21. METHODS OF DIFFERENTIATION AND APPLICATIONS OF DERIVATIVES

1. INTRODUCTION

The rate of change of one dependent quantity with respect to another dependent quantity has great importance. E.g. the rate of change of displacement of a particle with respect to time is called its velocity and the rate of change of velocity is called its acceleration. The rate of change of a quantity 'y' with respect to another quantity 'x' is known as the derivative or differentiable coefficient of 'y' with respect to 'x.'

According to the first principle of calculus, if y = f(x) is the derivative function, then the derivative of f(x) with respect to x is given by:

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Note: y', y_1, Dy can also be used to denote the derivative of y with respect to x. Differentiation is the process of finding the derivative of a function. $\Rightarrow \sin\beta > 0; \cos\alpha < 0$

2. DERIVATIVES OF SOME STANDARD FUNCTIONS

Different types of differentiation formulae

(a) $\frac{d}{dx}(\text{constant}) = 0$

(b) $\frac{d}{dx}(e^x) = e^x$

(c) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(d) $\frac{d}{dx}(\sin x) = \cos x$

(e) $\frac{d}{dx}(\tan x) = \sec^2 x$

(f) $\frac{d}{dx}(x^n) = nx^{n-1}$

(g) $\frac{d}{dx}(a^x) = a^x \log_e a$

(h) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$

(i) $\frac{d}{dx}(\cos x) = -\sin x$

(j) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(k) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(l) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(m) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, x \in R$

(n) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}} \, |x| > 1$

(o) $\frac{d}{dx}(\sinh x) = \cosh x$

(p) $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

(q) $\frac{d}{dx}(\text{sech} x) = -\text{sech} x \tanh x$

(r) $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}, \forall x \in R$

(s) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}, x \pm 1$

(t) $\frac{d}{dx}(\text{sech}^{-1}x) = -\frac{1}{|x|\sqrt{1-x^2}}, |x| < 1$

(u) $\frac{d}{dx}(e^{ax}\sin bx) = e^{ax}(a \sin bx + b \cos bx) = \sqrt{a^2+b^2}\, e^{ax}\sin(bx + \tan^{-1} b/a)$

(v) $\frac{d}{dx}(e^{ax}\cos bx) = e^{ax}(a \cos bx - b \sin bx) = \sqrt{a^2+b^2}\, e^{ax}\cos(bx + \tan^{-1} b/a)$

(w) $\frac{d}{dx}|x| = \frac{x}{|x|} \; (x \neq 0)$

(x) $\frac{d}{dx}[x] = 0, \forall x \in -I$ (where [.] denotes greatest integer function)

(y) $\frac{d}{dx}\{x\} = 1, \forall x \in R$ (where { . } denotes fractional part function)

(z) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(aa) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(ab) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}, \forall x \in R$

(ac) $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}} \, |x| > 1$

(ad) $\frac{d}{dx}(\cosh x) = \sinh x$

(ae) $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$

(af) $\frac{d}{dx}(\text{cosech} x) = -\text{cosech} x \coth x$

(ag) $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}, |x| > 1$

(ah) $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{x^2-1}, x \neq \pm 1$

(ai) $\frac{d}{dx}(\text{cosech}^{-1}x) = \frac{-1}{|x|\sqrt{x^2+1}}, \forall x \in R$

(aj) $\frac{d}{dx}\log|x| = \frac{1}{x}, (x \neq 0)$

PLANCESS CONCEPTS

If the function is continuous, you do not have to apply the first principle method to check differentiability. You can go directly for dy/dx and check whether dy/dx exists on both the left and right sides and are equal. If dy/dx does not exist for either one side or both the sides or if both the derivatives exist, but are not equal or finite, then the function is not differentiable.

E.g. Let y = sin(x) be a continuous function. Check differentiability at $x = \pi/2$.On checking for dy/dx = cos(x) on both the right and left sides, it is found to be equal and finite. Hence, y = sin(x) is differentiable at $x = \pi/2$.

PLANCESS CONCEPTS

Misconception:

(i) In dy/dx, dy or dx does not exist individually.

(ii) $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ only if both $\frac{dy}{dx}$ and $\frac{dx}{dy}$ exist.

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3. PRODUCT RULE

(a) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(b) $\frac{d}{dx}(uvw) = uv\frac{d(w)}{dx} + uw\frac{d(v)}{dx} + vw\frac{d(u)}{dx}$

4. DIVISION RULE

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, where v ≠ 0 (known as the quotient rule)

5. CHAIN RULE

$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$ or $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

Note: (a) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$, on condition that both f′(x) and g′(x)exist

(b) $\frac{d}{dx}(k\,f(x)) = k\frac{d}{dx}(f(x))$, where k is any constant

Illustration 1: If $y = \frac{x^4 + x^2 + 1}{x^2 + x + 1}$; $\frac{dy}{dx} = px + q$, find p and q. **(JEE MAIN)**

Sol: Differentiate and compare.

$$y = \frac{(x^2+1)^2 - x^2}{x^2 + x + 1} = \frac{\left(\left(x^2+1\right)+x\right)\left(x^2+1-x\right)}{x^2+x+1} = x^2 + 1 - x \Rightarrow \frac{dy}{dx} = 2x - 1 \Rightarrow p = 2 \text{ and } q = -1$$

$$y = \frac{((x^1+1)+x)(x^2+1-x)}{x^2+x+1}$$

Illustration 2: If $y = \frac{x^3 + 2^x}{e^x}$, then find $\frac{dy}{dx}$. **(JEE MAIN)**

Sol: Differentiate

$$y' = \frac{e^x(3x^2 + 2^x \ln 2) - (x^3 + 2^x)e^x}{e^{2x}} = \frac{(3x^2 + 2^x \ln 2) - (x^3 + 2^x)}{e^x}$$

Illustration 3: If y = $\dfrac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}$, find $\left(\dfrac{dy}{dx}\right)_{x=1}$. **(JEE MAIN)**

Sol: Differentiate and put x = 1.

$$y = \frac{2}{\pi}\left(\tan^{-1}x-\cot^{-1}x\right) \qquad ......\left(\tan^{-1}+\cot^{-1}x=\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\pi(1+x^2)}+\frac{2}{\pi(1+x^2)} = \frac{4}{\pi(1+x^2)} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = \frac{4}{2\times\pi} = \frac{2}{\pi}$$

Illustration 4: Differentiate the following functions with respect to x: **(JEE MAIN)**

(i) $\sqrt{3x+2}+\dfrac{1}{\sqrt{2x^2+4}}$ (ii) $e^{\sec^2 x}$ + 3cos^{-1}x (iii) $\log_7(\log x)$

Sol: (i) Let y = $\sqrt{3x+2}+\dfrac{1}{\sqrt{2x^2+4}}$ = $(3x + 2)^{1/2} + (2x^2 + 4)^{-1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(3x+2)^{\frac{1}{2}-1}\frac{d}{dx}(3x+2) + \left(-\frac{1}{2}\right)(2x^2+4)^{\frac{-1}{2}-1}\frac{d}{dx}(2x^2+4)$$

$$= \frac{1}{2}(3x+2)^{-\frac{1}{2}}.(3) - \left(\frac{1}{2}\right)(2x^2+4)^{-\frac{3}{2}}.4x = \frac{3}{2\sqrt{3x+2}}-\frac{2x}{(2x^2+4)^{3/2}}$$

(ii) Let y = $e^{\sec^2 x}$ + 3 cos^{-1}x

$$\frac{dy}{dx} = e^{\sec^2 x}.\frac{d}{dx}(\sec^2 x) + 3\left(-\frac{1}{\sqrt{1-x^2}}\right) = e^{\sec^2 x}.\left(2\sec x\frac{d}{dx}(\sec x)\right) + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= 2\sec x(\sec x\tan x)\,e^{\sec^2 x} + 3\left(-\frac{1}{\sqrt{1-x^2}}\right) = 2\sec^2 x\tan x\,e^{\sec^2 x} - 3\left(+\frac{1}{\sqrt{1-x^2}}\right)$$

(iii) Let y = $\log_7(\log x) = \dfrac{\log(\log x)}{\log 7}$ (using change of base formula)

$$\frac{dy}{dx} = \frac{1}{\log 7}\frac{d}{dx}(\log(\log x)) = \frac{1}{\log 7}\frac{1}{\log x}.\frac{d}{dx}(\log x) = \frac{1}{x\log 7\log x}$$

Illustration 5: Find $\dfrac{dy}{dx}$, if y =3 tan x + 5 $\log_a x$ + $\sqrt{x}$ – 3e^x + $\dfrac{1}{x}$. **(JEE MAIN)**

Sol: Chain rule.

We have y = 3 tan x + 5 $\log_a x$ + $\sqrt{x}$ – 3e^x + $\dfrac{1}{x}$

$$\text{Thus, } \frac{dy}{dx} = \frac{d}{dx}\left(3\tan x+5\log_a x+\sqrt{x}-3e^x+\frac{1}{x}\right)$$

$$= \frac{d}{dx}(3\tan x) + \frac{d}{dx}(5\log_a x) + \frac{d}{dx}(\sqrt{x}) - \frac{d}{dx}(3e^x) + \frac{d}{dx}\left(\frac{1}{x}\right) = 3\sec^2 x + \frac{5}{x}(\log_e a)^{-1} + \frac{1}{2}x^{-\frac{1}{2}} - 3e^x - x^{-2}.$$

Illustration 6: Let f, g and h be differentiable functions. If f(0) = 1, g(0) = 2, h(0) = 3 and the derivative pairwise products at x = 0 are (fg)′ (0) = 6, (gh)′(0) =4 and (hf)′ (0) = 5, then compute the value of (fgh)′ (0).

(JEE ADVANCED)

Sol: Product rule

$(fgh)' = f'gh + fhg' + fgh'$ ....(i)

$(fg)'(0) = 6 \Rightarrow (fg' + gf')(0) = 6$

$(gh)'(0) = 4 \Rightarrow (gh' + hg')(0) = 4$

$(hf)'(0) = 5 \Rightarrow (hf' + fh')(0) = 5$

$$(fgh)' = \frac{1}{2}(2f'gh + 2fg'h + 2fgh') = \frac{1}{2}(f'gh + f'gh + fg'h + fg'h + fgh' + fgh')$$

$$= \frac{1}{2}[h(f'g + fg') + g(f'h + fh') + f(g'h + gh')] = \frac{1}{2}[h(fg)' + g(fh)' + f(gh)']$$

$$\Rightarrow (fgh)'(0) = \frac{1}{2}[(3)(6) + (2)(5) + (1)(4)] = \frac{1}{2}[18 + 10 + 4] = 16$$

6. TRIGONOMETRIC TRANSFORMATIONS

In case of inverse trigonometric functions, it becomes very easy to differentiate a function by using trigonometric transformations.Given below are some important results on trigonometric and inverse trigonometric functions.

(a) $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

(b) $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(c) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(n) $\sin 3x = 3\sin x - 4\sin^3 x$

(d) $\cos 3x = 4\cos^3 x - 3\cos x$

(o) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

(e) $\sin^{-1} x + \cos^{-1} x = \tan^{-1} x + \cot^{-1} x = \sec^{-1} x + \mathrm{cosec}^{-1} x = \dfrac{\pi}{2}$

(f) $\sin^{-1}(-x) = -\sin^{-1} x$

(p) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(d) $\tan^{-1}(-x) = -\tan^{-1} x$

(q) $\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1} x$

(h) $\cot^{-1}(-x) = \pi - \cot^{-1} x$

(r) $\sec^{-1}(-x) = \pi - \sec^{-1} x$

(i) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

(s) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$

(j) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left|\dfrac{x \pm y}{1 \mp xy}\right|$

(t) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$ (Be aware of ranges for 'x')

(k) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$

(u) $2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(l) $\dfrac{\pi}{4} - \tan^{-1} x = \tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

(v) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

(m) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

(w) $3\tan^{-1} x = \tan^{-1}\left|\dfrac{3x - x^3}{1-3x^2}\right|$

PLANCESS CONCEPTS

Some useful substitutions in finding derivatives are given below.

Sl. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	x = a sinθ or a cosθ
(ii)	$\sqrt{x^2 + a^2}$	x = a tan θ or a cotθ
(iii)	$\sqrt{x^2 - a^2}$	x = a sec θ or a cosecθ
(iv)	$\sqrt{\frac{a-x}{a+x}}$	x = a cos 2θ
(v)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2 \cos 2\theta$
(vi)	$\sqrt{ax - x^2}$	$x = a\sin^2\theta$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x = a\tan^2\theta$
(viii)	$\sqrt{\frac{x}{a-x}}$	$x = a\sin^2\theta$
(ix)	$\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$
(x)	$\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$

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Illustration 7: If $y = \cot^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$, find $\frac{dy}{dx}$. **(JEE MAIN)**

Sol: Substitute a suitable trigonometric function in place of x and simplify.

Putting x = tan θ, we have

$$y = \cot^{-1}\left(\frac{\sec\theta + 1}{\tan\theta}\right) = \cot^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \cot^{-1}(\cot\theta/2) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+x^2}$$

PLANCESS CONCEPTS

To differentiate a complex function, put x in some trigonometric form so that the function can be easily differentiated and then put back x in the form of an inverse trigonometric function.

E.g. Find the derivatives of $\sec^{-1}[1/(2x^2-1)]$ with respect to $\sqrt{1-x^2}$ at x = 1/2.

Sol. Putting x = cosθ, we get

$$u = \sec^{-1}\frac{1}{2\cos^2\theta-1} = \sec^{-1}(\sec 2\theta) = 2\theta \text{ and } y = \sqrt{1-x^2} = \sin\theta$$

$$\therefore\ u = 2\sin^{-1}y \Rightarrow \frac{du}{dy} = \frac{2}{\sqrt{1-y^2}} = \frac{2}{\sqrt{x^2}} \qquad \text{Thus, } \left.\frac{du}{dy}\right|_{x=1/2} = 4$$

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Illustration 8: If $y = \sqrt{(a-x)(x-b)} - (a-b)\tan^{-1}\sqrt{\left(\frac{a-x}{x-b}\right)}$, find $\frac{dy}{dx}$. **(JEE ADVANCED)**

Sol: Use Substitution to simplify the given expression and then differentiate.

Let $x = a\cos^2\theta + b\sin^2\theta$

$\therefore\ a - x = a - a\cos^2\theta - b\sin^2\theta = (a-b)\sin^2\theta$ ... (i)

$x - b = a\cos^2\theta + b\sin^2\theta - b = (a-b)\cos^2\theta$ ... (ii)

$\therefore\ y = (a-b)\sin\theta\cos\theta - (a-b)\tan^{-1}(\tan\theta)$

$$y = \frac{(a-b)}{2}\sin 2\theta - (a-b)\theta$$

$$\text{Then, } \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{(a-b)\cos 2\theta-(a-b)}{(b-a)\sin 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta = \sqrt{\left(\frac{a-x}{x-b}\right)} \quad \text{[From (i) and (ii)]}$$

7. LOGARITHMIC DIFFERENTIATION

If differentiation of an expression is done after taking log on both the sides, then it is known as logarithmic differentiation. This method is used when a given expression is in one of the following forms:

(a) $(f(x))^{g(x)}$

Let $y = (f(x))^{g(x)}$

Taking logarithm of both the sides, we get log y = g(x) log f(x)

Differentiating with respect to x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = g(x)\cdot\frac{1}{f(x)}\cdot f'(x) + \log f(x)\cdot g'(x) \quad \Rightarrow \quad \frac{dy}{dx} = y\left(\frac{g(x)}{f(x)}f'(x) + \log f(x).g'(x)\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = (f(x))^{g(x)}\left(\frac{g(x)}{f(x)}f'(x) + \log f(x).g'(x)\right)$$

Short method: The derivative of $[f(x)]^{g(x)}$can be directly written as:

$$\frac{d}{dx}(f(x))^{g(x)} = f(x)^{g(x)}\left(\frac{d}{dx}\{g(x)\log f(x)\}\right)$$

(b) Product of three or more functions

If y = f(x).g(x).h(x), then y′ = $f(x).g(x).h(x).\left(\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}+\frac{h'(x)}{h(x)}\right)$

Illustration 9: If $f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$, find f′(101)/f(101). **(JEE ADVANCED)**

Sol: Use logarithms followed by differentiation.

$$f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$$

$$\ln f(x) = \sum_{n=1}^{100} n(101-n)\ln(x-n) \Rightarrow \frac{f'(x)}{f(x)} = \sum_{n=1}^{100}\frac{n(101-n)}{x-n}$$

$$\Rightarrow \frac{f'(101)}{f(101)} = \sum_{n=1}^{100}\frac{n(101-n)}{101-n} = \frac{100x101}{2} = 5050$$

Illustration 10: Find the derivative of $(\sin x)^{\cos x}$. **(JEE MAIN)**

Sol: Take logarithms on both sides and differentiate.

$$\frac{d}{dx}(\sin x)^{\cos x} = (\sin x)^{\cos x}\left[\frac{d}{dx}\{\cos x\log(\sin x)\}\right] = (\sin x)^{\cos x}\,[\cos x \,.\, \cot x - \sin x \,.\, \log \sin x]$$

Illustration 11: Find the derivative of x^x with respect to x. **(JEE MAIN)**

Sol: Use logarithms to find the derivative.

Let $y = x^x$ or $y = x^x$

log y = x log x = $e^{x \ln x}$

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x \quad \text{(or)}\ \frac{dy}{dx} = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\frac{d}{dx}(x\ln x)$$

$$\frac{dy}{dx} = x^x(1+\log x) \quad \text{(or)} \quad = e^{x\ln x}\left\{x.\frac{1}{x}+\ln x 1\right\}$$

$$\therefore \frac{d}{dx}x^x = x^x\left(1+\log_e x\right) \quad \text{(or)} \quad = x^x(1+\ln x)$$

Hence $\frac{d}{dx}(x^x) = x^x(1+\ln x)$

Illustration 12: Differentiate $x^{\sin x}$ with respect to x. **(JEE MAIN)**

Sol: Similar to the previous illustration.

First method: Let $y = x^{\sin x}$

$\therefore$ log y = log x^{sinx} = sinx log x

Differentiating we get, $\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x}$ + cos x log x

$\therefore \frac{dy}{dx} = \left(\frac{\sin x}{x} + \cos x \log x\right)x^{sinx}$

Second Method: y = x^{sinx} = $e^{sinx\ log\ x}$

Therefore, $\frac{dy}{dx} = e^{\sin x \log x}\left[\frac{\sin x}{x} + \cos x \log x\right] = \left(\frac{\sin x}{x} + \cos x \log x\right)x^{sinx}$

Illustration 13: Differentiate $e^{\cos^{-1}(x+1)}$ with respect to x. **(JEE ADVANCED)**

Sol: Similar to the previous illustration.

Let y = $e^{\cos^{-1}(x+1)}$ Then, $\frac{dy}{dx} = e^{\cos^{-1}(x+1)}.\frac{dy}{dx}[\cos^{-1}(x+1)]$

$= e^{\cos^{-1}(x+1)}.\frac{-1}{\sqrt{1-(x+1)^2}}.\frac{d}{dx}(x+1) = \frac{-1}{\sqrt{1-(x+1)^2}}e^{\cos^{-1}(x+1)}$

Illustration 14: Differentiate x^{sinx}, x > 0, with respect to x. **(JEE ADVANCED)**

Sol: Let y = $x^{\sin x}$

Taking logarithm on both the sides, we get log y = sin xlog x

Therefore, $\frac{1}{y}.\frac{dy}{dx} = \sin x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sin x)$

$\frac{1}{y}.\frac{dy}{dx} = (\sin x)\frac{1}{x} + \log x \cos x \frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$

Illustration 15: Find f′(x), if f(x) = $(\sin x)^{\sin x}$, for all 0 < x < π. **(JEE ADVANCED)**

Sol: Similar to the previous illustration.

log y = log$(\sin x)^{\sin x}$ = sin x log (sin x)

Then, $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\sin x \log(\sin x)) = \cos x \log(\sin x) + \sin x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)$

= cos x log (sin x) + cos x = (1 + log (sin x)) cos x

$\Rightarrow \frac{dy}{dx}$ = y((1 + log (sinx)) cos x) = (1 + log (sinx)) sin x^{sinx} cosx

Illustration 16: Differentiate $x^{\cos^{-1}x}$ with respect to x. **(JEE MAIN)**

Sol: (i) Let y = $x^{\cos^{-1}x}$

Then, y = $e^{\cos^{-1}x.\ln x}$

Differentiating both the sides with respect to x, we get

$\frac{dy}{dx} = e^{\cos^{-1}x.\log x}\frac{d}{dx}(\cos^{-1}x\ .\ \log x) \Rightarrow \frac{dy}{dx} = x^{\cos^{-1}x}\left\{\log x.\frac{d}{dx}(\cos^{-1}x) + \cos^{-1}x.\frac{d}{dx}(\log x)\right\}$

$\Rightarrow \frac{dy}{dx} = x^{\cos^{-1}x}\left(\frac{-\log x}{\sqrt{1-x^2}} + \frac{\cos^{-1}x}{x}\right)$

(ii) Let $(\sin x)^{\cos^{-1}x}$

Then, y = $e^{\cos^{-1}x.\log \sin x}$

Differentiating both the sides with respect to x, we get

$\frac{dy}{dx} = e^{\cos^{-1}x.\log\sin x}\frac{d}{dx}(\cos^{-1}x\ .\ \log\sin x)$

$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos^{-1}x}\left\{\cos^{-1}x.\frac{d}{dx}(\log\sin x) + \log\sin x\frac{d}{dx}(\cos^{-1}x)\right\}$

$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos^{-1}}x\left\{\cos^{-1}.\frac{1}{\sin x}\cos x + \log\sin x\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}$

$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos^{-1}x}\left\{\cos^{-1}x.\cot x - \frac{\log\sin x}{\sqrt{1-x^2}}\right\}$

Illustration 17: Find $\frac{dy}{dx}$, if y = $(\sin x)^{\sin x^{\sin x^{\ldots\ldots\infty}}}$ **(JEE ADVANCED)**

Sol: Write the given expression as y = $(\sin x)^y$ and proceed.

We have y = $(\sin x)^y$, Therefore, log y = y log sin x

Differentiating both the sides with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = y\frac{d}{dx}(\log\sin x) + (\log\sin x)\frac{dy}{dx} = y\frac{\cos x}{\sin x} + \log\sin x\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y} - \log\sin x\right)\frac{dy}{dx} = y\cot x \text{ or } \frac{dy}{dx} = \frac{y^2\cot x}{1 - y\log\sin x} = \frac{y^2\cot x}{1-\log y}$

8. DIFFERENTIATION OF IMPLICIT FUNCTION

If in an equation, both x and y occur together, i.e. f(x, y) = 0, and the equation cannot be solved for either x or y, then x (or y) is called the implicit function of y (or x).

E.g. $x^3 + y^3 + 3axy + c = 0$, $x^y + y^x = a^b$, etc.

Working rule for finding the derivative

First method:

(a) Every term of f(x, y) = 0 should be differentiated with respect to x.

(b) The value of dy/dx should be obtained by rearranging the terms.

Second method:

If f(x, y) = constant, then $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$, where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the partial differential coefficients of f(x, y) with respect to x and y, respectively.

Note: Partial differential coefficient of f(x, y) with respect to x can be defined as the ordinary differential coefficient of f(x, y) with respect to x keeping y constant.

E.g. $z = x^2y \Rightarrow \frac{\partial z}{\partial y} = x^2, \frac{\partial x}{\partial x} = 2xy$

Illustration 18: If y = $\cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+\cfrac{\cos x}{1+\ldots\text{to }\infty}}}}$, then prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x - \sin x}$. **(JEE ADVANCED)**

Sol: Write the R.H.S. in terms of x and y. Then differentiate the equation on both sides.

We have, $y = \frac{\sin x}{1+((\cos x)/(1+y))} = \frac{(1+y)\sin x}{1+y+\cos x} \Rightarrow y + y^2 + y\cos x = (1+y)\sin x$

On differentiating both the sides with respect to x, we get

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = \frac{dy}{dx}\sin x + (1+y)\cos x$$

$$\Rightarrow \frac{dy}{dx}\{1 + 2y + \cos x - \sin x\} = (1+y)\cos x + y\sin x \Rightarrow \frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x-\sin x}$$

Illustration 19: If f(x) = $x + \cfrac{1}{2x+\cfrac{1}{2x+\cfrac{1}{2x+\ldots\ldots\infty}}}$, then compute the value of f (100) . f′(100). **(JEE MAIN)**

Sol: Same as above $y - x = \cfrac{1}{2x+\cfrac{1}{2x+\cfrac{1}{2x+\ldots\ldots\infty}}}$

$$\Rightarrow y - x = \frac{1}{2x+y-x} \Rightarrow (y-x)(x+y) = 1 \Rightarrow y^2 - x^2 = 1$$

$$\Rightarrow (f(x))^2 = 1 + x^2 \Rightarrow 2(f(x)) \times f'(x) = 2x$$

$$\Rightarrow f(100) \,.\, f'(100) = 100$$

Illustration 20: If $y = \left((\ln x)^{\ln x}\right)^{(\ln x)^{(\ln x)^{\infty}}}$, then find $\frac{dy}{dx}$. **(JEE MAIN)**

Sol: Same as above

ln y = y ln(lnx)

$$\frac{1}{y} \times y' = \frac{y}{x\ln x} + \ln(\ln x).y' \quad \Rightarrow y'\left(\frac{1}{y} - \ln(\ln x)\right) = \frac{y}{x\ln x} \Rightarrow y'\left(\frac{1-y\ln(\ln x)}{y}\right) = \frac{y}{x\ln x}$$

$$\Rightarrow y' = \frac{y^2}{(x\ln x)(1-\ln(\ln x)y)}$$

Illustration 21: If $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, then prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$. **(JEE ADVANCED)**

Sol: Differentiating both the sides of the given relation with respect to x, We get, $\frac{d}{dx}[\log(x^2+y^2)] = 2\frac{d}{dx}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$

$$\Rightarrow \frac{1}{x^2+y^2}\cdot\frac{d}{dx}(x^2+y^2) = 2\cdot\frac{1}{1+(y/x)^2}\cdot\frac{d}{dx}\left(\frac{y}{x}\right) \Rightarrow \frac{1}{x^2+y^2}\left\{2x+2y\frac{dy}{dx}\right\} = 2.\frac{x^2}{x^2+y^2}\left\{\frac{x\frac{dy}{dx}-y.1}{x^2}\right\}$$

$$\Rightarrow 2.\left\{x+y\frac{dy}{dx}\right\} = 2\left\{x\frac{dy}{dx}-y\right\} \Rightarrow x+y\frac{dy}{dx} = x\frac{dy}{dx}-y \Rightarrow \frac{dy}{dx}(y-x) = -(x+y) \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

9. DIFFERENTIATION OF PARAMETRIC FORM

x and y are sometimes given as functions of a single variable, E.g. $x = \phi(t)$ and $y = \psi(t)$ are two functions, where t is a variable. Then in such cases, x and y are called parametric functions or parametric equations and t is called the parameter. To find $\frac{dy}{dx}$ in parametric functions, the relationship between x and y should be obtained by eliminating the parameter t and then it should be differentiated with respect to x. However, it is not convenient to eliminate the parameter every time. Therefore, $\frac{dy}{dx}$ can also be found by using the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

E.g. If $x = a(\theta+\sin\theta)$ and $y = a(1-\cos\theta)$, then find $\frac{dy}{dx}$.

Sol: Given that $\frac{dx}{d\theta} = a(1+\cos\theta)$, $\frac{dy}{d\theta} = a(\sin\theta)$

Therefore, $x = 0(\theta+\sin\theta)$. $y = a(1-\cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2}$$

Note: It may be noted here that $\frac{dy}{dx}$ can be expressed in terms of the parameter only without directly involving the main variables x and y.

Illustration 22: Find $\frac{dy}{dx}$, if $x = a\cos\theta$ and $y = a\sin\theta$. **(JEE MAIN)**

Sol: Differentiate the two equations w.r.t. θ and eliminate θ.

Given that $x = a\cos\theta$ and $y = a\sin\theta$

Therefore, $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = a\cos\theta$.

Hence, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$.

Illustration 23: If $x = a\sec^2\theta$ and $y = a\tan^3\theta$, where $q \in R$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{8}$. **(JEE MAIN)**

Sol: Differentiation of Parametric form.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\tan^2\theta\times\sec^2\theta}{2a\sec\theta\times\sec\theta\tan\theta} = \frac{3}{2}\tan\theta;\ \text{At } \theta = \frac{\pi}{8}, \frac{dy}{dx} = \frac{3}{2}\left(\sqrt{2}-1\right)$$

Illustration 24: If x = cosecθ – sinθ and y = cosec^{n}θ – sin^{n}θ, then find $\frac{dy}{dx}$. **(JEE ADVANCED)**

Sol: Differentiation of Parametric form.

$$x = \text{cosec}\,\theta - \sin\theta$$

$$\Rightarrow \quad x^2 + 4 = (\text{cosec}\,\theta - \sin\theta)^2 + 4 = (\text{cosec}\,\theta + \sin\theta)^2 \quad \text{... (i)}$$

$$\text{and} \quad y^2 + 4 = (\text{cosec}^n\theta - \sin^n\theta)^2 + 4 = (\text{cosec}^n\theta + \sin^n\theta)^2 \quad \text{... (ii)}$$

$$\text{Now, } \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{n(\text{cosec}^{n-1}\theta)(-\text{cosec}\theta\cot\theta) - n\sin^{n-1}\theta\cos\theta}{-\text{cosec}\theta\cot\theta - \cos\theta}$$

$$= \frac{n(\text{cosec}^n\theta\cot\theta + \sin^{n-1}\theta\cos\theta}{(\text{cosec}\theta\cot\theta + \cos\theta)} = \frac{n\cot\theta(\text{cosec}^n\theta + \sin^n\theta)}{\cot\theta(\text{cosec}\theta + \sin\theta)} = \frac{n(\text{cosec}^n\theta + \sin^n\theta)}{(\text{cosec}\theta + \sin\theta)}$$

Illustration 25: Find $\frac{dy}{dx}$ if x = at^{2}and y = 2at. **(JEE MAIN)**

Sol: Given that x = at^2, y = 2at

Therefore, $\frac{dx}{dt}$ = 2at and $\frac{dy}{dt}$ = 2a.

$$\text{Hence, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}.$$

Illustration 26: If x = cos^{3}t and y = sin^{3}t, then find $\frac{dy}{dx}$, for $\left(t \in \left(0, \frac{\pi}{2}\right)\right)$. **(JEE MAIN)**

Sol: $\frac{dx}{dt} = -3\cos^2 t \sin t\ (\neq 0) \Rightarrow \frac{dy}{dt} = 3\sin^2 t \cos t \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\sin^2 t\cos t}{-3\cos^2 t\sin t} = -\tan t$

Illustration 27: If y = sec 4x and t = tan x, then prove that $\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$. **(JEE ADVANCED)**

Sol: Write y in terms of t and differentiate.

$$y = \frac{1}{\cos 4x} = \frac{1+\tan^2 2x}{1-\tan^2 2x} = \frac{1+\left(2t/(1-t^2)\right)^2}{1-\left(2t/(1-t^2)\right)^2}$$

$$y = \frac{(1-t^2)^2+4t^2}{(1-t^2)^2-4t^2} = \frac{1+t^4-2t^2+4t^2}{1+t^4-2t^2-4t^2} = \frac{1+t^4+2t^2}{1+t^4-6t^2}; \qquad \frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$$

Illustration 28: If $x = \frac{1+\ln t}{t^2}$ and $y = \frac{3+2\ln t}{t}$, then show that $\frac{ydy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$ **(JEE MAIN)**.

Sol: Differentiation of Parametric form.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dt} = \frac{t\left(0+2(1/t)\right)-(3+2\ln t)1}{t^2} = \frac{2-3-2\ln t}{t^2} = -\left(\frac{1+2\ln t}{t^2}\right)$$

$$\frac{dx}{dt} = \frac{t^2\left(0+(1/t)\right)-(1+\ln t)2t}{t^4} = \frac{t-2t-2t\ln t}{t^4} = \frac{1-2-2\ln t}{t^3} = -\left(\frac{1+2\ln t}{t^3}\right)$$

$$\Rightarrow \frac{dy}{dx} = t \qquad \Rightarrow 2x\left(\frac{dy}{dx}\right)^2 + 1 = 2.\frac{1+\ln t}{t^2}.t^2 + 1 = 3+2\ln t = yt = y\frac{dy}{dx}$$

10. DIFFERENTIATING WITH RESPECT TOANOTHER FUNCTION

Suppose u = f(x) and v = g(x) are two functions of x. To find the derivative of f(x) with respect to g(x), i.e. to find $\frac{du}{dv}$, the formula $\frac{du}{dv} = \frac{du/dx}{dv/dx}$ is used. Thus, to find the derivative of f(x) with respect tog(x), both are differentiated with respect to x and then the derivative of f(x) with respect to x is divided by the derivative of g(x) with respect to x. The procedure is demonstrated in illustration 29.

Illustration 29: Differentiate $\sin^2 x$ with respect to $e^{\cos x}$. **(JEE MAIN)**

Sol: Differentiate both the functions with respect to the common variable and use parametric form.

Let $u(x) = \sin^2 x$ and $v(x) = e^{\cos x}$. We want to find the value of $\frac{du}{dv} = \frac{du/dx}{dv/dx}$

Clearly, $\frac{du}{dx} = 2\sin x\cos x$ and $\frac{dv}{dx} = e^{\cos x}(-\sin x) = -(\sin x)e^{\cos x}$

Hence, $\frac{du}{dv} = \frac{2\sin x\cos x}{(-\sin x)e^{\cos x}} = -\frac{2\cos x}{e^{\cos x}}$.

Illustration 30: Differentiate $\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ with respect to $\sqrt{1-x^4}$. **(JEE ADVANCED)**

Sol: Similar to the previous illustration.

$$\text{Let } u = \left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)^2}{(1+x^2)-(1-x^2)} = \frac{1+x^2+1-x^2+2\sqrt{1-x^4}}{2x^2}$$

$$\Rightarrow u = \frac{1+\sqrt{1-x^4}}{x^2} \Rightarrow \frac{du}{dx} = \frac{x^2\left(0+\left((-4x^3)/(2\sqrt{1-x^4})\right)\right)-\left(1+\sqrt{1-x^4}\right)2x}{x^4}$$

$$\Rightarrow \frac{du}{dx} = \frac{\left((-2x^5)/(\sqrt{1-x^4})\right)-2x\left(\sqrt{1-x^4}+1\right)}{x^4} \Rightarrow \frac{du}{dx} = \frac{-2x}{\sqrt{1-x^4}}\left(\frac{x^4+\left(\sqrt{1-x^4}\right)\left(1+\sqrt{1-x^4}\right)}{x^4}\right)$$

$$= \frac{-2x}{x^4\sqrt{1-x^4}}\left(x^4+\sqrt{1-x^4}+1-x^4\right) = \frac{-2x}{x^4\sqrt{1-x^4}}\left(\sqrt{1-x^4}+1\right) \qquad \text{....(i)}$$

Let $v = \sqrt{1-x^4}$

$$\frac{dv}{dx} = \frac{1}{2\sqrt{1-x^4}}(-4x^3) \Rightarrow \frac{dv}{dx} = \frac{-2x^3}{\sqrt{1-x^4}} \quad ....(ii)$$

$$\frac{du}{dv} = \frac{du}{dx}\Big/\frac{dv}{dx} \Rightarrow \frac{du}{dv} = \frac{-2x}{x^4\sqrt{1-x^4}}\left(\sqrt{1-x^4}+1\right)\frac{\sqrt{1-x^4}}{-2x^3} \Rightarrow \frac{du}{dv} = \frac{(1+\sqrt{1-x^4})}{x^6}$$

11. DIFFERENTIATION OF DETERMINANTS

To differentiate a determinant, one row (or column) at a time should be differentiated, keeping others unchanged, which is illustrated by the examples given below.

