Functions and Relations

NUMBERS & THEIR SETS

- Natural Number : N = {1, 2, 3, 4, 5, ...}
- Whole Number : W = {0, 1, 2, 3, ...}
- Integers : I or Z = {............, – 4, –3, – 2, – 1, 0, 1, 2, 3, 4, ...} = {0, ± 1, ± 2, ±3, ...}
- Rational Numbers : $Q = \left\{ \frac{p}{q} : p, q \in z, q \neq 0 \right\}$
- Irrational Numbers : The numbers which are not rational of which can not be written in the form of p/q called irrational numbers i.e.

  $\left\{ \sqrt{3}\, 2^{1/3}, 5^{1/4}, \pi, e .... \right\}$
- Real Numbers : {x, where x is rational and irrational number}

  R = {1, 1000, 20/6, π , $\sqrt{2}$, –10, –20/3, ...}
- Positive Real Numbers : $R^+ = (0, \infty)$
- Negative Real Numbers : $R^- = (-\infty, 0)$
- R_0 : all real number except 0(zero) = R – {0}
- Imaginary Numbers : Im = $\sqrt{-k}$, where $k \in R$

INTERVAL

The set of the numbers between any two real numbers is called interval.

- Closed Interval

  $[a, b] = \{x : a \le x \le b\}$
- Open Interval : (a, b) or]a, b[= {x : a < x < b}
- Semi Open or Semi Closed Interval

  $[a, b[$ or $[a, b) = \{x : a \le x < b\}$ (Left Closed)

  $]a, b]$ or $(a, b] = \{x : a < x \le b\}$ (Right Closed)

3. FUNCTION

Let A and B be two given sets and if a rule associates to each element a $\in$ A, a unique element b $\in$ B, then this rule f is called a function from A to B. It is denoted by $f : A \to B$.

If an element a $\in$ A is associated with b $\in$ B under the function f, then the element b is called the f image of a or the value of f at a. It is denoted by the symbols

b = f (a) or $f : a \to b$ or $(a, b) \in f$

Also a is called the pre-image of b under f.

Remarks :

Whether $f : A \to B$ is a function or not, test the following :

(i) existence of f-image of every element of A in the set B.

(ii) uniquenees of f-image of every element of the set A.

4. FUNCTION AS A SET OF ORDERED PAIRS

A function $f : A \rightarrow B$ can be expressed as a set of ordered pair is a member of A and second element is the member of B. Hence f is a set of ordered pairs (a, b) such that

(i) a is an element of A

(ii) b is an element of B

(iii) no two ordered pairs of f have the same first element

(iv) every member of A is a first element of one of the ordered pairs of f

$f : A \rightarrow B$ is a subset of $A \times B$. It is expressed in the form of ordered pairs as follows :

$f = \{(a, b)/b = f(a),\ a \in A \text{ and } b \in B\}$

DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Suppose that f is a function from A to B, i.e. $f : A \rightarrow B$, then set A is called the domain and set B is called the co-domain.

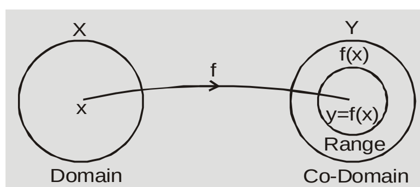
Also the set of all f-images of the elements of A is called the range of f and it is denoted by f(A).

Therefore $f(A) = \{f(a) \mid a \in A\} \subset B$. If f is expressed in terms of ordered pairs, then set of first element of ordered pairs of f will be domain and set of second elements of these ordered pairs will be range of f i.e.

Domain of $f = \{x \mid (x, y) \in f\}$

Range of $f = \{y \mid (x, y) \in f\}$

The set Y is also called the co-domain of f clearly $f(x) \subseteq Y$.



IDENTITY FUNCTION

The function $f : R \rightarrow R$ is called an identity function if $f(x) = x$, " $x \in R$. The domain of this identity function is R and its range is also R.

EQUAL FUNCTION

Two fucntion f and g are said to be equal fucntions, if and only if

(i) domain of f = domain of g

(ii) co-domain of f = co-domain of g

(iii) $f(x) = g(x)$ " $x \in$ their common domain

DOMAIN OF ALGEBRAIC FUNCTIONS

(i) If $f(x) = c$ (constant function), then $D_f = R$

(ii) If $f(x) = p(x)$, a polynomial function in x, $D_f = R$

(iii) If $f(x) = \dfrac{1}{p(x)}$, then $D_f = R - \{x : p(x) = 0,\ x \in R\}$

(iv) If $f(x) = \dfrac{p(x)}{q(x)}$, then $D_f = D_p \cap D_q - \{x : q(x) = 0,\ x \in R\}$

(v) If $f(x) = \sqrt{p(x)}$ then $D_f = \{x : p(x) \geq 0, x \in R\}$

(vi) If $f(x) = \dfrac{1}{\sqrt{p(x)}}$, then $D_f = \{x : p(x) > 0,\ x \in R\}$.

(vii) If $f(x) = p(x) \pm q(x)$

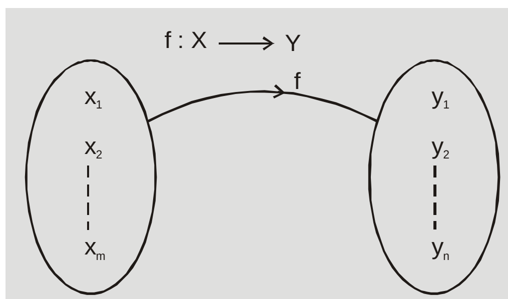
then $D_f = D_p \cap D_q$

(viii) If $f(x) = p(x) \times q(x)$ then $D_f = D_p \cap D_q$

(ix) If $f(x) = \sqrt{\dfrac{p(x)}{q(x)}}$ then $D_f = \left\{x : \dfrac{p(x)}{q(x)} \geq 0, q(x) \neq 0,\ x \in R\right\}$

