\ x=\cot\frac{\pi}{6}\Rightarrow x=\sqrt{3}$

37. L.H.S. = $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Putting $x = \sin\theta$, we get

L.H.S. $=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$=\sin^{-1}(2\sin\theta\cos\theta)$

$=\sin^{-1}(\sin 2\theta)=2\theta=2\sin^{-1}x=$ R.H.S.

38. Let $y=\tan^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

$$\Rightarrow y=\tan^{-1}\left[\frac{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$\Rightarrow y=\tan^{-1}\left[\frac{\sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}+\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}-\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}\right]$$

$$\Rightarrow\ y=\tan^{-1}\left[\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{2\sin\frac{x}{2}}{2\cos\frac{x}{2}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\tan\frac{x}{2}\right] \quad \Rightarrow \quad y = \frac{x}{2}.$$

39. $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \quad 2\tan^{-1}(\cos x) - \tan^{-1}(2\operatorname{cosec} x) = 0$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) - \tan^{-1}(2\operatorname{cosec} x) = 0$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right) - \tan^{-1}(2\operatorname{cosec} x) = 0$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{2\cos x}{\sin^2 x} - \frac{2}{\sin x}}{1+\left(\frac{2\cos x}{\sin^2 x}\right)\left(\frac{2}{\sin x}\right)}\right) = 0$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x\sin x - 2\sin^2 x}{\sin^3 x + 4\cos x}\right) = 0$$

$$\Rightarrow \quad \frac{2\cos x\sin x - 2\sin^2 x}{\sin^3 x + 4\cos x} = 0$$

$\Rightarrow \quad 2\cos x\sin x = 2\sin^2 x \;\Rightarrow\; \tan x = 1$

$$\Rightarrow \quad x = \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

40. L.H.S. $= \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5}\right) + \left(\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}}\right) + \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{8}{15}}{\frac{14}{15}}\right) + \tan^{-1}\left(\frac{\frac{15}{56}}{\frac{55}{56}}\right) = \tan^{-1}\left(\frac{4}{7}\right) + \tan^{-1}\left(\frac{3}{11}\right)$$

$$= \tan^{-1}\left[\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}\right] = \tan^{-1}\left(\frac{\frac{65}{77}}{\frac{65}{77}}\right)$$

$$= \tan^{-1} 1 = \frac{\pi}{4} = \text{R.H.S.}$$

41. $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$

$$\Rightarrow \sin^{-1}x + \frac{\pi}{2} - \cos^{-1}(1-x) = \cos^{-1}x \quad (\forall\, -1 \le x \le 1)$$

$$\Rightarrow \sin^{-1}x + \frac{\pi}{2} - \cos^{-1}x = \cos^{-1}(1-x)$$

$\Rightarrow \sin^{-1}x + \sin^{-1}x = \cos^{-1}(1-x)$

$\Rightarrow \quad 2\sin^{-1}x = \cos^{-1}(1-x) \;\Rightarrow\; \cos(2\sin^{-1}x) = (1-x)$

$\Rightarrow \quad 1 - 2\sin^2(\sin^{-1}x) = (1-x) \Rightarrow 2\sin^2(\sin^{-1}x) = x$

$\Rightarrow \quad 2x^2 = x \;\Rightarrow\; 2x^2 - x = 0 \;\Rightarrow\; x(2x-1) = 0$

$\Rightarrow x = 0$ or $2x - 1 = 0 \Rightarrow x = 0$ or $x = 1/2$

42. We have, $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$

$$\Rightarrow \cos^{-1}\left[\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right] = \alpha$$

$$\Rightarrow \frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha \qquad \text{... (1)}$$

Squaring on both sides, we get

$$\frac{x^2y^2}{a^2b^2} + \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right) - \frac{2xy}{ab}\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos^2\alpha$$

$$\Rightarrow \frac{x^2y^2}{a^2b^2} + 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} + \frac{x^2y^2}{a^2b^2} - \frac{2xy}{ab}\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = 1 - \sin^2\alpha$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\left[\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right] = \sin^2\alpha$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha \qquad \text{[From (1)]}$$