(i) If $F(x) = \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}$, then $\frac{d}{dx}\{F(x)\} = \begin{vmatrix} f'(x) & g'(x) \\ u(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ u'(x) & v'(x) \end{vmatrix}$

Also $\frac{d}{dx}\{F(x)\} = \begin{vmatrix} f'(x) & g(x) \\ u'(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g'(x) \\ u(x) & v'(x) \end{vmatrix}$

(ii) If $F(x) = \begin{vmatrix} f & g & h \\ \ell & m & n \\ u & v & w \end{vmatrix}$ Where f, g, h, ℓ, m, n, u, v, w are functions of x and differentiable, then

$$F'(x) = \begin{vmatrix} f' & g' & h' \\ \ell & m & n \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ \ell' & m' & n' \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ \ell & m & n \\ u' & v' & w' \end{vmatrix} \Rightarrow F(x) = \begin{vmatrix} f' & g & h \\ \ell' & m & n \\ u' & v & w \end{vmatrix} + \begin{vmatrix} f & g' & h \\ \ell & m' & n \\ u & v' & w \end{vmatrix} + \begin{vmatrix} f & g & h' \\ \ell & m & n' \\ u & v & w' \end{vmatrix}$$

Illustration 31: If $f(x) = \begin{vmatrix} \sec\theta & \tan^2\theta & 1 \\ \sec\theta & \tan x & x \\ 1 & \tan x - \tan\theta & 0 \end{vmatrix}$, then find $f'(\theta)$. **(JEE MAIN)**

Sol: Above discussed method.

$$f'(x) = \begin{vmatrix} 0 & 0 & 0 \\ \sec\theta & \tan x & x \\ 1 & \tan x - \tan\theta & 0 \end{vmatrix} + \begin{vmatrix} \sec\theta & \tan^2\theta & 1 \\ 0 & \sec^2 x & 1 \\ 1 & \tan x - \tan\theta & 0 \end{vmatrix} + \begin{vmatrix} \sec\theta & \tan^2\theta & 1 \\ \sec\theta & \tan x & x \\ 0 & \sec^2 x & 0 \end{vmatrix}$$

$$\Rightarrow \quad f'(\theta) = \begin{vmatrix} \sec\theta & \tan^2\theta & 1 \\ 0 & \sec^2\theta & 1 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \sec\theta & \tan^2\theta & 1 \\ \sec\theta & \tan\theta & \theta \\ 0 & \sec^2\theta & 0 \end{vmatrix} = (\tan^2\theta - \sec^2\theta) - \sec^2\theta\,(\theta\sec\theta - \sec\theta) = -1 - (\sec^3\theta)(\theta - 1)$$

12. SUCCESSIVE DIFFERENTIATION

If the first derivative $\frac{dy}{dx}$ of a function y = f(x) is also a differentiable function, then it can be further differentiated with respect to x. The derivative thus obtained is called the second derivative of y with respect to x and is denoted by $\frac{d^2y}{dx^2}$. If $\frac{d^2y}{dx^2}$ is also differentiable, then its derivative is called the third derivative of y and is denoted by $\frac{d^3y}{dx^3}$.

Similarly, $\frac{d^ny}{dx^n}$ denotes the n^{th} derivative of y. This process is known as successive differentiation and all these derivatives are called as successive derivatives of y.

The following symbols are also used to denote the successive derivatives of y = f(x):

$y_1, y_2, y_3, \ldots\ldots\ldots\ Y_n, \ldots\ldots\ldots$

$$y', y'', y''', \ldots\ldots\ y^n, \ldots\ldots \Rightarrow \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots \frac{d^ny}{dx^n} \ldots\ldots, \Rightarrow Dy, D^2y, D^3y, \ldots\ldots D^ny \ldots\ldots \text{ (where } D \equiv \frac{d}{dx}\text{)}$$

The following symbols are used to denote the value of the n^{th} derivative at x = a.

$$y_n(a), y^n(a), \left(\frac{d^ny}{dx^n}\right)_{x=a}, D^ny(a) \ \&\ f^n(a)$$

PLANCESS CONCEPTS

Misconception: $\frac{d^ny}{dx^n} \neq \left(\frac{dy}{dx}\right)^n$

Rohit Kumar (JEE 2012 AIR 79)

13. Nth DERIVAVES OF SOME STANDARD FUNCTIONS

(a) $D^n(ax + b)^m = m(m-1)(m-2)\ldots\ldots(m-n+1)\,a^n(ax+b)^{m-n}$

(b) If $m \in N$ and $m > n$, then $D^n(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$; $D^n(x^m) = \frac{m!}{(m-n)!}x^{m-n}$

(c) $D^n(ax+b)^n = n!\,a^n$; $D^n(x^n) = n!$

(d) $D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$; $D^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$

(e) $D^n\{\log(ax+b)\} = \frac{(-1)^{n-1}(n-1)!}{(ax+b)^n}a^n$; $D^n(\log x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$

(f) $D^n(e^{ax}) = a^ne^{ax}$

(g) $D^n(a^{mx}) = m^n(\log a)^n a^{mx}$

(h) $D^n\{\sin(ax+b)\} = a^n\sin\left(ax+b+n\frac{\pi}{2}\right)$; $D^n(\sin x) = \sin\left(x+n\frac{\pi}{2}\right)$

(i) $D^n\{\cos(ax+b)\} = a^n\cos\left(ax+b+n\frac{\pi}{2}\right)$; $D^n(\cos x) = \cos\left(x+n\frac{\pi}{2}\right)$

(j) $D^n\{e^{ax}\sin(bx+c)\} = (a^2+b^2)^{n/2}e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

(k) $D^n\{e^{ax}\cos(bx+c)\} = (a^2+b^2)^{n/2}e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

(l) $D^n\left(\tan^{-1}\frac{x}{a}\right) = \frac{(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta}{a^n}$, where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$

(m) $D^n(\tan^{-1}x) = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

14. LEIBNITZ THEOREM

If u and v are two functions such that their n^{th} derivative exists, then the n^{th} derivative of their product can be found by the following formula:

$D^n(uv) = (D^n u)v + {}^nC_1 D^{n-1}u.\ Dv + {}^nC_2 D^{n-2}u.\ D^2v + \ldots\ldots + \ldots\ldots + {}^nC_{n-1} Du.\ D^{n-1}v + \ldots\ldots.. + u.D^n v$

The n^{th} derivative of a product of two functions can be found out by using this theorem. While using this theorem, the second function in the product is the function whose successive derivative starts to vanish (if it is possible) after some stepsand the first function is a function whose n^{th} derivative is easily known.

Illustration 32: If $y = x^3 \cos x$, find $D^n y$. **(JEE MAIN)**

Sol: Leibnitz theorem

Choose cos x as the first function and x^3 as the second function

$D^n(\cos x, x^3) = D^n(\cos x)(x^3) + {}^nC_1 D^{n-1}(\cos x)(Dx^3) + {}^nC_2 D^{n-2}(\cos x).(D^2x^3) + {}^nC_3 D^{n-3}(\cos x).(D^3x^3)$

$$= x^3\cos\left(x+\frac{n\pi}{2}\right) + n.3x^2\cos\left(x+\frac{(n-1)\pi}{2}\right) + \frac{(n-1)}{1.2}6x.\cos\left(x+\frac{(n-2)\pi}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}.6.\cos\left(\frac{x+(n-3)\pi}{2}\right)$$

$$= x^3\cos\left(x+\frac{n\pi}{2}\right) + 3nx^2\sin\left(x+\frac{n\pi}{2}\right) - 3n(n-1)x\cos\left(x+\frac{n\pi}{2}\right) - n(n-1)(n-2)\sin\left(x+\frac{n\pi}{2}\right)$$

APPLICATION OF DERIVATIVES

1. THE INTERPRETATION OF THE DERIVATIVE

If y = f(x) be a given function, then the derivative/differential coefficient f′(x) or $\frac{dy}{dx}$ at the point $P(x_1, y_1)$ is called the trigonometric tangent of the angle ψ (say), which the positive direction of the tangent to the curve at P makes with the positive direction of the x-axis. Therefore, $\frac{dy}{dx}$ represents the slope of the tangent.

Thus, $f'(x) = \frac{dy}{dx}_{(x_1,y_1)} = \Psi$

Then,

(a) The inclination of the tangent with x-axis = $\tan^{-1}\frac{dy}{dx}$

(b) Slope of the tangent = $\frac{dy}{dx}$

(c) Slope of the normal = $-\frac{dx}{dy}$

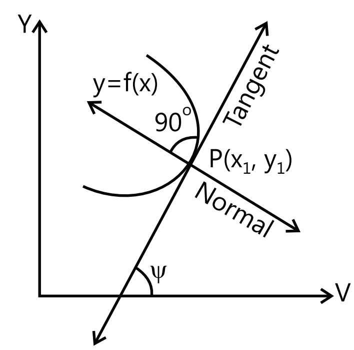


Figure 21.1

2. EQUATION OF TANGENT

(a) Equation of tangent to the curve y = f(x) at $A(x_1, y_1)$ is given by $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$

If the tangent at $P(x_1, y_1)$ of the curve y = f(x) is parallel to the x-axis (or perpendicular to the y-axis), then $\Psi = 0$, i.e its slop will be equal to zero.

$\Rightarrow \quad m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

The converse also holds true. Thus, the tangent at (x_1, y_1) is parallel to the x-axis.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

(b) If the tangent at P (x_1, y_1) of the curve y = f(x) is parallel to the y-axis (or perpendicular to the x-axis), then $\Psi = \pi / 2$ and its slope will be infinity, i.e.

$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty$

The converse also holds true. Thus, the tangent at (x_1, y_1) is parallel to the y-axis.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty$

(c) If at any point P (x_1, y_1) of the curve y = f(x) the tangent makes equal angles with both the axes, then at the point P, $\psi = x / 4$ or $3\pi / 4$.Therefore at P, $\tan \Psi = dy / dx = \pm 1$.

The converse of the result also holds true. Thus, at (x_1, y_1), the tangent line makes equal angles with both the axes.

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \pm 1$

(d) Concept of vertical tangent: y = f(x) has a vertical tangent at the point $x = x_0$ if

$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \infty$ or $-\infty$, but not both.

E.g. The functions $f(x) = x^{1/3}$ and f(x) = sin x both have a vertical tangent at x = 0

But $f(x) = x^{2/3}$, $f(x) = \sqrt{|x|}$ and $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ have no vertical tangents at x = 0.

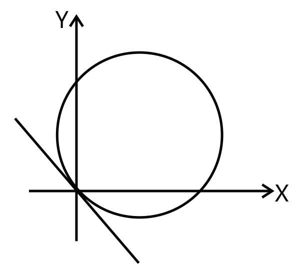


Figure 21.2

(e) If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating the lowest degree terms present in the equation of the curve to zero.

E.g.

(i) $x^2 + y^2 + 2gx + 2fy = 0$

Equation of tangent is gx + fy = 0

(ii) $x^3 + y^3 - 3x^2y + 3xy^2 + x^2 - y^2 = 0$

Equation of tangent at the origin is $x^2 - y^2 = 0$

(iii) $x^3 + y^3 - 3xy = 0$

Equation of tangent is xy = 0

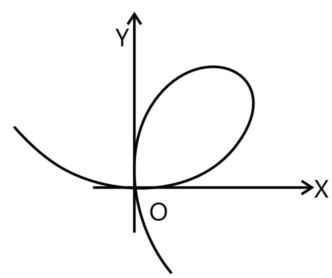


Figure 21.3

Note: This concept is valid only if the powers of x and y are natural numbers.

(f) Same line could be the tangent and normal to a given curve at a given point.

E.g. In $x^3 + y^3 - 3xy = 0$ (folium of Descartes), the line pair xy = 0 is both the tangent and normal at x = 0.

Some common parametric coordinates on a curve that are useful for differentiation

(a) For $x^{2/3} + y^{2/3} = a^{2/3}$, take parametric coordinates $x = a\cos^3$qand $y = a \sin^3\theta$.

(b) For $\sqrt{x} + \sqrt{y} = \sqrt{a}$, take $x = a\cos^4\theta$ and $y = a\sin^4\theta$.

(c) $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$, where $x = a(\sin\theta)^{2/n}$ and $y = b(\cos\theta)^{2/n}$.

(d) For $c^2(x^2 + y^2) = x^2y^2$, take $x = c\sec\theta$ and $y = c\,\text{cosec}\,\theta$.

(e) For $y^2 = x^3$, take $x = t^2$ and $y = t^3$.

Illustration 33: If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts off intercepts α and β on the coordinate axes, where $a^2 + b^2 = 61$, the value of $|a|$ is ____. **(JEE MAIN)**

(A) 16 (B) 28 (C) 30 (D) 31

Sol: (C) Write the equation of the tangent and find the value of α and β in terms of a. Then use $a^2 + b^2 = 61$ to find the value of a.

The slope of the tangent is given by $\frac{dy}{dx} = \frac{2ax + 3x^2}{6y^2}$ The value of this slope at (a, a) is 5/6.

Hence, the equation of tangent is $y - a = \frac{5}{6}(x - a) \Rightarrow \frac{x}{-a/5} + \frac{y}{a/6} = 1$

Thus, the x-intercept α is $-\frac{a}{5}$, and the y-intercept β is $\frac{a}{6}$.

From $a^2 + b^2 = 61$, we get $\frac{a^2}{25} + \frac{a^2}{36} = 61 \Rightarrow a^2 = 25 \times 36 \Rightarrow |a| = 30$

3. EQUATION OF NORMAL

Equation of normal at (x_1, y_1) to the curve $y = f(x)$ isgiven by the following formula:

$$(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x - x_1) \quad \Rightarrow \quad (y - y_1)\left(\frac{dy}{dx}\right)_{(x_1,y_1)} + (x - x_1) = 0$$

Some facts regarding the normal

(a) Slope of the normal drawn at point P (x_1,y_1) to the curve $y = f(x) = -\left(\frac{dx}{dy}\right)_{(x_1,y_1)}$

(b) If the normal makes an angle of θ with the positive direction of the x-axis, then $-\frac{dx}{dy} = \tan\theta$ or $\frac{dy}{dx} = -\cot\theta$

(c) If the normal is parallel to the x-axis, then $\frac{dx}{dy} = 0$ or $\frac{dy}{dx} = \infty$

(d) If the normal is parallel to the y-axis, then $\left(\frac{dx}{dy}\right) = \infty$ or $\frac{dy}{dx} = 0$

(e) If the normal is equally inclined from both the axes or cuts equal intercept, then $-\left(\frac{dx}{dy}\right) = \pm 1$ or $\left(\frac{dy}{dx}\right) = \pm 1$

(f) The length of the perpendicular from the origin to the normal is $P' = \left|\frac{x_1 + y_1\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$

(g) The length of the intercept made by the normal on the x-axis is $x_1 + y_1\left(\frac{dy}{dx}\right)$ and the length of the intercept on the y-axis is $y_1 + x_1\left(\frac{dx}{dy}\right)$.

Illustration 34: Find out the distance between the origin and the normal to the curve $y = e^{2x} + x^2$ at the point whose abscissa is 0. **(JEE MAIN)**

(A) $\frac{1}{\sqrt{5}}$ (B) $\frac{2}{\sqrt{5}}$ (C) $\frac{3}{\sqrt{5}}$ (D) $\frac{2}{\sqrt{3}}$

Sol: (B) Write the equation of the normal and find the distance of origin from the normal.

The point on the curve corresponding to x = 0 is (0, 1)

$$\frac{dy}{dx} = 2e^{2x} + 2x \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 2$$

Therefore, the equation of the normal at the point (0, 1) is

$y - 1 = (-1/2)(x - 0) \Rightarrow 2y + x - 2 = 0$

Hence, the distance of the point (0, 0) from this line is $\frac{2}{\sqrt{5}}$.

4. LENGTH OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL

4.1 Tangent

$$PT = MP\ \text{cosec}\ \Psi = y\sqrt{1+\cot^2\psi} = \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right|$$

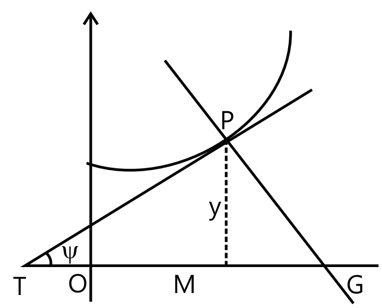


Figure 21.4

4.2 Subtangent

$$TM = MP\cot\Psi = \left|\frac{y}{(dy/dx)}\right|$$

4.3 Normal

$$GP = MP\sec\Psi = y\sqrt{1+\tan^2\psi} = \left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$$

4.4 Subnormal

$$MG = MP\tan\psi = \left|y\left(\frac{dy}{dx}\right)\right|$$

Illustration 35: For the parabola $y^2 = 16x$, the ratio of the length of the subtangent to the abscissa is _____.

(A) 2 : 1 (B) 1 : 1 (C) X : Y (D) X^2 : Y **(JEE MAIN)**

Sol: (A) The length of subtangent is $\left|\frac{y}{(dy/dx)}\right|$

Differentiating, $2y\frac{dy}{dx} = 16$ Hence, $\frac{dy}{dx} = \frac{8}{y}$

Thus, the length of the subtangent is $y\frac{dx}{dy} = \frac{y^2}{8} = \frac{16x}{8} = 2x$

Therefore, the ratio of the length of the subtangent to the abscissa = 2x : x = 2 : 1.

Illustration 36: Find out the length of the normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \pi/2$. **(JEE MAIN)**

Sol: Use differentiation of the Parametric form. Length of the normal = $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2} \Rightarrow \left(\frac{dy}{dx}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

Moreover, at $\theta = \frac{\pi}{2}$, $y = a\left(1-\cos\frac{\pi}{2}\right) = a$

Therefore, the required length of the normal = $y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = a\sqrt{1+1} = \sqrt{2}a$

Illustration 37: The length of the subtangent to the ellipse $x = a\cos t$, $y = b\sin t$ at $t = \pi/4$ is _____.

(A) A (B) B (C) B/ $\sqrt{2}$ (D) A/ $\sqrt{2}$ **(JEE MAIN)**

Sol: (D) Similar to the previous illustration.

$\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$; Therefore, $\left.\frac{dy}{dx}\right|_{t=\pi/4} = -\frac{b}{a}\cot(\pi/4) = -\frac{b}{a}$

Therefore, the length of the subtangent = $\left|y\frac{dx}{dy}\right|_{r=\pi/4} = \left|b\sin\frac{\pi}{4}\times-\frac{a}{b}\right| = \frac{a}{\sqrt{2}}$

5. ANGLE OF INTERSECTION OF TWO CURVES

The angle of intersection between two intersecting curves C_1 and C_2 is defined as the acute angle between their tangents (T_1 and T_2 or the normals) at the point of intersection of the two curves.

$\tan\psi = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$, where m_1 and m_2 are the slopes of the tangents T_1 and T_2 at the intersection point (x_1, y_1)

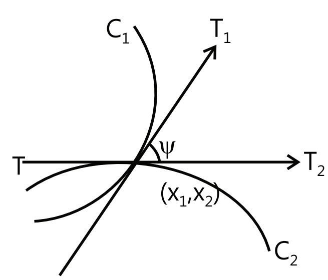


Figure 21.5

Note: If the two curves intersect orthogonally, i.e at right angle, then $\phi = \frac{\pi}{2}$. Hence, the condition will be

$$\left(\frac{dy}{dx}\right)_1 \cdot \left(\frac{dy}{dx}\right)_2 = -1$$

Illustration 38: Which of the following options represents the tangent of the angle at which the curves $y = a^x$ and $y = b^x (a \neq b > 0)$intersect? **(JEE ADVANCED)**

(A) $\frac{\log ab}{1+\log ab}$ (B) $\frac{\log a/b}{1+(\log a)(\log b)}$ (C) $\frac{\log ab}{1+(\log a)(\log b)}$ (D) None of these

Sol: (B) Differentiate the two curves and use the formula for angle between two lines.

Intersection of the two curves is given by $a^x = b^x$, which implies that x = 0. If α is the angle at which the two curves intersect, then

$$\tan\alpha = \frac{m_1 - m_2}{1+m_1m_2} = \frac{a^x \log a - b^x \log b}{1+a^x b^x (\log a)(\log b)} = \frac{(\log a/b)}{1+(\log a)(\log b)} \quad \text{(Putting x = 0)}$$

6. RATE MEASURE

Whenever a quantity y varies with another quantity x, satisfying the rule y = f(x), then $\frac{dy}{dx}$ (or f′(x)) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right|_{x=a}$ (or f′(a)) represents the rate of change of y with respect to x at x = a.

Illustration 39: The volume of a cube increases at the rate of 9 cm^3. How fast does the surface area increase when the length of an E.g. is 10 cm? **(JEE MAIN)**

Sol: Rate measurer.

Let x be the length of the side, V be the volume and S be the surface area of the cube.

Thus, $\frac{dV}{dt} = 9\ cm^3/s \Rightarrow 3x^2\frac{dx}{dt} = 9\ cm^3/s \Rightarrow \frac{dx}{dt} = \frac{3}{x^2}\ cm/s \Rightarrow \frac{dS}{dt} = \frac{d}{dt}(6x^2) = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x}\ cm^2/s$

$$\Rightarrow \left.\frac{dS}{dt}\right]_{x=10cms} = 3.6\ cm^2/s$$

Illustration 40: A man of height 2 meters walks away from a 5-meter lamppost at a uniform speed of 6 meters per minute. Find the rate at which the length of his shadow increases. **(JEE MAIN)**

Sol: Use similarity to establish the relation between the rate at which length of shadow increases and speed of the man.

Let AB be the lamp-post. Let at any time t, the man CD be at a distance x metres from the lamp-post and y metres be the length of his shadow CE.

Then, $\frac{dx}{dt}$ = 6 meters / minute [given] ... (i)

Clearly, the triangles ABE and CDE are similar

$\Rightarrow \frac{AB}{CD} = \frac{AE}{CE} \Rightarrow \frac{5}{2} = \frac{x+y}{y} \Rightarrow 3y = 2x$

$\Rightarrow 3\frac{dy}{dt} = 2\frac{dx}{dt} \Rightarrow 3\frac{dy}{dt} = 2(6)$ [Using (i)] $\Rightarrow \frac{dy}{dt}$ = 4 meters / minute

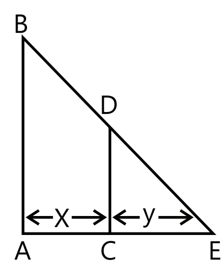


Figure 21.6

Illustration 41: An object has been moving in the clockwise direction along the unit circle $x^2 + y^2 = 1$. As it passes through the point (1/2, $\sqrt{3}$ /2), its y-coordinate decreases at the rate of 3 units per second. The rate at which the x-coordinate changes at this point is ______ units per second.

(A) 2 (B) $3\sqrt{3}$ (C) $\sqrt{3}$ (D) $2\sqrt{3}$ **(JEE MAIN)**

Sol: (B) Differentiate and proceed.

We find that $\frac{dx}{dt}$ when $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ given that $\frac{dy}{dt} = -3$ units/s and $x^2 + y^2 = 1$.

Differentiating $x^2 + y^2 = 1$, we get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

Putting $x = \frac{1}{2}$, $y = \sqrt{3}/2$ and $\frac{dy}{dt} = -3$, we get $\frac{1}{2}\frac{dx}{dt} + \frac{\sqrt{3}}{2}(-3) = 0 \Rightarrow \frac{dx}{dt} = 3\sqrt{3}$ (increasing)

Illustration 42: A given right circular cone has a volume p.The largest right circular cylinder that can be inscribed in the cone has a volume q. The ratio of p to q is _____. **(JEE MAIN)**

(A) 9 : 4 (B) 8 : 3 (C) 7 : 2 (D) None of these

Sol: (A) Let H be the height of the cone and α is its semi-vertical angle.

Let x be the radius of the inscribed cylinder and h be its height.

$h = QL = OL - OQ = H - x \cot a$

$p = \frac{1}{3}\pi(H \tan \alpha)^2 H$ ... (i)

V = volume of the cylinder $= \pi x^2 (H - x \cot \alpha)$

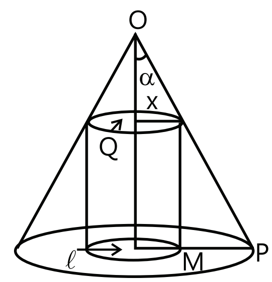


Figure 21.7

$\frac{dV}{dx} = \pi(2Hx - 3x^2 \cot \alpha)$

Hence, $\frac{dV}{dx} = 0 \Rightarrow x = 0$

$x = \frac{2}{3} H \tan \alpha$, $\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha} = -2\pi H < 0$, so

V is maximum when $x = \frac{2}{3} H \tan \alpha$ and $q = V_{max} = \pi\frac{4}{9}H^2 \tan^2\alpha \frac{1}{3} H = \frac{4}{9} p$ [using (i)]

Therefore, p : q = 9 : 4

7. APPROXIMATION USING DIFFERENTIALS

To calculate the approximate value of a function, differentials may be used, wherein the differential of a function is equal to its derivative multiplied by the differential of the independent variable.

$dy = f'(x)dx$ or $df(x) = f'(x)\, dx$

PLANCESS CONCEPTS

For the independent variable 'x', increment Δx and differential dx can be made equal, but the same cannot be applied in case of the dependent variable 'y,' i.e. $\Delta y \neq dy$.

Therefore, the approximate value of y when the increment Δ x is given to the independent variable x in $y = f(x)$ is

$y + \Delta y = f(x + \Delta x) = f(x) + \frac{dy}{dx} . \Delta x$

$\Rightarrow f(x + \Delta x) = f(x) + f'(x)\, \Delta x$

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 43: Find the approximate value of the square root of 25.2. **(JEE MAIN)**

Sol: Consider a function f(x) = $\sqrt{x}$ and differentiate to get the derivative. Then replace x by x+Dx and proceed.

Let f(x) = $\sqrt{x}$, so f′(x) = $\frac{1}{2\sqrt{x}}$ We can write 25.2 as 25 + 0.2

By taking x = 25 and $\Delta x = 0.2$, now $f(x + \Delta x) = f(x) + f'(x).\ \Delta x$

$$\Rightarrow \quad \sqrt{x} + \frac{1}{2\sqrt{x}}.\Delta x = \sqrt{25} + \frac{1}{2\sqrt{25}}.\ 0.2$$

$$= \quad 5 + \frac{0.2}{10} = 5 + 0.02 = 5.02$$

Illustration 44: What is the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%? **(JEE MAIN)**

Sol: Differentiate the equation $V = x^3$ and use the relation $\Delta V = \frac{dV}{dx}\Delta x$.

Let $\Delta(x)$ be the change in x and ΔV be the corresponding change in V

Given that $\frac{\Delta x}{x} \times 100 = 2$

We know that $V = x^3 \ \therefore \ \frac{dV}{dx} = 3x^2$

Therefore, $\Delta V = \frac{dV}{dx}\Delta x \quad \Rightarrow \quad \Delta V = 3x^2\Delta x = 3x^2 \times \frac{2x}{100} = 0.06\ x^3 m^3$

The approximate change in volume is $0.06\ x^3m^3$.

Illustration 45: What is the approximate value of cos 40º? **(JEE ANDANCED)**

(A) 0.7688 (B) 0.7071 (C) 0.7117 (D) 0.7

Sol: (A) Take a function f(x) = cos x and proceed.

Let f(x) = cos x. $40^\circ = 45^\circ - 5^\circ = \frac{\pi}{4} - \frac{\pi}{180} \times 5 = \frac{\pi}{4} - \frac{\pi}{36}$ radians

A differential is used to estimate the change in cos x

When x decreases from $\frac{\pi}{4}$ to $\frac{\pi}{4} - \left(\frac{\pi}{36}\right)$

$f'(x) = -\sin x$ and $df(x) = f'(x)\,h = -h\sin x$

With $x = \frac{\pi}{4}$ and $h = -\frac{\pi}{36}$, df is given by

$$df = -f'(x)h = -\left(-\frac{\pi}{36}\right)\sin\left(\frac{\pi}{4}\right) = \frac{\pi}{36}.\frac{1}{\sqrt{2}} = \frac{\pi\sqrt{2}}{72} = 0.0617$$

$\cos 40 \equiv \cos 45 + 0.0617 \equiv 0.7071 + 0.0617 = 0.7688.$

8. SHORTEST DISTANCE BETWEEN TWO CURVES

It has been found that the shortest distance between two non-intersecting curves is always along the common normal (wherever defined).

Illustration 46: Find out the shortest distance between the line y = x – 2 and the parabola $y = x^2 + 3x + 2$.

(JEE MAIN)

Sol: The distance would be minimum at the point on the parabola where the slope of the tangent is equal to the slope of the given line.

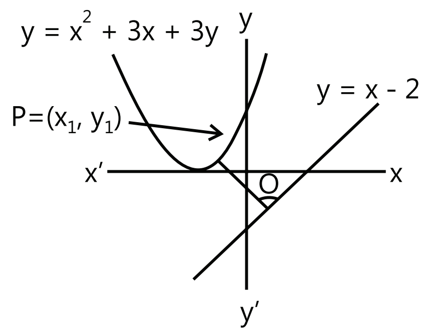


Figure 21.8

Let $P(x_1, y_1)$ is the point closest to the line y = x – 2

Then, $\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ = slope of the line

$\Rightarrow \quad 2x_1 + 3 = 1 \Rightarrow x_1 = -1$ and $y_1 = 0$

Therefore, point (–1, 0) is the closest and its perpendicular distance from the line y = x – 2 gives the shortest distance.

$\Rightarrow \quad$ Shortest distance = $\frac{3}{\sqrt{2}}$ units

Illustration 47: Which of the following points of the curve $y = x^2$ is closest to (4, – ½)? **(JEE MAIN)**

(A) (1, 1) (B) (2, 4) (C) (2/3, 4/9) (D) (4/3, 16/9)

Sol:(A) Using distance formula find the distance of the given point from the curve and find the minima.

Let the required point be (x, y) on the curve.

Hence, d = $\sqrt{(x-4)^2 + (y+1/2)^2}$ should be minimum, which is enough to consider.

$D = (x - 4)^2 + (y + 1/2)^2 = (x - 4)^2 + (x^2 + 1/2)^2$

$D' = 4x^3 + 4x - 8$

Now for critical points

$D' = 0$ so $x^3 + x - 2 = 0 \Rightarrow x = 1$

Clearly D″ at x = 1 is 16 > 0.

Thus, D is minimum when x = 1. Hence the required point is (1, 1).

PROBLEM-SOLVING TACTICS

- Reduce any fractions to be as basic as possible.
- Recognise when we can use the chain rule. it enables us to differentiate functions that often seem impossible to differentiate. Whenever you see a nested function, try to assess if the chain rule is needed (it usually is).
- We always want to start a long chain of differentiation by differentiating the last part of the function to touch the input - in short, the outermost part of the function.

FORMULAE SHEET

$\frac{dc}{dx} = 0$	$\frac{d}{dx}(cu) = c\frac{du}{dx}$
$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
$\frac{d}{dx}a^x = (\ln a)\, a^x$	$\frac{d}{dx}a^u = (\ln a)\, a^u\frac{du}{dx}$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}e^u = e^u\frac{du}{dx}$
$\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$	$\frac{d}{dx}\log_a u = \frac{1}{(\ln a)u}\frac{du}{dx}$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$
$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$	$\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u\frac{du}{dx}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx}$
$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$	$\frac{d}{dx}\operatorname{cosec} u = -\operatorname{cosec} u \cot u\frac{du}{dx}$
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$	$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

* Equation of tangent to the curve y = f(x) at $A(x_1, y_1)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1)$

* Equation of normal at (x_1, y_1) to the curve y = f(x) is $(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} (x - x_1)$

*** Length of Tangent, Normal, Subtangent and Subnormal**

Tangent: $PT = MP \operatorname{cosec} \Psi = y\sqrt{1+\cot^2\psi} = \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right|$

Subtangent: $TM = MP \cot \Psi = \left|\frac{y}{(dy/dx)}\right|$

Normal: $GP = MP \sec \Psi = y\sqrt{1+\tan^2\psi} = \left|y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$

Subnormal: $MG = MP \tan \Psi = \left|y\left(\frac{dy}{dx}\right)\right|$

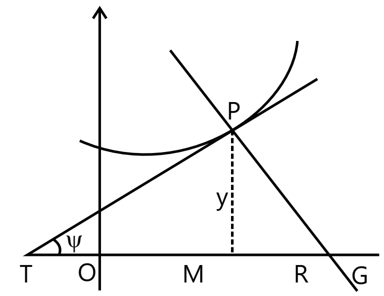


Figure 21.9

*** Angle of Intersection of Two Curves**

$\tan \Psi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$,

where m_1 and m_2 are the slopes of the tangents T_1 and T_2 at the intersection point (x_1, y_1).

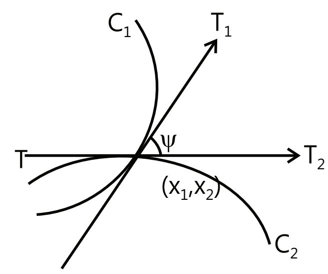


Figure 21.10

Solved Examples

JEE Main/Boards

Example 1: Show that the function f(x) = | x | is continuous at x = 0, but not differentiable at x = 0.

Sol: Evaluate $f'(0^+)$ and $f'(0^-)$.

We have $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0 = f(0)$

The function is continuous at x = 0

We also have

$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$

$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{-(-x)-0}{-x} = -1$

Since, $f'(0^+) \neq f'(0^-)$, the function is not differentiable at x = 0

Example 2: Find the derivative of the function f(x), defined by f(x) = sin x by 1[st]principle.

Sol: Use the first principle to find the derivative of the given function.

Let dy be the increment in y corresponding to an increment dx in x. We have

y = sin x

y +dy = sin (x + dx)

Subtracting, we get

dy = sin(x + dx) – sin x = 2 cos (x + dx/2) sin (dx / 2)

Dividing by dx, we obtain

$$\frac{\delta y}{\delta x} = \frac{\cos(x+\delta x/2)\sin(\delta x/2)}{(\delta x/2)}$$

Taking limits on both side, we get

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \cos(x + dx/2) \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)}$$

= cos x . 1 = cos x

Hence, we have d/dx(sin x) = cos x

Example 3: The derivative of log | x | is

Sol: Use the definition of the modulus to expand the given function. Then evaluate L.H.D. and R.H.D. at the critical point.

Let y = log | x | then

$$y = \begin{cases} \log x, & \text{when } x > 0 \\ \log(-x), & \text{when } x < 0 \end{cases}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \text{ when } x > 0$$

$$\text{and } \frac{dy}{dx} = \frac{1}{-x}(-1) = \frac{1}{x} \text{ when } x < 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \text{ when } x \neq 0$$

Example 4: If $y = \sqrt{\frac{1+x}{1-x}}$, then $\frac{dy}{dx}$ equals

Sol: Differentiate using u/v rule.

Differentiating w.r.t., we get

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \frac{(1-x)1-(1+x)(-1)}{(1-x)^2}$$

$$\Rightarrow \sqrt{\frac{1-x}{1+x}} \frac{1}{(1-x)^2} \Rightarrow \frac{1}{\sqrt{1+x}} \frac{1}{(1-x)^{3/2}}$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}\left(\frac{1}{1-x}\right)$$

Example 5: If $x^y = e^{x-y}$, then dy/dx equals -

Sol: Take logarithms on both sides and differentiate.