NUMBER OF FUNCTION (OR MAPPING) FROM X TO Y

Let $X = \{x_1, x_2, x_3, \ldots\ldots\ldots x_m\}$ (i.e. m elements)
and $Y = \{y_1, y_2, y_3 \ldots\ldots\ldots y_m\}$ (i.e. n elements)
Then each element in domain x_i (i = 1, 2, 3, ... m) corresponds n images

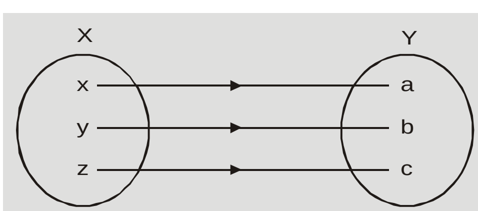


Thus, Total number of function from X to Y $\Rightarrow$ n × n × ..... m times = n^m = number of elements in domain.
i.e. $(\text{Number of elements in co-domain})^{\text{Number of elements in domain}}$

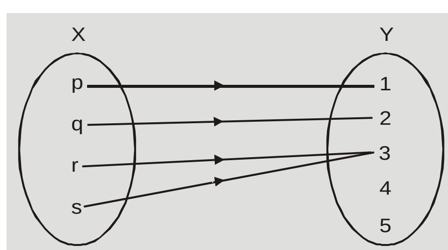
REPRESENTATION & TESTING FOR A FUNCTION

Mapping : It show the graphical aspect of the relation of the elements of X with the elements of Y.

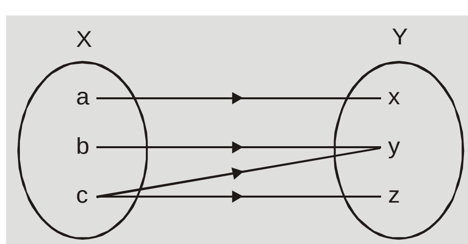
(a) $f_1 : X \to Y$



(b) $f_2 : X \to Y$



(c) $f_3 : X \to Y$



In the above given mapping rule f_1 and f_2 shows a function because each elements of X is associated with a unique element of Y. Where as f_3 is not function because in f_3 element c is associated with two elements of Y.

Algebraic Method : It show the relation between the elements of two sets in the from of two variables x and y where x is independent variable and y is depended variable.

If X and Y be two given sets X = {1, 2, 3}, Y = {5, 7, 9} then f : X ® Y, y = f(x) = 2x + 3

In the form of ordered pairs : A function f : X $\rightarrow$ Y can be expressed as a set of ordered piers in which first element of every ordered pair is a member of X and second element is the member of Y. So f is a set of order pairs (a, b) such that

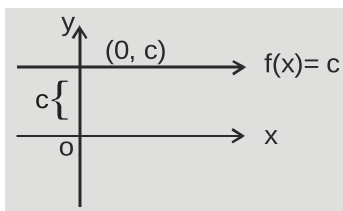
(i) a is an element of X (ii) b is an element of Y

(iii) Two ordered pairs should not have the same first element.

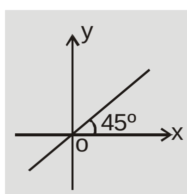
Vertical line test for a function : If we are given a graph of the relation. Then we can check whether the given relation is function or not. If it is possible to draw. a vertical line which cuts the given curve at more than one point then given relation is not a function and when this vertical line means line parallel to y-axis cuts the curve at only one point then it is a function. Fig represents a function.

CLASSIFICATION OF FUCNTION

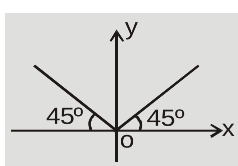
Constant function : If the range of a function f consists of only one number then f is called a constat function. e.g. let f(x) = c ; where c is constant number.



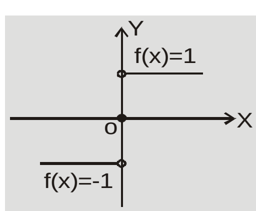
Identity function : The function defined by f(x) = x ; " x $\in$ R, is called the identity function.



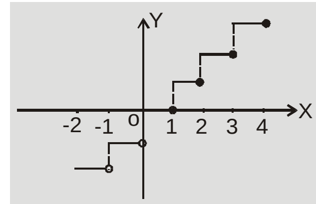
Modulus function : The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$, is called the modulus function.



Signum function : The function defined by $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases} = \begin{cases} 1, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -1, & \text{when } x < 0 \end{cases}$, is called signum function.



Greatest integer function : This function is denoted by [x]. Where [x] = greatest integer less than or equal to x. For example $\left[-\frac{5}{3}\right] = -2$ and $\left[\frac{5}{3}\right] = 1$



Note : Important Identities :

(i) [x] $\leq$ x (This is always true)

(ii) [x] + 1 > x

PROPERTIES OF GREATEST INTEGRAL FUNCTION :

(i) [x] = x, holds if x is integer.

(ii) [x + I] = [x] + I if I is integer.

(iii) [x + y] = [x] + [y], if {x} + {y} < 1

= [x] + [y] + 1, if {x} + {y} $\geq$ 1

(iv) If [ϕ (x)] ³ I, then ϕ(x) $\geq$ I.

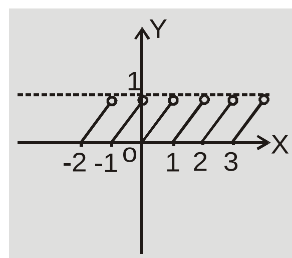
If [ϕ (x)] £ I, then ϕ(x) < I + I.