43. $\tan^{-1}\left(\frac{x-2}{x-1}\right) + \tan^{-1}\left(\frac{x+2}{x+1}\right) = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\left(\frac{x-2}{x-1}\right)+\left(\frac{x+2}{x+1}\right)}{1-\left(\frac{x-2}{x-1}\right)\left(\frac{x+2}{x+1}\right)}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{(x-2)(x+1)+(x+2)(x-1)}{(x^2-1)-(x^2-2^2)}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{x^2+x-2x-2+x^2-x+2x-2}{x^2-1-x^2+4}\right) = \frac{\pi}{4}$$

$\Rightarrow \left(\dfrac{2x^2-4}{3}\right) = \tan\left(\dfrac{\pi}{4}\right) \Rightarrow \dfrac{2x^2-4}{3} = 1$

$\Rightarrow\ 2x^2 - 4 = 3 \Rightarrow 2x^2 = 3 + 4$

$\Rightarrow x^2 = \dfrac{7}{2} \ \Rightarrow x = \pm\sqrt{\dfrac{7}{2}}$

44. L.H.S.

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\left(\frac{(1+\sin x)+(1-\sin x)+2\sqrt{1-\sin^2 x}}{(1+\sin x)-(1-\sin x)}\right)$$

$$= \cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right) = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}\right)$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} = \text{R.H.S.}$$

Hence, L.H.S. = R.H.S.

45. We have, $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1}x)$ ... (1)

Let $\cot^{-1}(x+1) = A$ and $\tan^{-1}x = B$

$\Rightarrow\ x + 1 = \cot A \Rightarrow \sin A = \dfrac{1}{\sqrt{(x+1)^2+1}}$

Also, $x = \tan B$ $\quad\therefore\ \cos B = \dfrac{1}{\sqrt{x^2+1}}$

Now, $\sin A = \cos B$ [From (1)]

$\Rightarrow\ \dfrac{1}{\sqrt{(x+1)^2+1}} = \dfrac{1}{\sqrt{x^2+1}} \Rightarrow (x+1)^2 + 1 = x^2 + 1$

$\Rightarrow\ 1 + 2x = 0 \Rightarrow\ x = -\dfrac{1}{2}$

46. $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \dfrac{5\pi^2}{8}$ [Given]

$\Rightarrow\ (\tan^{-1}x)^2 + \left(\dfrac{\pi}{2} - \tan^{-1}x\right)^2 = \dfrac{5\pi^2}{8}$

Putting $\tan^{-1}x = \theta$, we get

$\theta^2 + \left(\dfrac{\pi}{2} - \theta\right)^2 = \dfrac{5\pi^2}{8}$

$\Rightarrow\ \theta^2 + \dfrac{\pi^2}{4} + \theta^2 - \pi\theta = \dfrac{5\pi^2}{8}$

$\Rightarrow\ 2\theta^2 - \pi\theta + \left(\dfrac{\pi^2}{4} - \dfrac{5\pi^2}{8}\right) = 0$

$\Rightarrow\ 2\theta^2 - \pi\theta - \dfrac{3}{8}\pi^2 = 0$

$\Rightarrow\ 16\theta^2 - 8\pi\theta - 3\pi^2 = 0$

$\Rightarrow\ 4\theta(4\theta - 3\pi) + \pi(4\theta - 3\pi) = 0$

$\Rightarrow\ (4\theta + \pi)(4\theta - 3\pi) = 0$

$\Rightarrow$ Either $4\theta = 3\pi$ or $4\theta = -\pi$

$\Rightarrow\ \theta = \dfrac{3\pi}{4}$ or $\theta = -\dfrac{\pi}{4}$

Hence, $\tan^{-1}x = \dfrac{3\pi}{4}$ or $-\dfrac{\pi}{4}$

$\Rightarrow\ x = -1$

47. L.H.S.

$$= \cot^{-1}\left(\frac{xy+1}{x-y}\right) + \cot^{-1}\left(\frac{yz+1}{y-z}\right) + \cot^{-1}\left(\frac{zx+1}{z-x}\right)$$

$$= \tan^{-1}\left(\frac{x-y}{1+xy}\right) + \tan^{-1}\left(\frac{y-z}{1+yz}\right) + \tan^{-1}\left(\frac{z-x}{1+zx}\right)$$

$\left[\because\ \cot^{-1}x = \tan^{-1}\dfrac{1}{x}\right]$

$= (\tan^{-1}x - \tan^{-1}y) + (\tan^{-1}y - \tan^{-1}z) + (\tan^{-1}z - \tan^{-1}x)$

= 0 = R.H.S.