Taking log on both sides, we get

$$y \log x = x - y \Rightarrow y = \frac{x}{1+\log x}$$

$$\therefore \frac{dy}{dx} = \frac{(1+\log x).1 - x(1/x)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Example 6: If $y = \cot^{-1}\sqrt{x^2-1} + \sec^{-1}x$, then dy/dx equals

Sol: Use substitution to simplify the terms and then differentiate.

Put $x = \sec\theta$; $\cot^{-1}\sqrt{x^2-1} = \cot\sqrt{\sec^2\theta - 1}$

$$= \cot^{-1}(\tan\theta) = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1} x$$

$$\therefore y = \left(\frac{\pi}{2} - \sec^{-1} x\right) + \sec^{-1} x = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = 0$$

Example 7: If $x^2e^y + 2xye^x + 13 = 0$, then $\frac{dy}{dx}$ equals -

Sol: Use the formula for derivative of implicit function.

Using partial derivatives, we have

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{dy}{dx} = -\frac{2xe^y + 2ye^x + 2xye^x}{x^2e^y + 2xe^x}$$

$$= -\frac{2xe^{y-x} + 2y + 2xy}{x^2e^{y-x} + 2x} = -\frac{2xe^{y-x} + 2y(x+1)}{x(xe^{y-x} + 2)}$$

Example 8: $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right]$ equals -

Sol: Convert $\frac{\cos x}{1+\sin x}$ in terms of tan and proceed.

$$\because \frac{\cos x}{1+\sin x} = \frac{\sin(\pi/2 - x)}{1+\cos(\pi/2-x)}$$

$$= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \therefore \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) = \frac{\pi}{4} - \frac{x}{2} \Rightarrow \text{Derivative} = -\frac{1}{2}$$

Example 9: If $y = \frac{\sec x - \tan x}{\sec x + \tan x}$, then $\frac{dy}{dt}$ equals

Sol: Simplify the R.H.S. and differentiate.

$y = \frac{\sec x - \tan x}{\sec x + \tan x} \cdot \frac{\sec x - \tan x}{\sec x - \tan x}$

$y = (\sec x - \tan x)^2$

$\therefore \frac{dy}{dx} = 2(\sec x - \tan x)(\sec x \tan x - \sec^2 x)$

$\Rightarrow -2 \sec x (\sec x - \tan x)^2$

Example 10: If $y = \frac{1}{(t+2)(t+1)}$, then $\frac{dy}{dx}$ equals

Sol: Use the partial fraction method to find the derivative of given f^n

$y = \frac{1}{(t+2)(t+1)} = \frac{1}{t+1} - \frac{1}{t+2}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{(t+1)^2} + \frac{1}{(t+2)^2}$

Example 11: If $x = \theta - \frac{1}{\theta}$ and $y = \theta + \frac{1}{\theta}$, then $\frac{dy}{dx} = ?$

Sol: $x = \theta - \frac{1}{\theta} \Rightarrow \frac{dx}{d\theta} = 1 + \frac{1}{\theta^2}$

$y = \theta + \frac{1}{\theta} \Rightarrow \frac{dy}{d\theta} = 1 - \frac{1}{\theta^2}$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{1-(1/\theta^2)}{1+(1/\theta^2)} = \frac{\theta-(1/\theta)}{\theta+(1/\theta)} = \frac{x}{y}$

Example 12: Derivative of $\sin^{-1} x$ w.r.t. $\cos^{-1}\sqrt{1-x^2}$ is -

Sol: Substitute $\sin\theta$ in place of x.

Let $y = \sin^{-1}x$ and $z = \cos^{-1}\sqrt{1-x^2}$

Put $x = \sin\theta \Rightarrow z = \cos^{-1}(\cos\theta) = \theta$

$\therefore$ y = z and $\frac{dy}{dz} = 1$

Example 13: Derivative of $\sec^{-1}\left(\frac{1}{2x^2+1}\right)$ w.r.t. $\sqrt{1+3x}$ at $x = \frac{-1}{3}$ is

Sol: Differentiate the two functions and divide.

Let $y = \sec^{-1}\left(\frac{1}{2x^2+1}\right)$ and $z = \sqrt{1+3x}$

$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = (2x^2+1)\frac{1}{\sqrt{\left(1/(2x^2+1)\right)^2 - 1}}$

$\left(\frac{-4x}{2x^2+1}\right)\frac{2}{3}\sqrt{1+3x} \therefore \left(\frac{dy}{dz}\right)_{x=\frac{-1}{3}} = 0$

Example 14: Find $\frac{d}{dx}\left\{\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right\} =$

Sol: Use Substitution to simplify the inside the square root and then differentiate.

Let $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$.

Put $x = \cos 2\theta$.

$\therefore y = \sin^2\cot^{-1}\left\{\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right)\right\} = \sin^2\cot^{-1}(\cot\theta)$

$\therefore y = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2} - \frac{x}{2}$

$\therefore \frac{dy}{dx} = -\frac{1}{2}$

Example 15: Find the equation of the normal to the curve $y = x + \sin x \cos x$ at $x = \frac{\pi}{2}$.

Sol: Find a point on the curve slope of the normal at that point.

$x = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2} + 0 = \frac{\pi}{2}$, so the given point $= \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now from the given equation $\frac{dy}{dx} = 1 + \cos^2 x - \sin^2 x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} = 1 + 0 - 1 = 0$

$\therefore$ required equation of the normal is

$y - \frac{\pi}{2} = \frac{-1}{0}\left(x - \frac{\pi}{2}\right) \Rightarrow x - \frac{\pi}{2} = 0 \Rightarrow 2x = p$

Example 16: Find the point on the curve $y = x^2 - 3x$ at which tangent is parallel to x-axis.

Sol: Differentiate the given equation and put it equal to zero and proceed.

Let the point at which tangent is parallel to x-axis be $P(x_1, y_1)$

Then it must be on curve i.e.,$y_1 = x_1^3 - 3x_1$

Also differentiating w.r.t. x, we get, $\frac{dy}{dx} = 3x^2 - 3$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 3x_1^2 - 3 \qquad \text{... (i)}$$

since, the tangent is parallel to x-axis

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0 \Rightarrow 3x_1^2 - 3 = 0$$

$\Rightarrow x_1 = \pm 1$ ... (ii)

From (1) and (2); $y_1 = x_1^2 - 3x_1$

When $x_1 = 1$ when $x_1 = -1$

$y_1 = 1 - 3 = -2$, $y_1 = -1 + 3 = 2$

$\therefore$ points at which tangent is parallel to x-axis are (1, – 2) and (–1, 2).

Example 17: Find the equation of normal to the curve $x + y = x^y$, where it cuts x-axis.

Sol: Given curve is $x + y = x^y$ ... (i)

at x-axis y = 0,

$\therefore x + 0 = x^0 \Rightarrow x = 1$

$\therefore$ Point is A(1, 0)

Now to differentiation $x + y = x^y$

take log on both sides

$\Rightarrow \log(x + y) = y \log x$

$$\therefore \frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\} = y \cdot \frac{1}{x} + (\log x)\frac{dy}{dx}$$

Putting x = 1, y = 0, $\left\{1+\frac{dy}{dx}\right\} = 0$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -1 \quad \therefore \text{ slope of normal} = 1$$

Equation of normal is, $\frac{y-0}{x-1} = 1 \Rightarrow y = x - 1$

Example 18: If the curve $y^2 = 6x$, $9x^2 + by^2 = 16$, cut each other at right angles then the value of b is

Sol: Equate the product of $\frac{dy}{dx}$ from the two equations to -1.

The intersection of the two curves is given by $9x^2 + 6bx = 16$ ... (i)

Differentiating $y^2 = 6x$, We have $\frac{dy}{dx} = \frac{3}{y}$

Differentiating $9x^2 + 6y^2 = 16$

We have $\frac{dy}{dx} = -\frac{9x}{by}$

For curves to intersect at right angles, we must have at the point of intersection.

$$\frac{3}{y}\left(\frac{-9x}{by}\right) = -1 \Rightarrow 27x = by^2.$$

Thus we must have

$9x^2 + by^2 = 16 \Rightarrow 9x^2 + 27x - 16 = 0$ ... (ii)

(i) and (ii) must be identical so $27 = 6b \Rightarrow b = 9/2$.

Example 19: If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then P is

Sol: Solve the equation of the tangent with the equation of the curve.

$$2y\frac{dy}{dx} = (2 - x)^2 - 2x(2 - x)$$

$$= 3x^2 - 8x + 4. \text{ So } \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{1}{2}$$

An equation of tangent at (1, 1) is $Y - 1 = (-1/2)(X - 1)$.

i.e. $Y = (-1/2)x + 3/2$. The intersection of this line with the given curve is given by $((-x/2) + 3/2)^2 = x(2 - x)^2$

$\Rightarrow x^2 - 6x + 9 = 16x + 4x^3 - 16x^2$. So,

$4x^3 - 17x^2 + 22x - 9 = 0$

$\Rightarrow (x - 1)(4x - 9)(x + 1) = 0$

Thus x = 1, 9/4, –1. But x = – 1 cannot lie on the given curve so required point is (9/4, 3/8).

JEE Advanced/Boards

Example 1: Examine differentiability of f(x) at

$$x = 0 \text{ for } f(x) = \begin{cases} \frac{1-\cos x}{x\sin x}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$

Sol: Find the left and right hand derivative of the function f(x) about the point x = 0.

First we obtain L.f '(0)

$$= \lim_{h\to 0}\left[\frac{f(-h)-f(0)}{-h}\right] = \lim_{h\to 0}\left[\left(-\frac{1}{h}\right)\left\{\frac{1-\cos h}{h\sin h}-\frac{1}{2}\right\}\right]$$

$$= \lim_{h\to 0}\left[\frac{h\sin h + 2(1-\cos h)}{2h^2 \sin h}\right]; \left(\text{in}\frac{0}{0}\text{from}\right)$$

$$= \lim_{h\to 0}\left[\frac{h\left(h-\frac{h^3}{3!}+\frac{h^5}{5!}-.....\right)-2\left(\frac{h^2}{2!}-\frac{h^4}{4!}+\frac{h^6}{6!}-.....\right)}{2h^2\left(h-\frac{h^3}{3!}+.....\right)}\right]$$

$$= \lim_{h\to 0}\left[\frac{h^4\left\{\left(\frac{1}{12}-\frac{1}{3!}\right)+\left(\frac{1}{5!}-\frac{2}{6!}\right)h^2\right\}}{2h^3\left(h-\frac{h^2}{3!}+.....\right)}\right]$$

$$= \lim_{h\to 0}\left[\frac{h\left\{\left(\frac{1}{12}-\frac{1}{3!}\right)+\left(\frac{1}{5!}-\frac{2}{6!}\right)h+.....\right\}}{2\left(1-\frac{h^2}{3!}+.....\right)}\right] = 0 \text{ and}$$

$$Rf'(0) = \lim_{h\to 0}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$= \lim_{h\to 0}\left\{\frac{1}{h}\left(\frac{1-\cosh}{h\sinh}-\frac{1}{2}\right)\right\} = 0,$$

similarly as above i.e. L.f′(0) = Rf′(0)

$\Rightarrow$ f(x) is differentiable at x = 0

Example 2: Examine differentiability of the function f(x) = $\sin^{-1}(\cos x)$ at x = np + $\frac{\pi}{2}$. where n $\in$ I.

Sol: Similar to the previous example.

first, we obtain $Lf'\left(nx+\frac{\pi}{2}\right)$

$$= \lim_{h\to 0}\left(\frac{f\left(nx+(\pi/2)-h\right)-f\left(nx+(\pi/2)\right)}{-h}\right)$$

$$= \lim_{h\to 0}\left[\frac{\sin^{-1}\left\{\cos\left(nx+(\pi/2)-h\right)\right\}-\sin^{-1}\left\{\cos\left(nx+(\pi/2)\right)\right\}}{-h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin^{-1}\left\{(-1)^n\cos\left((\pi/2)-h\right)\right\}-\sin^{-1}\left\{(-1)^n\cos(\pi/2)\right\}}{-h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin^{-1}\left\{\sin(-1)^n h\right\}-\sin^{-1}0}{-h}\right]$$

$$= \lim_{h\to 0}\frac{(-1)^n\sin^{-1}\sin h-\sin^{-1}0}{-h}$$

$$= \lim_{h\to 0}\left[\frac{(-1)^n h}{-h}\right] = (-1)^{n-1}Rf'\left(n\pi+\frac{\pi}{2}\right)$$

$$= \lim_{h\to 0}\left\{\frac{f\left(nx+(\pi/2)+h\right)-f\left(nx+(\pi/2)\right)}{h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{\sin^{-1}\left[\left\{\cos\left(nx+(\pi/2)+h\right)\right\}-\sin^{-1}\left\{\cos\left(n\pi+(\pi/2)\right)\right\}\right]}{h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{\sin^{-1}\left\{(-1)^n\cos\left((\pi/2)+h\right)\right\}-\sin^{-1}\left\{(-1)^n\cos(\pi/2)\right\}}{h}\right\}$$

$$= \lim_{h\to 0}\left[\frac{\sin^{-1}\{(-1)^{n+1}\sinh\}}{h}\right] = \lim_{h\to 0}\left[\frac{\sin^{-1}\{\sin(-1)^{n+1}h\}}{h}\right]$$

$$= \lim_{h\to 0}\frac{(-1)^{n+1}\sin^{-1}\sin h}{h} = \lim_{h\to 0}\frac{(-1)^{n+1}h}{h} = (-1)^{n+1}$$

(Which is equal to $(-1)^{n-1}$)

Thus we find $Lf'\left(nx+\frac{\pi}{2}\right) = Rf'\left(nx+\frac{\pi}{2}\right)$

$\therefore$ f(x) is differentiable at $(nx + \pi/2)$

Example 3: If $x\sqrt{(1+y)} + y\sqrt{(1+x)} = 0$, then $\frac{dy}{dx}$ equals

Sol: Simplify the equation given and then differentiate it.

We have

$x\sqrt{(1+y)} + y\sqrt{(1+x)} = 0$ ... (i)

$\Rightarrow x\sqrt{(1+y)} = -y\sqrt{(1+x)}$

On squaring both sides $x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 - y^2 + x^2y - xy^2 = 0$

$\Rightarrow (x - y)(x + y + xy) = 0$

$x - y \neq 0$ [For y = x does not satisfy (1)]

$\therefore x + y + xy = 0 \Rightarrow y = -\frac{x}{(1+x)}$

$$\therefore \frac{dy}{dx} = -\left\{\frac{(1+x).1-x.1}{(1+x)^2}\right\} = -\frac{1}{(1+x)^2}.$$

Example 4: If $x^y y^x = 1$, then $\frac{dy}{dx}$ equals –

Sol: Use logarithms on both sides and then differentiate

Taking log on both sides, we have y log x + x log y = 0

Now using partial derivatives, we have

$$\frac{dy}{dx} = -\frac{y/x+\log y}{\log x + x/y} \Rightarrow -\frac{y(y+x\log y)}{x(x+y\log x)}$$

Example 5: If $\sqrt{1-x^2}+\sqrt{1-y^2}$ = a(x – y) then $\frac{dy}{dx}$ equals –

Sol: Use substitution for x and y.

Putting x = a sin A, y = a sin B, then given relation becomes

cos A + cos B = a(sin A – sin B)

$$\Rightarrow 2a\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$= 2a\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Divide and multiply by $\cos\left(\frac{A+B}{2}\right)$

$$\Rightarrow \cot\left(\frac{A-B}{2}\right) = a\left[\because \cos\left(\frac{A+B}{2}\right)\neq 0\right]$$

$\Rightarrow$ A – B = 2 $\cot^{-1}a$ $\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Example 6: If $x^2+y^2 = t-\frac{1}{t}$, $x^4+y^4 = t^2+\frac{1}{t^2}$ then $\frac{dy}{dx}$ equals

Sol: Eliminate t from the first and the second equation and then find the derivative.

Squaring the first equation, we have

$$x^4+y^4+2x^2y^2 = t^2+\frac{1}{t^2}-2$$

$$\Rightarrow t^2+\frac{1}{t^2}+2x^2y^2 = t^2+\frac{1}{t^2}-2 \text{ (from second equation)}$$

$$\Rightarrow x^2y^2 = -1 \Rightarrow y^2 = -\frac{1}{x^2}$$

$$\therefore 2y\frac{dy}{dx} = \frac{2}{x^3} \Rightarrow \frac{dy}{dx} = \frac{1}{x^3y}$$

Example 7: The derivation of

$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is

Sol: Differentiation w.r.t another function.

Putting x = tanθ

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right) = \theta$$

$$\& z = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = 2\theta$$

∴ Required derivative = ½

$$\Rightarrow \frac{dy}{dz} = \frac{\frac{dy}{d\theta}}{\frac{dz}{d\theta}} = \frac{\theta}{2\theta} = \frac{1}{2}$$

Example 8: If $y = \sin^{-1}\left(\sqrt{\sin x}\right)$ then $\frac{dy}{dx}$ equals–

Sol: Differentiation of function

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin x}}\cdot\frac{1}{2\sqrt{\sin x}}.\cos x$$

$$= \frac{\sqrt{1+\sin x}}{2\sqrt{\sin x}} = \frac{1}{2}\sqrt{1+\operatorname{cosec} x}$$

Example 9: If $\sqrt{\frac{v}{\mu}}+\sqrt{\frac{\mu}{v}}$ = 6 then $\frac{dv}{d\mu}$ = ?

Sol: Square the given equation and proceed.

$$\sqrt{\frac{v}{\mu}}+\sqrt{\frac{\mu}{v}} = 6 \Rightarrow \frac{v}{\mu}+\frac{\mu}{v}+2 = 36$$

$\Rightarrow \mu^2 + v^2 = 34\,\mu v$

Differentiating both sides w.r.t. μ we have

$$2\mu + 2v\frac{dv}{d\mu} = 34\,v + 34\,\mu\frac{dv}{d\mu}$$

$$\Rightarrow 2[17\mu - n]\frac{dv}{d\mu} = 2\,[\mu - 17\,n] \quad \therefore \frac{dv}{d\mu} = \frac{\mu-17v}{17\mu - v}$$

Example 10: If x = exp. $\tan^{-1}\left(\frac{y-x^2}{x^2}\right)$, then $\frac{dy}{dx}$ equals

Sol: Simplify the given equation and differentiate.

Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y-x^2}{x^2}\right)$$

$\Rightarrow \tan(\log x) = (y - x^2) / x^2 \quad \Rightarrow y = x^2 + x^2 \tan(\log x)$

$\therefore \frac{dy}{dx} = 2x + 2x \tan(\log x) + x \sec^2(\log x)$

$\Rightarrow 2x [1 + \tan(\log x)] + x \sec^2(\log x)$

Example 11: Find $\frac{d}{dx} \cos^{-1}\left(\frac{4x^3}{27} - x\right)$

Sol: Let $y = \cos^{-1}\left(\frac{4x^3}{27} - x\right) = \cos^{-1}\left(4\left(\frac{x}{3}\right)^3 - 3\left(\frac{x}{3}\right)\right)$

$\frac{x}{3} = \cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{x}{3}\right)$

$\therefore y = \cos^{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos 3\theta) = 3\theta$

$\therefore y = 3\cos^{-1}\left(\frac{x}{3}\right)$

$\therefore \frac{dy}{dx} = 3 \frac{-1}{\sqrt{1-(x^2/9)}} \cdot \frac{1}{3} = \frac{-3}{\sqrt{9-x^2}}$

Example 12: If $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log a$ then $\frac{dy}{dx} =$

Sol: Take cosine on both sides and then apply componendo and dividendo.

$\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log a$

$\Rightarrow \frac{x^2 - y^2}{x^2 + y^2} = \cos(\log a) = k \text{ (say)}$

by componendo and dividends,

$\therefore \frac{(x^2 - y^2) + (x^2 + y^2)}{(x^2 - y^2) - (x^2 + y^2)} = \frac{k+1}{k-1}$

$\Rightarrow \frac{2x^2}{-2y^2} = \frac{k+1}{k-1} \qquad \therefore \frac{x}{y} = \sqrt{\frac{k+1}{k-1}}$

Differentiating both sides w.r.t. 'x' we get

$\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx} = 0 \qquad \Rightarrow \frac{dy}{dx} = \frac{y}{x}$

Example 13: If $y^2 = p(x)$ is a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$ is equal to

Sol: Find first order, second order and third order derivative of p(x).

$p'(x) = 2yy' \Rightarrow p''(x) = 2yy'' + 2y'^2 \Rightarrow p'''(x) = 2yy''' + 4y'y''$

Also $2\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right) = 2\frac{d}{dx}(y^3y'')$

$= 2[y^3y''' + 3y'^2y'y''] = y^2[2yy''' + 6y'y''] = p(x)\, p'''(x)$

Example 14: If the tangent at the point $P(at^2, at^3)$ on the curve $ay^2 = x^3$ intersects the curve again at the point Q, find the point Q.

Sol: Solve the equation of the tangent and the equation of the curve.

$ay^2 = x^3 \Rightarrow 2ay\, dy/dx = 3x^2$

Slope of tangent at P is $\left(\frac{3x^2}{2ay}\right)_p = \frac{3a^2t^4}{2a^2t^3} = \frac{3}{2}t$

Let Q be (at_1^2, at_1^3). Slope of line

$PQ = \frac{at_1^3 - at^3}{at_1^3 - at^2} = \frac{t_1^2 + tt_1 + t^2}{t_1 + t}$

which must be the slope of tangent at P. Hence,

$\frac{t_1^2 + tt_1 + t^2}{t_1 + t} = \frac{3t}{2} \quad \Rightarrow 2t_1^2 - tt_1 - t^2 = 0$

$\Rightarrow (t_1 - t)(2t_1 + t) = 0 \qquad \Rightarrow t_1 = -\frac{t}{2}$

Thus, Q has coordinates $\left(\frac{at^2}{4}, -\frac{at^3}{8}\right)$

Example 15: Show that the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ cut orthogonally if, $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$

Sol: Equate the product of $\frac{dy}{dx}$ from the two equations to -1.

Let the two curves cut each other at the point (x_1, y_1); then

$ax_1^2 + by_1^2 = 1$ ... (i)

& $cx_1^2 + dy_1^2 = 1$ ... (ii)

From (i) and (ii), we get

$= (a - c)x_1^2 + (b - d)y_1^2 = 0$ ... (iii)

Slope of the tangent to the curve $ax^2 + by^2 = 1$, at (x_1, y_1) is given by, $\left[\frac{dy}{dx}\right]_{(x_1, y_1)} = -\frac{ax_1}{by_1}$

Slope of the tangent to the curve $cx^2 + dy^2 = 1$, at (x_1, y_1) is given by,

$$\left[\frac{dy}{dx}\right]_{(x_1,y_1)} = -\frac{cx_1}{dy_1}$$

If the two curves cut orthogonally, we must have, $\left(-\frac{ax_1}{by_1}\right)\left(-\frac{cx_1}{dy_1}\right)$

$\Rightarrow acx_1^2 + bdy_1^2 = 0$ ... (iv)

From (iii) and (iv), we have

$$\frac{a-c}{ac} = \frac{b-d}{bd} \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$$

Example 16: Find the acute angle between the curves y = | x² – 1 | and y = | x² – 3 | at their points of intersection when x > 0.

Sol: Solve the two curves and find the slope for the two tangents. Proceed to find the angle between the two lines.

For the intersection of the given curves

$|x^2-1| - |x^2-3| \Rightarrow (x^2-1)^2 = (x^2-3)^2$

$\Rightarrow (x^2-1)^2 - (x^2-3)^2 = 0$

$\Rightarrow [(x^2-1) - (x^2-3)]\,[(x^2-1) + (x^2-3)] = 0$

$\Rightarrow [2x^2 - 4] = 0 \Rightarrow 2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$

neglecting $x = -\sqrt{2}$ as x > 0

We have point of intersection as $x = -\sqrt{2}$

Here $y = |x^2-1| = (x^2-1)$ in the neighbouring of

$x = \sqrt{2}$ and $y = -(x^2-3)$ in the neighbouring of $x = \sqrt{2}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{c_1} = 2x = 2\sqrt{2}$ and $\left(\frac{dy}{dx}\right)_{c_1} = -2x = -2\sqrt{2}$

Hence, if θ is angle between them,

$$\Rightarrow \tan\theta = \left|\frac{2\sqrt{2}-(-2\sqrt{2})}{1+2\sqrt{2}(-2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{-7}\right| = \left(\frac{4\sqrt{2}}{7}\right)$$

$$\therefore \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

Example 17: At what points on the curve $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$, then tangent make equals angles with coordinate axes.

Sol.: Find dy/dx and equate it to ± 1.

Given curve is $y = \frac{2}{3}x^3 + \frac{1}{2}x^2$ ... (i)

Differentiating both sides w.r.t. x, then $\frac{dy}{dx} = 2x^2 + x$

∵ Tangents make equal angles with coordinate axes.

$\therefore \frac{dy}{dx} = \pm 1$ or $2x^2 + x = \pm 1$ or

$2x^2 + x + 1 \neq 0$ and $2x^2 + x - 1 = 0$

or $2x^2 + 2x - x - 1 = 0$

(If $2x^2 + x + 1 = 0$ then x is imaginary)

or $(2x-1)(x+1)$ $\therefore x = \frac{1}{2}, -1$

From (1), $x = \frac{1}{2}$, $y = \frac{2}{3}.\frac{1}{8} + \frac{1}{2}.\frac{1}{4} = \frac{5}{24}$

and for $x = -1$, $y = -\frac{2}{3} + \frac{1}{2} = -\frac{1}{6}$

hence point are $\left(\frac{1}{2}, \frac{5}{24}\right)$ and $\left(-1, -\frac{1}{6}\right)$

Example 18: The side of the rectangle of the greatest area, that can be inscribed in the ellipse $x^2 + 2y^2 = 8$, are given by

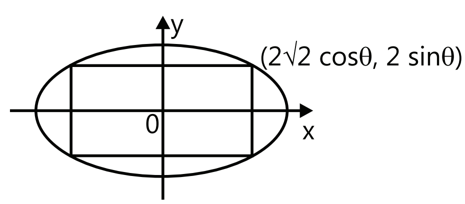
(A) $4\sqrt{2}$, 4 (B) 4, $2\sqrt{2}$

(C) 2, $\sqrt{2}$ (D) $2\sqrt{2}$, 2

Sol: (B) Consider a point on the ellipse and write the expression for the area of the rectangle. Then find the maximum area using first and second order derivative.

Any point on the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ is

$(2\sqrt{2}\cos\theta, 2\sin\theta)$ [see figure]



A = area of the inscribed rectangle

$= 4(2\sqrt{2}\cos\theta)(2\sin\theta) = 8\sqrt{2}\sin 2\theta$

$\frac{dA}{d\theta} = 16\sqrt{2}\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$

Also $\frac{d^2A}{d\theta^2} = -32\sqrt{2}\sin 2\theta < 0$ for $\theta = \frac{\pi}{2}$

Hence, the inscribed rectangle is of largest area if the sides are $4\sqrt{2}\cos\frac{\pi}{4}$ and $4\sin\left(\frac{\pi}{4}\right)$ i.e. 4 and $2\sqrt{2}$.

JEE Main/Boards

Exercise 1

Methods of Differentiation

Q.1 Find the derivative of $e^{\sqrt{x}+3}$, with respect to x.

Q.2 Differentiate, sin(log x), with the respect to x.

Q.3 If x = sinθ, y = – tanθ, find dy/dx.

Q.4 Differentiate, $\cos^{-1}\left(\sqrt{x}\right)$, with the respect to x.

Q.5 Differentiate, $e^{m\tan^{-1}x}$, with the respect to x.

Q.6 Differentiate, sin {log(x^3 – 1)}, with the respect to x.

Q.7 Differentiate, cos x, with the respect to e^x.

Q.8 Differentiate the following w.r.t., x : $\log_2(\sin x)$.

Q.9 Differentiate the following w.r.t., x : y = $5^{\log(\sin x)}$.

Q.10 Find $\frac{dy}{dx}$,when $\sqrt{x}+\sqrt{y}=5$ at (4, 9).

Q.11 y = $\cot^{-1}\left(\frac{1+x}{1-x}\right)$

Q.12 y = $\sqrt{\frac{1+\tan x}{1-\tan x}}$

Q.13 y = $\sin\left[\sqrt{\cos\sqrt{x}}\right]$

Q.14 y = $\tan^{-1}\left(\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}\right)$

Q.15 $y=(\sin x)^{\cos^{-1}x}$

Q.16 $y=\cos^{-1}\left[\left(2\cos x+3\sin x\right)\sqrt{13}\right]$

Q.17 y = $\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

Q.18 y = $\sin^{-1}\left[\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right]$

Q.19 y = $\sin^{-1}\left[2ax\sqrt{1-a^2x^2}\right]$

Q.20 y = $\sqrt{a+\sqrt{a+x}}$

Q.21 y = $\tan^{-1}\left[\frac{ax-b}{a+bx}\right]$

Q.22 y = $\log\left(x+\sqrt{x^2+a^2}\right)$

Q.23 y = $\log\left(\sin\sqrt{1+x^2}\right)$

Application of Derivatives

Q.1 Find the point on the curve y = x^2 – 4x + 5, where tangent to the curve is parallel to x-axis.

Q.2 If two curves cut orthogonally, then what can we say about the angle between tangents at the point of intersection of the curves.

Q.3 Find the slopes of tangent and normal to the curve $f(x)=3x^2-5$ at x = $\frac{1}{2}$.

Q.4 If the tangent of the curve y = f(x) at point (x, y) on the curve is parallel to y– axis, then what is the value of $\frac{dy}{dx}$.

Q.5 Find a point on the curve y = $2x^2$ – 6x – 4 at which the tangent is parallel to x-axis.

Q.6 Find a point on the curve y = x^2 – 4x – 32 at which the tangent is parallel to x-axis.

Q.7 Find the equations of tangent and normal to the curve y = $\sqrt[3]{5-x}$ at (–3, 2).

Q.8 Find equations of tangent to the curve y = $\sqrt{4x-3}$, if parallel to x-axis.

Q.9 Verify that the point (1, 1) is a point of intersection of the curves $x^2 = y$ and $x^3 + 6y = 7$ and show that these curves cut orthogonally at this point.

Q.10 Find the equation of tangent to the parabola $y^2 = 8x$ which is parallel to line $4x - y + 3 = 0$.

Q.11 Find the equation of tangent to the curve $y = -5x^2 + 6x + 7$ at the point $\left(\frac{1}{2}, \frac{35}{4}\right)$.

Q.12 Find the equation of tangent to the curve $xy = c^2$ at the point $\left(\frac{c}{k}, ck\right)$ on it.

Q.13 Prove that the tangents to the curve $y = x^3 + 6$ at the points (–1, 5) and (1, 7) are parallel.

Q.14 At what point on the curve $y = x^2$ does the tangent make an angle of 45° with x-axis?

Q.15 Find the point (s) on the curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$ parallel to y-axis.

Q.16 Find the slope of the normal to the curve $x = \frac{1}{t}$ $y = 2t$ at $t = 2$.

Q.17 Show that equation of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

Q.18 Find the equation of the normal lines to the curve $y = 4x^3 - 3x + 5$ which are parallel to the line $9y + x + 3 = 0$.

Q.19 Find the equation of normal line to the curve $y(x - 2)(x - 3) - x = 7 = 0$ at the point where it meets x-axis.

Q.20 Find the equation of tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point (x_1, y_1) and show that the sum of its intercepts on axis is constant.

Q.21 Find the equation of the normals to the curve $3x^2 + y^2 = 8$ parallel to the line $x + 3y = 4$.

Q.22 Find the equation of the tangents to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$.

Q.23 Find the points on the curve $y = x^3 - 2x^2 - 2x$ at which the tangent lines are parallel to the line $y = 2x - 3$.

Q.24 Find the angle between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at their point of intersection other than the origin.

Exercise 2

Methods of Differentiation

Single Correct Choice Type

Q.1 If $y = f\left(\frac{3x+4}{5x+6}\right)$ & $f'(x) = \tan x^2$ then $\frac{dy}{dx} =$

(A) $\tan x^3$

(B) $-2\tan\left[\frac{3x+4}{5x+6}\right]^2 \cdot \frac{1}{(5x+6)^2}$

(C) $f\left(\frac{3\tan x^2+4}{5\tan x^2+6}\right)\tan x^2$

(D) None

Q.2 Let g is the inverse function of f & $f'(x) = \frac{x^{10}}{(1+x^2)}$. If $g(2) = a$ then $g'(2)$ is equal to

(A) $\frac{5}{2^{10}}$ (B) $\frac{1+a^2}{a^{10}}$ (C) $\frac{a^{10}}{1+a^2}$ (D) $\frac{1+a^{10}}{a^2}$

Q.3 If $y = \frac{1}{1+x^{n-m}+x^{p-m}} + \frac{1}{1+x^{m-n}+x^{p-n}} + \frac{1}{1+x^{m-p}+x^{n-p}}$

Then $\frac{dy}{dx}$ at $e^{m^{n^p}}$ is equal to :

(A) e^{mnp} (B) $e^{mn/p}$ (C) $e^{np/m}$ (D) None

Q.4 Let f is differentiable in (0, 6) & $f'(4) = 5$ then $\lim_{x\to2}\frac{f(4)-f(x^2)}{2-x} =$

(A) 5 (B) 5/4 (C) 10 (D) 20

Q.5 Let $\ell = \lim_{x\to0} x^m(\ln x)^n$ where $m, n \in N$ then

(A) ℓ is independent of m and n

(B) ℓ is independent of m and depend on m

(C) ℓ is independent of n and depend on m

(D) ℓ is dependent on both m and n

Q.6 Let $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$ then $\lim_{a\to 0} \frac{f'(x)}{x} =$

(A) 2 (B) – 2 (C) – 1 (D) 1

Q.7 Let f(x) $\begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$ then $f'\left(\frac{\pi}{2}\right) =$

(A) 0 (B) – 12 (C) 4 (D) 1

Q.8 If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then dy/dx at x = π/2 is

(A) $\frac{-8}{\pi^2+4}$ (B) $\frac{4}{\pi^2+4}$

(C) $\frac{8}{\pi^2+4}$ (D) Does not exists

Q.9 If f(4) = g(4) = 2; f′(4) = 9 ; g′(4) = 6 then

$\lim_{a\to 4} \frac{\sqrt{f(x)}-\sqrt{g(x)}}{\sqrt{x}-2}$ is equal to :

(A) $3\sqrt{2}$ (B) $\frac{3}{\sqrt{2}}$ (C) 0 (D) None

Q.10 If $y = x + e^x$ then $\frac{d^2x}{dy^2}$ is :

(A) e^x (B) $-\frac{e^x}{(1+e^x)^3}$

(C) $-\frac{e^x}{(1+e^x)^2}$ (D) $\frac{-1}{(1+e^x)^3}$

Q.11 If f is twice differentiable such that f″(x) = – f(x), f′(x) = g(x) h′(x) = $[f(x)]^2 + [g(x)]^2$ and h(0) = 2, h(1) =4, then the equation y = h(x) represents :

(A) A curve of degree 2

(B) A curve passing through the origin

(C) A straight line with slope 2

(D) A straight line with y intercept equal to– 2

Q.12 Let f(x) = x + 3 ln(x – 2) & g(x) = x + 5 n(x – 1), then the set of x satisfying the inequality f′(x) < g′(x) is

(A) $\left(2, \frac{7}{2}\right)$ (B) $(1, 2) \cup \left(-\frac{7}{2}, \infty\right)$

(C) $(2, \infty)$ (D) $\left(\frac{7}{2}, \infty\right)$

Q.13 Let f(x) = sin x; g(x) = x^2& h(x) = $\log_e x$ & f(x) = h[g(f(x))] then $\frac{df(x)}{dx^2}$ is equal to :

(A) $2\ \text{cosec}^3 x$ (B) $2\ \text{cost}\ (x^2) - 4x^2\ \text{cosec}^2\ (x^2)$

(C) $2x \cot x^2$ (D) $-2\text{cosec}^2\ x$

Q.14 Let $f(x) = x^n$, n being a non-negative integer. The number of value of n for which f′ (p + q) = f′ $\frac{b}{ab+2ay}$ (p) + f′(q) is valid for all p, q > 0 is :

(A) 0 (B) 1 (C) 2 (D) None of these

Q.15 If $f(x) = \prod_{n=1}^{100} (x-n)^{n(101-n)}$; then $\frac{f(101)}{f'(101)} =$

(A) 5050 (B) $\frac{1}{5050}$ (C) 10010 (D) $\frac{1}{10010}$

Q.16 Let $f(x) = \begin{cases} \frac{3x^2+2x-1}{6x^2-5x+1} & \text{for } x \neq \frac{1}{3} \\ -4 & \text{for } x = \frac{1}{3} \end{cases}$ then $f'\left(\frac{1}{3}\right)$

(A) is equal to – 9 (B) is equal to – 27

(C) is equal to 27 (D) does not exist

Q.17 Let f(x) be a quadratic expression which is positive for all real x. If g(x) = f(x) + f′(x) + f″(x), then for any real x, which one is correct.