(v) [–x] = – [x] if x $\in$ integer

(vi) [–x] = – [x] – 1 if x $\notin$ integer

Fraction-Part Function :

{x} denotes frational part of 'x'. It is equal to x – [x]



e.g {2.7} = 0.7, {3} = 0

{–3.2} = 0.8

Domian = R

Range = [0, 1)

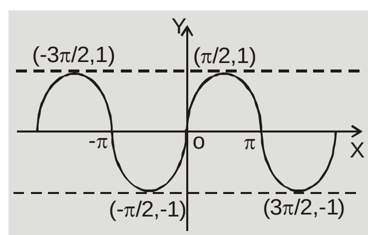
Trigonometric Functions :

(i) Sine function

f(x) = sin x

Domain = R

Range = [–1, 1].

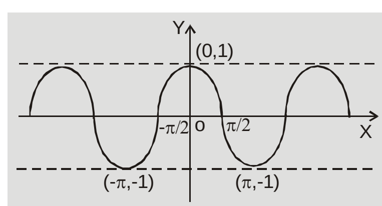


(ii) Cosine function

f(x) = cos x

Domain = R

Range = [–1, 1].

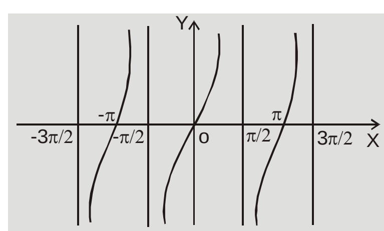


(iii) Tangent function

f(x) = tan x

Domain = $R - \left\{\frac{(2n+1)\pi}{2}\right\}$

Range = R.

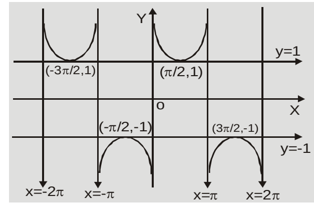


(iv) Cosecant function

$f(x) = \text{cosec } x$

Domain = $R - \{n\pi \ : n \in Z\}$;

Range = $R - (-1, 1)$

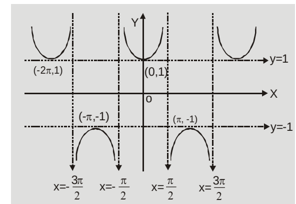


(v) Secant function

$f(x) = \sec x$

Domain = $R - \{(2n + 1)\ \pi/2\ : n \in Z\}$;

Range = $R - (-1, 1)$

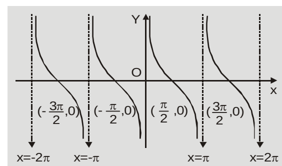


(vi) Cotangent function

$f(x) = \cot x$

Domain = $R - \{n\pi \ : n \in Z\}$;

Range = R.

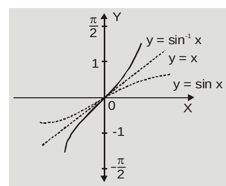


Inverse of trigonometric functions :

(i) $f(x) = \sin^{-1} x$

Domain = $[-1, 1]$.

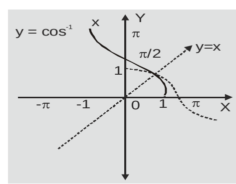
Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



(ii) $f(x) = \cos^{-1} x$

Domain = $[-1, 1]$.

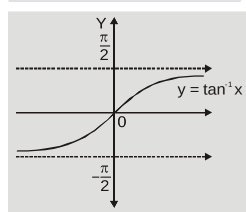
Range = $[0, \pi]$



(iii) $f(x) = \tan^{-1} x$

Domain = R

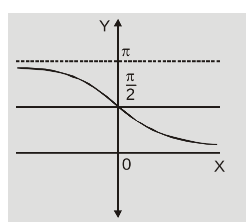
Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



(iv) $f(x) = \cot^{-1} x$

Domain = R ;

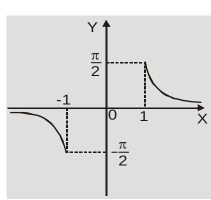
Range = $(0, \pi)$



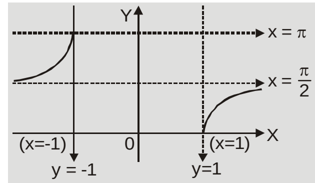
(v) $f(x) = \text{cosec}^{-1} x$

Domain = $R - (-1, 1)$

Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$



(vi) $f(x) = \sec^{-1} x$

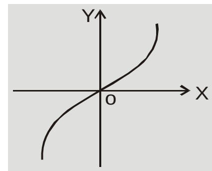


Domain = R – (–1, 1).

Range = $[0, \pi] - \{\pi/2\}$

11.9 Hyperbolic function :

(i) $f(x) = \sinh x$

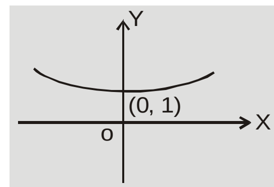


Domain = $(-\infty, \infty)$

Range = $(-\infty, \infty)$

It is a continuous and one-one function,

(ii) $f(x) = \cosh x$

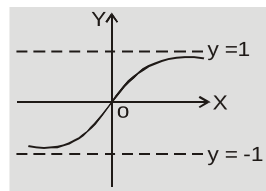


Domain = $(-\infty, \infty)$

Range = $[1, \infty)$

It is a continuous and many one function.

(iii) $f(x) = \tanh x$



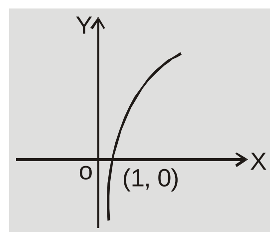
Domain = $(-\infty, \infty)$

Range = $(-1, 1)$

It is a continuous and one-one function.

Logarithmic function :

(i) $f(x) = \log_a x \ (a > 1)$

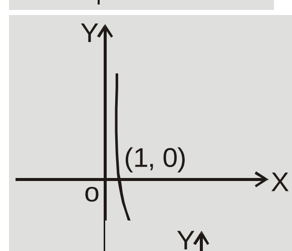


Domain = R^+

Range = R

It is a continuous and one-one function.

