CHAPTER 13

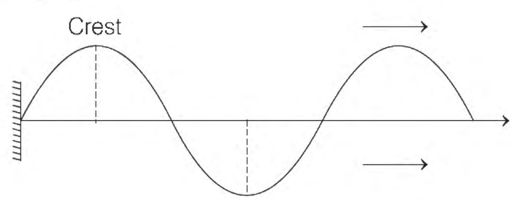
Wave Motion

In any type of wave, oscillations of a physical quantity y are produced at one place and these oscillations (along with energy and momentum) are transferred to other places also.

Classification of Waves

A wave may be classified in following three ways

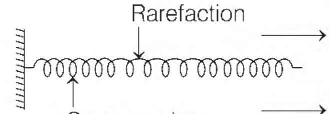
(i) **Transverse waves** A wave in which the particles of the medium vibrate at right angles to the direction of propagation of wave, is called a transverse wave.



These waves travel in the form of crests and troughs.

It is one dimensional. Transverse waves on the surface of water are two dimensional. Sound wave due to a point source is three dimensional.

(ii) **Longitudinal waves** A wave in which the particles of the medium vibrate in the same direction in which wave is propagating, is called a longitudinal wave.



These waves travel in the form of compressions and rarefactions.

(iii) **Mechanical waves** It require medium for their propagation. Sound waves are mechanical in nature. Non-mechanical waves do not require medium for their propagation. Electromagnetic waves are non-mechanical in nature.

Wave Equation

In any wave equation, value of y is a function of position and time. In case of one dimensional wave, position can be represented by one co-ordinate (say x) only. Hence,

$$y = f(x, t)$$

Only those functions of x and t represent a wave equation which satisfy following condition.

$$\frac{\partial^2 y}{\partial x^2} = (\text{constant}) \frac{\partial^2 y}{\partial t^2}$$

Here $\text{constant} = \frac{1}{v^2}$

where, v is the wave speed.

All functions of x and t of type, $y = f(ax \pm bt)$ satisfy above mentioned condition of wave equation, provided value of y should be finite for any value of x and t. If $y(x, t)$ function is of this type, then following two conclusions can be drawn.

(i) Wave speed, $v = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{b}{a}$

(ii) Wave travels along positive x-direction, if ax and bt have opposite signs and it travels along negative x-direction, if they have same signs.

Sine Wave

- If oscillations of y are simple harmonic in nature, then wave is called sine wave. General equation of this wave is,

$$y = A\sin(\omega t \pm kx \pm \phi) \quad \text{or} \quad y = A\cos(\omega t \pm kx \pm \phi)$$

In these equations,

- A is amplitude of oscillation,
- ω is angular frequency,

$$T = \frac{2\pi}{\omega}, \omega = 2\pi f \quad \text{and} \quad f = \frac{1}{T} = \frac{\omega}{2\pi}$$

- k is wave number, $k = \frac{2\pi}{\lambda}$ (where, λ = wavelength)
- Wave speed, $v = \frac{\omega}{k} = f\lambda$
- ϕ is initial phase angle of the particle at $x = 0$ and
- $(\omega t \pm kx \pm \phi)$ is phase angle at time t of the particle at coordinate x.
- Alternate expressions of a sine wave travelling along positive x-direction are

$$y = A\sin k(x - vt) = A\sin(kx - \omega t)$$
$$= A\sin\frac{2\pi}{\lambda}(x - vt) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

Similarly, the expression,

$$y = A\sin k(x + vt) = A\sin(kx + \omega t)$$

$$= A \sin 2\pi\left(\frac{x}{\lambda} + \frac{t}{T}\right) \textit{ etc.}$$

represent a sine wave travelling along negative x-direction.

- **Difference between two equations**

  $$y = A\sin(kx - \omega t) \;\; \textit{and} \;\; y = A\sin(\omega t - kx)$$

  Both equations represent a wave travelling in positive x-direction with speed $v = \frac{\omega}{k}$.

  The phase difference between them is π. It means, if a particle at $x = 0$ and at time $t = 0$ is at mean position and moving upwards (represented by first wave), then the same particle will be at its mean position but moving downwards (represented by the second wave).

Particle Speed (v_p) and Wave Speed (v) in Case of Sine Wave

- In $y = f(x, t)$ equation, x and t are two variables.