(A) g(x) < 0 (B) g(x) > 0 (C) g(x) = 0 (D) g(x) ≥ 0

Q.18 If $y = \frac{x^4+4}{x^2-2x+2}$ then $\left.\frac{dy}{dx}\right]_{x=1/2}$ is :

(A) 3 (B) – 1 (C) 4 (D) None

Q.19 A function f, defined for all positive real numbers, satisfies the equation $f(x^2) = x^3$ for every x > 0. Then the value of f′(4)

(A) 12 (B) 3 (C) 3/2 (D) Cannot be determined

Q.20 If x = sin t and y = sin 3t, then the value of 'K' for which $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + Ky = 0$ is

(A) 3 (B) 6 (C) 12 (D) 9

Q.21 If $x = \ln t$ & $y = t^2 - 1$ then $y''(1)$ at $t = 1$ is

(A) 2 (B) 4 (C) 3 (D) None

Application of Derivatives

Single Correct Choice Type

Q.1 The angle at which the curve $y = ke^{kx}$ intersects the y-axis is

(A) $\tan^{-1}k^2$ (B) $\cot^{-1}(k^2)$

(C) $\sec^{-1}\left(\sqrt{1+k^4}\right)$ (D) None

Q.2 The angle between the tangent lines to the graph of the function $f(x)= \int_2^x (2t-5)\,dt$ at the point where the graph cuts the x-axis is -

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

Q.3 If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x and y axis respectively then the value of a^2b is

(A) $27c^3$ (B) $\frac{4}{27}c^3$ (C) $\frac{27}{4}c^3$ (D) $\frac{4}{9}c^3$

Q.4 Consider the function $f(x) = \begin{cases} x\sin\frac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$

then the number of points in (0, 1) where the derivative

$f'(x)$ vanishes, is

(A) 0 (B) 1 (C) 2 (D) infinite

Q.5 The tangent to the graph of the function $y = f(x)$ at the point with abscissa $x = a$ forms with the x-axis an angle of $\pi/3$ and at the point with abscissa $x = b$ at an angle of $\pi/4$, then the value of the integral, $\int_a^b f'(x).f''(x)dx$ is equal to

(A) 1 (B) 0 (C) $-\sqrt{3}$ (D) -1

[assume $f''(x)$ to be continuous]

Q.6 Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. If the gradient at B is K times the gradient at A then K is equal to

(A) 4 (B) 2 (C) -2 (D) ¼

Q.7 The subnormal at any point on the curve $xy^n = a^{n+1}$ is constant for :

(A) $n = 0$ (B) $n = 1$

(C) $n = -2$ (D) No value of n

Q.8 Equation of the line through the point (1/2, 2) and tangent to the parabola $y = \frac{-x^2}{2} + 2$ and secant to the curve $y = \sqrt{4-x^2}$ is

(A) $2x + 2y - 5 = 0$ (B) $2x + 2y - 3 = 0$

(C) $y - 2 = 0$ (D) None of these

Q.9 Two curves C_1: $y = x^2 - 3$ and C_2 : $y = kx^2$, $k \in R$ intersect each other at two different point. The tangent drawn to C_2 at one of the point of intersection $A \equiv (a, y_1)$, $(a > 0)$ meets C_1 again at $B(1, y_2)$ $(y_1 \neq y_2)$. The value of 'a' is

(A) 4 (B) 3 (C) 2 (D) 1

Q.10 Number of roots of the equation $x^2.e^{2-|x|} = 1$ is:

(A) 2 (B) 4 (C) 6 (D) Zero

Q.11 The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to

(A) Square of the abscissa of the point of tangency

(B) Square root of the abscissa of the point of tangency

(C) Cube of the abscissa of the point of tangency

(D) Cube root of the abscissa of the point of tangency

Q.12 The line which is parallel to x-axis and crosses the curve $y = \sqrt{x}$ at an angle of $\frac{\pi}{4}$ is

(A) $y = -1/2$ (B) $x = ½$ (C) $y = 1/4$ (D) $y = ½$

Q.13 The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

Q.14 Consider $f(x) = \int_0^x \left(t + \frac{1}{t}\right)dt$ and $g(x) =$ 'f' for $x \in \left[\frac{1}{2}, 3\right]$

If P is a point on the curve y = g(x) such that the tangent to this curve at P is parallel to a chord joining the points $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$ and (3, g(3)) of the curve, then the coordinates of the point P

(A) can't be found out (B) $\left(\frac{7}{4}, \frac{65}{28}\right)$

(C) (1, 2) (D) $\left(\sqrt{\frac{3}{2}}, \frac{5}{\sqrt{6}}\right)$

Q.15 The co-ordinates of the point on the curve $9y^2 = x^3$ where the normal to the curve makes equal intercepts with the axes is

(A) $\left(1, \frac{1}{3}\right)$ (B) $\left(3, \sqrt{3}\right)$ (C) $\left(4, \frac{8}{3}\right)$ (D) $\left(\frac{6}{5}, \frac{2}{5}\sqrt{\frac{6}{5}}\right)$

Previous Years' Questions

Q.1 The normal to the curve x = a (cos θ + θ sin θ), y = a (sin θ – θ cos θ) at any / point "θ" is such that ***(1983)***

(A) It makes a constant angle with the x-axis

(B) It passes through the origin

(C) It is at a constant distance from the origin

(D) None of the above

Q.2 The slope of tangent to a curve y = f(x) at [x, f(x)] is 2x + 1. If the curve passes through the point (1, 2),then the area bounded by the curve, the x-axis and the line x = 1 is ***(1995)***

(A) 5/6 (B) 6/5 (C) 1/6 (D) 6

Q.3 If the normal to the curve y = f(x) at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then f′(3) is equal to ***(2000)***

(A) – 1 (B) –3/4 (C) 4/3 (D) 1

Q.4 The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is (are) ***(2002)***

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm\sqrt{\frac{11}{3}}, 0\right)$

(C) (0, 0) (D) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

Q.5 The equation of the common tangent to the curves $y^2 = 8x$ and xy = – 1 is ***(2002)***

(A) 3y = 9x + 2 (B) y = 2x + 1

(C) 2y = x + 8 (D) y = x + 2

Q.6 Tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the mid point of the intercept made by the tangents between the coordinate axes is ***(2004)***

(A) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (B) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$

(C) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (D) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

Q.7 The angle between the tangent drawn from the point (1, 4) to the parabola $y^2 = 4x$ is ***(2004)***

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Q.8 The tangent at (1, 7) to the curve $x^2 - y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at ***(2005)***

(A) (6, 7) (B) (– 6, 7) (C) (6, –7) (D) (–6, – 7)

Q.9 The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the point $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ ***(2007)***

(A) On the left of x = c (B) On the right of x = c

(C) At no point (D) At all points

Q.10 Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right), & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$.

Then which one of the following is true ? ***(2008)***

(A) f is neither differentiable at x = 0 not at x = 1

(B) f is differentiable at x = 0 and at x = 1

(C) f is differentiable at x = 0 but not at x = 1

(D) f is differentiable at x = 1 but not at x = 0

Q.11 The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to ***(2008)***

(A) $\frac{5}{3}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

JEE Advanced/Boards

Exercise 1

Methods of Differentiation

Q.1 Let y = x sin kx. Find the possible value of k for which the differential equation $\frac{d^2y}{dx^2}+y = 2k\cos kx$ holds true for all $x \in R$.

Q.2 Find a polynomial function f(x) such that f(2x) = f(x) f″(x).

Q.3 Let f and g be two real-valued differentiable function on R If f′(x) = g(x) and g′(x) = f(x) "x ∈ R and f(3) = 5, f′(3) = 4 then find the value of $\left(f^2(\pi)-g^2(\pi)\right)$.

Q.4 Find the value of the expression $y^3\frac{d^2y}{dx^2}$ on the ellipse $3x^2 + 4y^2 = 12$.

Q.5 The function f : R → R satisfies $f(x^2) . f''(x) = f'(x) . f'(x^2)$ for all real x. Given that f(1) = 1 and f‴(1) = 8, compute the value of f′(1) + f″(1).

Q.6 If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ then $(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = ky$, then find the value of 'k'.

Q.7 If the dependent variable y is changed to 'z' by the substitution y = tan z then the differential equation $\frac{d^2y}{dx^2} = 1+\frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2$ is changed to $\frac{d^2z}{dx^2} = \cos^2 z + k\left(\frac{dz}{dx}\right)^2$, then find the value of k.

Q.8 Show that the substitution $z = \ln\left(\tan\frac{x}{2}\right)$ changes the equation

$$\frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} + 4y\,\text{cosec}^2x = 0$$

to $\left(\frac{d^2y}{dz^2}\right) + 4y = 0$

Q.9 Let $f(x) = \frac{\sin x}{x}$ if x ≠ 0 and f(0) = 1. Define the function f′(x) for all x and find f″(0) if it exist.

Q.10 Show that $R = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$ can be reduced to the form

$$R^{2/3} = \frac{1}{(d^2y\ dx^2)^{2/3}} + \frac{1}{(d^2x\ dy^2)^{2/3}}$$

Q.11 Suppose f and g are two functions such that f, g: R → R, $f(x) = \ln(x+\sqrt{1+x^2})$ then find the value of $xe^{g(x)}\left(f\left(\frac{1}{x}\right)\right)' + g'(x)$ at x = 1.

Q.12 Let f(x) be a derivative function at x = 0 & $f\left(\frac{x+y}{k}\right) = \frac{f(x)+f(y)}{k}$ (k ∈ R, k ≠ 0, 2). Show that f(x) is either a zero or an odd linear function.

Q.13 If $f(x) = \begin{vmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{vmatrix}$ then

$$f'(x) = \lambda . \begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix}.$$

Find the value of λ.

Q.14 Let P(x) be a polynomial of degree 4 such that P(1) = P(3) = P(5) = P′(7) = 0. If the real number x ≠ 1, 3, 5 is such that P (x) = 0 can be expressed as x = p/q where 'p' and 'q' are relatively prime, then find (p + q).

Q.15 If f : R → R is a function such that $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$ for all x ∈ R, then prove that f(2) = f(1) – f(0).

Q.16 If $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$

then find f′(x)

Q.17 Let $f(x) = \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$. Show that f″(x) = 0 and that f(x) = f(0) + kx where k denotes the sum of all the co-factors of the elements in f(0)

Q.18 If $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13} = + \ldots$ to n terms.

Find dy/dx, expressing your answer in 2 terms.

Q.19 If Y = sX and Z = tX, where all the letter denotes the functions of x and suffixes denotes the differentiation w.r.t. then prove that $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3\begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$

Q.20 If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ &

$x = \sec^{-1}\frac{1}{2u^2-1}, u \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{\sqrt{2}},1\right)$

prove that $2\frac{dy}{dx} + 1 = 0$

Q.21 If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$,

The find $\frac{dy}{dx}$ for $x \in (-1, 1)$

Q.22 If $y = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$,

find $\frac{dy}{dx}$ if $x \in \left(0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right)$.

Q.23 Prove that the second order derivative of a single valued function parametrically represented by $x = \phi(t)$ and $y = \Psi(t)$, $\alpha < t < \beta$ where $\phi(t)$ and $\Psi(t)$ are differentiable functions and $\phi'(t) \neq 0$ is given

by $$\frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^3}$$

Q.24 (a) If y = y(x) and it follows the relation $e^{xy} + y\cos x = 2$, then find (i) y′(0) and (ii) y″(0).

(b) A twice differentiable function f(x) is defined for all real numbers and satisfies the following conditions f(0) = 2; f′(0) = –5 and f″(0) = 3

The function g(x) is defined by $g(x) = e^{ax} + f(x)\ \forall\ x \in R$, where 'a' is any constant. If g′(0) + g″(0) = 0. Find the value(s) of 'a'.

Application of derivatives

Q.1 Find the equations of the tangents drawn to the curves $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2).

Q.2 The tangent to $y = ax^2 + bx + \frac{7}{2}$ at (1,2) is parallel to the normal at the point (– 2, 2) on the curvey $= x^2 + 6x + 10$. Find the value of a and b.

Q.3 Find the point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is intersected by the curve $xy = 1 - y$.

Q.4 Find the equation of the normal to the curve $y = (1+x)^y + \sin^{-1}(\sin^2 x)$ at x = 0.

Q.5 A function is defined parametrically by the equation

$$f(t) = x = \begin{cases} 2t + t^2\sin\frac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \text{ and}$$

$$g(t) = y = \begin{cases} \frac{1}{t}\sin t^2 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Find the equation of the tangent and normal at the point for t = 0 is exist.

Q.6 A line is tangent to the curve $f(x) = \frac{41x^3}{3}$ at the point P in the first quadrant, and has a slope of 2009. This line intersects the y-axis at(0, b). Find the value of 'b'.

Q.7 Find all the tangents to the curve y = cos (x + y), $-2\pi \le x \le 2\pi$, that are parallel to the line x + 2y = 0

Q.8 There is a point (p, q) on the graph of $f(x) = x^2$ and a point (r, s) on the graph of $g(x) = \frac{-8}{x}$ where p > 0 and r > 0. If the line through (p,q) and (r, s) is also tangent to both the curves at these points respectively, then find the value of (p + r)

Q.9 (i) Use differentials to approximate the values of; (a) $\sqrt{36.6}$ and (b) $\sqrt[3]{26}$.

(ii) If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Q.10 The chord of the parabola $y=-a^2x^2+5ax-4$ touches the curve $y=\frac{1}{1-x}$ at the point x = 2 and is bisected by that point. Find 'a'.

Q.11 Tangent at a point P_1 [other than (0, 0)] on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3& so on. Show that the abscissae of P_1, P_2, P_3 .... P_n, form a GP. Also find the ratio $\frac{\text{area}(P_1P_2P_3)}{\text{area}(P_2P_3P_4)}$.

Q.12 Determine a differentiable function y = f(x) which satisfies $f'(x) = [f(x)]^2$ and $f(0) = -\frac{1}{2}$. Find also the equation of the tangent at the point where the curve crosses the y-axis.

Q.13 The curve $y = ax^3 + bx^2 + cx + 5$, touches the x-axis at P(–2, 0) & cuts the y-axis at a point Q where its gradient is 3. Find a, b, c

Q.14 Find the gradient of the line passing through the point (2, 8) and touching the curve $y = x^2$.

Q.15 Let $f:\{0,\infty)\to R$ be a continuous, strictly increasing function such that$f^3(x) = \int_0^x tf^2(t)dt$. If a normal is drawn to the curve y=f(x) with gradient $\frac{-1}{2}$, then find the intercept made by it on the y-axis.

Q.16 The graph of a certain function f contains the point (0, 2) and has the property that for each number'p the line tangent to y = f(x) at (p, f(p)) intersect the x-axis at p + 2. Find f(x)

Q.17 (a) Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant(b) Show that in the curve $y = a.\ln(x^2 - a^2)$sum of the length of tangent & subtangent varies as the product of the coordinates of the point of contact

(c) If the two curve $C_1 : x = y^2$ and $C_2 : xy = k$ cut at right angles find the value of k.

Q.18 Let the function $f : [-4, 4] \to [-1, 1]$ be defined implicitly by the equation $x=5y-y^5=0$.

Find the area of triangle formed by tangent and normal to f(x) at x = 0 and the line y = 5.

Q.19 The normal at the point $P\left(2,\frac{1}{2}\right)$ on the curve xy = 1, meets the curve again at Q. If m is the slope of the curve at Q, then find | m |.

Q.20 Let C be the curve $f(x) = \ln^2x + 2\ln x$ and A(a, f(a), B(b, (f(b)) where (a < b) are the points of tangency of two tangents drawn from origin to the curve C.

(i) Find the value of the product ab.

(ii) Find the number of values of x satisfying the equation $5x f'(x) - x \ln 10 - 10 = 0$.

Q.21 A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate

Q.22 A man 1.5 m tall walks away from a lamp post 4.5 m high at the rate of 4 km/hr.

(i) How fast is the father end of the shadow moving on the pavement?

(ii) How fast is his shadow lengthening?

Q.23 A water tank has the shape of a right circular cone with its vertex down. Its altitude is 10 cm and the radius of the base is 15 cm. Water leaks out of the bottom at a constant rate of 1 cu. cm/sec. Water is poured into the tank at a constant rate of C cu. cm/sec. Compute C so that the water level will be rising at the rate of 4 cm/sec at the instant when the water is 2 cm deep.

Q.24 Water is dripping out from a conical funnel of semi vertical angle $\frac{\pi}{4}$, at the uniform rate of 2 cm^3/sec through a tiny hole at the vertex at the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

Q.25 Sand is pouring from a pipe at the rate of 12 cc/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Q.26 A circular ink blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\frac{6}{11}$ seconds. Use $\pi = \frac{22}{7}$

Q.27 A variable ΔABC in the xy plane has its orthocentre at vertex 'B', a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when $t = \frac{7}{2}$ sec.

Q.28 At time t > 0, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At t = 0, the radius of the sphere is 1 unit and at t = 15 the radius is 2 units.

(a) Find the radius of the sphere as a function of time t

(b) At what time t will the volume of the sphere be 27 times its volume at t = 0

Q.29 Water is flowing out at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius R = 13 m. The volume of water in the hemispherical bowl is given by $V = \frac{\pi}{3} \cdot y^2 (3R - y)$ when the water is y meter deep . Find

(a) At what rate is the water level changing when the water is 8 m deep?

(b) At what rate is the radius of the water surface changing when the water is 8 m deep?

Exercise 2

Methods of Differentiation

Single Correct Choice Type

Q.1 If $y = \frac{x}{a+}\frac{x}{b+}\frac{x}{a+}\frac{x}{b+}\frac{x}{a+}\frac{x}{b+}....\infty$, then $\frac{dy}{dx}$

(A) $\frac{a}{ab+2ay}$ (B) $\frac{b}{ab+2by}$

(C) $\frac{a}{ab+2by}$ (D) $\frac{b}{ab+2ay}$

Q.2 The function $f(x) = e^x + x$, being differentiable and one to one to one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at the point f(ln2) is

(A) $\frac{1}{\ln 2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) None

Q.3 $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all real x and $f(5) = 2 = f'(5)$ then $f^2(10) + g^2(10)$ is-

(A) 2 (B) 4 (C) 8 (D) None

Q.4 Differential coefficient of

$$\left(x^{\frac{\ell+m}{m-n}}\right)^{\frac{1}{n-\ell}} \cdot \left(x^{\frac{m+n}{n-\ell}}\right)^{\frac{1}{\ell-m}} \cdot \left(x^{\frac{n+\ell}{\ell-m}}\right)^{\frac{1}{m-n}}$$ w.r.t. is

(A) 1 (B) 0 (C) –1 (D) x^{lmn}

Q.5 Let $f(x)=(x^x)^x$ and $g(x)=x^{(x^x)}$ then:

(A) $f'(1) = 1$ and $g'(1) = 2$ (B) $f'(1) = 2$ and $g'(1) = 1$

(C) $f'(1) = 1$ and $g'(1) = 0$ (D) $f'(1) = 1$ and $g'(1) = 1$

Q.6 If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, then the value of $\frac{(x^2-1)y''+xy'}{y}$ is equal to value equal to

(A) 4 m^2 (B) 2 m^2 (C) m^2 (D) – m^2

Q.7 If $y^2 = P(x)$, is a polynomial of degree 3, then $2\left(\frac{d}{dx}\right)\left(y^3.\frac{d^2y}{dx^2}\right)$ equals:

(A) $p'''(x) + P'(x)$ (B) $P''(x).P'''(x)$

(C) $P(x).P'''(x)$ (D) a constant

Q.8 Given $f(x) = \frac{-x^3}{3} + x^2 \sin 1.5a - x \sin a . \sin 2a - 5$ are $\sin(a^2 - 8a + 17)$ then :

(A) f(x) is not defined at x = sin 8

(B) f ' (sin 8) > 0

(C) f ' (x) is not defined at x = sin 8

(D) f ' (sin 8) < 0

Q.9 If $y = \frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$, then $\frac{dy}{dx} =$

(A) 2 ain x + cos x (B) – 2 sin x

(C) cos 2x (D) sin 2x

Q.10 A curve is parametrically represented by y = R(1 – cos θ) & x = R(θ – sinθ), then $\frac{d^2y}{dx^2}$ at θ = π is -

(A) $-\frac{1}{2R}$ (B) $\frac{1}{4R}$ (C) $\frac{1}{2R}$ (D) $-\frac{1}{4R}$

Q.11 If f(x) = $(1 + x)^n$ then the value of f(0) + f ′ (0) + ... + $\frac{f^n(0)}{n!}$ is –

(A) n (B) 2^n (C) 2^{n-1} (D) None

Q.12 If the function y = $e^{4x} + 2e^{-x}$ is a solution of the differential equation $\frac{\frac{d^3y}{dx^3}-13\frac{dy}{dx}}{y}$ = K. then the value of K

(A) 4 (B) 6 (C) 9 (D) 12

Q.13 $x^4 + 3x^2y^2 + 7xy^3 + 4x^3y - 15y^4 = 0$, then $\frac{d^2y}{dx^2}$ at (1, 1) is -

(A) 2 (B) 1 (C) 7 (D) 0

Q.14 If f(x) = e^{e^x}. Let g(x) be it's inverse then g′ (x) at x = 2 is -

(A) $\frac{\ell n2}{2}$ (B) $\frac{1}{2\ell n2}$ (C) $2\ell n2$ (D) e^2

Q.15 $y=\tan^{-1}\left(\frac{1-2\ell n|x|}{1+2\ell n|x|}\right)+\tan^{-1}\left(\frac{3+2\ell n|x|}{1-6\ell n|x|}\right)$, then $\frac{d^2y}{dx^2}$ equals

(A) 2 (B) 1 (C) 0 (D) –1

Q.16 $\lim_{x\to 0^+}(x^{x^x}-x^x)$ equals -

(A) 1 (B) –1 (C) 0 (D) None of these

Q.17 $\lim_{x\to 0}$ {$(\cot x)^x + (1 - \cos x)^{\text{cosecx}}$} is equal to -

(A) 2 (B) +1

(C) 0 (D) None of these

Multiple Correct Choice Type

Q.18 Let f(x) = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, then

(A) f(x) = 112 $\tan^{-1}x$ Δ x∈R – {0}

(B) f ′ (x) = $\frac{1}{2(1+x^2)}$ ζ x∈R – {0}

(C) f(x) is an odd function

(D) f(x) + f(– x) = p

Q.19 If y = tan x tan 2x tan 3x then $\frac{dy}{dx}$ has the value to:

(A) 3 $\sec^2$ 3x tan x tan 2x + $\sec^2$ x tan 2x tan 3x + 2 $\sec^2$ 2x tan 3x tan x

(B) 2y (cosec 2x + 2 cosec 4x + 3 cosec 6x)

(C) 3 $\sec^2$ 3x – 2 $\sec^2$2x – $\sec^2$ x

(D) $\sec^2$ x + 2 $\sec^2$ 2x + 3 $\sec^2$ 3x

Q.20 Let y = √x + Öx + $\sqrt{x+......\infty}$ then $\frac{dy}{dx}$

(A) $\frac{1}{2y-1}$ (B) $\frac{x}{x-2y}$ (C) $\frac{1}{\sqrt{1+4x}}$ (D) $\frac{y}{2x+y}$

Q.21 If $2^x + 2^y = 2^{x+y}$ then $\frac{dy}{dx}$ has the value equal to

(A) $-\frac{2^y}{2^x}$ (B) $\frac{1}{1-2^x}$ (C) $1-2^y$ (D) $\frac{2^x(1-2^y)}{2^y(2^x-1)}$

Q.22 If $\sqrt{y+x}+\sqrt{y-x}$ = c, then $\frac{dy}{dx}$ is equal to

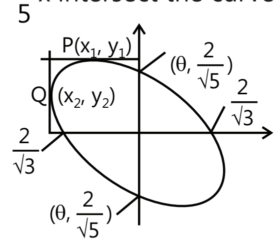
(A) $\frac{2x}{c^2}$ (B) $\frac{x}{y+\sqrt{y^2-x^2}}$

(C) $\frac{y-\sqrt{y^2-x^2}}{x}$ (D) $\frac{c^2}{2y}$

Application of Derivatives

Single Correct Choice Type

Q.1 The line y = $-\frac{3}{2}$x and y = $-\frac{2}{5}$x intersect the curve $3x^2 + 4xy + 5y^2 - 4 = 0$ at the point P and O respectively. The tangent drawn to the curve at P and Q



(A) Intersect each other at angle of 45°

(B) Are parallel to each other

(C) Are perpendicular to each other

(D) None of these

Q.2 A curve is represented by the equations $x = \sec^2 t$ and $y = \cot t$ where t is a parameter. If the tangent at the point P on the curve where $t = \pi/4$ meets the curve again at the point Q then | PQ | is equal to

(A) $\frac{5\sqrt{3}}{2}$ (B) $\frac{5\sqrt{5}}{2}$

(C) $\frac{2\sqrt{5}}{3}$ (D) $\frac{3\sqrt{5}}{2}$

Q.3 Let $f(x) = \begin{cases} x^{35} & \text{if } x \le 1 \\ -(x-2)^3 & \text{if } x > 1 \end{cases}$ then the number of critical points on the graph of the function is

(A) 1 (B) 2 (C) 3 (D) 4

Q.4 At any two points of the curve represented parametrically by $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$ the tangents are parallel to the axis of x corresponding to the values of the parameter t differeing from each other by

(A) $2\pi/3$ (B) $3\pi/4$ (C) $\pi/2$ (D) $\pi/3$

Q.5 At the point P (a, a") on the graph of $y = x^n (n \in N)$ in the first quadrant a normal is drawn. The normal intersects the y-axis at the point (0, b). If $\lim_{a\to 0} b = \frac{1}{2}$, then n equal

(A) 1 (B) 3 (C) 2 (D) 4

Q.6 Let $f(x) = \begin{cases} -x^2 & \text{for } x > 0 \\ x^2 + 8 & \text{for } x \ge 0 \end{cases}$. Then x intercept of the line that is tangent to the graph of f(x) is

(A) zero (B) – 1 (C) – 2 (D) – 4

Q.7 The ordinate of all points on the curve

$y = \frac{1}{2\sin^2 x + 3\cos^2 x}$ where the tangent is horizontal, is -

(A) Always equal to 1/2

(B) Always equal to 1/3

(C) 1/2 or 1/3 according as n is an even or an odd integer

(D) 1/2 or 1/3 according as n is an odd or an odd integer

Q.8 The equation of the tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ ($n \in N$) at the point with abscissa equal to 'a' can be

(A) $\left(\frac{x}{a}\right) + \left(\frac{y}{b}\right) = 2$ (B) $\left(\frac{x}{a}\right) - \left(\frac{y}{b}\right) = 2$

(C) $\left(\frac{x}{a}\right) - \left(\frac{y}{b}\right) = 0$ (D) $\left(\frac{x}{a}\right) + \left(\frac{y}{b}\right) = 0$

Multiple Correct Choice Type

Q.9 If $\frac{x}{a} + \frac{x}{b} = 1$ is a tangent to the curve $x = Kt$, $y = \frac{1}{t}$, $K > 0$ then

(A) a > 0, b > 0 (B) a > 0, b < 0

(C) a < 0, b > 0 (D) a < 0, b < 0

Q.10 The abscissa of the point on the curve $\frac{dy}{dx} = a + x$, the tangent at which cuts off equal intersects from the co-ordinate axes is (a > 0)

(A) $\frac{a}{\sqrt{2}}$ (B) $-\frac{a}{\sqrt{2}}$ (C) $a\sqrt{2}$ (D) $-a\sqrt{2}$

Q.11 The parabola $y = x^2 + px + q$ cuts the straight line $y = 2x - 3$ at a point with adscissa 1. If the distance between the vertex of the parabola and the x-axis is least then

(A) p = 0 & q = – 2

(B) p = – 2 & q = 0

(C) least distance between the parabola and x–axis is 2

(D) least distance between the parabola and x–axis is 1

Q.12 The co-ordinates of the point(s) on the graph ...... function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cut off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign, is

(A) (2, 8/3) (B) (3, 7/2)

(C) (1, 5/6) (D) None of these

Q.13 Equation of a tangent to the curve $y \cot x = y^3 \tan x$ at the point where the abscissa is $\frac{\pi}{4}$ is

(A) $4x + 2y = \pi + 2$ (B) $4x - 2y = \pi + 2$

(C) x = 0 (D) y = 0

Q.14 The angle made by the tangent of the curve x = a(t + sin t cos t) ;y = a(1 + sin t)^2 with the x-axis at any point on it is -

(A) $\frac{1}{4}(\pi + 2t)$ (B) $\frac{1-\sin t}{\cos t}$ (C) $\frac{1}{4}(2t - \pi)$ (D) $\frac{1+\sin t}{\cos 2t}$

Q.15 Consider the curve represented parametrically by the equation $x = t^3 - 4t^2 - 3t$ and $y = 2t^2 + 3t - 5$ where $t \in R$

If H denotes the number of point on the curve where the tangent is horizontal and V the number of point where the tangent is vertical then

(A) H = 2 and V = 1 (B) H = 1 and V = 2

(C) H = 2 and V = 2 (D) H = 1 and V = 1

Previous Years' Questions

Q.1 If the line ax + bx + c = 0 is a normal to the curve xy = 1, then ***(1986)***

(A) a > 0 , b > 0 (B) a > 0, b < 0

(C) a < 0, b > 0 (D) a < 0, b < 0

Q.2 On the ellipse $4x^2 + 9y^2 = 1$, the point at which the tangents are parallel to the line 8x = 9y, are ***(1999)***

(A) $\left(\frac{2}{5},\frac{1}{5}\right)$ (B) $\left(-\frac{2}{5},\frac{1}{5}\right)$ (C) $\left(-\frac{2}{5},-\frac{1}{5}\right)$ (D) $\left(\frac{2}{5},-\frac{1}{5}\right)$

Q.3 Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of points on the curve C where the tangent is vertical, then H = ... and V = ...***[1997]***

Q.4 A swimmer S is in the sea at a distance d km from the closest point A on a straight shore. The house of the swimmer is on the shore at a distance L km from A. He can swim at a speed of u km/h and walk at a speed of v km/h (v > u). At what point on the shore should be land so that he reaches his house in the shortest possible time? ***(1983)***

Q.5 Find the coordinates of the point on the curve $y = \frac{x}{1+x^2}$, where the tangent to the curve has the greatest slope. ***(1997)***

Q.6 Find all the tangents to the curve y = cos (x + y), $-2\pi \le x\ 2\pi$, that are parallel to the line x + 2y = 0 ***(1997)***

Q.7 Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point (0, –2). ***(1987)***

Q.8 Three normals are drawn from the point (c, 0) to the curve $y^2 = x$. Show that c must be greater than $\frac{1}{2}$. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other ***(1991)***

Q.9 What normal to the curve $y = x^2$ forms the shortest chord? ***(1992)***

Q.10 Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at x = 0 ***(1993)***

Q.11 Tangent at a point P_1\{other than (0, 0)\} on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissae of $P_1, P_2, P_3, \ldots < P_n$, form a GP. Also find the ratio. [Area $(dP_1P_2P_3)$] / [area $(dP_2P_3P_4)$] ***(1993)***

Q.12 A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted an open rectangular box by folding after removing square of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are ***(2013)***

(A) 24 (B) 32 (C) 45 (D) 60

Q.13 Match List I with List II and select the correct answer using the code given below the lists : ***(2013)***

	List - I		List – II
P	$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1} y) + y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}\right) + y^4\right)^{1/2}$	1.	$\frac{1}{2}\sqrt{\frac{5}{3}}$
Q	If cos x + cos y + cos z = 0 = sin x + sin y + sin z then possible value of cos $\frac{x-y}{2}$ is	2.	$\sqrt{2}$

R	If $\cos\left(\frac{\pi}{4}-x\right)\cos 2x+\sin x\sin 2x\sec x = \cos x\sin 2x\sec x+\cos\left(\frac{\pi}{4}+x\right)\cos 2x \cos 2x$ then possible value of sec x is	3.	$\frac{1}{2}$
S	If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right)=\sin\left(\tan^{-1}(x\sqrt{6})\right)$, $x \neq 0$, then possible value of x is	4.	1

(A) $P \to 4; Q \to 3; R \to 1; S \to 2$

(B) $P \to 4; Q \to 3; R \to 2; S \to 1$

(C) $P \to 3; Q \to 4; R \to 2; S \to 1$

(D) $P \to 3; Q \to 4; R \to 1; S \to 2$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Methods of Differentiation

Q.9 Q.12 Q.15 Q.23

Application of Derivatives

Q.7 Q.11 Q.15 Q.21

Exercise 2

Methods of Differentiation

Q.2 Q.11 Q.12 Q.15
Q.17 Q.20

Application of Derivatives

Q.5 Q.7 Q.8 Q.10

Previous Years' Questions

Q.2 Q.5 Q.7 Q.9

JEE Advanced/Boards

Exercise 1

Methods of Differentiation

Q.5 Q.8 Q.11 Q.12
Q.14 Q.18 Q.23

Application of Derivatives

Q.8 Q.11 Q.15 Q.17
Q.20 Q.23 Q.24 Q.27

Exercise 2

Methods of Differentiation

Q.1 Q.3 Q.5 Q.9
Q.11 Q.15 Q.20

Application of Derivatives

Q.2 Q.4 Q.7 Q.11

Previous Years' Questions

Q.4 Q.9 Q.11

Answer Key

JEE Main/Boards

Exercise 1

Methods of Differentiation

Q.1 $\frac{1}{2\sqrt{x+3}}e^{\sqrt{x+3}}$

Q.2 $\frac{1}{x}$ cos (log x)

Q.3 – $\sec^3\theta$

Q.4 $\frac{-1}{2\sqrt{x}\sqrt{1-x}}$

Q.5 $\frac{me^{m\tan^{-1}x}}{1+x^2}$

Q.6 $\frac{3x^2\cos\{\log(x^3-1)\}}{x^3-1}$

Q.7 – $e^{-x}\sin x$

Q.8 cot x . $\log_2 e$

Q.9 $5^{\ln \sin x}$ (cot x) (ln 5)