(ii) $f(x) = \log_a x \ (a < 1)$



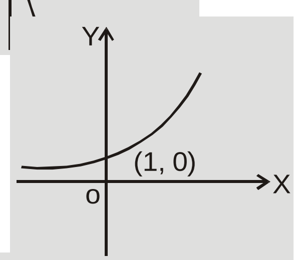
Domain = R^+

Range = R

It is a continuous and one-one function.

11.11 Exponential function :

(i) $f(x) = a^x \ (a > 1)$

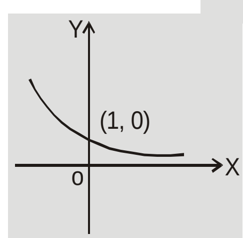


Domain = R

Range = R^+

It is a continuous and one-one function.

(ii) $f(x) = a^x \ (a < 1)$



Domain = R

Range = R^+

It is a continuous and one-one function.

Polynomial function : A function $y = f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$, where $a_0, a_1, a_2 .... a_n$ are real numbers and n is non negative integer, then f(x) is called polynomial if $a_0 \neq 0$, then n is the degree of polynomial function.

For Ex. $f(x) = x^{1920} + 5x^{1919} + 6x$, ... $g(x) = x^2 + 9x + 3$

· The domain of a Polynomial Function is R.

Rational function : A function is said to be rational function, if it is of the form $\frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomial functions and $q(x) \neq 0$. i.e.,

The quotient of two polynomial functions is called the Rational function.

For example, $f(x) = \frac{x^2 - 1}{2x^3 + x^2 + 1}$ is a Rational Function.

Irrational function : A function involving one or more radicals of polynomial or polynomials is called an irrational function.

For Ex. $f(x) = x + \sqrt{x} + 6, g(x) = \frac{x^3 - \sqrt{x}}{1 + x^{1/4}}$ are Irrational Functions.

Algebraic function : An algebraic function is one which consists of a finite number of terms involving power and roots of variable x and the four simple operations – addition, subtraction, multiplication and division. Obviously, all polynomial, rational and irrational functions are algebraic functions.

Transcendental function : All functions which are not algebraic are called transcendental functions. These functions include -

(i) Trigonometric function as sinx, cosx tanx etc.

(ii) Inverse trigonometric functions as $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ etc.

(iii) Logarithmic function as $\log_a x$, $\log_e (1 + x)$, $\log\left(\frac{1+x^2}{2+x^2}\right)$, $\log (x + \sqrt{1+x^2})$ ..... etc.

(iv) Exponential function as e^x, e^{-x}, a^x $\frac{e^x - 1}{e^x + 1}$, etc.

(v) Mixed function as $\sin^2 x + e^x + 3 \log x$ etc.

(vi) The general exponential functions as $x^{\log x}, x^{\cos x}$ etc.

EXPLICIT AND IMPLICIT FUNCTION

Explicit Function : A function is said to be explicit if its rule is directly expressed (or can be expressed) in terms of the independent variable. Such a function is generally written as y = f(x), x = g(y).

e.g. $y = 2x + 3, x^2 = y + 1$ etc.

Implicit Function : A function is said to be implicit if its rule cannot be expressed directly in terms of the independent variable symbolically we write such a function as f(x, y) = 0, f(x, y) = 0

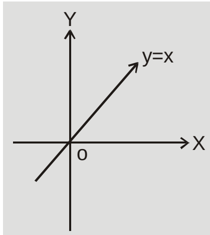
e.g. $x^3 + y^3 + 3xy = 0, x^y + y^x = a^b$ etc.

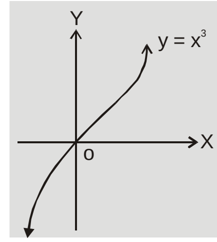
ODD AND EVEN FUNCTION

Odd Function : A function f(x) is said to be an odd function if, f(–x) = – f(x) for all x.

The graph of odd function is symmetrical in opposite quadrants.

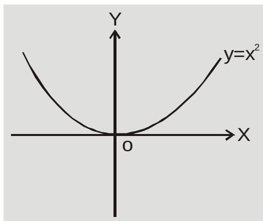
For Ex. $x^3, x^5, \sin^3 x$

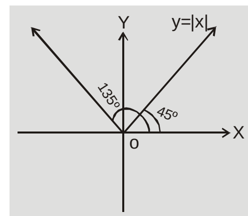




Even Function : A function f(x) is said to be an even of f(–x) = f(x) for all x. The graph is always symmetrical about y-axis.

For Ex. $|x|$, x^2, cosx, $\sin^2 x$.





Properties of Odd and Even Functions :

(i) The product of two odd functions or two even functions is an even function.

(ii) The product of odd and even function is an odd function.

(iii) Every function y = f(x) can be expressed as the sum of an even and odd function.

(iv) The derivative of an odd function is an even function and derivative of even function is an odd function.

(v) A function which is even or odd, when squared becomes even function.

(vi) It is not necessary that every function is either even or odd, i.e. there are functions which are neither even nor odd. For example $2x^4 + x^3$, e^{-x} etc.

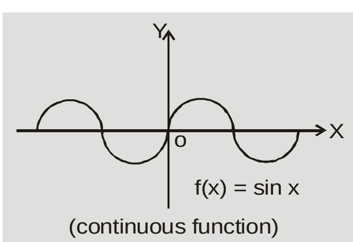
(vii) For odd function f(–x) + f(x) = 0

(viii)For even function f(–x) – f(x) = 0

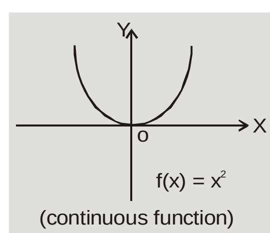
CONTINUOUS & DISCONTINUOUS FUNCTION

Continuous Function : A function is said to be continuous. If we are not required to lift the pen or pencil off the paper while plotting the graph. i.e there is no gap or break or jump in the graph.

e.g. $f(x) = x^2$, f(x) = sin x, f(x) = | x |, f(x) = cos x all continuous function.