48. We have, $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\dfrac{8}{31}$

$\Rightarrow\ \tan^{-1}\left(\dfrac{(x+1+x-1)}{1-(x+1)(x-1)}\right) = \tan^{-1}\dfrac{8}{31}$

for $(x+1)(x-1) < 1$

$\Rightarrow\ \dfrac{2x}{1-(x^2-1)} = \dfrac{8}{31} \Rightarrow \dfrac{2x}{2-x^2} = \dfrac{8}{31}$

$\Rightarrow\ 31x = 8 - 4x^2 \quad \Rightarrow \quad 4x^2 + 31x - 8 = 0$

$\Rightarrow\ (4x-1)(x+8) = 0$

$\Rightarrow\ 4x - 1 = 0$ or $x + 8 = 0$

$\Rightarrow\ x = \dfrac{1}{4}$ or $x = -8$

But $x = -8$ does not satisfy the equation.

Hence, $x = \dfrac{1}{4}$ is the only solution.

49. $\tan^{-1}\left(\frac{1}{1+1\cdot 2}\right)+\tan^{-1}\left(\frac{1}{1+2\cdot 3}\right)+\ldots$

$$..+\tan^{-1}\left(\frac{1}{1+n\cdot(n+1)}\right)=\tan^{-1}\theta$$

$$\Rightarrow \tan^{-1}\left(\frac{2-1}{1+1\cdot 2}\right)+\tan^{-1}\left(\frac{3-2}{1+2\cdot 3}\right)$$

$$+\ldots+\tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)=\tan^{-1}\theta$$

$\Rightarrow \quad \tan^{-1} 2 - \tan^{-1} 1 + \tan^{-1} 3 - \tan^{-1} 2 +$

$+ \ldots + \tan^{-1}(n+1) - \tan^{-1}(n) = \tan^{-1}\theta$

$\Rightarrow \quad \tan^{-1}(n+1) - \tan^{-1} 1 = \tan^{-1}\theta$

$$\Rightarrow \tan^{-1}\left(\frac{(n+1)-1}{1+(n+1)(1)}\right)=\tan^{-1}\theta$$

$$\Rightarrow \tan^{-1}\left(\frac{n}{n+2}\right)=\tan^{-1}\theta \Rightarrow \frac{n}{n+2}=\theta$$

50. We have, $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right)=\frac{\pi}{4} \text{ (for } 2x\cdot 3x<1)$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right)=\frac{\pi}{4}$$

Therefore, $\frac{5x}{1-6x^2}=\tan\frac{\pi}{4}=1$

$\Rightarrow \quad 6x^2+5x-1=0 \Rightarrow (6x-1)(x+1)=0$

which gives $x=\frac{1}{6}$ or $x=-1$.

Since $x=-1$ does not satisfy the equation as the L.H.S. of the equation becomes negative.

$\therefore \; x=\frac{1}{6}$ is the only solution of the given equation.

51. Consider R.H.S. $=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

$$=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}.\frac{4}{3}}\right)$$

$$=\tan^{-1}\left(\frac{15+48}{36-20}\right)=\tan^{-1}\left(\frac{63}{16}\right)=\text{L.H.S.}$$

52. L.H.S. $=2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$$=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}.\frac{1}{2}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$=\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}\right)=\tan^{-1}\left(\frac{28+3}{21-4}\right)=\tan^{-1}\left(\frac{31}{17}\right)$$

Now, $\tan^{-1}\left(\frac{31}{17}\right)=\theta$ (say) ...(1)

$$\Rightarrow \tan\theta=\frac{31}{17}$$

$$\therefore \quad \sin\theta=\frac{1}{\operatorname{cosec}\theta}=\frac{1}{\sqrt{1+\cot^2\theta}}$$

$$=\frac{1}{\sqrt{1+\left(\frac{17}{31}\right)^2}}=\frac{31}{\sqrt{31^2+17^2}}=\frac{31}{\sqrt{1250}}=\frac{31}{25\sqrt{2}}$$

$\Rightarrow \quad \theta=\sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$ ...(2)

From (1) & (2), L.H.S $=\sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)=$ R.H.S.