Q.10 $-\frac{3}{2}$

Q.11 $\frac{-1}{1+x^2}$

Q.12 $\frac{\sec^2 x}{(1-\tan x)^2}\sqrt{\frac{1-\tan x}{1+\tan x}}$

Q.13 $\frac{-\cos\sqrt{\cos\sqrt{x}}\,\sin\sqrt{x}}{4\sqrt{x}\sqrt{\cos\sqrt{x}}}$

Q.24 $\frac{1}{2\sqrt{x}}\left(\frac{1}{1+x^2}\right)$

Q.15 $(\sin x)^{\cos^{-1}x}\left[\cos^{-1}x.\cot x-\frac{\log(\sin x)}{\sqrt{1+x^2}}\right]$

Q.16 1

Q.17 $\frac{2}{1+x^2}$

Q.18 $\frac{dy}{dx}=-\frac{1}{2\sqrt{1-x^2}}$

Q.19 $\frac{-2}{\sqrt{1-a^2x^2}}$

Q.20 $\frac{1}{4}\times\frac{1}{\sqrt{a+\sqrt{a+x}}}\times\frac{1}{\sqrt{a+x}}$

Q.21 $\frac{1}{1+x^2}$

Q.22 $\frac{1}{\sqrt{a^2+x^2}}$

Q.23 $\frac{x\cot\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}$

Application of Derivatives

Q.1 (2, 1)

Q.2 90°

Q.3 – $\frac{1}{3}$

Q.4 $\frac{dy}{dx}$ is not defined

Q.5 $\left(\frac{3}{2},\frac{-17}{2}\right)$

Q.6 (2 – 36)

Q.7 x + 12y – 21 = 0; 12x – y + 38 = 0

Q.8 $\sqrt{2}$ bx + $\sqrt{2}$ ay – ab

Q.9 M_1M_2= –1

Q.10 8x – 2y + 1 = 0

Q.11 4x – 4y + 33 = 0

Q.12 $k^2x + y - 2ck = 0$

Q.13 3

Q.14 $\left(\frac{1}{2},\frac{1}{4}\right)$

Q.15 (±3, 0)

Q.161/818.x + 9y – 55 = 0 ; x + 9y – 35 = 0

Q.17 -1

Q.18 55

Q.19 20 x + y – 140 = 0

Q.20 $\frac{x}{\sqrt{x_1}}+\frac{x}{\sqrt{y_1}}=\sqrt{a}$

Q.21 x + 3y = 8; x + 3y = – 8

Q.22 $2x + 2y = a^2$

Q.23 (2, –4); $\left(\frac{-2}{3},\frac{4}{27}\right)$

Q.24 $\tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}$

Exercise 2

Methods of Differentiation

Single Correct Choice Type

Q.1 B	**Q.2** B	**Q.3** D	**Q.4** D	**Q.5** A	**Q.6** B
Q.7 C	**Q.8** A	**Q.9** A	**Q.10** B	**Q.11** C	**Q.12** D
Q.13 D	**Q.14** D	**Q.15** B	**Q.16** B	**Q.17** C	**Q.18** A
Q.19 B	**Q.20** D	**Q.21** B			

Application of Derivatives

Single Correct Choice Type

Q.1 B	**Q.2** D	**Q.3** C	**Q.4** D	**Q.5** D	**Q.6** A
Q.7 C	**Q.8** A	**Q.9** D	**Q.10** B	**Q.11** C	**Q.12** D
Q.13 D	**Q.14** D	**Q.15** C			

Previous Years' Questions

Q.1 C	**Q.2** A	**Q.3** D	**Q.4** D	**Q.5** D	**Q.6** A
Q.7 C	**Q.8** D	**Q.9** A	**Q.10** A	**Q.11** D	

JEE Advanced/Boards

Exercise 1

Methods of Differentiation

Q.1 k =1, –1 or 0

Q.2 $\frac{4x^3}{9}$

Q.3 9

Q.4 $\frac{-9}{4}$

Q.5 6

Q.6 25

Q.7 k = 2

Q.9 $f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$; $f''(0) = -\frac{1}{3}$

Q.11 Zero

Q.13 3

Q.14 100

Q.15 6

Q.16 $2(1 + 2x) . \cos 2(x + x^2)$

Q.17 $f(0) + kx$

Q.18 $\frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$

Q.19 $= X[S_1t_2X^2 - S_2t_1X^2] + X_3\begin{vmatrix} S_1 & t_1 \\ S_2 & t_2 \end{vmatrix}$

Q.20 0

Q.21 $\frac{1-2x}{2\sqrt{1-x^2}}$

Q.22 $\frac{1}{2}$ or $-\frac{1}{2}$

Q.23 L.H.S = R.H.S

Q.24 (a) (i) $y'(0) = -1$; (ii) $y''(0) = 2$; (b) a = 1, – 2

Application of Derivatives

Q.1 $2\sqrt{3}\,x - y = 2(\sqrt{3}-1)$ or $2\sqrt{3}\,x + y = 2(\sqrt{3}+1)$

Q.2 $a = 1,\ b = \frac{-5}{2}$

Q.3 (0, 1)

Q.4 $x = y - 1 = 0$

Q.5 T : $x - 2y = 0$; N : $2x + y = 0$

Q.6 $-\frac{82.7^3}{3}$ **Q.7** x + 2y = π/2 & x + 2y = –3π/2 **Q.8** 5

Q.9 (i) (a) 6.05, (b) $\frac{80}{27}$; (ii) 9.72 πcm^2 **Q10** a = 1 **Q.11** 1/16

Q.12– $\frac{1}{x+2}$; x – 4y = 2 **Q.13** a = –1/2; b = – ¾; c = 3 **Q.14** 3, 12

Q.15 9 **Q.16** $2e^{\frac{-x}{2}}$ **Q.17** (a) n = –2, (c) $\pm\frac{1}{2\sqrt{2}}$

Q.18 65 **Q.19** 64 **Q.20** (i) 1, (ii) 2

Q.21 (4, 11) & (– 4, – 31/3) **Q.22** (i) 6 km/h; (ii) 2 km/hr **Q.23** 1 + 36 π cu. cm / sec

Q.24 $\frac{\sqrt{2}}{4\pi}$ cm/sec **Q.25** 1/48 π cm/s **Q.26** $\frac{1}{4}$ cm/sec

Q.27 $\frac{66}{7}$ **Q.28** (a) $r = (1 + t)^{1/4}$, (b) t = 80

Q.29 (a) $-\frac{1}{24\pi}$ m/min, (b) $-\frac{5}{288\pi}$ m/min

Exercise 2

Methods of Differentiation

Single Correct Choice Type

Q.1 D **Q.2** B **Q.3** C **Q.4** B **Q.5** D **Q.6** C

Q.7 C **Q.8** D **Q.9** B **Q.10** D **Q.11** B **Q.12** D

Q.13 C **Q.14** C **Q.15** C **Q.16** B **Q.17** B

Multiple Correct Choice Type

Q.18 B, C **Q.19** A, B, C **Q.20** ACD **Q.21** A, B, C, D **Q.22** A, B, C

Application of Derivatives

Single Correct Choice Type

Q.1 C **Q.2** D **Q.3** C **Q.4** A **Q.5** C **Q.6** B

Q.7 D **Q.8** A

Multiple Correct Choice Type

Q.9 A, D **Q.10** A, B **Q.11** B, D **Q.12** A, B **Q.13** A, B, D **Q.14** A, B

Q.15 B, D

Previous Years' Questions

Q.1 B, C **Q.2** B, D **Q.3** H = ∅, V = {1.1} **Q.4** $\frac{ud}{\sqrt{v^2-u^2}}$ **Q.5** (0, 0)

Q.6 $x + 2y = \frac{\pi}{2}$ and $x + 2y = -\frac{3\pi}{2}$ **Q.7** (0, 2) **Q.8** 3/4 **Q.9** $\sqrt{2}x - 2y + 2 = 0$

Q.10 dy/dx = 1 **Q.11** 1/16 **Q.12** A C **Q.13** B

Solutions

JEE Main/Boards

Exercise 1

Methods of Differentiation

Sol 1: $y = e^{\sqrt{x+3}}$

$$\frac{dy}{dx} = \frac{de^{\sqrt{x+3}}}{\sqrt{x+3}}.\frac{d\sqrt{x+3}}{dx}.\frac{d(x+3)}{dx} \quad \text{[Chain Rule]}$$

$$= e^{\sqrt{x+3}}.\frac{1}{2\sqrt{x+3}}.1 = \frac{1}{2\sqrt{x+3}}e^{\sqrt{x+3}}$$

Sol 2: y = sin(logx)

$$\frac{dy}{dx} = \frac{d\sin(\log x)}{d(\log x)}\frac{d\log x}{dx} \quad \text{[Chain rule]}$$

$$= \cos(\log x)\left(\frac{1}{x}\right) = \frac{1}{x}\cos(\log x)$$

Sol 3: y = –tanθ, x = sinθ

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d(-\tan\theta)}{d\theta}}{\frac{d(\sin\theta)}{d\theta}} = \frac{-\sec^2\theta}{\cos\theta} = -\sec^3\theta$$

Sol 4: $y = \cos^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{d\cos^{-1}\sqrt{x}}{d\sqrt{x}}\frac{d\sqrt{x}}{dx} = \frac{-1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}} = \frac{-1}{2\sqrt{x-x^2}}$$

Sol 5: $y = e^{m\tan^{-1}x}$

$$\frac{dy}{dx} = \frac{de^{m\tan^{-1}x}}{d(m\tan^{-1}x)}.\frac{d(m\tan^{-1}x)}{dx}$$

$$= e^{m\tan^{-1}x}.\left(m.\frac{1}{1+x^2}\right) = \frac{me^{m\tan^{-1}x}}{1+x^2}$$

Sol 6: $y = \sin\{\log(x^3-1)\}$

$$\frac{dy}{dx} = \frac{d\sin\{\log(x^3-1)\}}{d\{\log(x^3-1)\}} \times \frac{d\{\log(x^3-1)\}}{d\{x^3-1\}} \times \frac{d(x^3-1)}{dx}$$

$$= \cos\{\log(x^3-1)\}.\frac{1}{x^3-1}.3x^2 = \frac{3x^2\cos\{\log(x^3-1)\}}{x^3-1}$$

Sol 7: $\frac{d\cos x}{de^x} = \frac{d\cos x}{dx}.\frac{1}{\frac{de^x}{dx}} = \frac{-\sin x}{e^x} = -(\sin x)e^{-x}$

Sol 8: $y = \log_2(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\log 2}\frac{d\log(\sin x)}{d(\sin x)}.\frac{d(\sin x)}{dx} = \frac{\cos x}{\sin x}\log_2 e = (\cot x)\log_2^e$$

Sol 9: $y = 5^{\log(\sin x)}$

logy = (log5) log(sinx)

$$\frac{1}{y}\frac{dy}{dx} = (\log 5)\frac{d.\log(\sin x)}{d(\sin x)}.\frac{d(\sin x)}{dx} = (\log 5)\frac{\cos x}{\sin x}$$

$$\therefore \frac{dy}{dx} = 5^{\log(\sin x)}\big((\cot x)\log 5\big)$$

Sol 10: $\sqrt{x}+\sqrt{y} = 5$

Differentiate w.r.t x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

For (4, 9)

$$\frac{dy}{dx} = -\sqrt{\frac{9}{4}} = -\frac{3}{2}$$

Sol 11: $y = \cot^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

Take x = tanθ

$$\therefore y = \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right) = \frac{\pi}{4}-\theta = \frac{\pi}{4}-\tan^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{d\tan^{-1}x}{dx} = -\frac{1}{1+x^2}$$

Sol 12: $y = \sqrt{\frac{1+\tan x}{1-\tan x}} = \sqrt{\tan\left(x+\frac{\pi}{4}\right)}$

$$\therefore \frac{dy}{dx} = \frac{d\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}{d\tan\left(x+\frac{\pi}{4}\right)}.\frac{d\tan\left(x+\frac{\pi}{4}\right)}{dx}$$

$$= \frac{1}{2\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}.\sec^2\left(x+\frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{1-\tan x}{1+\tan x}}\right)\left(1+\tan^2\left(x+\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{1-\tan x}{1+\tan x}}\right)\left(\frac{1+\tan^2 x}{(1-\tan x)^2}\right)^2$$

$$= \frac{\sec^2 x}{(1-\tan x)^2}\sqrt{\frac{1-\tan x}{1+\tan x}}$$

Sol 13: $\frac{-\cos\sqrt{\cos\sqrt{x}}\sin\sqrt{x}}{4\sqrt{x}\sqrt{\cos\sqrt{x}}}$

Sol 14: $\frac{2}{x(\log x^2)\log(\log x^2)}$

Sol 15: $y = (\sin x)^{\cos^{-1}x}$

Taking log both sides

Log y = $\cos^{-1}x\log(\sin x)$

Differentiate both side w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{d\cos^{-1}x\log(\sin x)}{dx}$$

$$= \cos^{-1}x\frac{d\log(\sin x)}{dx} + \log(\sin x)\frac{d\cos^{-1}x}{dx}$$

$$= \cot x\cos^{-1}x - \frac{\log(\sin x)}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = (\sin x)^{\cos^{-1}x}\left[\cot x\cos^{-1}x - \frac{\log(\sin x)}{\sqrt{1-x^2}}\right]$$

Sol 16: $y = \cos^{-1}\left[(2\cos x + 3\sin x)\sqrt{13}\right]$

$$= \cos^{-1}\left[\frac{2}{\sqrt{13}}\cos x + \frac{3}{\sqrt{13}}\sin x\right]$$

where $\frac{2}{\sqrt{13}} = \cos\theta$ and $\frac{3}{\sqrt{13}} = \sin\theta$

$$= \cos^{-1}\left[\cos\theta\cos x + \sin\theta\sin x\right] = \cos^{-1}\left[\cos(x-\theta)\right]$$

$\because \cos^{-1}(\cos x) = x \quad \therefore y = x-\theta$

$\Rightarrow \frac{dy}{dx} = 1$

Sol 17: $y = \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Take $x = \tan\theta$

$\therefore y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$

$\Rightarrow y = 2\theta \quad \therefore y = 2\tan^{-1}x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$

Sol 18: $y = \sin^{-1}\left[\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right]$

Put $x = \cos 2\theta$

$\therefore y = \sin^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{2}\right]$

$= \sin^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{2}\right]$

$= \sin^{-1}\left[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right]$

$= \sin^{-1}\left[\sin\left(\theta+\frac{\pi}{4}\right)\right] = \theta + \frac{\pi}{4} \qquad \therefore y = \frac{1}{2}\cos^{-1}x + \frac{\pi}{4}$

$\frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}}$

Sol 19: $y = \sin^{-1}\left[2ax\sqrt{1-a^2x^2}\right]$

Put $x = \frac{1}{a}\cos\theta \quad \therefore y = \sin^{-1}\left[2\cos\theta\sqrt{1-a^2\left(\frac{\cos^2\theta}{a^2}\right)}\right]$

$= \sin^{-1}\left[2\cos\theta\sqrt{1-\cos^2\theta}\right]$

$= \sin^{-1}[2\cos\theta\sin\theta] = \sin^{-1}[\sin 2\theta] = 2\theta = 2\cos^{-1}ax$

$\therefore \frac{dy}{dx} = -2\times\frac{1}{\sqrt{1-a^2x^2}} = \frac{-2}{\sqrt{1-a^2x^2}}$

Sol 20: $y = \sqrt{a+\sqrt{a+x}}$

$$\frac{dy}{dx} = \frac{d\sqrt{a+\sqrt{a+x}}}{d(a+\sqrt{a+x})} \times \frac{d\left(a+\sqrt{a+x}\right)}{dx}$$

$$= \frac{1}{2}\times\frac{1}{\sqrt{a+\sqrt{a+x}}}\times\left(\frac{1}{2\sqrt{a+x}}\right)\times 1$$

$$= \frac{1}{4}\times\frac{1}{\sqrt{a+\sqrt{a+x}}}\times\frac{1}{\sqrt{a+x}}$$

Sol 21: $y = \tan^{-1}\left[\frac{ax-b}{a+bx}\right] = \tan^{-1}\left[\frac{x-\frac{b}{a}}{1+\frac{b}{a}x}\right]$

Let $\tan\alpha = \frac{b}{a}$, $x = \tan t$

$\therefore\ y = \tan^{-1}\left[\frac{\tan t-\tan\alpha}{1+\tan\alpha\tan t}\right]$ or $y = \tan^{-1}\tan(t-\alpha)$

$= t-\alpha$, α is constant

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ or $y = t-\alpha = \tan^{-1}x-\alpha$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$

Sol 22: $y = \log\left(x+\sqrt{x^2+a^2}\right)$

$$\frac{dy}{dx} = \frac{d\log\left(x+\sqrt{x^2+a^2}\right)}{d\left(x+\sqrt{x^2+a^2}\right)}.\frac{d\left(x+\sqrt{x^2+a^2}\right)}{dx}$$

$$= \frac{1}{\left(x+\sqrt{x^2+a^2}\right)}.\left[1+\frac{d\left(\sqrt{x^2+a^2}\right)}{d(x^2+a^2)}.\frac{d(x^2+a^2)}{dx}\right]$$

$$= \frac{1}{\left(x+\sqrt{x^2+a^2}\right)}\left[1+\frac{1}{2\sqrt{x^2+a^2}}\times 2x\right]$$

$$= \frac{1}{\left(x+\sqrt{x^2+a^2}\right)}.\left[\frac{\sqrt{x^2+a^2}+x}{\sqrt{a^2+x^2}}\right] = \frac{1}{\sqrt{a^2+x^2}}$$

Sol 23: $y = \log\left(\sin\sqrt{1+x^2}\right)$

$$\frac{dy}{dx} = \frac{d\log\left(\sin\sqrt{1+x^2}\right)}{d\left(\sin\sqrt{1+x^2}\right)} \times \frac{d\left(\sin\sqrt{1+x^2}\right)}{d\left(\sqrt{1+x^2}\right)} \times \frac{d\sqrt{1+x^2}}{dx}$$

$$= \frac{1}{\sin\left(\sqrt{1+x^2}\right)}\left(\cos\sqrt{1+x^2}\right)\times\frac{1}{2\sqrt{1+x^2}}\times\frac{d(1+x^2)}{dx}$$

$$= \frac{x\cot\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}$$

Sol 24: $y = \tan^{-1}\left(\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{a}\sqrt{x}}\right)$

Put $\sqrt{x} = \tan t$

$\sqrt{a} = \tan\alpha$

$\therefore y = \tan^{-1}\left(\frac{\tan t+\tan\alpha}{1-\tan t\tan\alpha}\right) = \tan^{-1}\tan(t+\alpha) = t+\alpha$

$y = \tan^{-1}\sqrt{x} + \alpha$

$$\therefore\ \frac{dy}{dx} = \frac{d\left(\tan^{-1}\sqrt{x}\right)}{d\sqrt{x}}.\frac{d\sqrt{x}}{dx}+\frac{d\alpha}{dx}$$

$$= \frac{1}{1+\left(\sqrt{x}\right)^2}.\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}\left(\frac{1}{1+x^2}\right)$$

Application of Derivatives

Sol 1: $y = x^2 - 4x + 5$

Tangent is parallel to x axis $\Rightarrow \frac{dy}{dx} = 0 = 2x-4$

$\Rightarrow x = 2;\ y = 4-8+5 = 1$

Point A (2, 1)

Sol 2: If two curves cut orthogonally, then the tangents at point of intersection are perpendicular.

Sol 3: $f(x) = 3x^2 - 5$

Tangent at $\left[x = \frac{1}{2}, y = \left(\frac{3}{4}-5\right)\right]$

$$\left(\frac{dy}{dx}\right)_{x=1/2} = 6x = 3 = (m)_{tangent}$$

$$\left(-\frac{dx}{dy}\right) = -\frac{1}{3} = (m)_{normal}$$

Sol 4: $y = f(x)$, $\frac{dy}{dx} = \infty$ i.e. not defined

Sol 5: $y = 2x^2 - 6x - 4$

$$\frac{dy}{dx} = 4x - 6 = 0 \Rightarrow x = \frac{3}{2}$$

$$y = 2\times\frac{9}{4} - 6\times\frac{3}{2} - 4 \Rightarrow \frac{9}{2} - 9 - 4 = -\frac{17}{4}$$

$$\left(\frac{3}{2}, \frac{-17}{2}\right)$$

Sol 6: $y = x^2 - 4x - 32$

$$\frac{dy}{dx} = 2x - 4 = 0$$

$x = 2;\ y = 4 - 8 - 32 = -36$

Point (2, – 36)

Sol 7: $y = (5 - x)^{1/3}$

At P (–3, 2)

$y = (8)^{1/3} = 2$

$$\left(\frac{dy}{dx}\right)_{x=-3} = \frac{1}{3}(5-x)^{-2/3} = \frac{-1}{3(8)^{2/3}} = \frac{-1}{12}$$

$$\frac{y-2}{x+3} = \frac{-1}{12} \Rightarrow 12y - 24 = -(x+3) \Rightarrow x + 12y = 21$$

Equation of normal is $\frac{y-2}{x+3} = +12$

$\Rightarrow y - 12x = 38$

Sol 8: $y = (4x - 3)^{1/2}$ $\frac{dy}{dx} = \frac{4}{2\sqrt{4x-3}} = \frac{2}{\sqrt{4x-3}} \neq 0$, it can never be parallel to x axis.

Sol 9: $x^2 = y$ & $x^3 + 6y = 7$

$x^3 + 6x^2 = 7$

$(x^2 + 7x + 7)(x - 1) = 0$

$\Rightarrow x = 1$

Point of intersection (1, 1)

$P_1 (x^2 = y) P_2 (x^3 + 6y = 7)$

$$\frac{dy}{dx} = 2x\ 3x^2 + \frac{6dy}{dx} = 0$$

$$\left(\frac{dy}{dx}\right)_{x=1} = 2\ \left(\frac{dy}{dx}\right)_{x=1} = \frac{-1}{2}$$

$M_1M_2 = -1$ i.e., tangents are orthogonal at (1, 1)

Sol 10: $y^2 = 8x$

$$\Rightarrow 2y\frac{dy}{dx} = 8 \Rightarrow \frac{dy}{dx} = 4 = \frac{8}{2y}$$

This give $y = 1;\ x = \frac{1}{8}$

$$\Rightarrow \frac{y-1}{x-\frac{1}{8}} = 4 \Rightarrow \frac{y-1}{8x-1} = \frac{1}{2} \Rightarrow 2y - 2 = 8x - 1$$

$\Rightarrow 8x - 2y + 1 = 0$

Sol 11: $y = -5x^2 + 6x + 7$

$$y' = -10x + 6 \Rightarrow (y')_{x=\frac{1}{2}} = -5 + 6 = 1$$

$$y - \frac{35}{4} = x - \frac{1}{2}$$

$4y - 35 = 4x - 2 \Rightarrow 4x - 4y + 33 = 0$

Sol 12: $y = \frac{c^2}{x}$

$$\frac{dy}{dx} = \frac{-c^2}{x^2} \Rightarrow \left(\frac{dy}{dx}\right)_{\frac{c}{k}} = -k^2$$

Equation: – y – ck

$$= -k^2\left(x - \frac{c}{k}\right) \Rightarrow y + k^2x = 2x$$

Sol 13: $y = x^3 + 6 \Rightarrow \frac{dy}{dx} = 3x^2$

$$(y')_{x=-1y=5} = 3;\ (y')_{x=-1y=7} = 3$$

So the tangents are parallel

Sol 14: $y = x^2 \Rightarrow \left(\frac{dy}{dx}\right) = 2x = 1$

$$x = \frac{1}{2};\quad y = \frac{1}{4};\quad P\left(\frac{1}{2}, \frac{1}{4}\right)$$

Sol 15: $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$$\frac{2x}{9} + \frac{2y}{4}y' = 0 \Rightarrow y' = -\frac{4x}{9y}$$

This will be parallel to y axis if y = 0

$x = \pm 3$

P (+ 3, 0), (– 3, 0)

Sol 16: $x = \frac{1}{t}$; $y = 2t \Rightarrow xy = 2$

$y = \frac{2}{x} \Rightarrow y' = \frac{-2}{x^2} = -8$

At t = 2 i.e. $x = \frac{1}{2}$

$y' = -8$

Slope of normal = $\frac{1}{8}$

Sol 17: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow y' = -\frac{b^2x}{a^2y}$

At (x_0, y_0); $y' = -\frac{b^2x_0}{a^2y_0}$

$\frac{y-y_0}{x-x_0} = -\frac{b^2x_0}{a^2y_0}$

$a^2yy_0 - a^2y_0^2 = b^2x_0^2 - b^2xy_0$

$xx_0b^2 + yy_0a^2 = a^2y_0^2 + b^2x_0^2 = a^2b^2$

$\Rightarrow \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$

Sol 18: $y = 4x^3 - 3x + 5$

$9y + x + 3 = 0$ [Given]

$M_{normal} = \frac{-1}{9}$

$y' = 12x^2 - 3 = 9 \Rightarrow x = \pm 1$; $y = 6, 4$

$\frac{y-4}{x+1} = \frac{-1}{9}, \frac{y-6}{x-1} = -\frac{1}{9}$

$x + 9y = 35, x + 9y = 55$

Sol 19: $y(x-2)(x-3) = x-7$

$y = \frac{x-7}{(x-2)(x-3)} = 0 \Rightarrow x = 7$

$\frac{dy}{dx} = \frac{(x-2)(x-3)-(x-7)(2x-5)}{(x-2)^2(x-3)^2}$

$\left(\frac{-dx}{dy}\right)_{x=7} = \frac{(x-2)^2(x-3)^2}{(x-7)(2x-5)-(x-2)(x-3)} = \frac{5^2(4)^2}{0-(5)(4)} = -20$

$\frac{y}{x-7} = -20 \Rightarrow 20x + y = 140$

Sol 20: $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0 \Rightarrow y' = -\sqrt{\frac{y}{x}} = -\sqrt{\frac{y_1}{x_1}}$

$\Rightarrow \frac{y-y_1}{x-x_1} = -\sqrt{\frac{y_1}{x_1}} \Rightarrow y\sqrt{x_1} - y_1\sqrt{x_1} = -x\sqrt{y_1} + x_1\sqrt{y_1}$

$\Rightarrow x\sqrt{y_1} + y\sqrt{x_1} = y_1\sqrt{x_1} + x_1\sqrt{y_1}$

$\frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{a}$

Sol 21: $3x^2 + y^2 = 8 \Rightarrow 6x + 2yy' = 0 \Rightarrow y' = \frac{-3x}{y}$

$(m)_{normal} = \frac{y}{3x} = \frac{-1}{3}$

$y = -x$

$x = \pm\sqrt{2}$, $y = \mp\sqrt{2}$

$\frac{y-\sqrt{2}}{x+\sqrt{2}} = -\frac{1}{3} \Rightarrow 3y + x = 2\sqrt{2}$

$\frac{y+\sqrt{2}}{x-\sqrt{2}} = \frac{-1}{3} \Rightarrow 3y + x = -2\sqrt{2}$

Sol 22 : $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$\frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0$

$y' = -\sqrt{\frac{y}{x}} = -\sqrt{1} = -1$

$\frac{y - \frac{a^2}{4}}{x - \frac{a^2}{4}} = 1 \Rightarrow y + x = \frac{a^2}{2}$

Sol 23: $y = x^3 - 2x^2 - 2x$

$y' = 3x^2 - 4x - 2 = 2$

$3x^2 - 6x + 2x - 4 = 0$

$(3x + 2)(x - 2) = 0$

$x = 2, \frac{-2}{3}$; $y = 8 - 8 - 4 = -4,$

$y = \frac{-8}{27} - \frac{2x4}{9} + \frac{4}{3} = -\frac{8}{27} - \frac{8}{9} + \frac{4}{3}$

$\Rightarrow \frac{36-24-8}{27} = \frac{4}{27}$

Therefore, the points are

$(2, -4)\left(\frac{-2}{3}, \frac{4}{27}\right)$

Sol 24: $x^2 = 4$ by

$y^2 = 4axx^2 = 16b\,(a^2\,b)^{1/3} = 16b^{4/3}a^{2/3}$

$y^4 = 16a^2\,4byx = 4\,(b^2a)^{1/3}$

$$y^3 = 64a^2by' = \frac{\frac{x}{2b}}{\frac{2a}{y}}$$

$$y = 4(a^2b)^{1/3}y' = 2\,\frac{\left(\frac{a}{b}\right)^{1/3}\cdot 2a}{4\left(a^2b\right)^{1/3}} = \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}$$

$$\tan\theta = \frac{2\left(\frac{a}{b}\right)^{1/3} - \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}}{1+\left(\frac{a}{b}\right)^{2/3}} = \frac{\frac{3}{2}\left(\frac{a}{b}\right)^{1/3}}{1-\left(\frac{a}{b}\right)^{2/3}}$$

$$= \frac{3a^{1/3}\,b^{1/3}}{2\left(b^{2/3}+a^{2/3}\right)}$$

Exercise 2

Methods of Differentiation

Single Correct Choice Type

Sol 1: (B) $y = f\left(\frac{3x+4}{5x+6}\right)$ $f'(x) = \tan x^2$

$$\frac{dy}{dx} = \frac{df\left(\frac{3x+4}{5x+6}\right)}{d\left(\frac{3x+4}{5x+6}\right)}\cdot\frac{d\left(\frac{3x+4}{5x+6}\right)}{dx}$$

$$= \left[\tan\left(\frac{3x+4}{5x+6}\right)^2\right]\frac{(5x+6)3-(3x+4)5}{(5x+6)^2}$$

$$= -2\tan\left(\frac{3x+4}{5x+6}\right)^2\cdot\frac{1}{(5x+6)^2}$$

Sol 2: (B) $f'(x) = \frac{x^{10}}{(1+x^2)}$ $g(x) = f^{-1}(x)$

$\because f(g(x)) = x$

$\therefore f'(g(x))g'(x) = 1$

$$\therefore g'(x) = \frac{1}{f'(g(x))}$$

$$\therefore g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(a)} = \frac{1}{\frac{a^{10}}{1+a^2}} = \frac{1+a^2}{a^{10}}$$

Sol 3: (D) $y = \frac{1}{1+x^{n-m}+x^{p-m}} + \frac{1}{1+x^{m-n}+x^{p-n}}$

$$+\frac{1}{1+x^{m-p}+x^{n-p}}$$

$$= \frac{x^m}{x^p+x^m+x^n} + \frac{x^n}{x^n+x^m+x^p} + \frac{x^p}{x^p+x^m+x^n}$$

$$= \frac{x^m+x^n+x^p}{x^m+x^n+x^p} = 1$$

$$\therefore \frac{dy}{dx} = 0$$

Sol 4: (D) $\lim\limits_{x\to 2}\frac{f(4)-f(x^2)}{2-x} = \lim\limits_{x\to 2}\frac{\frac{d\left(f(4)-f(x)^2\right)}{dx}}{\frac{d(2-x)}{dx}}$

$$= \lim_{x\to 2}\frac{f'(x^2)[2x]}{+1} = f'(2^2).2.2 = 4f'(4) = 20$$

Sol 5: (A) $\ell = \lim\limits_{x\to 0} x^m(\ln x)^n$ $m, n \in N$

$$\ell = \lim_{x\to 0}\frac{(\ln x)^n}{\left(\frac{1}{x}\right)^m} = \lim_{x\to 0}\frac{n(\ln x)^{n-1}\frac{1}{x}}{-m\left(\frac{1}{x}\right)^{m+1}}$$

(using L-Hospital rule)

$$= \lim_{x\to 0}\left(-\frac{n}{m}\frac{(\ln x)^{n-1}}{\left(\frac{1}{x}\right)^m}\right) = \lim_{x\to 0}\frac{-n(n-1)(n-2)......1\ln x}{m^{n-1}\left(\frac{1}{x}\right)^m}$$

$$= \lim_{x\to 0}\frac{-(n)!\frac{1}{x}}{m^n\left(\frac{1}{x}\right)^{m+1}} = \lim_{x\to 0}\frac{-(n)!}{m^n}(x)^m = 0$$

$\therefore$ Independent of n and m

Sol 6: (B) $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

$$f'(x)=\begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix}$$

$$+\begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \sec^2 x & 1 & 0 \end{vmatrix}$$

$f'(x) = [x^2\sin x\ (2x\tan x - 2\sin x)]$

$+ x[2\tan x - 2\cos x] + [2x\cos x - 2x\tan x] + (-2x\cos x) + 2x^2\sec^2 x + 2\sin x - x^2\sec^2 x$

$= x^2\sin x + x^2\sec^2 x + 2x\tan x - 2x\cos x$

$$\lim_{x\to 0}\frac{f'(x)}{x} = \lim_{x\to 0}\frac{x^2(\sin x+\sec^2 x)(2x(\tan x-\cos x)}{x}$$

$$= \lim_{x\to 0} x(\sin x + \sec^2 x) + 2(\tan x - \cos x) = -2$$

Sol 7: (C) $f'(x) = \begin{vmatrix} -\sin x & +\cos x & -\sin x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$

$$+\begin{vmatrix} \cos x & \sin x & \cos x \\ -2\sin 2x & 2\cos 2x & -4\sin 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$$

$$+\begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ -3\sin 3x & +3\cos 3x & -9\sin 3x \end{vmatrix}$$

$$f'\left(\frac{\pi}{2}\right) = \begin{vmatrix} -1 & 0 & -1 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$+\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 3 & 0 & 9 \end{vmatrix} = (2-1) + (0) + (3) = 4$$

Sol 8: (A) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

$\therefore y = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$

$y = \sin^{-1}\sin 2\theta$

$-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \Rightarrow \theta \le \frac{\pi}{4}$

$\because$ as $x = \frac{\pi}{2}$

$\therefore \theta = \tan^{-1}\frac{\pi}{2}$

$\therefore 2\theta = 2\tan^{-1}\frac{\pi}{2} > \frac{\pi}{2}$

$\therefore y = \sin^{-1}\sin(\pi - 2\theta)$

$y = \pi - 2\theta$

$y = \pi - 2\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{-2}{1+x^2}$

$\therefore \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = \frac{-8}{\pi^2+4^2}$

Sol 9: (A) $\lim_{x\to 4}\frac{\sqrt{f(x)}-\sqrt{g(x)}}{\sqrt{x}-2}$

$$= \lim_{x\to 4}\frac{\frac{d\left[\sqrt{f(x)}-\left(\sqrt{g(x)}\right)\right]}{dx}}{\frac{d\left(\sqrt{x}-2\right)}{dx}}$$

$$= \lim_{x\to 4}\frac{\frac{1}{2\sqrt{f(x)}}f'(x)-\frac{1}{2\sqrt{g(x)}}g'(x)}{\frac{1}{2\sqrt{x}}}$$

$$= \frac{\frac{1}{2\sqrt{f(4)}}f'(4)-\frac{1}{2\sqrt{g(4)}}g'(4)}{\frac{1}{2\sqrt{4}}} = \frac{\frac{9}{\sqrt{2}}-\frac{6}{\sqrt{2}}}{\frac{1}{2}}$$

$= 3\sqrt{2}$

Sol 10: (B) $y = x + e^x \Rightarrow \frac{dy}{dx} = 1 + e^x$

$\Rightarrow \frac{dx}{dy} = \frac{1}{1+e^x}$

$$\frac{d^2x}{dy^2} = \frac{d(1+e^x)^{-1}}{dx}\times\frac{1}{(1+e^x)}$$

$$= \frac{1}{(1+e^x)}\frac{d(1+e^x)^{-1}}{d(1+e^x)}\frac{d(1+e^x)}{dx}$$

$$= \frac{1}{(1+e^x)}\left(-\frac{1}{(1+e^x)^2}\right)e^x = \frac{-e^x}{(1+e^x)^3}$$