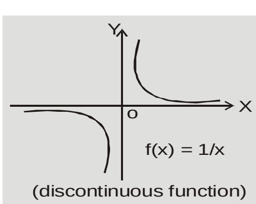
(continuous function)



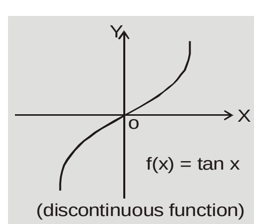
(continuous function)

Discontinuous function : A function is said to be discontinuous if there is a break or gap or jump in the graph of the function at any point.

e.g. f(x) = 1/x, f(x) = tanx, f(x) = [x] are discontinuous functions.



(discontinuous function)



(discontinuous function)

INCREASING FUNCTION

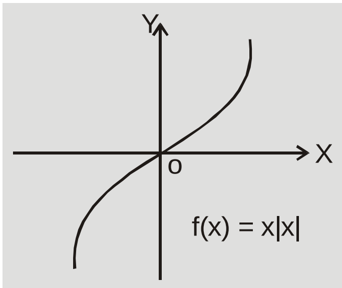
A function f(x) is called increasing function in the domain D if the value of the function does not decreases by increasing the value of x.

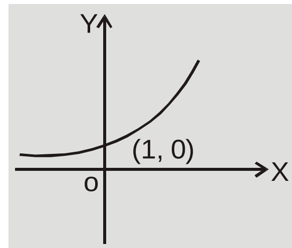
So $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2) \ \forall \ x_1, x_2 \in$ Domain

or $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \ \forall \ x_1, x_2 \in$ Domain

e.g. $f(x) = e^x$, $f(x) = a^x$, $f(x) = x^2, x \geq 0$, $f(x) = x|x|$ are increasing functions.

The graph of these functions rises from left to right.





A function is called strictly increasing if

If $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \ \forall \ x_1, x_2 \in$ Domain

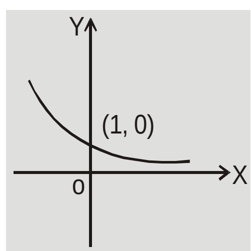
DECREASING FUNCTION

A function f(x) is said to be decreasing function in the domain D if the value of the function does not increase by increasing the value of x (variable).

So if $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2) \ \forall \ x_1, x_2, \in D$

or $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \ \forall \ x_1, x_2 \in D$

e.g. $f(x) = \log_a x (a < 1)$, $f(x) = e^{-x}$ are decreasing function. The graph of these functions is downward from left to right.



A function is called strictly decreasing if

If $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

Or $x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \ \forall \ x_1, x_2 \in D$

Note : It is not essential for any function to be increasing or decreasing. There are some functions which are neither increasing nor decreasing i.e. function is increasing in one part of given interval and decreasing in second part.

e.g. $f(x) = \sin x$, $f(x) = |x|$, $f(x) = e^x + e^{-x}$

PERIODIC FUNCTION

A fuction f(x) is periodic if there is a positive number T such that $f(x + T) = f(x)$ for all $x \in D$.The smallest value of such T is called the principal or fundamental period of function f(x).If we draw graph of a periodic function f(x), we find graph gets repeated after each interval of length T.

- Obviously, if T is the period of f(x), then f(x) = f (x + T) = f (x + 2T) = f (x + 3T) = ...
  e.g. y = sin x is periodic with period 2 p as sin(x + 2p) = sin x. Graphically.

RULES FOR FINDING PERIOD OF A PERIODIC FUNCTION

(i) If f(x) is periodic with period T, then af(x) + b, where a,b $\in$ R (a $\neq$ 0) is also a periodic function with period T.

(ii) If f(x) is periodic with period T, then f(nx + b), is also periodic with period $\frac{T}{|n|}$.

(iii) If f(x) is periodic with T_1 as the period and g(x) is periodic with T_2 as the period and L.C.M. of T_1 & T_2 is possible, then f(x) + g(x) is periodic with period equal to L.C.M. of T_1 & T_2, provided f(x) and g(x) connot be interchanged by adding a positive number in x which is less than L.C.M. of T_1 & T_2 in this case this number becomes period of f(x) ± g(x).

(iv) If f(x) is periodic with period T, then $\frac{1}{f(x)}$ is also periodic with same period T.

(v) If f(x) is periodic with period T, $\sqrt{f(x)}$ is also periodic with same period T.

(vi) If f(x) is a periodic function with period T and g(x) is a strictly monotonic function. Then g(f((x)) will also be periodic with period T.

(vii) Constant function is periodic with no-fundamental period.

VALUE OF THE FUCNTION

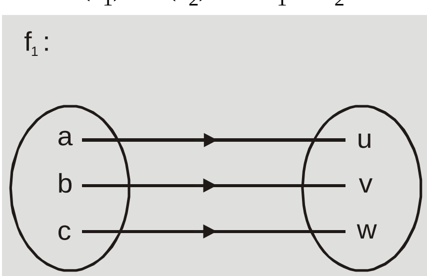
If y = f(x) is any function defined in R, then for any given value of x(say x = a), the value of the function f(x) can be obtained by substituting x = a in it and it is denoted by f(a).