53. We have,

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, \quad (x>0)$$

$$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \frac{3}{2}\tan^{-1}x=\tan^{-1}1=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}x=\frac{\pi}{4}\times\frac{2}{3}=\frac{\pi}{6} \Rightarrow x=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow x=\frac{1}{\sqrt{3}}$$

54. L.H.S.

$$=2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)$$

$$=2\left[\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right]+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{\frac{13}{40}}{\frac{39}{40}}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\sqrt{\left(\frac{5}{7}\sqrt{2}\right)^2-1}$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)+\tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S.}$$

55. Putting $x = \cos\theta$, we get

$$\text{L.H.S.} = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2(\theta/2)}-\sqrt{2\sin^2(\theta/2)}}{\sqrt{2\cos^2(\theta/2)}+\sqrt{2\sin^2(\theta/2)}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos(\theta/2)-\sin(\theta/2)}{\cos(\theta/2)+\sin(\theta/2)}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}\right\}$$

[Dividing numerator and denominator by $\cos(\theta/2)$]

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\} = \frac{\pi}{4}-\frac{\theta}{2}$$

$$= \left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right) = \text{R.H.S.}$$

56. *Refer to answer 43.*

57. We have $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

Let $\tan^{-1}x = \theta \Rightarrow \tan\theta = x$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{1+x^2}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

Also, let $\cot^{-1}\frac{3}{4} = \beta \Rightarrow \cot\beta = \frac{3}{4}$

$$\Rightarrow \sin\beta = \frac{4}{5} \Rightarrow \beta = \sin^{-1}\frac{4}{5}$$

So, $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$$\Rightarrow \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right) = \sin\left(\sin^{-1}\frac{4}{5}\right)$$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}} = \frac{4}{5} \Rightarrow 16+16x^2 = 25 \Rightarrow x = \pm 3/4.$$

Hence values of x are 3/4, –3/4

58. L.H.S. $= \cot^{-1}7+\cot^{-1}8+\cot^{-1}18$

$$= \tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right)+\tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\frac{3}{11}+\tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right) = \tan^{-1}\left(\frac{65}{195}\right) = \tan^{-1}\frac{1}{3}$$

$= \cot^{-1}3 = \text{R.H.S.}$

Hence proved.

59. $\text{L.H.S} = \cos^{-1}x+\cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right\}$

$$= \cos^{-1}x+\cos^{-1}\left\{\frac{1}{2}\cdot x+\frac{\sqrt{3}}{2}\cdot\sqrt{1-x^2}\right\}$$

$$= \cos^{-1}x+\cos^{-1}\left\{\frac{1}{2}\cdot x+\sqrt{1-\left(\frac{1}{2}\right)^2}\cdot\sqrt{1-x^2}\right\}$$

$$= \cos^{-1}x+\cos^{-1}\frac{1}{2}-\cos^{-1}x.$$

$$= \cos^{-1}\frac{1}{2} = \frac{\pi}{3} = \text{R.H.S.}$$

60. $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$

$\Rightarrow \quad \frac{\pi}{2} - \cot^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$

$\Rightarrow \quad \cot^{-1}x = \frac{2\pi}{3} - \frac{\pi}{2} \Rightarrow \cot^{-1}x = \frac{4\pi - 3\pi}{6}$

$\Rightarrow \quad \cot^{-1}x = \frac{\pi}{6} \Rightarrow x = \cot\frac{\pi}{6} \Rightarrow x = \sqrt{3}$

61. L.H.S. $= \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$

$= \sin^{-1}\left(\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right)$

$= \sin^{-1}\left(\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right)$

$= \sin^{-1}\left(\frac{45}{85} + \frac{32}{85}\right) = \sin^{-1}\left(\frac{77}{85}\right)$

$= \cos^{-1}\sqrt{1-\left(\frac{77}{85}\right)^2} = \cos^{-1}\sqrt{\frac{7225-5929}{7225}}$

$= \cos^{-1}\sqrt{\frac{1296}{7225}} = \cos^{-1}\frac{36}{85} = \text{R.H.S.}$

62. $\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$

$= \tan\frac{1}{2}[2\tan^{-1}x + 2\tan^{-1}y] \quad = \tan(\tan^{-1}x + \tan^{-1}y)$

$= \tan\left\{\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\} = \frac{x+y}{1-xy}, y > 0 \,\&\, xy < 1$

63. L.H.S. $= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right) + \tan^{-1}\frac{1}{8}$

$= \tan^{-1}\left(\frac{5+2}{10-1}\right) + \tan^{-1}\frac{1}{8}$

$= \tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8} \quad = \tan^{-1}\left[\frac{\left(\frac{7}{9}+\frac{1}{8}\right)}{\left(1-\frac{7}{9}\times\frac{1}{8}\right)}\right]$