Sol 11: (C) $h'(x) = [f(x)]^2 + [g(x)]^2$

$h''(x) = 2f(x)f'(x) + 2g(x)g'(x)$

Also $f'(x) = g(x)$

$\therefore f''(x) = g'(x) = -f(x)$

$\therefore h''(x) = 2f(x)g(x) + 2g(x)(-f(x)) = 0$

$\therefore$ h(x) = ax + b

h(0) = b = 2; h(1) = a + b = 4

$\therefore$ a = 2, b = 2

$\therefore$ h(x) is a straight line with slope 2 and y intercept 2

Sol 12: (D) f(x) = x + 3ln(x – 2)

$F'(x) = 1 + \frac{3}{(x-2)}$

g(x) = x + 5ln(x – 1)

$g(x) = 1 + \frac{5}{(x-1)}$

$\therefore f'(x) < g'(x)$

$\Rightarrow \frac{3}{(x-2)} < \frac{5}{(x-1)} \Rightarrow \frac{3(x-1)-5(x-2)}{(x-2)(x-1)} < 0$

$\therefore \frac{7-2x}{(x-2)(x-1)} < 0 \Rightarrow x \in (1, 2) \cup \left(\frac{7}{2}, \infty\right)$

Also x – 2 > 0 and x – 1 > 0

$\therefore x > 2 \Rightarrow x \in \left(\frac{7}{2}, \infty\right)$

Sol 13: (D) g(x) = x^2, f(x) = sinx, h(x) = $\log_e x$

$\therefore g(f(x)) = (\sin x)^2$

$h(g(f(x))) = \log_e(\sin x)^2 = F(x)$

$\therefore$ F(x) = 2logsinx

$\therefore \frac{dF}{dx} = \frac{2}{\sin x}.\cos x = 2\cot x$

$\therefore \frac{d^2t}{dx^2} = -2\text{cosec}^2 x$

Sol 14: (D) f(x) = x^n

$f'(p+q) = n(p+q)^{n-1}$

$f'(p) = n(p)^{n-1}$

$f'(q) = n(q)^{n-1}$

for f′(p+q) = f′(p) + f′(q)

$(p + q)^{n-1} = [(p)^{n-1} + (q)^{n-1}]$ $n \neq 0$

$\Rightarrow \left(1 + \frac{q}{p}\right)^{n-1} - 1 = \left(\frac{q}{p}\right)^{n-1}$

This condition satisfies of n – 1 =1

$\Rightarrow$ n = 2

Also if n = 0

$\therefore$ f(x) = 1

$\therefore$ f′(p + q) = 0 = f′(p) + f′(q)

$\therefore$ n = 0, 2 (two values)

Sol 15: (B) $f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$

$\ln f(x) = \sum_{n=1}^{100} n(101-n)\ln(x-n)$

$\Rightarrow \frac{1}{f(x)}f'(x) = \sum_{n=1}^{100}\frac{n(101-x)}{(x-n)}$

$\therefore \frac{f'(101)}{f(101)} = \sum_{n=1}^{100}\frac{n(101-n)}{(101-n)} = \sum_{n=1}^{100} n = \frac{100\times101}{2} = 5050$

$\therefore \frac{f(101)}{f'(101)} = \frac{1}{5050}$

Sol 16: (B) f(x) is continuous and differentiableat $x = \frac{1}{3}$

$f(x) = \frac{3x^2+2x-1}{6x^2-5x+1} = \frac{x+1}{2x-1}$

$\therefore f'(x) = \frac{(2x-1)-2(x+1)}{(2x-1)^2} = \frac{-3}{(2x-1)^2}$

$f'\left(\frac{1}{3}\right) = \frac{-3}{\left(2\left(\frac{1}{3}\right)-1\right)^2} = 27$

Sol 17: (C) f(x) = $ax^2 + bx + c$

For x $\in$ R f(x) is always positive

$\therefore$ a > 0 and $b^2 - 4ac < 0$

$\therefore$ g(x) = f(x) + f′(x) + f″(x)

= $ax^2 + bx + c + 2ax + b + 2a$

= $ax^2 + (2a + b)x + (2a + b + c)$

D = $(2a + b)^2 - 4(2a + b + c)a$

= $4a^2 + b^2 + 4ab - 8a^2 - 4ab - 4ac$

= $b^2 - 4a^2 - 4ac = (b^2 - 4ac) - 4a^2$

Q$b^2 - 4ac < 0$

$\therefore$ D < 0 $\therefore$ g(x) = 0

Sol 18: (A) $y = \frac{x^4+4}{x^2-2x+2}$

$\frac{dy}{dx} = \frac{(x^2-2x+2)(4x^3)-(x^4+4)(2x-2)}{(x^2-2x+2)^2}$

$$\frac{dy}{dx}=\frac{4x^5-8x^4+8x^3-2x^5+2x^4-8x+8}{(x^2-2x+2)^2}$$

$$\frac{dy}{dx}=\frac{2x^5+8x^3-8x+8-6x^4}{(x^2-2x+2)^2}$$

$$\left.\frac{dy}{dx}\right|_{1/2}=\frac{2\times\left(\frac{1}{2}\right)^5+8\left(\frac{1}{2}\right)^3-8\times\frac{1}{2}+8-6\times\frac{1}{16}}{\left(\frac{1}{4}-1+2\right)^2}$$

$$=\frac{\frac{1}{16}+1-4+8-\frac{6}{16}}{\left(\frac{5}{4}\right)^2}=\frac{\frac{81}{16}-\frac{6}{16}}{\frac{25}{16}}=\frac{75/16}{25/16}=3$$

Sol 19: (B) $f(x^2) = x^3$

$\therefore f(x) = x^{3/2}$

$\therefore f'(x) = \frac{3}{2}x^{1/2}$

$f'(4) = \frac{3}{2}\times 4^{1/2} = 3$

Sol 20: (D) x = sint, y = sin3t

$$\frac{dy}{dx}=\frac{3\cos 3t}{\cos t}=\frac{3(4\cos^3 t-3\cos t)}{\cos t}$$

$= 12\cos^2 t - 9$

$$\frac{d^2y}{dx^2}=\frac{24\cos t(-\sin t)}{\cos t}=-24\sin t$$

$\therefore (1-\sin^2 t)(-24\sin t) - (\sin t)(12\cos^2 t - 9) + k(\sin 3t) = 0$

$= -24\sin t + 24\sin^3 t - 12\sin t(1-\sin^2 t) + 9\sin t + k(3\sin t - 4\sin_3 t) = 0$

$\Rightarrow (3k-36)\sin t + (36-4k)\sin^3 t = 0$

$\therefore 3k - 27 = 0$ and $36 = 4k$

$\Rightarrow k = 9$

Sol 21: (B) $x = \ln t$, $y = t^2 - 1$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2t}{1/t}=2t^2$$

$$\frac{d^2y}{dx^2}=\frac{\dfrac{d\left(\dfrac{dy}{dx}\right)}{dt}}{\dfrac{dx}{dt}}=\frac{4t}{1/t}=4t^2$$

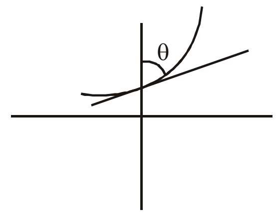
$\therefore y''(1) = 4$

Application of Derivatives

Single Correct Choice Type

Sol 1: (B) $y = ke^{kx} \Rightarrow y' = k^2e^{kx} \Rightarrow y = k, x = 0$

$y' = k^2$



$\Rightarrow \tan\theta=\frac{1}{k^2} \Rightarrow \cot\theta = k^2$

$\Rightarrow \theta = \cot^{-1}k^2$

Sol 2: (D) $f(x)=\int_2^x(2t-5)=t^2-5t\Big|_2^x=x^2-5x-4+10$

$=x^2-5x+6=(x-2)(x-3)$

$f^1(x)=2x-5$

$[x=2, f(x)=0]\ \ f'(x)=-1$

$[x=3, f(x)=0]\ \ f'(x)=1$

Angle between the 2 tangents is 90°

(as $m_1m_2 = -1$)

Sol 3: (C) $x^2y=c^3$

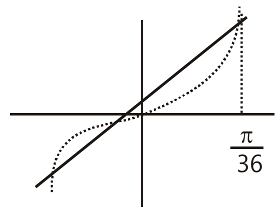
$y=\frac{c^3}{x^2} \Rightarrow y'=\frac{-2c^3}{x^3}$

$y-\frac{c^3}{t^2}=-\frac{2c^3}{t^3}(x-t)\left[x=t\ y=\frac{c^3}{t^2}\right]$

x intercept $=\frac{3t}{2}=a$ y intercept

$=\frac{3c^3}{t^2}=b \Rightarrow a^2b=\frac{9t^2}{4}\times\frac{3c^3}{t^2}=\frac{27c^3}{4}$ [C]

Sol 4: (D) $f(x)=\begin{cases}x\sin\left(\frac{\pi}{x}\right) & x>0\\ 0 & x=0\end{cases}$



$f'(x)=x\cos\left(\frac{\pi}{x}\right)\left(\frac{-\pi}{x^2}\right)+\sin\frac{\pi}{x}=0\ \ \frac{\pi}{x}=\tan\left(\frac{\pi}{x}\right)$

$x\in[0,1]$ infinite solution

Sol 5: (D) $y = f(x) \Rightarrow \int_a^b \underset{I}{f'(x)}\,\underset{II}{f''(x)}$

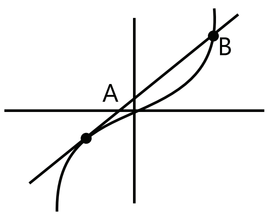
$\Rightarrow I = f'(x)\int_a^b f''(x) - \int_a^b f''(x)\, f'(x)$

$\Rightarrow 2I = \left[f'(x)\right]^2$

$\Rightarrow 2I = \left[f'b\right]^2\left[f'a\right]^2$

$\Rightarrow 2I = -2 \Rightarrow I = -1$

Sol 6: (A) $y = x^3$



$y' = 3x^2$

$3(x_B)^2 = k^3(x_A)^2$

$\frac{x_B}{x_A} = \pm\sqrt{k}$

$\Rightarrow (y - t^3) = 3t^2(x - t)\ [\text{at } x = t]$

$\Rightarrow x^3 - t^3 = 3t^2(x - t)$

$\Rightarrow x^2 + t^2 + xt = 3t^3$

$\Rightarrow (x_B)^2 + (x_B x_A) = 2(x_A)^2$

$\Rightarrow kx_A^2 \pm \sqrt{k}x_A^2 = 2x_A^2$

$\Rightarrow 2 - k = \pm\sqrt{k}$

$\Rightarrow k = 4$

Sol 7: (C) Subnormal = $y\frac{dy}{dx}$

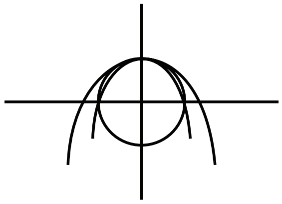
$xny^{n-1}y' + y^n = 0$

$y' = \frac{-y}{nx}$

$\left|y\frac{dy}{dx}\right| = \frac{y^2}{nx}$ it is constant for $\Rightarrow \frac{y^{2+n}}{a^{n+1}n}$

Constant for n = – 2

Sol 8: (A) $y = -\frac{x^2}{2} + 2$



$y' = -x$

$\Rightarrow \frac{y-2}{x - \frac{1}{2}} = m$

$\Rightarrow y - 2 = m\left(x - \frac{1}{2}\right) \Rightarrow \frac{-x^2}{2} = \frac{2mx}{2} - \frac{m}{2}$

$\Rightarrow x^2 + 2mx - m = 0$

$\Rightarrow x = \frac{-2m \pm \sqrt{4m^2 + 4m}}{2} = 0$

For D = 0

$\Rightarrow 4m^2 + 4m = 0 \Rightarrow m = 0, -1$

$\Rightarrow y - 2 = -x + \frac{1}{2}$

$\Rightarrow x + y = \frac{5}{2}$

Sol 9: (D) $c_1 y = x^2 - 3$ and $c_2 y = kx^2$

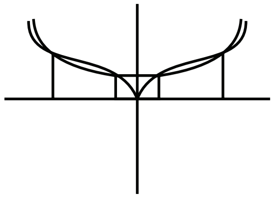
$\Rightarrow kx^2 = x^2 - 3$

$\Rightarrow x = \pm\sqrt{\frac{3}{1-k}} = a \Rightarrow y = \frac{3}{1-k} - 3 = \frac{3k}{1-k}$

$\Rightarrow \frac{y - y_1}{x - a} = 2ka \Rightarrow y - y_1 = 2ka(x - a)$

$\Rightarrow x^2 - 3 - y_1 = 2ka(x - a)$

$\Rightarrow -2 - y_1 = 2ka(1 - a)$



$y_2 = -2$

$y_1 = ka^2$

$\Rightarrow -2 - ka^2 = 2ka - 2ka^2$

$\Rightarrow ka^2 - 2ka - 2 = 0$

$\Rightarrow k\left(\frac{3}{1-k}\right) - 2 = 2k\sqrt{\frac{3}{1-k}} \Rightarrow \frac{5k-2}{\sqrt{1-k}} = 2k\sqrt{3}$

$\Rightarrow 5k - 2 = 2k\sqrt{3 - 3k}$

$k = \frac{2}{3}$, a = 1

Sol 10: (B) $x^2 = e^{|x|-2}$

No. of roots are 4

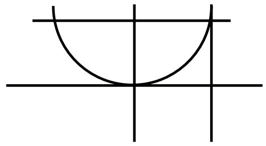
Sol 11: (C) $-\frac{a}{x^3} - \frac{b}{y^3}y' = 0$ at any general point $\left(t, t\sqrt{\frac{b}{t^2 - a}}\right)$

$y' = -\frac{ay^3}{bx^3}$

$y - t\sqrt{\frac{b}{t^2 - a}} = \frac{-a}{b}\frac{t^3}{t^3}\left(\frac{b}{t^2 - a}\right)^{3/2}(x - t)$

$\Rightarrow$ x intercept $\frac{t - b(t^2 - a) + t}{ab} = \frac{bt^3}{ab} = \frac{t^3}{a} = [B]$

Sol 12: (D) m = 0



y = c

$y = \sqrt{x}$

$y^1 = \frac{1}{2\sqrt{x}} = 1$; $x = \frac{1}{4} y = \frac{1}{2} = c$

Sol 13: (D) $y^3 - x^2y + 5y - 2x = 0$

$\Rightarrow x^4 - x^3y^2 + 5x + 2y = 0$

$\Rightarrow 3y^2y' - x^2y' - 2xy + 5y' - 2 = 0$

$\Rightarrow y' = \frac{2+2xy}{3y^2 - x^2 + 5}$, $y'(0, 0) = \frac{2}{5}$

$\Rightarrow 4x^3 - 3x^2y^2 - 2x^3 yy' + 5 + 2y' = 0$

$\Rightarrow y' = \frac{4x^3 - 3x^2y^2 + 5}{-2 + 2x^3y}$, $y'(0,0) = -\frac{5}{2}$

$\Rightarrow \tan\theta = \frac{\frac{2}{5} + \frac{5}{2}}{1-1} = \infty \qquad \therefore \theta = \frac{\pi}{2}$

Sol 14: (D) $f(x) = \int_0^x \left(t + \frac{1}{t}\right) dt$

$g(x) = f'(x) = x + \frac{1}{x}$

$m = \frac{g(3) - g\left(\frac{1}{2}\right)}{3 - \frac{1}{2}} = \frac{3 + \frac{1}{3} - \frac{1}{2} - 2}{\frac{5}{2}} = \frac{\frac{5}{2} - \frac{5}{3}}{\frac{5}{2}} = \frac{2}{6 \times 1} = \frac{1}{3}$

$y' = 1 - \frac{1}{x^2} = \frac{1}{3}$

$x = x = \sqrt{\frac{3}{2}}\ y = \frac{5}{\sqrt{6}}$

Sol 15: (C) $y = \frac{x^{3/2}}{3}$

$18yy' = 3x^2 \Rightarrow y' = \frac{x^2}{6y}$

$m_{normal} = \left(\frac{-1}{y^1}\right) = \frac{-6y}{x^2} = \pm 1$

$6y = \pm x^2 \Rightarrow y = \frac{x^2}{6}$

$\Rightarrow 9 \times \frac{x^4}{36} = x^3 \Rightarrow x = 4$ and $y = \frac{8}{3}$

Previous Years' Questions

Sol 1: (C) Given, $x = a(\cos\theta + \theta\sin\theta)$ and

$y = a(\sin\theta - \theta\cos\theta)$

$\therefore \frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$

$= a\theta\cos\theta$ and $\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$

$\frac{dy}{d\theta} = a\theta\ \sin\theta \Rightarrow \frac{dy}{dx} = \tan\theta$

Thus, equation of normal is

$\frac{y - a(\sin\theta - \theta\cos\theta)}{x - a(\cos\theta + \theta\sin\theta)} = \frac{-\cos\theta}{\sin\theta}$

$\Rightarrow -x\cos\theta + a\theta\sin\theta\cos\theta + a\cos^2\theta$

$= y\sin\theta + \theta a\sin\theta\cos\theta - a\sin^2\theta$

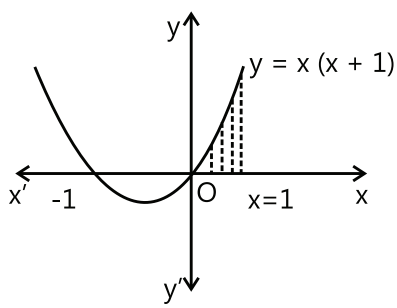
$\Rightarrow x\cos\theta + y\sin\theta = a$

Whose distance from origin is,

$\frac{|0 + 0 - a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = a$

Sol 2: (A) Given, $\frac{dy}{dx} = 2x + 1$

on integrating both sides



$\int dy = \int (2x+1)dx$

$\Rightarrow y = x^2 + x + c$ which passes through (1, 2)

$\therefore 2 = 1 + 1 + c \Rightarrow c = 0$

$\therefore y = x^2 + x$

Thus, the required area bounded by x axis, the curve and x = 1

$= \int_0^1 (x^2 + x)dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ sq unit

Sol 3: (D) Slope of tangent y = f(x) is

$\frac{dy}{dx} = f'(x)_{(3,4)}$

Therefore, slope of normal

$= -\frac{1}{f'(x)_{(3,4)}} = -\frac{1}{f'(3)}$

But $-\frac{1}{f'(3)} = \tan\left(\frac{3\pi}{4}\right)$ (given)

$\Rightarrow -\frac{1}{f'(3)} = \tan\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = -1$

$f'(3) = 1$

Sol 4: (D) Given $y^3 + 3x^2 = 12y$ ...(i)

$\Rightarrow 3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{6x}{12-3y^2} \Rightarrow \frac{dx}{dy} = \frac{12-3y^2}{6x}$

For vertical tangent, $\frac{dx}{dy} = 0$

$\Rightarrow 12 - 3y^2 = 0 \Rightarrow y = \pm 2$

On putting y = 2 in Eq. (i), we get $x = \pm\frac{4}{\sqrt{3}}$ and again putting y = – 2 in Eq. (i), we get $3x^2 = -16$, no real solution

$\therefore$ The required point $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

Sol 5: (D) Tangent to the curve $y^2 = 8x$ is,

$y = mx + \frac{2}{m}$.

So it must satisfy xy = –1

$\Rightarrow x\left(mx+\frac{2}{m}\right) = -1$

$\Rightarrow mx^2 + \frac{2}{m}x + 1 = 0,$

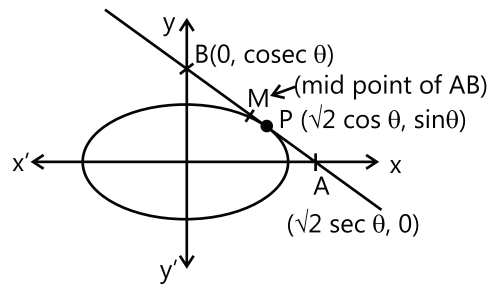
Since it has equal roots, therefore D = 0

$\Rightarrow \frac{4}{m^2} - 4m = 0 \Rightarrow m^3 = 1 \Rightarrow m = 1$

So equation of common tangent is y = x + 2

Sol 6: (A) Let the point be $P(\sqrt{2}\cos\theta, \sin\theta)$

on $\frac{x^2}{2} + \frac{y^2}{1} = 1$



Equation of tangent is,

$\frac{x\sqrt{2}}{2}\cos\theta + y\sin\theta = 1$

whose intercept on coordinate axes are A $(\sqrt{2}\sec\theta, 0)$ and B $(0, \text{cosec}\,\theta)$

$\therefore$ Mid point of its intercept between axes $\left(\frac{\sqrt{2}}{2}\sec\theta, \frac{1}{2}\text{cosec}\,\theta\right) = (h, k)$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}h}$ and $\sin\theta = \frac{1}{2k}$

Thus, locus of mid point M is

$(\cos^2\theta + \sin^2\theta) = \frac{1}{2h^2} + \frac{1}{4k^2}$

$\Rightarrow \frac{1}{2x^2} + \frac{1}{4y^2} = 1$, is required locus

Sol 7: (C) We know, tangent to parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

$\therefore$ Tangent to $y^2 = 4x$ is $y = mx + \frac{1}{m}$

Since, tangent passes through (1, 4)

$\therefore 4 = m + \frac{1}{m}$

$\Rightarrow m^2 - 4m + 1 = 0$

(Whose roots are m_1 and m_2)

$\therefore m_1 + m_2 = 4$ and $m_1m_2 = 1$

and $|m_1 - m_2| = \sqrt{(m_1+m_2)^2 - 4m_1m_2}$

$= \sqrt{12} = 2\sqrt{3}$

Thus, angle between tangents

$\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right| = \left|\frac{2\sqrt{3}}{1+1}\right| = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

Sol 8: (D) The tangent at (1, 7) to the parabola $x^2 = y - 6$ is

$x(1) = \frac{1}{2}(y + 7) - 6$

(replacing $x^2 \to x x_1$ and $2y \to y + y_1$)

$\Rightarrow 2x = y + 7 - 12$

$\Rightarrow y = 2x + 5$

Which is also tangent to the circle

$x^2 + y^2 + 16x + 12y + c = 0$

i.e., $x^2 + (2x + 5)^2 + 16x$

$+ 12(2x + 5) + c = 0$

must have equal roots i.e., $\alpha = b$

$\Rightarrow 5x^2 + 60x + 85 + c = 0$

$\Rightarrow \alpha + \beta = -\frac{60}{5}$

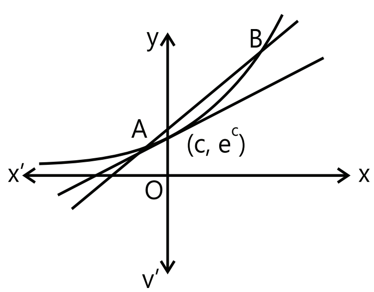
$\Rightarrow a = -6 \Rightarrow x = -6$

and $y = 2x + 5 = -7$

$\therefore$ Point of contact is $(-6, -7)$

Sol 9: (A) Slope of the line joining the point

$(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ is equal to $\frac{e^{c+1} - e^{c-1}}{2} > e^c$



$\Rightarrow$ Tangent to the curve $y = e^x$ will intersect the given line to the left of the line $x = c$.

Alternate Solution

The equation of the tangent to the curve $y = e^x$ at (c, e^c) is

$y - e^c = e^c(x - c)$ ....(i)

Equation of the line joining the given points is

$y - e^{c-1} = \frac{e^c(e - e^{-1})}{2}[x - (c - 1)]$ ... (ii)

Eliminating y from equation (i) and (ii), we get

$[x - (c - 1)]\,[2 - (e - e^{-1})] = 2e^{-1}$

$\Rightarrow x - c = \frac{e + e^{-1} - 2}{2 - (e - e^{-1})} < 0 \Rightarrow x < c$

Sol 10: (A)

$$f'(1) = \lim_{n\to 0}\frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \quad f'(1) = \lim_{n\to 0}\frac{(1+h-1)\sin\left(\frac{1}{1+h-1}\right) - 0}{h}$$

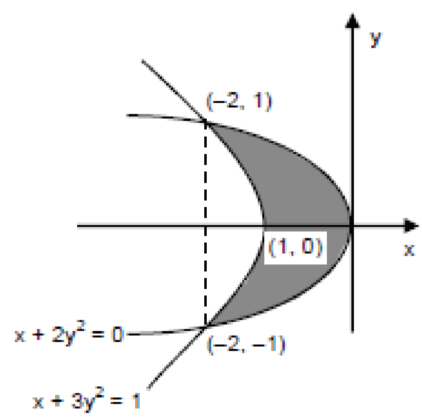
$$= \lim_{n\to 0}\frac{h}{h}\sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \quad f'(1) = \lim_{n\to 0}\sin\left(\frac{1}{h}\right)$$

$\therefore$ f is not differentiable at x = 1.

Similarly, $f'(0) = \lim_{h\to 0}\frac{f(h) - f(0)}{h}$

$$\Rightarrow \quad f'(0) = \lim_{h\to 0}\frac{(h-1)\sin\left(\frac{1}{h-1}\right) - \sin(1)}{h}$$



$\Rightarrow$ f is also not differentiable at x = 0.

Sol 11: (D)

Solving the equation we get the points of intersection (-2, 1) and (-2, -1)

The bounded region is shown as shaded region.

The required area $= 2\int_0^1 (1 - 3y^2) - (-2y^2)$

$$= 2\int_0^1 (1 - y^2)dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = 2\times\frac{2}{3} = \frac{4}{3}$$

JEE Advanced/Boards

Exercise 1

Methods of Differentiation

Sol 1: $y = x \sin kx$

$\frac{dy}{dx} = \sin kx + xk \cos kx$

$\frac{d^2y}{dx^2} = k \cos kx + k \cos kx - xk^2 \sin kx$

$\therefore \frac{d^2y}{dx^2} + y = 2k \cos kx - xk^2 \sin kx + x \sin kx = 2k \cos kx$

$\Rightarrow x \sin kx(1 - k^2) = 0$

$\therefore k = 1, -1$ when $1 - k^2 = 0$

and $k = 0$ when $\sin kx = 0$

Sol 2: Let $f(x) = ax^3 + bx^2 + cx + d$

$f(2x) = f'(x)f''(x)$

$a(2x)^3 + b(2x)^2 + c(2x) + d$

$= (3ax^2 + 2bx + c)(6ax + 2b)$

$8ax^3 + 4bx^2 + 2cx + d$

$= 18a^2x^3 + (6ab + 12ab)x^2$

$+ 4b^2x + 6acx + 2bc$

$\therefore 8a = 18a^2 \Rightarrow a = 0, \frac{4}{9}$

$\therefore$ f(x) should be a cubic equation

$\therefore a \neq 0$ and $a = \frac{4}{9}$

also $18ab = 4b$

$b(18a - 4) = 0 \Rightarrow b\left(18 \times \frac{4}{9} - 4\right) = 8b = 0 \Rightarrow b = 0$

and $4b^2 + bac = 2c$

$\Rightarrow c(6a - 2) = 0 \Rightarrow c = 0$

$\therefore f(x) = \frac{4}{9}x^3$ and $d = 2bc = 0$

Sol 3: $f'(x) = g(x)$; $g'(x) = f(x)$

Let $h(x) = f^2(x) - g^2(x)$

$h'(x) = 2f(x)f'(x) - 2g(x)g'(x)$

$= 2f(x)\,g(x) - 2g(x)f(x) = 0$

$\therefore$ h(x) is a constant function whose value is constant for every value of x

$\therefore h(3) = f^2(3) - g^2(3) = (5)^2 - \left[f'(3)\right]^2 = 5^2 - 4^2 = 9$

$\therefore f^2(\pi) - g^2(\pi) = 9$

Sol 4: $3x^2 + 4y^2 = 12 \Rightarrow y^2 = 3 - \frac{3}{4}x^2$

Differentiating both sides, we get

$2y\frac{dy}{dx} = -\frac{3}{2}x \quad \Rightarrow \frac{dy}{dx} = \frac{-3}{4}\frac{x}{y}$

$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = -\frac{3}{2}$

or $2\left(\frac{-3}{4}\frac{x}{y}\right)^2 + 2y\frac{d^2y}{dx^2} = -\frac{3}{2}$

$\frac{18}{16}\frac{x^2}{y^2} + 2y\frac{d^2y}{dx^2} = -\frac{3}{2}$

$18x^2 + 2y^3 \times 16\frac{d^2y}{dx^2} = -24y^2$

$y^3\frac{d^2y}{dx^2} = \frac{-24y^2 - 18x^2}{32}$

$= -(3x^2 + 4y^2) \times \frac{6}{32} = \frac{-12 \times 6}{32} = -\frac{9}{4}$

Sol 5: $f(x^2)f''(x) = f'(x)\,f'(x^2)$ ... (i)

$f(1) = 1, f'''(1) = 8$

Find $f'(1) + f''(1)$

Differentiate the given equation

$f(x^2)f'''(x) + f'(x^2)f''(x)2x = f''(x)f'(x^2) + f'(x)f''(x^2)2x$ ... (ii)

Put x = 1 in equation (1)

$\Rightarrow f(1)\,f''(1) = f'(1)f'(1) \Rightarrow f''(1) = [f'(1)]^2$ ... (iii)

Put x = 1 in equation (2)

$f(1)\,f'''(1) + f'(1)f''(1) \times 2 = f''(1)f'(1) + f'(1)f''(1)2$

$\Rightarrow f''(1)f'(1) = 8$ ... (iv)

From equation (3) and (4)

$[f'(1)]^3 = 8$

$\therefore f'(1) = 2$ and $f''(1) = (2)^2 = 4$

$\therefore f'(1) + f''(1) = 2 + 4 = 6$

Sol 6: $2x = y^{1/5} + y^{\frac{1}{1/5}}$

Take $y^{1/5} = t$

$t^2 - 2xt + 1 = 0$

$\therefore t = \dfrac{2x+\sqrt{4x^2-4}}{\alpha} = x + \sqrt{x^2-1}$

$\therefore y^{1/5} = x + \sqrt{x^2-1}$

$\frac{1}{5}y^{-4/5}\frac{dy}{dx} = 1+\frac{x}{\sqrt{x^2-1}} = \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}} = \frac{y^{1/5}}{\sqrt{x^2-1}}$

$\therefore \frac{dy}{dx} = \frac{5y}{\sqrt{x^2-1}}$

$$\frac{d^2y}{dx^2} = \frac{5\sqrt{x^2-1}\frac{dy}{dx} - \frac{5yx}{\sqrt{x^2-1}}}{(x^2-1)}$$

$$= \frac{5\left[(x^2-1)\frac{dy}{dx} - xy\right]}{(x^2-1)^{3/2}} \quad (x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$$

$$= 5\sqrt{x^2-1}\frac{dy}{dx} - \frac{5yx}{\sqrt{x^2-1}} + \frac{5xy}{\sqrt{x^2-1}} = 25y$$

$\therefore k = 25$

Sol 7: $\frac{d^2y}{dx^2} = 1+\frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2$

$y = \tan z$

$\frac{dy}{dx} = \sec^2 z\frac{dz}{dx} \Rightarrow \frac{dz}{dx} = \cos^2 z\frac{dy}{dx}$

$$\therefore \frac{d^2z}{dx^2} = \cos^2 z\frac{d^2y}{dx^2} - 2\cos z\sin z\frac{dy}{dx}.\frac{dz}{dx}$$

$$= \cos^2 z\left[1+\frac{2(1+\tan z)}{\sec^2 z}\left(\sec^4 z\left(\frac{dz}{dx}\right)^2\right)\right]$$

$$- 2\cos z\sin z\sec^2 z\left(\frac{dz}{dx}\right)^2$$

$$= \cos^2 z + [2(1+\tan z) - 2\tan z]\left(\frac{dz}{dx}\right)^2$$

$$= \cos^2 z + 2\left(\frac{dz}{dx}\right)^2 \Rightarrow k = 2$$

Sol 8: $z = \ln\left(\tan\frac{x}{2}\right)$

$$\frac{dz}{dy} = \frac{1}{\tan\frac{x}{2}}\sec^2\frac{x}{2}\times\frac{1}{2}\frac{dx}{dy} = \operatorname{cosec}x\frac{dx}{dy} \Rightarrow \sin x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\frac{d^2y}{dz^2} = \left[\cos x\frac{dy}{dx} + \sin x\frac{d^2y}{dx^2}\right]\frac{dx}{dz}$$

$$\therefore \frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} = \operatorname{cosec}x\frac{d^2y}{dz^2}\times\frac{dz}{dx} = \operatorname{cosec}^2x\frac{d^2y}{dz^2}$$

$$\therefore \operatorname{cosec}^2x\frac{d^2y}{dx^2} + 4y\operatorname{cosec}^2x = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + 4y = 0$$

Sol 9: $f(x) = \frac{\sin x}{x}, x \neq 0, f(0) = 1$

$$f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f''(0) = \lim_{h\to 0}\frac{\frac{h\cos h - \sin h}{h^2} - 0}{h}$$

$$= \lim_{h\to 0}\left(\frac{h\cos h - \sin h}{h^3}\right) \text{ (L-Hospital's rule)}$$

$$= \lim_{h\to 0}\left(\frac{\cos h - h\sin h - \cos h}{3h^2}\right) = \lim_{h\to 0} -\frac{1}{3}\left(\frac{\sin h}{h}\right) = -\frac{1}{3}$$

$$\therefore f''(0) = -\frac{1}{3}$$

Sol10: $R = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \Rightarrow R^{2/3} = \dfrac{1}{\left(\frac{d^2y}{dx^2}\right)^{2/3}} + \dfrac{1}{\left(\frac{d^2x}{dy^2}\right)^{2/3}}$

Let $\frac{dy}{dx} = t \Rightarrow \frac{dx}{dy} = \frac{1}{t}$

$$\therefore R^{2/3} = \frac{1}{\left(\frac{dt}{dx}\right)^{2/3}} + \frac{1}{\left(\frac{d(1/t)}{dy}\right)^{2/3}}$$

$$= \frac{1}{\left(\frac{dt}{dx}\right)^{2/3}} + \frac{1}{\left[\frac{-1}{t^2}\frac{\left(\frac{dt}{dx}\right)}{\left(\frac{dy}{dx}\right)}\right]^{2/3}} = \frac{1}{\left(\frac{dt}{dx}\right)^{2/3}} + \frac{1}{\left[-\frac{1}{t^3}\frac{dt}{dx}\right]^{2/3}}$$

$$\therefore \frac{dy}{dx} = t = \frac{1}{\left(\frac{dt}{dx}\right)^{2/3}}\left[1+\frac{1}{1/t^2}\right] = \frac{1}{\left(\frac{dt}{dx}\right)^{2/3}}\left[1+t^2\right]$$

$$\therefore R^{2/3} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\left(\frac{d^2y}{dx^2}\right)^{2/3}} \Rightarrow R = \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)}\right]^{3/2}$$

Sol 11: $f(x) = \ell n\left(1+\sqrt{1+x^2}\right)$

$g(x) = \ell n\left(x+\sqrt{1+x^2}\right) \Rightarrow e^{g(x)} = \left(x+\sqrt{1+x^2}\right)$

$$f\left(\frac{1}{x}\right) = \ell n\left(1+\sqrt{1+\frac{1}{x^2}}\right) = \ell n\left(x+\sqrt{1+x^2}\right) - \ell n x$$

$$f'\left(\frac{1}{X}\right) = \frac{1}{\sqrt{1+x^2}} - \frac{1}{x}$$

$$\therefore xe^{g(x)}\left(f\left(\frac{1}{x}\right)' + g'(x)\right)$$

$$= x\left(x+\sqrt{1+x^2}\right)\left[\frac{x-\sqrt{1+x^2}}{x\sqrt{1+x^2}}\right] + \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{x^2-(1+x^2)}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} = 0$$