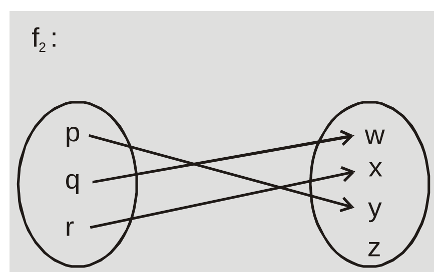
TYPE OF FUNCTION

One-one or injective Function : A function f : X $\rightarrow$ Y is said to be one-one or injective if distinct elements of X have distinct image in Y. Therefore for any two elements x_1, x_2 of a set X,

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then function is one-one



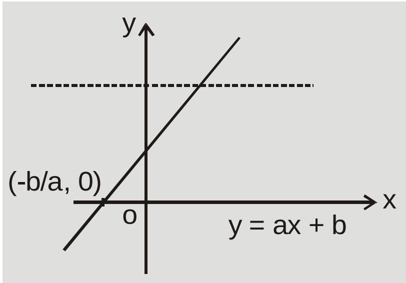


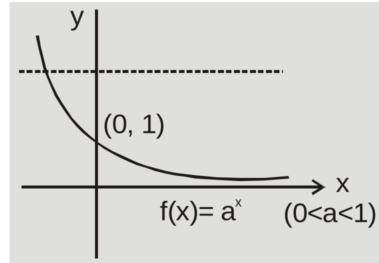
The above given diagram f_1 & f_2 shows one-one function.

Note :

(i) If function is given in the from of ordered pairs and if no two ordered pairs have same second element then function is one-one.

(ii) If the graph of the function y = f(x) is given, and each line parallel to x-axis cuts the given curve at maximum one point then function is one-one.





EXAMPLES OF ONE-ONE FUNCTIONS

(i) $f : R \to R$, $f(x) = x$,

(ii) $f : R \to R$, $f(x) = ex$,

(iii) $f : R \to R$, $f(x) = ax^n + b$, n is odd positive integer

(iv) $f : R \to R$, $f(x) = x\,|x|$

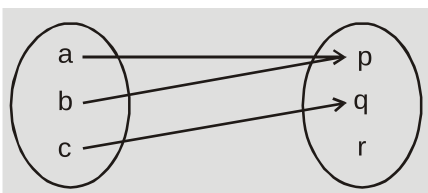
Note : Let f be a one-one function from a set A to the set B. Let O(A) = m and O(B) = n. If m > n no one-one function can be defined from A to B. So, we have $m \leq n$.

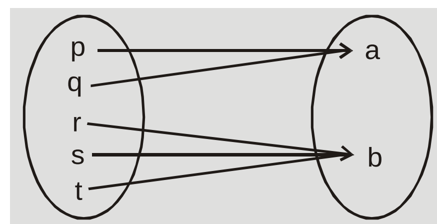
$\therefore$ Total number of one-one functions from A to B = $n(n-1)$ .... $[n-(m-1)] = {}^nP_m$

Many-One Function : A function $f : X \to Y$ is said to be many-one if there exists at least two distinct elements in X whose images are same in Y.

Therefore $f : X \to Y$ is many-one if

$$x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$$

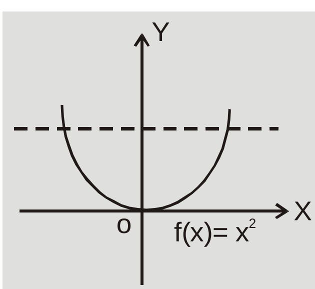


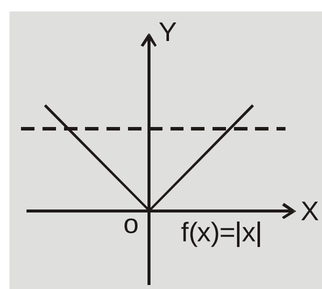


The above given arrow-diagrams show many one-function.

(i) If function is given in the form of set of ordered pairs and the second element of at least two ordered pairs are same then function is many-one.

(ii) If the graph of y = f(x) is given and the line parallel to x-axis cuts the curve at more than one point then function is many-one.





Example of many-one function :

(i) $f : R \to R$, $f(x) = C$, where C is a constat

(ii) $f : R \to R$, $f(x) = x^2$

(iii) $f : R \to R$, $f(x) = ax^2 + b$

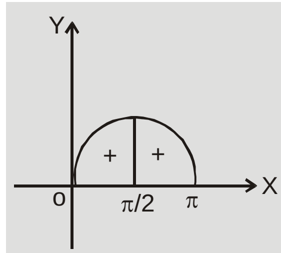
(iv) $f : R \to R$, $f(x) = |\,x\,|$

Methods to check trigonometrical function to be one-one many-one :

(i) If the domain of the function is in one quadrant then trigonometrical functions are always one-one.

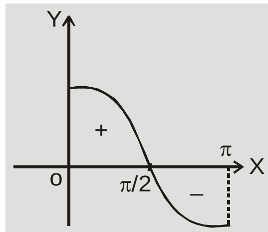
(ii) If trigonometrical function changes its sign in two consecutive quadrants then it is one-one but if it does not change the sign then it is many one.

$f : (0, \pi), f(x) = \sin x$



many one

$f : (0, \pi), f(x) = \cos x$

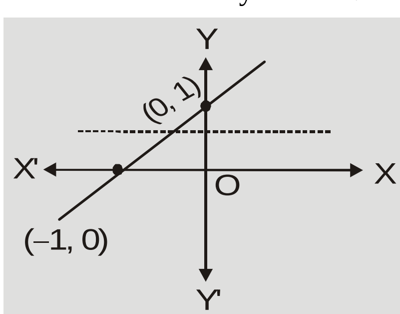


one one

(iii) In three consecutive quadrants trigonometrical functions are always many one.