  So, $$v_p = \frac{\partial y}{\partial t}$$

- In sine wave, particles are in SHM. Therefore, all equations of SHM can be applied for particles also.
- **Relation between v_p and v**

  $$v_p = -v \cdot \frac{\partial y}{\partial x}$$

Phase Difference ($\Delta\phi$)

***Case* I** $\Delta\phi = \omega(t_1 \sim t_2)$

or $$\Delta\phi = \frac{2\pi}{T} \cdot \Delta t$$

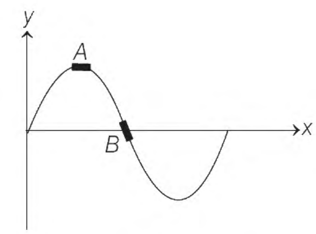
= phase difference of one particle at a time interval of Δt.

***Case* II** $\Delta\phi = k(x_1 \sim x_2) = \frac{2\pi}{\lambda} \cdot \Delta x$

= phase difference at one time between two particles at a path difference of Δx.

Energy Density (u), Power (P) and Intensity (I) in Sine Wave

- Energy density, $u = \frac{1}{2}\rho\omega^2 A^2$ = energy of oscillation per unit volume.
- Power, $P = \frac{1}{2}\rho\omega^2 A^2 S v$ = energy transferred per unit time.
- Intensity, $I = \frac{1}{2}\rho\omega^2 A^2 v$ = energy transferred per unit time per unit area.
- For a string segment, the potential energy depends on the slope of the string and is maximum when the slope is maximum, which is at the equilibrium position of the segment, the same position for which the kinetic energy is maximum.



At *A* : Kinetic energy and potential energy both are zero.
At *B* : Kinetic energy and potential energy both are maximum.

Note ***Intensity due to a point source*** *If a point source emits wave uniformly in all directions, the energy at a distance r from the source is distributed uniformly on a spherical surface of radius r and area* $S = 4\pi r^2$.

$$I = \frac{P}{S} = \frac{P}{4\pi r^2} \quad \text{or} \quad I \propto \frac{1}{r^2}$$

Sound Waves

- Sound waves are the mechanical waves that occur in nature.
- Sound waves are of three types
  - (i) **Infrasonic Waves** The sound waves of frequency lying between 0 to 20 Hz are called infrasonic waves.
  - (ii) **Audiable Waves** The sound waves of frequency lying between 20 Hz to 20000 Hz are called audiable waves.
  - (iii) **Ultrasonic Waves** The sound waves of frequency greater than 20000 Hz are called ultrasonic waves.

  Sound waves are mechanical longitudinal waves and require medium for their propagation. Sound waves can travel through any material medium (i.e. solids, liquid and gases) with speed that depends on the properties of the medium.
- Sound waves cannot propagate through vacuum.
- If v_s, v_l and v_g are speed of sound waves in solid, liquid and gases, then

$$v_s > v_l > v_g$$

- Sound waves (longitudinal waves) can reflect, refract, interfere and diffract but cannot be polarised as only transverse waves can be polarised.

Characteristics of Musical Sound

Musical sound has three characteristics

(i) **Intensity or Loudness** Intensity of sound is energy transmitted per second per unit area by sound waves. Its SI unit is watt/metre^2. Loudness which is related to intensity of sound is measured in decibel (dB).

(ii) **Pitch or Frequency** Pitch of sound directly depends upon frequency.

A shrill and sharp sound has higher pitch and a grave and dull sound has lower pitch.

(iii) **Quality or Timbre** Quality is the characteristic of sound that differentiates between two sounds of same intensity and same frequency.

Longitudinal Wave

There are three equations associated with any longitudinal wave

$$y(x, t), \Delta p(x, t) \quad \text{and} \quad \Delta \rho(x, t)$$

y represents displacement of medium particles from their mean position parallel to direction of wave velocity.

From $y(x, t)$ equation, we can make $\Delta p(x, t)$ or $\Delta \rho(x, t)$ equations by using the fundamental relation between them,

$$\Delta p = -B \cdot \frac{\partial y}{\partial x}$$

and

$$\Delta \rho = -\rho \cdot \frac{\partial y}{\partial x}$$

$\Delta p_0 = BAk$ and $\Delta \rho_0 = \rho A k$

$\Delta p(x, t)$ and $\Delta \rho(x, t)$ are in same phase. But $y(x, t)$ equation has a phase difference of $\frac{\pi}{2}$ with other two equations.