$\therefore$ For every x, given function is zero

Sol 12: $f\left(\frac{x+y}{k}\right) = \frac{f(x)+f(y)}{k}$

Put x, y = 0

$f(0) = \frac{2f(0)}{k}$

$\Rightarrow f(0)\left[\frac{k-2}{k}\right] = 0 \quad \because k \neq 2, 0$

$\therefore f(0) = 0$

Put y = –x

$f(0) = \frac{f(x)+f(-x)}{k} = 0$

$\therefore f(x) = -f(-x)$ or $f(-x) = -f(x)$

$\therefore$ If above function is satisfying the given condition then the function should be odd or f(x) = 0

Sol 13: $f'(x) = 4\begin{vmatrix}(x-a)^3 & (x-a)^3 & 1\\(x-b)^3 & (x-b)^3 & 1\\(x-c)^3 & (x-c)^3 & 1\end{vmatrix}$

$$+ 3\begin{vmatrix}(x-a)^4 & (x-a)^2 & 1\\(x-b)^4 & (x-b)^2 & 1\\(x-c) & (x-c)^2 & 1\end{vmatrix} + \begin{vmatrix}(x-a)^4 & (x-a)^3 & 0\\(x-b)^4 & (x-b)^3 & 0\\(x-c)^4 & (x-c)^3 & 0\end{vmatrix}$$

$$\therefore f'(x) = 3\begin{vmatrix}(x-a)^4 & (x-a)^2 & 1\\(x-b)^4 & (x-b)^2 & 1\\(x-c)^4 & (x-c)^2 & 1\end{vmatrix}$$

$\therefore \lambda = 3$

Sol 14: $P(x) = ax^4 + bx^3 + cx^2 + dx + e$

P(1) = a + b + c + d + e = 0 ... (i)

P(3) = 81a + 27b + 9c + 3d + e = 0 ... (ii)

P(5) = 625a + 125b + 25c + 5d + c = 0 ... (iii)

$P'(7) = 4a(7)^3 + 3b(7)^2 + 2c(7) + d = 0$

= 1372a + 147b + 14c + d = 0 ... (iv)

(2) – (1)$\Rightarrow$80a + 26b + 8c + 2d = 0

$\Rightarrow$ 40a + 13b + 4c + d = 0 ... (v)

(3) – (1)$\Rightarrow$624a + 124b + 24c + 4d = 0

$\Rightarrow$ 156a + 31b + 6c + d = 0 ... (vi)

(6) – (5)$\Rightarrow$116a + 18b + 2c = 0 ... (vii)

(4) – (6)$\Rightarrow$1216a + 116b + 8c = 0

$\Rightarrow$ 304a + 29b + 2c = 0 ... (viii)

(8) – (7)$\Rightarrow$188a + 11b = 0

$\therefore -\frac{b}{a} = \frac{188}{11}$

also $\frac{-b}{a} = 1 + 3 + 5 + x = \frac{188}{11}$ (sum of roots)

$\therefore x = \frac{188}{11} - 9 = \frac{89}{11}$

$\therefore \left(x - \frac{89}{11}\right)$ is a root of 4 degree polynomial

$\therefore$ p = 89 q = 11

$\therefore$ p + q = 100

Alternate:

Take P(x) = (x – 1) (x – 3) (x – 5) (qx – p)

Now apply condition that P′(7) = 0

Sol 15: $f(x) = x^3 + x^2 f(1) + xf''(2) + f'''(3)$

T.P. f(2) = f(1) – f(0)

$f'(x) = 3x^2 + 2x f'(1) + f''(2)$

$f'(1) = 3 + 2f'(1) + f''(2)$

$\therefore f'(1) + f''(2) + 3 = 0$ ... (i)

$f''(x) = 6x + 2f'(1)$

$f''(2) = 12 + 2f'(1)$ ... (ii)

$f'''(x) = 6$

$\therefore f'''(3) = 6$ ... (iii)

$f(2) = 8 + 4f'(1) + 2f''(2) + f'''(3)$

From (1), (2) and (3)

$f'(1) = -5, f''(2) = 2$

$f'''(3) = 6$

$f(1) = 1 + f'(1) + f''(2) + f'''(3) = 1 - 5 + 2 + 6 = 4$

$f(2) = 8 - 20 + 4 + 6 = -2$

$f(0) = f'''(3) = 6$

$\therefore f(2) = f(1) - f(0)$ Hence proved

Sol 16: $f(x) = \sin 2x[\sin(x + x^2)\sin(x - x^2) + \cos(x + x^2)\cos(x - x^2)] + \sin 2x^2[\cos(x + x^2)\cos(x - x^2) - \sin(x + x^2)\sin(x - x^2)]$

$= \sin 2x\cos(x+x^2 + x^2 - x) + \sin 2x^2[\cos(x+x^2 + x - x^2)]$

$= \sin 2x\cos 2x^2 + \sin 2x^2\cos 2x = \sin(2x + 2x^2)$

$\therefore f'(x) = 2(2x+1)\cos 2(x^2 + x)$

Sol 17: $f(0) = \begin{vmatrix} a & b & c \\ \ell & m & n \\ p & q & r \end{vmatrix}$

$$F'(x) = \begin{vmatrix} 1 & b+x & c+x \\ 1 & m+x & n+x \\ 1 & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & 1 & c+x \\ \ell+x & 1 & n+x \\ p+x & 1 & r+x \end{vmatrix}$$

$$+ \begin{vmatrix} a+x & b+x & 1 \\ \ell+x & m+x & 1 \\ p+x & q+x & 1 \end{vmatrix}$$

$f'(x) = (m - b)(r - c) - (n - c)(q - b)$

$+ (-1)[(\ell - a)(r - c) - (n - c)(p - a)]$

$\therefore f''(x) = 0$

$$f(x) = \begin{vmatrix} a+x & b+x & c+x \\ \ell-a & x-b & n-c \\ p-a & q-b & r-c \end{vmatrix}$$

$(a+x)[(m-b)(r-c) - (q-b)(n-c)]$

$+ (b+x)[(\ell-a)(r-c) - (n-c)(p-a)]$

$+ (c+x)[(\ell-a)(q-b) - (m-b)(p-a)]$

$= a[(m-b)(r-c) - (q-b)(n-c)]$

$+ b[(\ell-a)(r-c) - (n-c)(p-a)]$

$+ c[(\ell-a)(q-b) - (m-b)(p-a)]$

$+ x[\{(m - b(r-c) - (q-b)(n-c)]$

$+ \{(\ell-a)(r-c) - (n-c)(p-a)\}$

$+ \{(\ell-a)(a, 0) - (m-b)(p-a)\}]$

$= f(0) + kx$

k= sum of all the co-factor of elements of f(0)

Sol 18: $y = \sum \tan^{-1} \dfrac{1}{x^2 + (2n-1)x + \{(n)(n-1)+1\}}$

$= \sum \tan^{-1} \dfrac{1}{(x+n)(x+n-1)+1}$

$= \sum \tan^{-1} \dfrac{(x+n)-(x+n-1)}{(x+n)(x+n-1)+1}$

Let $\tan\alpha = x + n$

$\tan\beta = x + n - 1$

$\therefore y = \sum \tan^{-1}\tan(\alpha - \beta) = S\alpha - S\beta$

$y = \sum_{n=1}^{n} \tan^{-1}(x+n) - \sum_{n=1}^{n} \tan^{-1}(x+n-1)$

$\therefore y = \tan^{-1}(x + n) - \tan^{-1}x$

$y' = \dfrac{1}{1+(x+n)^2} - \dfrac{1}{x^2+1}$

$= \dfrac{1+x^2-1-(x+n)^2}{(1+x^2)(1+(x+n)^2} = \dfrac{x^2-(x+n)^2}{(1+x^2)(1+(x+n)^2)}$

Sol 19: Y = SX

Z = tX

$Y_1 = SX_1 + S_1X$

$Z_1 = tX_1 + t_1X$

$Y_2 = SX_2 + S_1X_1 + S_2X + S_1X_1$

$= SX_2 + 2S_1X_1 + S_2X$

$Z_2 = tX_2 + 2t_1X_1 + t_2X = 5X_2 + 2S_1X_1 + S_2X$

$Z_2 = tX_2 + 2t_1X_1 + t_2X$

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$= \begin{vmatrix} X & SX & tX \\ X_1 & SX_1 + S_1X & tX_1 + t_1X \\ X_2 & SX_2 + 2S_1X_1 + S_2x & tX_2 + 2t_1X_1 + t_2X \end{vmatrix}$$

$R_2 \to R_2 - SX_1,\ R_3 \to R_3 - tX_1$

$$\begin{vmatrix} X & 0 & 0 \\ X_1 & S_1X & t_1X \\ X_2 & 2S_1X_1 + S_2X & 2t_1X_1 + t_2X \end{vmatrix}$$

$$= X[S_1t_2X^2 - S_2t_1X^2] + X_3\begin{vmatrix} S_1 & t_1 \\ S_2 & t_2 \end{vmatrix}$$

Sol 20: $y = \tan^{-1}\frac{x}{\sqrt{1-u^2}}$, $x = \sec^{-1}\frac{1}{2u^2-1}$

$$\mu = \left(0, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$$

To prove $2\frac{dy}{dx} + 1 = 0$, take $u = \cos\theta$

$\therefore y = \tan^{-1}\tan\theta = \theta = \cos^{-1}u$

$$x = \sec^{-1}\frac{1}{2\cos^2\theta - 1} = \sec^{-1}\sec 2\theta = \pi - 2\theta = \pi - 2\cos^{-1}u$$

$$\therefore \frac{dy}{dx} = \frac{-d\cos^{-1}u}{2d\cos^{-1}u} = \frac{-1}{2}$$

$$\Rightarrow \frac{2dy}{dx} + 1 = 0$$

Sol 21: $\sin\left(2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right)$

$$= \frac{2\tan\tan^{-1}\sqrt{\frac{1-x}{1+x}}}{1+\left(\tan\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)^2} = \frac{2\sqrt{\frac{1-x}{1+x}}}{\frac{2}{(1+x)}} = \sqrt{1-x^2}$$

$$\therefore\ y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}+1}\right) + \sqrt{1-x^2}$$

$\therefore$ Put $x = \sin\theta$

$$\therefore \frac{x}{\left(\sqrt{1-x^2}\right)+1} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$$

$$\therefore y_1 = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\sin^{-1}x$$

$$\therefore y = \frac{1}{2}\sin^{-1}x + \sqrt{1-x^2}$$

$$y' = \frac{1}{2\sqrt{1-x^2}} + \frac{1(-2x)}{2\sqrt{1-x^2}}$$

$$= \frac{1}{2\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \frac{1-2x}{2\sqrt{1-x^2}}$$

Sol 22: $y = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$= \cot^{-1}\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{1+\sin x-(1-\sin x))}$$

$$= \cot^{-1}\frac{\left(2+2\sqrt{1-\sin^2 x}\right)}{2\sin x} = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cos^{-1}\cot\frac{x}{2},\ x \in \left(0, \frac{\pi}{2}\right) = \frac{x}{2},\ x \in \left(0, \frac{\pi}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2} \text{ or } \cot^{-1}\cot\left(\frac{\pi}{2}-\frac{x}{2}\right) = \frac{\pi}{2}-\frac{x}{2}\ x \in \left(\frac{\pi}{2}, \pi\right)$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}$$

Sol 23: $x = \phi(t)$, $y = \psi(t)$

$$\frac{dx}{dy} = \phi'(t)\ \frac{dy}{dt} = \psi'(t)$$

$$\frac{d^2x}{dt^2} = \phi''(t)\ \frac{d^2y}{dt^2} = \psi''(t)$$

$$\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)} - \frac{d^2y}{dx^2} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx} = \frac{\frac{d\frac{\psi'(t)}{\phi'(t)}}{dt}}{\frac{dx}{dt}} = \frac{\phi'(t)\psi''(t) = \psi'(t)\phi''(t)}{[\phi'(t)^2\phi'(t)]}$$

$$= \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{d^2x}{dt^2}\right)\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

Sol 24: (a) $e^{xy} + y\cos x = 2$

Differentiate the equation w.r.t. x

$$e^{xy}\left[y + x\frac{dy}{dx}\right] + \cos x\frac{dy}{dx} - y\sin x = 0$$

$$\frac{dy}{dx}\left(xe^{xy} + \cos x\right) = y\sin x - ye^{xy}$$

$$\therefore \frac{dy}{dx} = \frac{y\sin x - ye^{xy}}{xe^{xy} + \cos x}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = -y$$

at x = 0; y = 1

(i) $\quad \therefore \left.\frac{dy}{dx}\right|_{x=0} = -1$

also

$$\frac{d^2y}{dx^2}\left(xe^{xy} + \cos x\right) + \frac{dy}{dx}\left(e^{xy} + xe^{xy}\left(y + x\frac{dy}{dx}\right) - \sin x\right)$$

$$= y\cos x + \frac{dy}{dx}\sin x - ye^{xy}\left(y + x\frac{dy}{dx}\right) - \frac{dy}{dx}e^{xy}$$

(ii) $\quad \left.\frac{d^2y}{dx}\right|_{x=0} = \frac{y - y^2 - \frac{dy}{dx} - \frac{dy}{dx}(1)}{(0+1)}$

$$= \frac{1 - 1 - (-1) - (-1)}{1} = 2$$

(b) $g(x) = e^{ax} + f(x)$

$g'(x) = ae^{ax} + f'(x)$

$g''(x) = a^2e^{ax} + f''(x)$

$g'(x) + g''(x) = (a + a^2)e^{ax} + f'(x) + f''(x)$

$\therefore g'(0) + g''(0) = a + a^2 + f'(0) + f''(0) = 0$

$\Rightarrow a + a^2 - 5 + 3 = 0 \Rightarrow a^2 + a - 2 = 0 \Rightarrow a = 1, -2$

Application of Derivatives

Sol 1: $y - 2 = m(x - 1)$

curve $(y - 2)^2 = 2x^3 - 4$

tangent $2(y-2)\frac{dy}{dx} = 6x^2$

$$\left.\frac{dy}{dx}\right|_{h,k} = \frac{3h^2}{k-2}$$

$$\frac{k-2}{h-1} = \frac{3h^2}{k-2}$$

$$(k-2)^2 = 3h^2(h-1)$$

$\Rightarrow 2h^3 - 4 = 3h^3 - 3h^2$

$\Rightarrow 3h^2 - 4 = h^3 \Rightarrow h^3 - 3h^2 + 4 = 0$

$\Rightarrow (h + 1)(h^2 - 4h + 4) = 0$

$h = -1, 2$

$h = 2, \Rightarrow k = 2 \pm \sqrt{12}$

Eq. of tangent

$$y - 2 = \pm\frac{\sqrt{12}}{1}(x-1)$$

Sol 2: $y = ax^2 + bx + \frac{7}{2}$ at (1, 2)

Now at (1, 2), we will get

$2 = a + b + \frac{7}{2}$ ....(i)

The tangent will be

$y' = 2ax + b$

$y = x^2 + 6x + 10$

$y^1_{(-2,2)} = 2(-2) + 6 = 2$

$m_{normal} = \frac{-1}{2}$

$2a(1) + b = \frac{-1}{2} \Rightarrow 4a + 2b = -1$

$\Rightarrow a + b = \frac{-3}{2}$

$\Rightarrow 2a = 2;\ a = 1$

$b = -\frac{5}{2}$

Sol 3: $xy = 1 - y$

$x^2y = xy \Rightarrow y(x^2 - x) = 0$

$y = 0 \mid x = 0 \mid x = 1$

$A(0, 1), B\left(1, \frac{1}{2}\right)$

$$x^2y' + 2xy = -y' \Rightarrow y' = -\frac{2xy}{x^2+1}$$

$$y'_{(0,1)} = 0,\ y^1_{\left(1,\frac{1}{2}\right)} = \frac{-1}{2}$$

Equation of tangents

$$\frac{y-\frac{1}{2}}{x-1} = \frac{-1}{2} \Rightarrow y = -\frac{x}{2}+1$$

This gives y = 1, x = 0 or (0,1)

Sol 4:

$$y' = y\,(1+x)^{y-1}\left[\begin{array}{l}\text{if } y = (+x)^y \\ \ln y = y\ln 1 + y \\ y' = yy'\ln(1+x) + \frac{y^2}{1+x}\end{array}\right]$$

$$y' = \frac{(1+x)^{2y}}{(1+x)\left[1-(1+x)^y\ln(1+x)\right]} + \frac{2\sin x\cos x}{\sqrt{1+\sin^4 x}}$$

at x = 0, y′ = 1 + 0 = 1

$m_{normal} = -1$

Equation of normal $\Rightarrow \frac{y-1}{x-0} = -1 \Rightarrow x+y=1$

Sol 5: $x = 2t + t^2\sin\left(\frac{1}{t}\right), \qquad t \neq 0$

$= 0 \qquad t = 0$

$y = \frac{\sin t^2}{t} \qquad t \neq 0$

$= 0 \qquad t = 0$

$$x' = 2 + t^2\cos\left(\frac{1}{t}\right)\left(\frac{-1}{t^2}\right) + 2t\sin\left(\frac{1}{t}\right)$$

$$= 2 - \cos\frac{1}{t} + 2t\sin\left[\frac{1}{t}\right]$$

$$y' = \frac{2t^2\cos t^2 - \sin t^2}{t^2} = 2\cos t^2 - \frac{\sin t^2}{t^2}$$

$$\frac{y'}{x'} = \frac{2t^2\cos t^2 - \sin t^2}{t^2\left(2-\cos\frac{1}{t}+2t\sin\frac{1}{t}\right)}$$

$$\text{Slope at } (t=0) = \frac{2t^2\cos t^2 - \sin t^2}{t^2\left[1+2t\sin\frac{1}{t}+2\sin^2\frac{1}{2t}\right]}$$

Does not exist

Sol 6: $y' = 41x^2 = 2009$

$$x^2 = \frac{2009}{41} = t^2 = 49$$

$$\left(y - \frac{41t^3}{3}\right) = 2009(x-t)$$

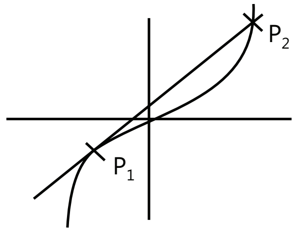
$$b - \frac{41t^3}{3} = -2009t$$

$$b = \frac{41t^3}{3} - 2009t = \frac{t}{3}\left(41t^2 - 2009.3\right)$$

$$= 7\times\left(\frac{41\times 49}{3} - 2009\right) = -9375.33$$

Sol 7: $y^2 = -\sin(x+y)\left[1+y'\right]$

$$y^1 = -\frac{\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2}$$



$\sin(x+y) = 1, \cos(x+y) = 0$

i.e. y = 0

$$x = \frac{\pi}{2}, -\frac{3\pi}{2}$$

$$\frac{y}{x-\pi/2} = -\frac{1}{2}\ \&\ \frac{y}{x+\frac{3\pi}{2}} = -\frac{1}{2}$$

$$2y + x = \frac{\pi}{2}\ 2y + x = -\frac{3\pi}{2}$$

Sol 8: $q = p^2 \quad P > 0$

$s = -\frac{8}{r} \quad r > 0, s < 0$

$y - t^2 = 2t\,(x-t)$ – tangent to curve (1) at x = t

$y + \frac{8}{z} = \frac{8}{z^2}(x-z)$ – tangent to curve (2) at x – z

Both pass through (p, q) (r, s)

$y = 2t\,x - t^2\ t = p$

Same tangent

$y = \frac{8x}{z^2} - \frac{16}{z}\ z = r$

$tz^2 = 4t^2z = 16$

$t^2z^4 = 16$

z = 1, t = 4

z + t = p + r = 5

Sol 9: (a) $y = \sqrt{36.6}$

$y = \sqrt{x} \Rightarrow y' = \dfrac{1}{2\sqrt{x}}$

$x = 36,\ \Delta x = 0.6$

$\Rightarrow f(x + \Delta x) = f(x) + f'x\ \Delta x$

$\Rightarrow f(36.6) = f(36) + \dfrac{1}{2\sqrt{36}} \times 0.6 = 6 + \dfrac{0.6}{120} = 6.05$

(b) $(26)^{1/3}$

$y = (x)^{1/3}$

$y' = \dfrac{1}{3} x^{-2/3}\ x = 27\ \Delta x = -1$

$f(26) = f(27) + \dfrac{1\ x(-1)}{3x^{2/3}} = 3 - \dfrac{1}{3 \times 9} = 3 - \dfrac{1}{27} = \dfrac{80}{27}$

(ii) $r = 9 \pm 0.03$

$v = \dfrac{4}{3}\pi r^3$

$\dfrac{dv}{v} = \dfrac{4\pi r^2 dr}{v} = \dfrac{3dr}{R} = \dfrac{3 \times 0.03}{9} = \dfrac{1}{100}$

$dv = \dfrac{\frac{4}{3}\pi r^3}{100} = \dfrac{4\pi}{300} \times 9 \times 9 \times 9 = 9.72\ \pi$

Sol 10: Mid point was (2, –1)

$\dfrac{y+1}{x-2} = 1 \Rightarrow x - y = 3$ (equation of tangent)

$x - 3 = -a^2x^2 + 5ax - 4$

$\Rightarrow a^2x^2 + x(1 - 5a) + 1 = 0$

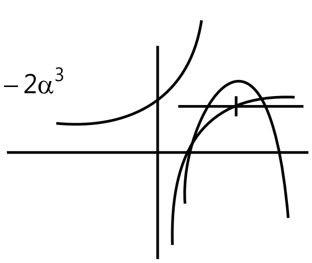
$\alpha + \beta\left(\dfrac{1-5a}{-a^2}\right) = \dfrac{5a-1}{a^2}$

$\dfrac{\alpha+\beta}{2} = 2 = \dfrac{5a-1}{2a^2} \Rightarrow 4a^2 - 5a + 1 = 0$

$a = 1, + \dfrac{1}{4}$

Sol 11: $P_1(\alpha, \alpha^3)$

$y - \alpha^3 = 3\alpha^2(x - \alpha)\ y - 3\alpha^2 x - 2\alpha^3$



$x^3 - 3\alpha^2 x + 2\alpha^3 = 0$

$x_1 + x_2 + x_3 = 0$

$x_1 = x_2 = \alpha$

$x_3 = 2\alpha \Rightarrow P_2(-2\alpha, -8\alpha^3)$

$(\alpha, -2\alpha, 4\alpha\ldots\ldots)$ forms a G.P.

Tangent at P_2

$\dfrac{y + 8\alpha^3}{x + 2\alpha} = 12\alpha^2$

$y = 12\alpha^2 x + 16\alpha^3$

$\alpha_1 + \alpha_2 + \alpha_3 = 0$

$\alpha_1 + \alpha_2 = -2\alpha$

$\alpha_3 = 4\alpha\ P_3 = (4\alpha, 64\alpha^3)$

Area of $P_1P_2P_3 = \dfrac{1}{2}\begin{vmatrix} \alpha & \alpha^3 & 1 \\ -2\alpha & -8\alpha^3 & 1 \\ 4\alpha & 64\alpha^3 & 1 \end{vmatrix}$

Area of $P_2P_3P_1 = \dfrac{1}{2}\begin{vmatrix} -2\alpha & -8\alpha^3 & 1 \\ +4\alpha & 64\alpha^3 & 1 \\ -8\alpha & -512\alpha^3 & 1 \end{vmatrix}$

$\dfrac{P_1P_2P_3}{P_2P_3P_4} = \left|\dfrac{\alpha(-72\alpha^3) - \alpha^3(-6\alpha) + 1(-128 + 32)\alpha^4}{-2\alpha(576) + 8\alpha^3(12\alpha) + 1(-2048 + 512)\alpha^4}\right|$

$= \dfrac{-72 + 6 - 96}{96 - 1536 - 1152} = \dfrac{162}{2592} = \dfrac{1}{16}$

Sol 12: $f^1 x = (fx)^2$

$\int \dfrac{dy}{y^2} = \int dx\ \ -\dfrac{1}{y} = x + c$

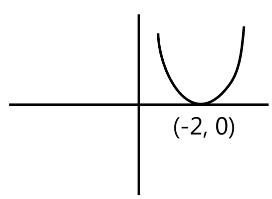
For f (0) = $\dfrac{-1}{2}$, We have c = 2

$\Rightarrow y = \dfrac{-1}{x+2}$

$\dfrac{y + \frac{1}{2}}{x} = \dfrac{1}{4}$

$4y + 2 = x$

Sol 13: $y = ax^3 + bx^3 + cx + 5$



$y' = 3ax^2 + 2bx + c$

$y'(-2) = 12a - 4b + c = 0$

$-8a + 4b - 2c + 5 = 0$

c = 3 [y^1 (x = 0) = 3)]

12a – 4b + 3 = 0

– 8a + 4b = 1

$4a = -2 \Rightarrow a = -\frac{1}{2}$ and $b = \frac{-3}{4}$

Sol 14: $y - t^3 = 3t^2(x - t)$

$\Rightarrow 8 - t^3 = 3t^2(2 - t) \Rightarrow 8 - t^3 = 6t^2 - 3t^3$

$\Rightarrow 2t^3 - 6t^2 + 8 = 0$

t = – 1, 2, 2

$m = 3t^2$, $(m)_{-1} = 3$, $(m)_2 = 12$

Sol 15: $f^3(x) = \int_0^x t\, f^2(t)dt$

$3f^2(x) = f'x = xf^2x$

$f^2x(3f'x - x) = 0$

Either f(x) = 0 (not possible)

or $f'x = \frac{x}{3}$ $M_{normal} = \frac{-3}{x} - \frac{1}{2} \Rightarrow x = 6$

$7x = \frac{x^2}{6} + c$

Equation of normal

$y - 6 - c = \frac{-1}{2}(x - 6)$

Intercept on y axis

y = 9 + c

$\left(\frac{x^2}{6} + c\right)^3 = \int_0^x \left(\frac{x^2}{6} + c\right)^2 x = \left(\frac{x^4}{36} + c^2 + \frac{x^2}{3}\right)x$

$= \int \frac{x^5}{36} + c^2x + \frac{x^3}{3}\Big|_0^x$

$\frac{x^6}{6.36} + c^3 + \frac{3c^2x^2}{6} + \frac{3x^4c}{36} = \frac{x^6}{6.36} + \frac{c^2x^2}{2} + \frac{x^4}{12} = c = 0$

Intercept is 9

Sol 16: $[y - f(p)] = f'(p)(x - p)$

$-f(p) = f'(p)2$

$\Rightarrow f'p = -\frac{1}{2}f(p) \Rightarrow \frac{f'p}{fp} = -\frac{1}{2}$

$\ln fp = \frac{-x}{2} + c \Rightarrow fp = c\, e^{-x/2}$

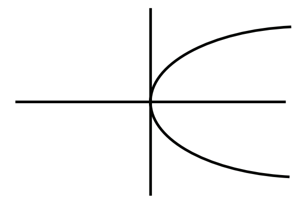
Passes through (0, 2)

f(o) = c = 2

$f(p) = 2e^{-x/2}$

Sol 17: (a) Similar to exercise (3) q.7

(b) $y = a \ln (x^2 - a^2)$



$\left|\frac{y\sqrt{1 + (y')^2}}{y'}\right| + \left|\frac{y}{y'}\right|$

$y' = \frac{2ax}{x^2 - a^2}$

$\sqrt{1 + (y')^2} = \frac{x^2 + a^2}{x^2 - a^2}$

$\frac{y}{y'} = \frac{\ln(x^2 - a^2)(x^2 - a^2)}{(2x)}$

$\Rightarrow \frac{x^2 - a^2}{2x}\ln(x^2 - a^2)\left[\frac{x^2 + a^2}{x^2 - a^2} + 1\right]$

$\Rightarrow x\ln(x^2 - a^2) = \frac{xy}{a}$ i.e. $\boxed{\alpha xy}$

$x = y^2, xy = k$

$y' = \frac{1}{2y}$ $y' = \frac{-k}{x^2}$

$y^3 = k$ intersection is $(k^{2/3}, k^{1/3})$

$y_1' \times y_2' = -1$

$\frac{-k}{2x^2y} = -1$

$k = 2x^2y = 2y^5$

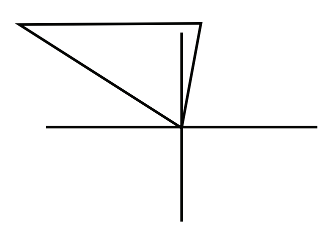
$k = 2k^{5/3}$

$k^{-2/3} = 2 \Rightarrow k^{-2} = 8$

$k = \pm \frac{1}{2\sqrt{2}}$

Sol 18: $x + 5y - y^5 = 0$ (0, 0) (0, $5^{1/4}$)

$1 + 5y' - 5y^4y' = 0$



$y' = \frac{1}{5y^4 - 5}$,

Equation of tangent

$= y = \frac{-1x}{5}$

Equation of normal y = 5x

Coordinates are (0, 0) (–25, 5), (1, 5)

Area $= \frac{1}{2}\times 5\times 26 = 65$

Sol 19: $\frac{y-\frac{1}{2}}{x-2} = 4$

$y = 4x - \frac{15}{2} \Rightarrow \frac{1}{x} = 4x - \frac{15}{2}$

$\Rightarrow 8x^2 - 15x - 2 = 0 \Rightarrow 8x^2 - 16x + x - 2 = 0$

$\Rightarrow (8x + 1)(x - 2) = 0 \Rightarrow x = -\frac{1}{8}$

$y' = \frac{-1}{x^2} = -64$

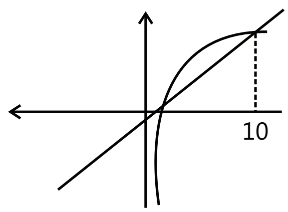
$|y'| = 64$

Sol 20: $f(x) = \ln^2 x + 2\ln x$

$y = m_1 x$

$y^1 = \frac{2\ln x + 2}{x}$

$y - (\ln^2 t + 2\ln t)$



$= \frac{2}{t}(\ln t + 1)[x - t]$

It passes through (0, 0)

$-\ln^2 t - 2\ln t = -2(1 + \ln t)$

$\ln^2 t = 2$

$\ln t = \pm\sqrt{2}$

$t = e^{\sqrt{2}}, e^{-\sqrt{2}}$ ab = 1

(ii) $5x\left[\frac{2\ln x + 2}{x}\right] - x\ln 10 - 10 = 0$

$10\ln x + 10 - x\ln 10 - 10 = 0$

$\boxed{10\ln x = x\ln 10}$

2 solution from graph

1 is x = 10

Sol 21: Given that $6y = x^3 + 2$ and also dy = 8dx

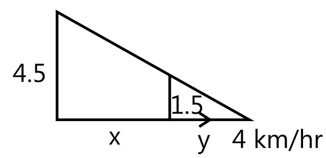
$6dy = 3x^2 dx$ (Differentiating the given equation)

$\Rightarrow \frac{x^2}{2} = 8 \Rightarrow x = \pm 4$

$y = \pm\frac{64+2}{6} = \frac{-62}{6}$ or 11

Hence (4, 11) $\left(-4, \frac{-31}{3}\right)$

Sol 22:



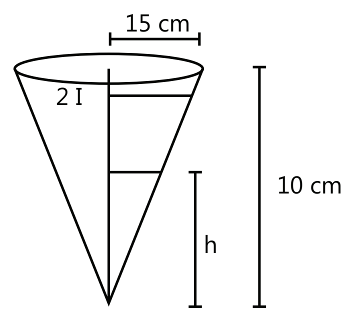
$\frac{1.5}{y} = \frac{4.5}{y+x} \Rightarrow 3y = 1.5x$

$y = \frac{x}{2}\ \frac{dy}{dx} = \frac{1}{2}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = 2$

$\frac{dx}{dt} + \frac{dy}{dt} = \frac{12}{2} = 6$

hcc shadow is lightening at rate $\frac{dy}{dt}$ i.e. 2 km/ hr

Sol 23: $\frac{dv}{dt} = -1\text{cm}^3/\text{sec}$



$\frac{dv}{dt} = c\ \text{cm}^3/\text{sec}\,c,\ \tan\theta = \frac{3}{2} = \frac{r}{h}$

$\frac{dh}{dt} = 4\ \frac{dr}{dt} = \frac{3}{2}\frac{dh}{dt} = 6$

$v = \frac{1}{3}\pi r^2 h$

$\frac{dv}{dt} = \frac{\pi}{3}\frac{r^2 dh}{dt} + \frac{\pi h}{3}(2r)\frac{dr}{dt}$

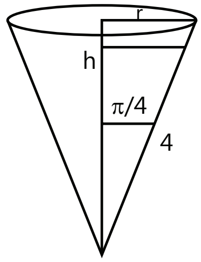
$c-1 = \frac{\pi r^2 4}{3} + \frac{2\pi}{3}hr\frac{dr}{dt}$

h = 2r = 3

$c - 1 = \frac{\pi}{3}\times 9\times 4 + \frac{2\pi}{3}\times 6.6 = 12\pi + 24\pi = 36\pi$

$c = 1 + 36\pi$

Sol 24: $\frac{dv}{dt} = -2$



$v = \frac{1}{3}\pi r^2 h$

$r = h = \frac{l}{\sqrt{2}}$

$v = \frac{\pi}{3}\frac{\ell^3}{2\sqrt{2}}$

$\frac{dv}{dt} = \frac{\pi}{6\sqrt{2}} 3\ell^2 \frac{d\ell}{dt}$

$-2 = \frac{\pi \times 16}{2\sqrt{2}}\frac{d\ell}{dt}$

$\frac{d\ell}{dt} = -\frac{1}{\pi 2\sqrt{2}} = \frac{-\sqrt{2}}{4\pi}$

Sol 25: $h = \frac{1}{6}r$

$\Rightarrow \int dv = \frac{1}{3}\pi r^2 h \Rightarrow v = \frac{1}{3}\pi\ 36h^3$

$\int dv = 12\pi h^3$

$\frac{dv}{dt} = 36\pi h^2 \frac{dh}{dt}$

$\Rightarrow 12 = 36\pi h^2 \times \frac{dh}{dt}$

$\Rightarrow \frac{1}{3\pi \times 16} = \frac{dh}{dt} = \frac{1}{48\pi} cm/sec$

Sol 26: $\frac{dA}{dt} = 2cm^2 / sec$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \Rightarrow 2 = 2\pi r\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{\pi r} \Rightarrow rdr = \frac{dt}{\pi}$

$\Rightarrow \frac{r^2}{2} = \frac{1}{\pi}\frac{28}{\pi} \Rightarrow \frac{7.28}{11.11} = r^2$

$\Rightarrow r = \frac{14}{\pi}$

$\frac{dr}{dt} = \frac{7 \times 11}{22 \times 14} = \frac{1}{4}$ cm/sec

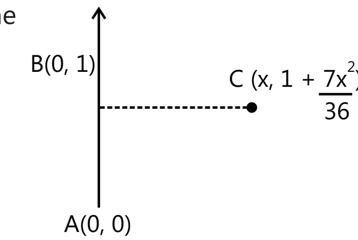
Sol 27: A (0, 0)

$C\left(t, 1 + \frac{7t^2}{36}\right)$

B (0, 1) initially

Co-ordinate of B at time (0, 1 + 2t)