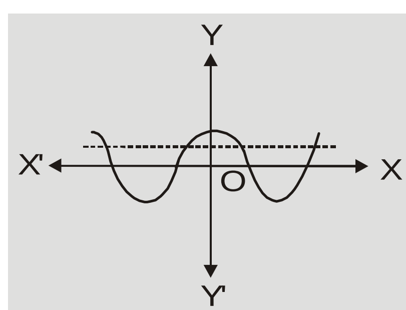
Remarks :

1. Some functions are one-one in every domain. The examples of such functions are as follows :
   (i) $ax^n + b$, when n is odd. In particular $3x^3 + 1$, $2x + 1$, x^5 etc.
   (ii) Identity function $f(x) = x$; conjugate function $g(z) = \bar{z}$, Reciprocal function $h(x) = 1/x$.
   (iii) Exponential functions e^x, a^x, 2^x and logarithmic functions $\log_a x$, $\log_e x$.
   (iv) Hyperbolic functions sinh x, tanh x, cosech x, coth x.
   (v) $x\,|x|$.
2. Some functions are usually many-one functions. Examples are as follows :
   (i) $ax^n + b$, where n is even and domain contains both positive and negative elements
   For ex. x^2, $x^4 - 1$ etc.
   (ii) Constant function $f(x) = c$
   (iii) Modulus function $f(x) = |x|$, $g(x) = x + |x|$, $h(x) = x - |x|$, $\phi(x) = \frac{x + |x|}{x}$ etc.
   (iv) Greatest integer function [x].
   (v) Trigonometric functions sin x, cos x, tan x, sec x, cosec x. Hyperbolic functions sech x, cosh x.
   (vi) Even function x^2, cos x etc.
3. If the graph of the given function can be easily constructed, then draw its graph. When a line parallel to x-axis meet it in two or more than two points, then the function is many-one otherwise it is one-one. Obviously even functions are usually one-one.



$y = x + 1$

(One one)



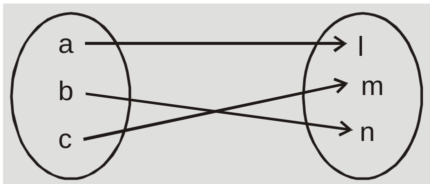
$y = \cos x$

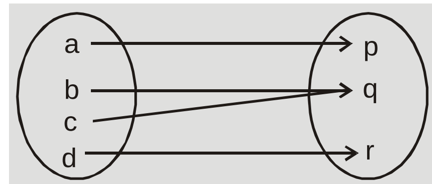
(Many one)

19.3 Onto or Surjective Function : A function $f : X \to Y$ is said to be onto or surjective if every element of Y is the image of some element of X under the map f.

In other words. Range of f = Co-domain of f.

The following arrow-diagram shows onto function





Example of onto function :

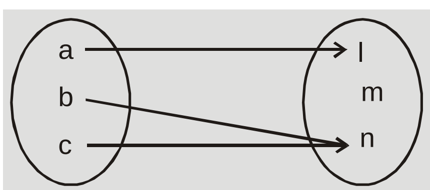
(i) $f : R \to R$, $f(x) = x$,   (ii) $f : R \to R$, $f(x) = ax + b$, $a \neq 0$, $b \in R$

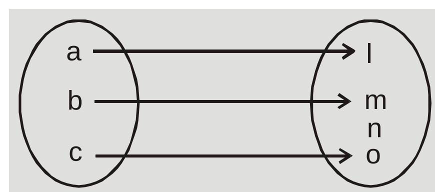
Note : Let f be an onto function from a set A to the set B. Let O(A) = m and O(B) = n, we suppose $1 \leq n \leq m$.

The number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r}\, {}^{n}C_r\, r^m$.

19.4 Into function : A function $f : X \to Y$ is into if there exist at least one element in Y which is not the f-image of any element in X. Therefore, at least one element of Y such that $f^{-1}(y) = \phi$ then function is into. In other words. Range of $f \neq$ co-domain of f.

The following arrow-diagram shows into function.





Examples of into function :

(i) $f : R \to R$, $f(x) = x^2$ (ii) $f : R \to R$, $f(x) = |x|$

(iii) $f : R \to R$, $f(x) = c$ (c is constant)   (iv) $f : R \to R$, $f(x) = \sin x$

Note : For a function to be onto or into depends mainly on their co-domain.

20. Composite of Functions

Let $f : X \to Y$ and $g : Y \to Z$ be two functions, the composite of the function f and g denoted by gof, is a function from X to Z given by

$gof : X \to Z$, $(gof)(x) = g[f(x)]$.

Obviously gof is defined if

range of $f \subset$ domain of g.

Also domain of gof = f-primage of $\{R_f \cap D_g\}$.

In particular when $R_f = D_g$, then $D_{gof} = D_f$

Properties of Composite function :

The following properties of composite functions can easily be established.

(i) Composite of functions is not commutative i.e. $fog \neq gof$

(ii) Composite of functions is associative i.e. $(fog)oh = fo(goh)$

(iii) Composite of two bijection is also a bijection

INVERSE FUNCTION

If $f : X \to Y$ be a one-one onto (bijection) function, then the mapping $f^{-1} : Y \to X$ which associates each element $y \in Y$ with element $x \in X$, such that $f(x) = y$, is called the inverse function of the function $f : X \to Y$

$$f^{-1} : Y \to X,\ f^{-1}(y) = x \Rightarrow f(x) = y$$

In terms of ordered pair inverse function is defined as-

$$f^{-1} = \{(y, x) \mid (x, y) \in f\}$$

Note : For the existence of inverse function, it should be one-one and onto.

Properties :

(i) Inverse of a bijection is also a bijection function.

(ii) Inverse of a bijection a unique.

(iii) $(f^{-1})^{-1} = f$

(iv) If f and g are two bijections such that (gof) exists then

$$(gof)^{-1} = f^{-1}og^{-1}$$

(v) If $f : X \to Y$ is a bijection then $f^{-1} : Y \to X$ is an inverse function of f.

$$f^{-1}of = I_x \text{ and } fof^{-1} = I_y$$

Here I_x is an identity function on set X, and I_y, is an identity function on set Y.

RELATION

Let A and B be two non-empty sets. A relation from set A to set B is a subset of A × B.

Thus, if R is a relation from A to B then $R \subseteq A \times B$. Also, if $(a, b) \in R$, then we say that a is R-related to b and denote this by aRb.