Wave Speed

- Speed of transverse wave on a stretched wire,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho S}}$$

- Speed of longitudinal wave, $v = \sqrt{\frac{E}{\rho}}$

  (i) In solids, $E = Y =$ Young's modulus of elasticity

  $\therefore \quad v = \sqrt{\frac{Y}{\rho}}$

  (ii) In liquids, $E = B =$ bulk modulus of elasticity

  $\therefore \quad v = \sqrt{\frac{B}{\rho}}$

  (iii) In gases, according to Newton,

  $E = B_T =$ isothermal bulk modulus of elasticity $= p$

  $\therefore \quad v = \sqrt{\frac{p}{\rho}}$

  But results did not match with this formula.

  Laplace made correction in it. According to him,

  $E = B_S =$ adiabatic bulk modulus of elasticity $= \gamma p$

  $\therefore \quad v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{\gamma kT}{m}}$

Effect of Temperature, Pressure and Relative Humidity in Speed of Sound in Air

- **With temperature** $v \propto \sqrt{T}$

$$\Rightarrow \qquad \frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}}$$

If v_0 and v_t are velocities of sound in air at 0°C and t°C, then

$$v_t = v_0 \left(1 + \frac{t}{273}\right)^{1/2}$$

or

$$v_t = v_0 + 0.61\, t$$

- **With pressure** Pressure has no effect on speed of sound as long as temperature remains constant.
- **With relative humidity** With increase in relative humidity in air, density decreases. Hence, speed of sound increases.

Sound Level (L)

$$L = 10 \log_{10} \frac{I}{I_0} \qquad \text{(in dB)}$$

Here, I_0 = intensity of minimum audible sound = 10^{-12} Wm^{-2}.

On comparing loudness of two sounds we may write,

$$L_2 - L_1 = 10 \log_{10} \frac{I_2}{I_1}$$

In case of point source, $I \propto \frac{1}{r^2}$ or $\frac{I_2}{I_1} = \left(\frac{r_1}{r_2}\right)^2$.

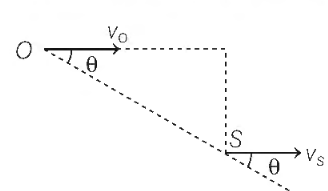
In case of line source, $I \propto \frac{1}{r}$ or $\frac{I_2}{I_1} = \left(\frac{r_1}{r_2}\right)$.

Doppler Effect in Sound

- If v_s and v_o are the velocities along the line joining S and O, then

$$f' = f\left(\frac{v \pm v_m \pm v_o}{v \pm v_m \pm v_s}\right)$$

- If velocities v_s and v_o are along some other direction, the components of velocities along the line joining source and observer are taken.



For example, in the figure shown,

$$f' = \left(\frac{v + v_o \cos\theta}{v + v_s \cos\theta}\right) f$$

- Change in frequency depends on the fact that whether the source is moving towards the observer or the observer is moving towards the source. But when the speed of source and observer are much lesser than that of sound, then the change in frequency becomes independent of the fact whether the source is moving or the observer.

  For example, suppose a source is moving towards a stationary observer with speed u and the speed of sound is v, then

$$f' = \left(\frac{v}{v-u}\right) f = \left(\frac{1}{1-\frac{u}{v}}\right) f$$

$$= \left(1 - \frac{u}{v}\right)^{-1} f$$

  Using the binomial expansion, we have

$$\left(1 - \frac{u}{v}\right)^{-1} \approx 1 + \frac{u}{v} \qquad [\text{if } u << v]$$

$$\therefore \qquad f' = \left(1 + \frac{u}{v}\right) f$$

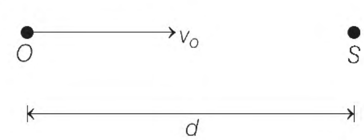
  On the other hand, if an observer moves towards a stationary source with same speed u, then

$$f' = \left(\frac{v+u}{v}\right) f = \left(1 + \frac{u}{v}\right) f$$

  which is same as above.

- As long as v_s and v_o are along the line joining S and O, so doppler's effect (or change in frequency) does not depend upon the distance between S and O.

  For example in the given figure,



$$f' = f\left(\frac{v + v_o}{v}\right)$$

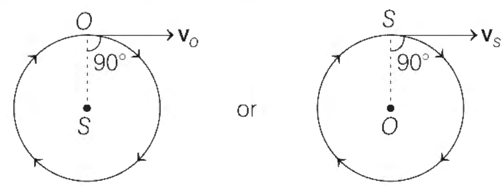
  $f' > f$ but it is constant and independent of d.