$= \tan^{-1}\left(\frac{56+9}{72-7}\right) = \tan^{-1}\left(\frac{65}{65}\right)$

$= \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$

64. Put $\sin^{-1}\frac{3}{4} = \theta \Rightarrow \sin\theta = \frac{3}{4}$

$\Rightarrow \quad \cos\theta \; = \sqrt{1-\left(\frac{3}{4}\right)^2} \; = \frac{\sqrt{7}}{4}$

Now, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \; = \tan\frac{\theta}{2}$

$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{\sqrt{7}}{4}}{1+\frac{\sqrt{7}}{4}}} = \sqrt{\frac{4-\sqrt{7}}{4+\sqrt{7}}\times\frac{4-\sqrt{7}}{4-\sqrt{7}}}$

$= \frac{4-\sqrt{7}}{3}.$

65. $\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{a-b}{a+b}\right)$

$= \tan^{-1}\left[\frac{\frac{a}{b}-\frac{a-b}{a+b}}{1+\frac{a}{b}\cdot\frac{a-b}{a+b}}\right] \; = \tan^{-1}\left[\frac{a(a+b)-b(a-b)}{b(a+b)+a(a-b)}\right]$

$= \tan^{-1}\left[\frac{a^2+b^2}{a^2+b^2}\right] \; = \tan^{-1}(1) = \frac{\pi}{4}$

66. *Refer to answer 61.*

67. We have, $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$ ...(1)

$\Rightarrow \quad \sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$

$\Rightarrow \quad 1-x = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right)$

$\Rightarrow \quad 1 - x = \cos(2\sin^{-1}x) = \cos 2\theta,$

where $\theta = \sin^{-1}x \;\Rightarrow\; x = \sin\theta$

$\therefore \quad 1 - x = 1 - 2\sin^2\theta \;\Rightarrow\; 1 - x = 1 - 2x^2$

$\Rightarrow \quad x = 2x^2 \Rightarrow x(2x-1) = 0 \;\Rightarrow\; x = 0, \frac{1}{2}$

For $x = \frac{1}{2}$,

L.H.S. of (1) $= \sin^{-1}\left(1-\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$

$= -\sin^{-1}\frac{1}{2} \; = -\frac{\pi}{6} \neq \frac{\pi}{2}$

$\therefore \quad x = \frac{1}{2}$ is not a solution of (1).

Hence, $x = 0$ is the only solution of (1).

68. L.H.S. $= \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$= \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right)$

$= \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) = \frac{\pi}{4}-\frac{x}{2} = \text{R.H.S.}$

69. Let $\sin^{-1}\frac{3}{5} = \theta$ and $\cot^{-1}\frac{3}{2} = \phi$

$\Rightarrow \quad \sin\theta = \frac{3}{5}$ and $\cot\phi = \frac{3}{2}$

$\Rightarrow \quad \tan\theta = \frac{3}{4}$ and $\tan\phi = \frac{2}{3}$

$\therefore \quad \theta = \tan^{-1}\frac{3}{4}$ and $\phi = \tan^{-1}\frac{2}{3}$

Thus, $\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}$

$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right) = \tan^{-1}\left[\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right]$

$= \tan^{-1}\left(\frac{17}{6}\right) = \alpha$ (say) ...(1)

$\Rightarrow \quad \tan\alpha = \frac{17}{6} \Rightarrow \cos\alpha = \frac{6}{\sqrt{6^2+17^2}} = \frac{6}{5\sqrt{13}}$

Now, L.H.S. $= \cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

$= \cos\alpha = \frac{6}{5\sqrt{13}} = \text{R.H.S.}$ [Using (1)]

70. *Refer to answer 43.*

71. L.H.S. $= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\left[\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right] - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\left(\frac{15+12}{20-9}\right) - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\frac{8}{19} = \tan^{-1}\left[\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right]$

$= \tan^{-1}\left(\frac{513-88}{209+216}\right)$

$= \tan^{-1}\left(\frac{425}{425}\right) = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$

72. *Refer to answer 65.*

73. *Refer to answer 52.*

74. L.H.S. $= 2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$

$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{4}}{1-\frac{3}{4}\cdot\frac{3}{4}}\right) - \tan^{-1}\left(\frac{17}{31}\right)$

$= \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\cdot\frac{17}{31}}\right)$