In particular, if B = A then the subsets of A × A are called relations from the set A to the set A or simply as relations in the set A.

Illustrations (i). Let A = {1, 3, 5, 7}, B = {6, 8}. Let R be the relation "is less than" from A to B.

∴ 1R6, 1R8, 3R6, 3R8, 5R6, 5R8, 7R8.

Equivalently, R = {(1, 6), (1, 8), (3, 6), (3, 8), (5, 6), (5, 8), (7, 8)}.

(ii) Let A = {1, 2, 3, ..., 32}.

Let R be the relation "is one fourth of" in A.

∴ 1R4, 2R8, 3R12, 4R16, 5R20, 6R24, 7R28, 8R32

Equivalently, R = {(1, 4), (2, 8), (3, 12), (4, 16), (5, 20), (6, 24), (7, 28), (8, 32)}.

DOMAIN AND RANGE

If R is a relation from A to B, then the set of first elements of elements in R is called the domain of R and the set of second elements of elements in R is called the range of R.

Symbolically,

Domain of $R = \{x : (x, y) \in R\}$; Range of $R = \{y : (x, y) \in R\}$.

Domain of R is a subset of A and range of R is a subset of B.

For example,

R = {(4, 7), (5, 8), (6, 10)} is relation from A = {1, 2, 3, 4, 5, 6} to the set B = {6, 7, 8, 9, 10}.

∴ Domain of R = {4, 5, 6}. This is a subset of A..

Range of R = {7, 8, 10}. This is a subset of B.

NUMBER OF RELATIONS

Let A and B be two non-empty finite sets consisting of m and n elements respectively.

$\therefore$ A × B contains mn ordered pairs.

$\therefore$ Total number of subsets of A × B is 2^{mn}.

Since each relation from A to B is a subset of A × B, then total number of relations from A to B is 2^{mn}.

TYPES OF RELATIONS

(i) A relation R in a set A is called the universal relation in A if R = A × A.
For example, if A = {2, 6}, then the universal relation in A is the set {(2, 2), (2, 6), (6, 2) (6, 6)}.

(ii) A relation R in a set A is called the identity relation in A if R = {(a, a); a $\in$ A}.
For example, if A = {a, b, c}, then the identity relation in A is the set {(a, a), (b, b), (c, c)}.

(iii) A relation R in a set A is called a void relation in A if R = ϕ
For example, if A = {1, 2, 3} and let R be the relation defined by aRb iff a – b = $\sqrt{2}$. The relation
R = $\phi \subseteq$ A × A is void relation.

(iv) A relation R in a set A is called a reflexive relation if (a, a) $\in$ R $\forall$ a $\in$ A. i.e., aRa $\forall$ a $\in$ A
For example, if A = {2, 4, 7}, then the relation {(2, 2), (2, 4), (4, 4), (7, 7)} is reflexive.

(v) A relation R in a set A is called a symmetric relation if (a, b) $\in$ R then (b, a) $\in$ R, i.e., aRb implies bRa.
For example, if A = {2, 4, 7}, then the relation {(2, 4), (4, 2), (7, 7)} is symmetric.

(vi) A relation R in a set A is called a transitive relation if (a, b), (b, c) $\in$ R then (a, c) $\in$ R, i.e., aRb, bRc implies aRc.
For example, if A = {2, 4, 7}, then the relation {(2, 4), (4, 7), (2, 7), (4, 4)} is transitive.

(vii) A relation R in a set A is called an equivalence relation if R is reflexive, symmetric and transitive.

$\therefore$ For an equivalence relation R in A, we have

(i) aRa $\forall$ a $\in$ A (ii) aRb $\Rightarrow$ bRa (iii) aRb and bRc $\Rightarrow$ aRc

For example, if A = {1, 3, 4, 7}, then the relation {(1, 1), (1, 3), (3, 1), (3, 3) , (4, 4), (7, 7),(4, 7) (7,4)} is an equivalence relation.

(viii) A relation R in a set A is called an anti-symmetric relation if (a, b), (b, a) $\in$ R then a = b.

(ix) If R is a relation from A to B, then the inverse relation of R is a relation from B to A and is defined as {(y, x) : (x, y) $\in$ R}. The inverse relation of R is denoted by R^{-1}.
For example, if R = {(1, 4), (9, 15), (10, 2)} is a relation in A, then R^{-1} = {(4, 1), (15, 9), (2, 10)} is the inverse relation of R.

(x) Let R and S be two relations from the sets A to B and B to C respectively, we define a relation SoR from A to C as follows :
(a, c) $\in$ SoR iff (a, b) $\in$ R and (b, c) $\in$ S for some b $\in$ B. This relation SoR is called the composition of the relation R and S.
For example, let A = {2, 3, 4}, B = {1, 5, 7}, C = {1, 3, 6, 8}.
Let R = {(2, 1), (2, 5), (3, 5), (4, 1), (4, 7)} be a relation from A to B.
Let S = {(1, 3), (5, 3), (5, 6), (7, 6), (7, 8)} be a relation from B to C.

$\therefore$ SoR = {(2, 3), (5, 3), (3, 3), (4, 3), (4, 6), (4, 8)} is a relation from A to C.

Note : (4, 6), (4, 8) $\in$ SoR because (4, 7) $\in$ R and (7, 6), (7, 8) $\in$ S.

IMPORTANT TIPS :

1. Total number of relations from set A to set B is equal to $2^{n(A)\,n(B)}$.
2. The universal relation on a non-empty set is always reflexive, symmetric and transitive.
3. The identity relation on a non-empty set is always reflexive, symmetric and transitive.
4. The identity relation on a non-empty set is always anti-symmetric.
5. If R is a relation from A to B and S is a relation from B to C then $(RoS)^{-1} = S^{-1}oR^{-1}$.
6. For two relations R and S, the composite relations RoS, SoR may be void relations.