- Frequency is given by $f = \frac{v}{\lambda}$

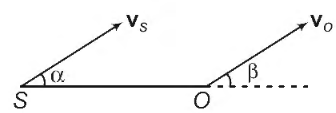
  By the motion of source, λ changes, therefore frequency changes. From the motion of observer, relative velocity v between sound and observer changes, therefore frequency changes.

- Despite the motion of source or observer (or both), Doppler's effect is not observed (or $f' = f$) under the following four conditions.

***Condition* 1** $\mathbf{v}_s$ or $\mathbf{v}_o$ is making an angle of 90° with the line joining S and O. This is illustrated in the following figure.



***Condition* 2** Source and observer both are in motion but their velocities are equal or relative motion between them is zero.



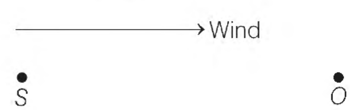
In the figure shown, $\mathbf{v}_s = \mathbf{v}_o$ if $v_s = v_o$ and $\alpha = \beta$.
Taking the components along SO, we have

$$f' = f\left(\frac{v - v_o \cos\beta}{v - v_s \cos\alpha}\right)$$

$\Rightarrow$ $f' = f$ because $v_s = v_o$ and $\alpha = \beta$

***Condition* 3** Source and observer both are at rest. Only medium is in motion.

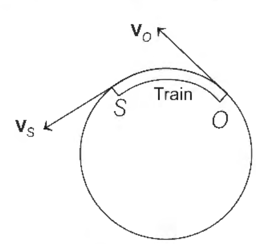
In the figure shown, source and observer are at rest. Only wind is blowing in a direction from source to observer.



Then, change in frequency,

$$f' = f\left(\frac{v + v_{\text{wind}}}{v + v_{\text{wind}}}\right) = f$$

***Condition* 4** A train is moving on a circular track. Engine is the source of sound and guard is the observer. Although, yet f' comes out to be f.



$$\mathbf{v}_s \neq \mathbf{v}_o$$

Principle of Superposition and Interference

- $y = y_1 + y_2$
- $A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos\phi}$ ...(i)
- $I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\phi$ ...(ii)
- In the above equations, ϕ is the constant phase difference at that point, as the sources are coherent. Value of this constant phase difference will be different at different points.
- The special case of above two equations is, when the individual amplitudes (or intensities) are equal

  or $A_1 = A_2 = A_0$ (say) $\Rightarrow$ $I_1 = I_2 = I_0$ (say)

  In this case, Eqs. (i) and (ii) become

  $$A = 2A_0 \cos\frac{\phi}{2} \quad \text{...(iii)}$$

  and $$I = 4I_0 \cos^2\frac{\phi}{2} \quad \text{...(iv)}$$

- From Eqs. (i) to (iv), we can see that, for given values of A_1, A_2, I_1 and I_2 or the resultant amplitude and the resultant intensity are the functions of only ϕ.
- If three or more than three waves (due to coherent sources) meet at some point, then there is no direct formula for finding resultant amplitude or resultant intensity.

  In this case, first of all we will find resultant amplitude by vector method (either by using polygon law of vector addition or component method) and then by the relation $I \propto A^2$, we can also determine the resultant intensity.

  For example, if resultant amplitude comes out to be $\sqrt{2}$ times, then resultant intensity will become two times.

- In interference, two or more than two waves from coherent sources meet at several points. At different points, Δx, $\Delta\phi$ or ϕ, resultant amplitude and therefore resultant intensity will be different (varying from I_{max} to I_{min}).

  But whatever is the resultant intensity at some point, it remains constant at that point.

- Most of the problems of interference can be solved by calculating the net path difference Δx and then by putting

  $$\Delta x = 0, \lambda, 2\lambda, \ldots \quad \text{(For constructive interference)}$$

  $$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2}, \ldots \quad \text{(For destructive interference)}$$

  provided the waves emitted from S_1 and S_2 are in phase.

- If two waves emitted from S_1 and S_2 have already a phase difference of π, the conditions of maximas and minimas are interchanged, i.e. path difference

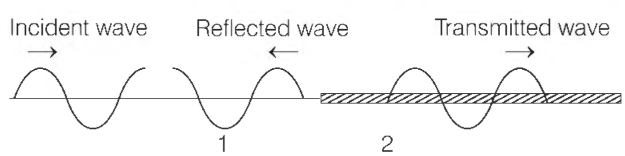
  $$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \ldots \quad \text{(For constructive interference)}$$

  and $$\Delta x = \lambda, 2\lambda, \ldots \quad \text{(For destructive interference)}$$

Reflection and Transmission of a Wave

Wave property	Reflection	Transmission (Refraction)
v	does not change	changes
f, T, ω	do not change	do not change
λ, k	do not change	change
A, I	change	change
ϕ	$\Delta\phi = 0$, from a rarer medium	$\Delta\phi = 0$
	$\Delta\phi = \pi$, from a denser medium	

- Amplitude in reflection as well as transmission, changes.