$= \tan^{-1}\left(\frac{24\times31-17\times7}{7\times31+24\times17}\right)$

$= \tan^{-1}\left(\frac{744-119}{217+408}\right) = \tan^{-1}\left(\frac{625}{625}\right)$

$= \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S.}$

75. We have, $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$

$\left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{6} \Rightarrow \frac{2x}{1-x^2} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad 2\sqrt{3}x = 1-x^2 \Rightarrow x^2 + 2\sqrt{3}x - 1 = 0$

$\Rightarrow \quad x = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = -\sqrt{3} \pm 2$

$= 2-\sqrt{3}$ (Reject $-\sqrt{3}-2$ as $-1 < x < 1$).

76. To prove : $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\tan^{-1}\frac{4}{3}$

$\Rightarrow \quad 2\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\right)=\tan^{-1}\frac{4}{3}$

Now L.H.S. $=2\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\right)$

$=2\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)=2\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{2}$

$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}\times\frac{1}{2}}\right)=\tan^{-1}\frac{4}{3}=$ R.H.S.

77. The given equation is

$\cos(2\sin^{-1}x)=\frac{1}{9} \qquad (x>0) \qquad ...(1)$

Put $\sin^{-1}x=\theta \Rightarrow x=\sin\theta$

$\therefore$ Eq. (1) $\Rightarrow \cos 2\theta=\frac{1}{9}\Rightarrow 1-2\sin^2\theta=\frac{1}{9}$

$\Rightarrow \quad 2\sin^2\theta=1-\frac{1}{9}=\frac{8}{9}$

$\Rightarrow \quad x^2=\frac{4}{9}\Rightarrow x=\frac{2}{3} \quad (\because x>0)$

78. Putting $x=\tan^2\theta$, we get

R.H.S. $=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\frac{1}{2}\times 2\theta=\theta$

$=\tan^{-1}\sqrt{x}=$ L.H.S.

$[\because \ x=\tan^2\theta\Rightarrow\tan\theta=\sqrt{x}\Rightarrow\theta=\tan^{-1}\sqrt{x}]$

$\therefore \quad \tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

79. *Refer to answer 63.*

80. L.H.S. $=\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]$

Let $\cot^{-1}x=\theta\Rightarrow x=\cot\theta\Rightarrow x^2=\cot^2\theta$

$\Rightarrow \quad \operatorname{cosec}^2\theta-1=x^2 \Rightarrow \operatorname{cosec}^2\theta=1+x^2$

$\Rightarrow \quad \operatorname{cosec}\theta=\sqrt{1+x^2}\Rightarrow\sin\theta=\frac{1}{\sqrt{1+x^2}}$

Now, L.H.S. $=\cos\left(\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)$

Let $\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)=\phi$

$\Rightarrow \quad \frac{1}{\sqrt{1+x^2}}=\tan\phi\Rightarrow\frac{1}{1+x^2}=\tan^2\phi$

$\Rightarrow \quad \sec^2\phi-1=\frac{1}{1+x^2}\Rightarrow\sec^2\phi=1+\frac{1}{1+x^2}$

$\Rightarrow \quad \sec^2\phi=\frac{1+x^2+1}{1+x^2}\Rightarrow\cos^2\phi=\frac{1+x^2}{2+x^2}$

$\therefore \quad$ L.H.S. $=\cos\phi=\sqrt{\frac{1+x^2}{2+x^2}}=$ R.H.S.

81. L.H.S. $=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$=\tan^{-1}\left[\frac{\left(x+\frac{2x}{1-x^2}\right)}{\left(1-\frac{x(2x)}{1-x^2}\right)}\right]=\tan^{-1}\left(\frac{x(1-x^2)+2x}{(1-x^2)-2x^2}\right)$

$=\tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=$ R.H.S.

82. $\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x}{2}+\frac{x}{3}}{1-\frac{x}{2}\times\frac{x}{3}}\right]=\frac{\pi}{4}\left(\text{for } \frac{x}{2}\cdot\frac{x}{3}<1\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{3x+2x}{6-x^2}\right)=\frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{5x}{6-x^2}\right)=\frac{\pi}{4}\Rightarrow\frac{5x}{6-x^2}=\tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{5x}{6-x^2}=1 \Rightarrow x^2+5x-6=0$

$\Rightarrow \quad (x+6)(x-1)=0 \Rightarrow x=-6, x=1$

But $x=-6$ does not satisfy the equation.