If amplitude of incident wave in medium-1 is A_i, it is partly reflected and partly transmitted at the boundary of two media-1 and 2. Wave speeds in two media are v_1 and v_2. If amplitudes of reflected and transmitted waves are A_r and A_t, then

$$A_r = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_i$$

and

$$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i$$

From the above two expressions, we can make the following conclusions :

***Conclusion* 1** If $v_1 = v_2$, then $A_r = 0$ and $A_t = A_i$

Basically $v_1 = v_2$ means both media are same from wave point of view. So, in this case there is no reflection ($A_r = 0$), only transmission ($A_t = A_i$) is there.

***Conclusion* 2** If $v_2 < v_1$, then A_r comes out to be negative. Now, $v_2 < v_1$ means the second medium is denser. A_r in this case is negative means, there is a phase change of π.

***Conclusion* 3** If $v_2 > v_1$, then $A_t > A_i$. This implies that amplitude always increases as the wave travels from a denser medium to rarer medium (as $v_2 > v_1$, so second medium is rarer).

- **Power** At the boundary of two media,

energy incident per second = energy reflected per second + energy transmitted per second

or power incident = power reflected + power transmitted

or $$P_i = P_r + P_t$$

Stationary Waves

- Stationary waves are formed by the superposition of two identical waves travelling in opposite directions.
- Formation of stationary waves is really the interference of two waves in which coherent (same frequency) sources are required.
- By the word 'Identical waves' we mean that they must have same value of v, ω and k. Amplitudes may be different, but same amplitudes are preferred.
- In stationary waves, all particles oscillate with same value of ω but amplitudes vary from $A_1 + A_2$ to $A_1 - A_2$.

  Points where amplitude is maximum (or $A_1 + A_2$) are called antinodes (or points of constructive interference) and points where amplitude is minimum (or $A_1 - A_2$) are called nodes (or points of destructive interference).
- If $A_1 = A_2 = A$, then amplitude at antinode is $2A$ and at node is zero. In this case points at node do not oscillate.
- Points at antinodes have maximum energy of oscillation and points at nodes have minimum energy of oscillation (zero when $A_1 = A_2$).
- Points lying between two successive nodes are in same phase. They are out of phase with the points lying between two neighbouring successive nodes.
- Equation of stationary wave is of type,

  $$y = 2A \sin kx \cos \omega t$$

  or $$y = 2A \cos kx \sin \omega t, \text{ etc.}$$

  This equation can also be written as,

  $$y = A_x \sin \omega t \text{ or } y = A_x \cos \omega t$$

  If $x = 0$ is a node then, $A_x = A_0 \sin kx$

  If $x = 0$ is an antinode then, $A_x = A_0 \cos kx$

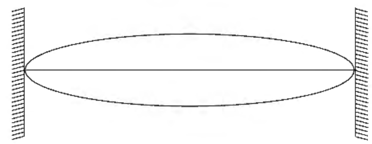
  Here, A_0 is maximum amplitude at antinode.
- Energy of oscillation in a given volume can be obtained either by adding energies due to two individual waves travelling in opposite directions or by integration. Because in standing wave amplitude and therefore energy of oscillation varies point to point.

Travelling waves	Stationary waves
In these waves, all particles of the medium oscillate with same frequency and amplitude.	In these waves, all particles except nodes (if $A_1 = A_2$) oscillate with same frequency but different amplitudes. Amplitudes is zero at nodes and maximum at antinodes
At any instant phase difference between any two particles can have any value between 0 and 2π.	At any instant phase difference between any two particles can be either zero or π.
In these waves, at no instant all the particles of the medium pass through their mean positions simultaneously.	In these waves, all particles of the medium pass through their mean positions simultaneously twice in each time period.
These waves transmit energy in the medium.	These waves do not transmit energy in the medium, provided $A_1 = A_2$.