$\therefore \quad x=1$ is the only solution.

83. *Refer to answer 48.*

84. *Refer to answer 52.*

85. We have $\cos^{-1}x+\sin^{-1}\left(\frac{x}{2}\right)=\frac{\pi}{6}$

$\Rightarrow \cos^{-1}x=\frac{\pi}{6}-\sin^{-1}\left(\frac{x}{2}\right)$

$\Rightarrow x=\cos\left(\frac{\pi}{6}-\sin^{-1}\frac{x}{2}\right)$

$=\cos\frac{\pi}{6}\cos\left(\sin^{-1}\frac{x}{2}\right)+\sin\frac{\pi}{6}\sin\left(\sin^{-1}\frac{x}{2}\right)$

$\Rightarrow x=\frac{\sqrt{3}}{2}\cos\left(\cos^{-1}\sqrt{1-\frac{x^2}{4}}\right)+\frac{1}{2}.\frac{x}{2}$

$\Rightarrow x=\frac{\sqrt{3}}{2}\sqrt{1-\frac{x^2}{4}}+\frac{x}{4} \Rightarrow x-\frac{x}{4}=\frac{\sqrt{3}}{2}\sqrt{1-\frac{x^2}{4}}$

$\Rightarrow \frac{3x}{4}=\frac{\sqrt{3}}{2}\sqrt{1-\frac{x^2}{4}}\Rightarrow\frac{9x^2}{16}=\frac{3}{4}\left(1-\frac{x^2}{4}\right)$

$\Rightarrow \frac{3x^2}{4}=1-\frac{x^2}{4}\Rightarrow\frac{3x^2}{4}+\frac{x^2}{4}=1 \Rightarrow \frac{4x^2}{4}=1$

$\Rightarrow x^2=1 \quad \therefore\ x=\pm 1.$

86. L.H.S. $=\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}$

$=\sin^{-1}\left(\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right)+\sin^{-1}\frac{16}{65}$

$=\sin^{-1}\left(\frac{4}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{3}{5}\right)+\sin^{-1}\frac{16}{65}$

$=\sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)+\sin^{-1}\frac{16}{65}$

$=\sin^{-1}\left(\frac{63}{65}\right)+\sin^{-1}\frac{16}{65}$

$=\cos^{-1}\left(\sqrt{1-\left(\frac{63}{65}\right)^2}\right)+\sin^{-1}\frac{16}{65}$

$\left[\because \sin^{-1}x=\cos^{-1}\sqrt{1-x^2},\ \forall\ 0\le x\le 1\right]$

$=\cos^{-1}\frac{16}{65}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}=$ R.H.S.

87. *Refer to answer 52.*

88. $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x$

$\Rightarrow \tan^{-1}(1)+\tan^{-1}x=\frac{\pi}{4}+\tan^{-1}x$

$\Rightarrow \frac{\pi}{4}+\tan^{-1}x=\frac{\pi}{4}+\tan^{-1}x$

$\therefore\ x\in(0,1).$

89. Let $y=\cot^{-1}\left(\sqrt{1+x^2}-x\right)$

Let $x=\cot\theta\Rightarrow\theta=\cot^{-1}x$

$\therefore y=\cot^{-1}\left(\sqrt{1+\cot^2\theta}-\cot\theta\right)$

$=\cot^{-1}(\operatorname{cosec}\theta-\cot\theta)=\cot^{-1}\left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right)$

$=\cot^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\cot^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right)$

$=\cot^{-1}\left(\tan\frac{\theta}{2}\right)=\cot^{-1}\left(\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)$

$=\frac{\pi}{2}-\frac{\theta}{2}=\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x.$

90. L.H.S. $=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$

$=\frac{9}{4}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{1}{3}\right)\right]$ $\left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

$=\frac{9}{4}\cos^{-1}\left(\frac{1}{3}\right)$

$=\frac{9}{4}\sin^{-1}\sqrt{1-\left(\frac{1}{3}\right)^2}$ $\left[\because \cos^{-1}x=\sin^{-1}\sqrt{1-x^2}\ \text{for}\ 0\le x\le 1\right]$

$=\frac{9}{4}\sin^{-1}\sqrt{1-\frac{1}{9}}=\frac{9}{4}\sin^{-1}\sqrt{\frac{8}{9}}$

$=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)=$ R.H.S.