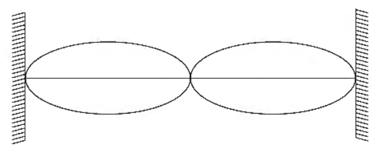
Oscillations of Stretched Wire or Organ Pipes

- **Stretched wire**

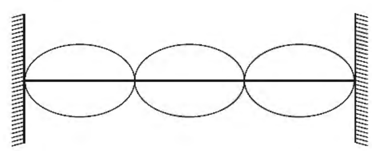
  Fundamental tone or first harmonic ($n = 1$)



First overtone or second harmonic ($n = 2$)



Second overtone or third harmonic ($n = 3$)



$$f = n\left(\frac{v}{2l}\right)$$

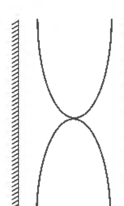
Here, $n = 1, 2, 3, ...$

Even and odd both harmonics are obtained.

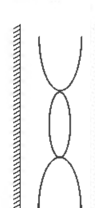
Here, $$v = \sqrt{\frac{T}{\mu}} \text{ or } \sqrt{\frac{T}{\rho S}}$$

- **Open organ pipe**

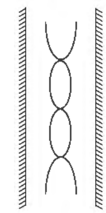
  Fundamental tone or first harmonic ($n = 1$)



First overtone or second harmonic ($n = 2$)



Second overtone or third harmonic ($n = 3$)



$$f = n\left(\frac{v}{2l}\right)$$

Here, $n = 1, 2, 3, \ldots..$

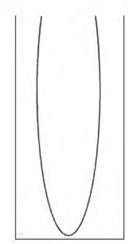
Even and odd both harmonics are obtained.

Here, v = speed of sound in air.

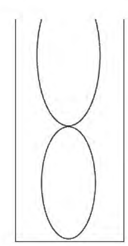
v will be either given in the question, otherwise calculate from $v = \sqrt{\frac{\gamma RT}{M}}$.

- **Closed organ pipe**

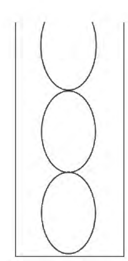
  Fundamental tone or first harmonic ($n = 1$)



First overtone or third harmonic ($n = 3$)



Second overtone or fifth harmonic ($n = 5$)



$$f = n\left(\frac{v}{4l}\right)$$

$$n = 1, 3, 5, \ldots$$

- Open end of pipe is displacement antinode, but pressure and density are nodes. Closed end of pipe is displacement node, but pressure and density are antinodes.
- Laplace correction $e = 0.6r$ (in closed pipe)

  and $2e = 1.2r$ (In open pipe)

  Hence, $f = n\left[\frac{v}{2(l + 1.2r)}\right]$ (In open pipe)

  with $n = 1, 2, 3 \ldots$

  and $f = n\left[\frac{v}{4(l + 0.6r)}\right]$ (In closed pipe)

  with $n = 1, 3, 5 \ldots$
- If an open pipe and a closed pipe are of same lengths, then fundamental frequency of open pipe is two times the fundamental frequency of closed pipe.

Note *Stationary transverse waves are formed in stretched wire and longitudinal stationary waves are formed in organ pipes.*

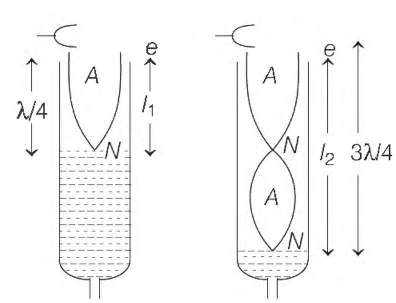
Beats Frequency

$$f_b = f_1 - f_2 \quad (f_1 > f_2)$$

The difference of frequencies should not be more than 10. Sound persists on human ear drums for 0.1 s. Hence, beats will not be heard if the frequency difference exceeds 10.

Resonance Tube

Resonance tube is a closed organ pipe in which length of air column can be changed by changing height of liquid column in it.



For first resonance, $\frac{\lambda}{4} = l_1 + e$

For second resonance, $\frac{3\lambda}{4} = l_2 + e$

$$\frac{3\lambda}{4} - \frac{\lambda}{4} = (l_2 - l_1)$$

$$\Rightarrow \quad \frac{\lambda}{2} = (l_2 - l_1)$$

$$\text{or} \quad \lambda = 2(l_2 - l_1)$$

Velocity of sound, $v = f\lambda = 2f(l_2 - l_1)$

End correction, $e = \dfrac{l_2 - 3l_1}{2}$

Here, f = frequency of tuning fork

Echo

The repetition of sound caused by the reflection of sound waves is called an echo.

Since, sound persists on ear for 0.1 s, so the minimum distance from a sound reflecting surface to hear an echo is 16.5 m.