Co-ordinate of C at time



$1 + \frac{7x^2}{36} = 1 + 2t$

$x = 6\sqrt{\frac{2t}{7}}$

$BC = x = 6\sqrt{\frac{2t}{7}}$

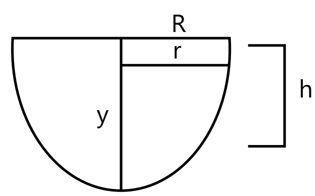
AB = 1 + 2t

$\text{Area} = \frac{1}{2}(AB)(BC) = 3(1+2t)\sqrt{\frac{2t}{7}} = 3\sqrt{\frac{2t}{7}} + 6t\sqrt{\frac{2t}{7}}$

$\frac{dA}{dt} = \frac{3}{\sqrt{7t}} + \frac{3}{2} \times 6\sqrt{\frac{2t}{7}}$

$\left.\frac{dA}{dt}\right|_{t=\frac{7}{2}} = 3\sqrt{2} + 9$

Sol 28:



$\frac{dv}{dt} = \frac{k}{r}$

$v = \frac{4}{3}\pi r^3$

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$4\pi r^2 \frac{dr}{dt} = \frac{k}{r} \Rightarrow \frac{dr}{dt} = \frac{k}{4\pi r^3}$

$\Rightarrow \pi r^4\Big|_1^2 = kt\Big|_0^{15} \Rightarrow \pi 15 = 15k$

$\Rightarrow k = \pi$

$\Rightarrow \pi r^4\Big|_1^t = \pi\Big|_0^t$

$\Rightarrow \pi\left(r^4 - 1\right)\pi t$

$\Rightarrow r = (1+t)^{1/4}$

$\Rightarrow \frac{dv}{dt} = \frac{k}{(1+t)^{1/4}}$

$\Rightarrow dv = \pi(1+t)^{-1/4}\,dt$

$\Rightarrow \int_v^{27v} v = \frac{4\pi}{3}(1+t)^{3/4}\Big|_0^t$

$\Rightarrow 26v = \frac{4\pi}{3}\left[(1+t)^{3/4}-1\right]$

$\Rightarrow v = \frac{4\pi}{3} \Rightarrow 27 = (1+t)^{3/4}$

$\Rightarrow (1+t)^{1/4} = 3 \Rightarrow t = 80$ sec

Sol 29: $\frac{dv}{dt} = 6m^3$ / min

$v = \frac{\pi}{3}y^2(3R-y)$

$\frac{dv}{dt} = \frac{\pi}{3}\left[6Ry\frac{dy}{dt} - 3y^2\frac{dy}{dt}\right] - 6 = \pi\left[2Ry - y^2\right]y'$

For $y = 8$, $y' = \frac{-6}{8\pi(2R-y)} = \frac{-6}{8\pi(18)} = -\frac{1}{24\pi}$ m / min

$\tan\theta = \frac{r}{y}$

$\frac{dr}{dt} = \frac{dy}{dt}\tan\theta = \frac{-1\times 5}{24\pi\times 12} = -\frac{5}{288\pi}$

Exercise 2

Methods of Differentiation

Single Correct Choice Type

Sol 1: (D) $y = \frac{x}{a+\frac{x}{b+y}}$

$\Rightarrow ay + \frac{xy}{b+y} = x$

$aby + ay^2 + xy = xb + xy$

$\therefore aby + ay^2 = xb$

$\Rightarrow ab\frac{dy}{dx} + 2ay\frac{dy}{dx} = b$

$\therefore \frac{dy}{dx} = \frac{b}{2ay+ab}$

Sol 2: (B) $f(x) = e^x + x$

$f'(x) = 1 + e^x$

Also $f(f^{-1}(x)) = x$

$f'(f^{-1}(x))\,(f^{-1}(x))' = 1$

$\therefore (f^{-1}(x))' = \frac{1}{f'\left[f^{-1}(x)\right]}$

$\therefore f(\ln 2) = y$

$\therefore f^{-1}(y) = \ln 2$

$\because f'(\ln 2) = 1 + e^{\ln 2} = 1 + 2 = 3$

$\therefore [f^{-1}(y)] = \frac{1}{f'\left[f^{-1}(y)\right]} = \frac{1}{f'(\ell n2)} = \frac{1}{3}$

Sol 3: (C) $y(x) = f^2(x) + g^2(x)$

$y'(x) = 2f(x)\,g(x) + 2g(x)\,g'(x)$

$= 2f(x)\,g(x) - 2f(x)g(x) = 0$

$\therefore y(x) = a = f^2(x) + g^2(x)$

$y(5) = a = f^2(5) + g^2(5) = (2)^2 + (2)^2$

$\therefore a = 8$

$\therefore y(10) = f^2(10) + g^2(10) = 8$

Sol 4: (B) $y(x) = x^{\left[\left(\frac{\ell+m}{m-n}\right)\frac{1}{n-\ell}+\left(\frac{m+n}{n-\ell}\right)\frac{1}{\ell-m}+\left(\frac{n+\ell}{\ell-m}\right)\frac{1}{m-n}\right]}$

$= x^{\left(\frac{1}{n-\ell}\right)\left[\frac{\ell+m}{m-n}+\frac{m+n}{\ell-m}\right]+\left(\frac{n+\ell}{\ell-m}\right)\frac{1}{m-n}}$

$= x^{\left(\frac{1}{n-\ell}\right)\left[\frac{\ell^2-m^2+m^2-n^2}{(m-n)(\ell-m)}\right]+\frac{(n+\ell)}{(\ell-m)}\times\frac{1}{(m-n)}}$

$= x^{-\frac{-(\ell+n)}{(m-n)(\ell-m)}+\frac{(n+\ell)}{(\ell-m)(m-n)}} = x^0 = 1$

$\therefore \frac{dy}{dx} = 0$

Sol 5: (D) $f(x) = (x^x)^x$ $g(x) = x^{(x^x)}$

$\log f(x) = x\log x^x = x^2\log x$

$\frac{1}{f(x)}f'(x) = 2x\log x + x$

$f'(x) = (x^x)^x[2xc\log x + x]$

$\log(x) = x^x\log x$

$\frac{1}{g(x)}g'(x) = \frac{x^x}{x} + \log x\frac{dx^x}{dx}$

$= x^{x-1} + (\log x)\,(x^x(\log x+1))$

$\therefore g'(x) = x^{\left(x^x\right)}\left[x^{x-1} + x^x(\log x + 1)\log x\right]$

$f'(1) = [2.1.\log 1 + 1]\,(1^1)^1 = 1$

$g'(1) = (1)^{(1^1)}[1^{1-1} + 1^1(\log 1+1)\log 1] = 1$

Sol 6: (C) $y^{1/m} + y^{-1/m} = 2x$

Let $y^{1/m} = a$

$\therefore a + \frac{1}{a} = 2x \Rightarrow a^2 - 2ax + 1 = 0$

$\Rightarrow a = x + \sqrt{x^2-1}$

$y^{1/m} = x + \sqrt{x^2-1}$

$\therefore y = \left(x+\sqrt{x^2-1}\right)^m$

$\Rightarrow y' = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right) = \frac{m\left(x+\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}}$

$\Rightarrow y'\sqrt{x^2-1} = my$

$\Rightarrow y''y'\sqrt{x^2-1} + \frac{y'2x}{2\sqrt{x^2-1}} = my'$

$\Rightarrow y''(x^2-1) + xy' = my'\sqrt{x^2-1}$

$\therefore \frac{y''(x^2-1)+xy'}{y} = m\sqrt{x^2-1}\,\frac{y'}{y}$

$= m\sqrt{x^2-1} \times \frac{m}{\sqrt{x^2-1}} = m^2$

Sol 7: (C) $y^2 = P(x)$

$2.y\frac{dy}{dx} = P'(x)$

$\Rightarrow 2.yy'' + 2(y')^2 = P''(x)$

Multiply this equation by y^2

$\Rightarrow 2y^3y'' + 2(yy')^2 = P''(x)y^2 = P''(x)\,P(x)$

$\therefore 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = [P''(x)\,P(x) - 2(yy')^2]'$

$= [P''(x)\,P(x)]' - 2\left[\frac{[P'(x)]^2}{4}\right]'$

$= P'''(x)\,P(x) + P''(x)P'(x) - P'(x)\,P''(x) = P'''(x)P(x)$

Sol 8: (D) $f(x) = -\frac{x^3}{3} + x^2\sin 1.5a - x.\sin a.\sin 2a$

$- 5\sin^{-1}(a^2 - 8a + 17)$

$f'(x) = -x^2 + 2x\sin 1.5a - \sin a\sin 2a$

$f'(\sin 8) = -\sin^2 8 + 2\sin 8\sin 1.5a - \sin a\sin 2a$

Also $-1 \le a^2 - 8a + 17 \le 1$

$-1 \le (a-4)^2 + 1 \le 1$

$-2 \le (a-4)2 \le 0$

$\Rightarrow a = 4$

$\therefore f'(\sin 8) = -\sin^2 8 + 2\sin 8\sin 6$

$- \sin 4\sin 8$

$f'(\sin 8) = -\sin^2 8 + \sin 8(2\sin 6 - \sin 4)$

$\because |2\sin 6| < |\sin 4|\ \pi < 4 < 6 < 2p$

and $\sin 8 < 0$

$\therefore f'(\sin 8) < 0$

Sol 9: (B) $y = \frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$

$\cos 6x + \cos 4x + 5\cos 4x + 5\cos 2x$

$+ 10(\cos 2x + 1)$

$\Rightarrow 2\cos\frac{10x}{2}\cos\frac{2x}{2}$

$+2\times 5\cos\frac{(4+2)x}{2}\cos\frac{x}{2} + 10 \times 2\cos^2 x$

$= 2\cos x[\cos 5x + 5\cos 3x + 10\cos x]$

$\therefore y = 2\cos x$

$\therefore \frac{dy}{dx} = -2\sin x$

Sol 10: (D) $y = R(1 - \cos\theta)$

$x = R(\theta - \sin\theta)$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{(1-\cos\theta)}$

$= \frac{1+\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta$

$\frac{d^2y}{dx^2} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{d\theta}}{\frac{dx}{d\theta}}$

$= \frac{-\operatorname{cosec}\theta\cot\theta - \operatorname{cosec}^2\theta}{R(1-\cos\theta)}$

$= -\operatorname{cosec}\theta\frac{(1+\cos\theta)}{\sin\theta} \times \frac{1}{R}\frac{(1+\cos\theta)}{\sin^2\theta}$

$= -\frac{1}{\sin^4\theta}(1+\cos\theta)^2 \times \frac{1}{R}$

$= -\frac{1}{R}\left(\frac{1+\cos\theta}{\sin^2\theta}\right)^2 = \frac{-1}{R}\left(\frac{1}{1-\cos\theta}\right)^2$

$\therefore \left.\frac{d^2y}{dx^2}\right|_{\theta=\pi} = -\frac{1}{R}\left(\frac{1}{1-(-1)}\right)^2 = -\frac{1}{4R}$

Sol 11: (B) $f(x) = (1 + x)^n$

$f'(x) = n(1 + x)^{n-1}$

$f''(x) = n(n-1)(1 + x)^{n-2}$

$f^n(x) = n(n-1) \ldots\ldots 2.1\ (1+x)^0$

$f(0) = 1,\ f'(0) = n,\ f''(0) = n(n-1)$

$f(0) + f'(0) + \ldots + \frac{f^n 0)}{n!}$

$= 1 + n + \frac{n(n-1)}{2!} \ldots\ldots \frac{n(n-1)\ldots(2.(1)}{n!}$

$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

Sol 12: (D) $y = e^{4x} + 2e^{-x}$

$\frac{dy}{dx} = 4e^{4x} - 2e^{-x}$

$\frac{d^2y}{dx^2} = 16e^{4x} + 2e^{-x}$

$\frac{d^3y}{dx^3} = 64e^{4x} - 2e^{-x}$

$\frac{d^3y}{dx^3} - 13\frac{dy}{dx} = 64e^{4x} - 2e^{-x} - 13(4e^{4x} - 2e^{-x})$

$= 12e^{4x} + 24e^{-x} = 12y$

$\therefore K = 12$

Sol 13: (C) $x^4 + 3x^2y^2 + 7xy^3 + 4x^3y - 15y^4 = 0$

$\Rightarrow (x-y)(x^3 + 5x^2y + 8xy^2 + 5y^3) = 0$

This is in the form f(x, y) g(x, y) = 0 where f(x, y) = 0 and g(x, y) ≠ 0 at $P(x_1, y_1)$

$\Rightarrow f'(x, y)\, g(x, y) + f(x, y)\, g'(x, y) = 0 \Rightarrow f'(x_1, y_1) = 0$

Also

$f''(x, y)\, g(x, y) + 2 f'(x, y)\, g'(x, y) + f(x, y) g''(x, y) = 0$

$\Rightarrow f''(x_1, y_1) = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$ at (1, 1)

Sol 14: (C) $f(x) = e^{e^x}$ $g(x) = f^{-1}(x),\ f(g(x)) = x$

$f'(g(x)\, g'(x) = 1$

$g'(x) = \frac{1}{f'(g(x))}$

$g(2) \Rightarrow 2 = e^{e^x} \Rightarrow x = \ln(\ln 2)$

$\therefore g(2) = \ln\ln 2$

$\therefore g'(2) = \frac{1}{f'(g(2)} = \frac{1}{f'(\ln\ln 2)}$

$f'(x) = e^x . e^{e^x}$

$\therefore f'(\ln\ln 2) = e^{\ln\ln} . e^{e^{\ln\ln 2}} = 2\ln 2$

Sol 15: (C)

$y = \tan^{-1}\left(\frac{1-2\ln|x|}{1+2\ln|x|}\right) + \tan{-1}\left(\frac{3-2\ln|x|}{1-6\ln|x|}\right)$

Let $3 = \tan\alpha$, $2\ln|x| = \tan\beta$

$\therefore y = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\beta\right)\right) + \tan^{-1}(\tan(\alpha+\beta))$

$= \frac{\pi}{4} - \beta + \alpha + \beta$

$y = \frac{\pi}{4} + \alpha = \frac{\pi}{4} + \tan^{-1}3$

$\therefore \frac{dy}{dx} = 0$

Sol 16: (B) $\lim\limits_{x\to 0^+}(x^{x^x} - x^x)$

$x^x = e^{x\ln x}$

$\therefore \lim\limits_{x\to 0} x^x = \lim\limits_{x\to 0} e^{x\ln x} = \lim\limits_{x\to 0} e^{\frac{\ln x}{\left(\frac{1}{x}\right)}}$

$= \lim\limits_{x\to 0} e^{\frac{(\ln x)'}{\left(\frac{1}{x}\right)'}} = \lim\limits_{x\to 0} e^{\frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}} = \lim\limits_{x\to 0} e^{-x} = 1$

$\lim\limits_{x\to 0^+} x^{x^x} = \lim\limits_{x\to 0^+}(x)^{(x^x)} = \lim\limits_{x\to 0^+}(x)^1 = 0$

$\therefore \lim\limits_{x\to 0^{-1}}\left(x^{x^x} - x^x\right) = 0 - 1 = -1$

Sol 17: (B) $\lim\limits_{x\to 0}\left\{(\cot x)^x + (1-\cos x)^{\operatorname{cosec} x}\right\}$

$\lim\limits_{x\to 0} e^{x\ln\cot x} + \lim\limits_{x\to 0} e^{\frac{\ln(1-\cos x)}{\sin x}}$

$\lim\limits_{x\to 0} e^{\frac{\ln\cot x}{\left(\frac{1}{x}\right)}} + \lim\limits_{x\to 0} e^{\text{cosec}x\ln(1-\cos x)}$

$\lim\limits_{x\to 0} e^{\text{cosec}x\ln(1-\cos x)}$ doesn't exist as LHL ≠ RHL

Multiple Correct Choice Type

Sol 18: (B, C)

$f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{\tan\theta}\right)$

Put x = tanθ

$\therefore f(x) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$,

$\tan\theta \neq 0$

$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\tan\left(\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{\tan^{-1}x}{2}$

$\therefore f(-x) = \frac{\tan^{-1}(x)}{2} = -\frac{\tan^{-1}x}{2} = -f(x)$

∴ f(x) is a odd function

also $f'(x) = \left(\frac{1}{2}\tan^{-1}x\right) = \frac{1}{2(1+x^2)}$, $x \in R - \{0\}$

Sol 19: (A, B, C)

y = tanx tan2xtan3x

x+2x–3x = 0

⇒ tanx+tan2x–tan3x = tanxtan2xtan3x = y

$\Rightarrow y' = 3\sec^2 3x - \sec^2 x - 2\sec^2 2x$

or y′ = (tan3x)′tanxtan2x + (tanx)′tan2xtan3x

+ (tan2x)′tanxtan3x

$= 3\sec^2 3x \tan x \tan 2x + \sec^2 x \tan 2x\tan 3x$

$+ 2\sec^2 2x\tan x\tan 3x$

$= \frac{3\tan x\tan 2x\tan 3x}{\sin 3x\cos 3x} + \frac{\tan x\tan 2x\tan 3x}{\sin x\cos x}$

$+ \frac{2\tan x\tan 2x\tan 3x}{\sin 2x\cos 2x}$

= 2tanxtan2xtan3x ×

$\left[\frac{3}{2\sin 3x\cos 3x} + \frac{1}{2\sin x\cos x} + \frac{2}{2\sin 2x\cos 2x}\right]$

$= 2y[2\text{cosec}6x + 2\text{cosec}4x + \text{cosec}2x]$

Sol 20: (A, C, D)

$y = \sqrt{x\sqrt{x+\sqrt{x+.....\infty}}}$

$\Rightarrow y = \sqrt{x+y}$

$\Rightarrow y^2 - y = x \Rightarrow (y-1) = \frac{x}{y}$

$\therefore 2y\frac{dy}{dx} - \frac{dy}{dx} = 1$

$\therefore \frac{dy}{dx} = \frac{1}{2y-1} = \frac{1}{2(y-1)+1} = \frac{1}{2.\frac{x}{y}+1} = \frac{y}{2x+y}$

also $y = \frac{1\pm\sqrt{1+4x}}{2}$

$\therefore \frac{dy}{dx} = \frac{1}{2\frac{(1\pm\sqrt{1+4x})}{2}-1} = \frac{1}{\pm\sqrt{1+4x}}$

Sol 21: (A, B, C, D)

$2^x + 2^y = 2^{x+y}$

$2^x \ln 2 + 2^y \ln 2\frac{dy}{dx} = 2^{x+y}\ln 2\left(1+\frac{dy}{dx}\right)$

$(2^y - 2^{x+y})\frac{dy}{dx} = 2^{x+y} - 2^x$

$\frac{dy}{dx} = \frac{2^x(2^y-1)}{2^y(1-2^x)} = \frac{2^x(1-2^y)}{2^y(2^x-1)}$

Also $2^{x+y} - 2^y = 2^y(2^x-1) = 2^x$

or $2^{x+y} - 2^x = 2x(2^y-1) = 2^y$

$\therefore \frac{dy}{dx} = -\frac{2^x}{2y}$ or $\frac{2^y(2^x-1)}{2^x} = 1$

$\therefore \frac{dy}{dx} = (1-2^y)$ or $\frac{2^x(2^y-1)}{2^y} = 1$

$\therefore \frac{dy}{dx} = -\frac{1}{(2^x-1)} = \frac{1}{1-2^x}$

Sol 22: (A, B, C)

$\sqrt{y+x} + \sqrt{y-x} = c$

$$\frac{1}{2\sqrt{y+x}}\left(\frac{dy}{dx}+1\right) + \frac{1}{2\sqrt{y-x}}\left(\frac{dy}{dx}-1\right) = 0$$

$$\frac{dy}{dx}\left(\frac{\sqrt{y-x}+\sqrt{y+x}}{\sqrt{y^2-x^2}}\right) = \frac{\sqrt{y+x}-\sqrt{y-x}}{\sqrt{y^2-x^2}}$$

Application of Derivatives

Single Correct Choice Type

Sol 1: (C) $3x^2 + 4xy + 5y^2 - 4 = 0$

$$y' = -\frac{(3x+2y)}{(2x+5y)}$$

$y' = 0$ when $y = -\frac{3}{2}x$

and $y' = \infty$ when $y = -\frac{2}{5}x$

So angle is 90º.

Sol 2: (D) $x = \sec^2 t$, $y = \cot t$

$\Rightarrow x = 1+\frac{1}{y^2}$

At $t = \frac{\pi}{4}$, x = 2, y = 1

P (2, 1)

$\Rightarrow \frac{y-1}{x-2} = y' = \frac{-1}{2} \Rightarrow 2y-2 = 2-x \Rightarrow x + 2y = 4$

$\Rightarrow 4-2y = \frac{y^2+1}{y^2}$ putting the value of x in first equation

$\Rightarrow 4y^2-2y^3 = y^2+1 \Rightarrow 2y^3-3y^2+1 = 0$

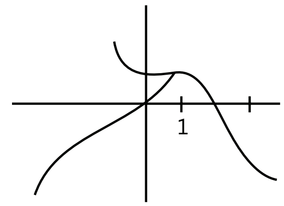
$\Rightarrow (y-1)(2y^2-y-1) = 0 \Rightarrow (y-1)(2y+1)(y-1)$

$y = -\frac{1}{2}$; x = 5; $\left(5,-\frac{1}{2}\right)$

$$PQ = \sqrt{9+\frac{9}{4}} = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$$

Sol 3: (C) $f(x) = \begin{cases} x^{3/5} & x \le 1 \\ -(x-2)^3 & x > 1 \end{cases}$

$f'(x) = \begin{cases} \frac{3}{5}x^{-3/5} & x \le 1 \\ -3(x-2)^2 & x > 1 \end{cases}$



$x = 1 \left.\frac{3}{5}\right|, -3$

x = 1 is critical point

$$f''(x) = \begin{Bmatrix} \frac{-9}{25}x^{-8/5} & x \le 1 \\ -6(x-2) & x > 1 \end{Bmatrix}$$

$\left.\begin{matrix} x = 0 \\ x = 2 \end{matrix}\right\}$ critical point

at x = 2, f″ (2) changes its sign

Sol 4: (A) x = a (2 cos t – cos 2t)

y = a (2 sin t – sin 2t)

$$\frac{dy}{dt} = a(2\cos t - 2\cos 2t)$$

$$\frac{dx}{dt} = a(-2\sin t + 2\sin 2t)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos t - \cos 2t}{\sin 2t - \sin t} = \frac{\cos t - 2\cos^2 t + 1}{2\sin t\cos t - \sin t} = 0$$

$\Rightarrow 2\cos^2 t - \cos t - 1 = 0$

$\Rightarrow 2\cos^2 t - 2\cos t + \cos t - 1 = 0$

$\Rightarrow (2\cos t + 1)(\cos t - 1)$

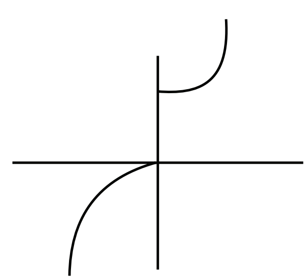
cos t = 1, cos t = $\frac{-1}{2}$

t = 0, 2t, t = $\frac{2\pi}{3}, \frac{4\pi}{3}$

t = 0 is not possible as $\frac{dy}{dx}$ is not defined

$$t = \frac{4\pi}{3} - \frac{2\pi}{3} = \frac{2\pi}{3}$$

Sol 5: (C)



$$\frac{y-a^n}{x-a} = m = \frac{-1}{y^1} = \frac{-1}{nx^{n-1}}$$

$y^1 = nx^{n-1}$

$y = a^n + \frac{a}{na^{n-1}}$

$b = a^n + \frac{aa^{1-n}}{n}\ a^n + \frac{a^{2-n}}{n} = \frac{na^{2n}+a^2}{na^n}$

$\lim_{a\to 0} b = \frac{2n^2a^{2n-1}+2a}{n^2a^{n-1}} = \frac{2n^2(2n-1)a^{2n-2}+2}{n^2(n-1)a^{n-2}}$,

value of b exist & equal to $\frac{1}{2}$

only if (n = 2)

Sol 6: (B) $f(x) = \begin{bmatrix} -x^2 & x<0 \\ x^2+8 & x\ge 0 \end{bmatrix}$

For points $(a, -a^2)$

$\frac{y+a^2}{x-a} = -2a$

$\Rightarrow y + a^2 = 2a^2 - 2ax$

$\Rightarrow y = a^2 - 2ax = x^2 + 8$

$\Rightarrow x^2 + 2ax - a^2 + 8 \Rightarrow 4a^2 = -4(8 - a^2)$

At a = – 2

$\Rightarrow \frac{y+4}{x+2} = +4$

Therefore x intercept = – 1

Sol 7: (D) $y = \frac{1}{2+\cos^2 x}$

$\Rightarrow y' = -\frac{[2\cos x \sin x]}{(2+\cos^2 x)^2}$

$x = \frac{\pi}{2}|0|\pi$ For x=0 or x=$\frac{\pi}{2}$

$\Rightarrow y = \frac{1}{3}(\text{if } x=0), \frac{1}{2}\left(\text{if} x = \frac{\pi}{2}\right)$

Sol 8: (A) At P (a, b), the equation will be

$\frac{y-b}{x-a} = M$ where M is the slope

$\Rightarrow \frac{n}{a}\left(\frac{x}{a}\right)^{n-1} + \frac{n}{b}\left(\frac{y}{b}\right)^{n-1} y' = 0$

$\Rightarrow \frac{n}{a} + \frac{n}{a}y' = 0$

$\Rightarrow y' = \frac{-b}{a}$

$\Rightarrow \frac{y-b}{x-a} = \frac{-b}{a}$

$\Rightarrow ay + bx = 2ab$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = 2$

Multiple Correct Choice Type

Sol 9: (A, D) We can write

$xy = k$ ... (i)

$\frac{x}{a} + \frac{y}{b} = 1$ ... (ii)

Solving these two equations, we get

$x^2b + abx + ka = 0$

For D = 0, $(ab)^2 - 4abk = 0$

ab (ab – 4k) = 0

$\Rightarrow ab + 4k$

$\Rightarrow ab > 0$

Hence a > 0, b > 0 or a < 0, b < 0

Sol 10: (A, B) $\sqrt{xy} = a + x$

$\frac{(xy'+y)}{2\sqrt{xy}} = 1$

$y' = \frac{2\sqrt{xy}-y}{x}$

$\frac{y - \frac{(a+t)^2}{t}}{x-t} = \frac{2\sqrt{t}\left(\frac{a+t}{\sqrt{t}}\right) - \frac{(a+t)^2}{t}}{t}$

$\frac{y - \frac{(a+t)^2}{t}}{x-t} = \frac{2\frac{(a+t)t}{t} - \frac{(a+t)(a+t)}{t}}{t}$

$= \frac{a+t}{t}\left[1 - \frac{a}{t}\right]$

$= \frac{a+t}{t^2}(t-a) = \frac{t^2-a^2}{t^2}$

x intercept will be

$\Rightarrow t - \frac{(t+a)^2 t^2}{t(t^2-a^2)} = \frac{t-(t+a)t}{(t-a)} = \frac{-2at}{t-a}$

y intercept will be

$$= \frac{(a+t)^2}{t} - \frac{(t^2-a^2)}{t} = \frac{2a^2+2at}{t} = \frac{2at}{a-t}$$

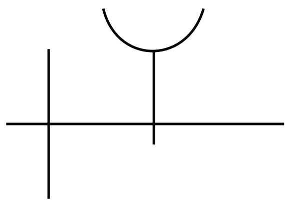
$= a^2 - t^2 = t^2 \Rightarrow a^2 = 2t^2$

$t = \frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}$ [A, B]

Sol 11: (B, D) $2x - 3 = x^2 + px + q$

$p + q + 1 = -1$

$p + q = -2$



$y = \left[q - \frac{p^2}{4}\right]$ is minimum

$= -2 - p - \frac{p^2}{4}$

$\Rightarrow -p - \frac{2p}{4} = 0$

Therefore $\boxed{\begin{array}{l} p=-2 \\ a=0 \end{array}}$

Least distance is $\left|0 - \frac{4}{4}\right| = 1$ D is correct

Sol 12: (A, B) Given that $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$

$f'(x) = x^2 - 5x + 7$

Equation of triangle will be

$$\frac{y - \left(\frac{t^3}{3} - \frac{5t^2}{2} + 7t + 4\right)}{x-t} = t^2 - 5t + 7$$

x intercept will be

$$x - t = \frac{-\left(\frac{t^3}{3} - \frac{5t^2}{2} + 7t - 4\right)}{t^2 - 5t + 7}$$

$$\Rightarrow x = -\frac{\left(\frac{t^3}{3} - \frac{5t^2}{2} + 7t - 4\right)}{t^2 - 5t + 7} + t \quad \text{... (i)}$$

y intercept will be

$$y = -t(t^2 - 5t + 7) + \left(\frac{t^3}{3} - \frac{3t^2}{2} + 7t - 4\right) \quad \text{... (ii)}$$

Equating (i) and (ii) and keeping them opposite in sign

$$\frac{-\left(\frac{t^3}{3} - \frac{5t^2}{2} + 7t - 4\right)}{t^2 - 5t + 7} + t = t(t^2 - 5t + 7) - \left(\frac{t3}{3} + \frac{5t^2}{2} - 7t + 4\right)$$

Solving above, we get t = 2,3

Therefore, co-ordinates are $\left(2, \frac{8}{3}\right)\left(3, \frac{7}{2}\right)$

Sol 13: (A, B, D)

$y \cot x = y^3 \tan x$

$y^2 = \cot^2 x$

$\Rightarrow y = \cot x, -\cot x$

$\left(-\frac{\pi}{4}, -1\right)\left(-\frac{\pi}{4}, +1\right)$

$$\frac{y+1}{x+\frac{\pi}{4}} = \frac{-1}{\frac{1}{2}} \Rightarrow y + 1 = -2x - \frac{\pi}{2} \Rightarrow 4x + 2y = 2 + \pi$$

$$\frac{y-1}{x+\frac{\pi}{4}} = \frac{1}{\frac{1}{2}} \Rightarrow y - 1 = 2x + \frac{\pi}{2} = 4x - 2y = 2 + \pi$$

Sol 14: (A, B)

$x = a(t + \sin t \cos t)$

$y = a(1 + \sin t)^2$

$$y' = \frac{2a(1+\sin t)\cos t}{a(1 - \sin^2 t + \cos^2 t)}$$

$$= \frac{2(1+\sin t)\cos t}{2\cos^2 t}$$

$$\tan\theta = \frac{1+\sin t}{\cos t}$$

$$\theta = \tan^{-1}\left(\frac{1+\sin t}{\cos t}\right) = \frac{\pi + 2t}{4}$$

Sol 15: (B, D) $y = t^3 - 4t^2 - 3t$

$y = 2t^2 + 3 - 5$

$$\frac{dy}{dt} = 4t + 3$$

$$\frac{dx}{dt} = 3t^2 - 8t - 3$$

$\frac{dy}{dt} = \frac{4t+3}{3t^2 - 8t - 3}$

$\frac{dy}{dx} = 0, t = \frac{-3}{4}$, H=1

$\frac{dy}{dx}$ = not defined at $3t^2 - 8t - 3 = 0$

$3t^2 - 9t + t - 3 = 0$

$(3t + 1)(t - 3) = 0$

$t = 3, \frac{-1}{3}$

v = 2

Previous Years' Questions

Sol 1: (B, C)

Given, $xy = 1 \Rightarrow y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$

Thus, slope of normal = x^2 (Which is always positive) and it is given ax + by + c = 0 is normal whose slope $= -\frac{a}{b}$

$\Rightarrow -\frac{a}{b} > 0$ or $\frac{a}{b} < 0$,

$\therefore$ a and b are of opposite sign.

Sol 2: (B, D) Given, $4x^2 + 9y^2 = 1$

On differentiating w.r.t. x, we get

$8x + 18y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{8x}{18y} = -\frac{4x}{9y}$

The tangent at point (h, k) will be parallel to 8x = 9y, then $-\frac{4h}{9k} = \frac{8}{9}$

$\Rightarrow h = -2k$

Point (h, k) also lies on the ellipse

$\therefore 4h^2 + 9k^2 = 1$ ... (ii)

On putting value of h in Eq. (ii), we get

$4(-2k)^2 + 9k^2 = 1$

$\Rightarrow 16k^2 + 9k^2 = 1$

$\Rightarrow 25k^2 = 1$

$\Rightarrow k^2 = \frac{1}{25}; k = \pm\frac{1}{5}$

Thus, the point where the tangents are parallel to 8x = 9y are $\left(-\frac{2}{5}, \frac{1}{5}\right)$ and $\left(\frac{2}{5}, -\frac{1}{5}\right)$.

Therefore, (b) and (d) are true answers.

Sol 3: Given, $y^3 - 3xy + 2 = 0$

$\Rightarrow 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$

$\Rightarrow \frac{dy}{dx}(3y^2 - 3x) = 3y$

$\Rightarrow \frac{dy}{dx} = \frac{3y}{3y^2 - 3x}$

Thus, the point where tangent is horizontal. The slope of tangent is 0

$\therefore \frac{dy}{dx} = 0 \Rightarrow \frac{3y}{3y^2 - 3x} = 0$

$\Rightarrow$ y = 0 but y = 0 does not satisfy the given equation of the curve therefore y cannot lie on the curve.

So, H = ϕ(null set)

For the point where tangent is vertical, then $\frac{dy}{dx} = \infty$

$\Rightarrow \frac{y}{y^2 - x} = \infty$

$\Rightarrow y^2 - x = 0 \Rightarrow y^2 = x$

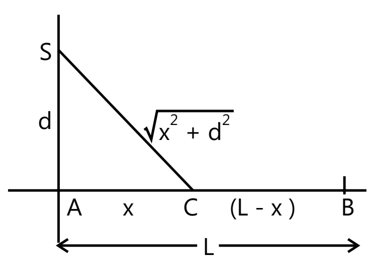
On putting this value in the given equation of the curve, we have

Sol 4: Let the house of the swimmer be at B

$\therefore$ AB = L km

Let the swimmer land at C, on the shore and let

AC = x km



$\therefore SC = \sqrt{x^2 + d^2}$ and CB = (L – x)

$\therefore \text{Time} = \frac{\text{Distance}}{\text{Speed}}$

Time from S to B = time from S to C + time from C to B.

$\therefore$ T = $\dfrac{\sqrt{x^2+d^2}}{u} + \dfrac{L-x}{v}$

Let f(x) = T = $\dfrac{1}{u}\sqrt{x^2+d^2} + \dfrac{L}{v} - \dfrac{x}{v}$

$\Rightarrow f'(x) = \dfrac{1}{u}.\dfrac{1.2x}{2\sqrt{x^2+d^2}} + 0 - \dfrac{1}{v}$

For maximum or minimum,

Put f ′ (x) = 0

$\Rightarrow v^2x^2 = u^2(x^2 + d^2)$

$\Rightarrow x^2 = \dfrac{u^2d^2}{v^2-u^2}$

$\therefore$ f ′(x) = 0

at $x = \pm\dfrac{ud}{\sqrt{v^2-u^2}}$, (v > u)

But $x \neq \dfrac{-ud}{\sqrt{v^2-u^2}}$

$\therefore$ We consider $x = \dfrac{ud}{\sqrt{v^2-u^2}}$

Now, $f''(x) = \dfrac{1}{u}\dfrac{d^2}{\sqrt{x^2+d^2}(x^2+d^2)} > 0$ for all x

$\therefore$ f has minimum at $x = \dfrac{ud}{\sqrt{v^2-u^2}}$

Sol 5: Given $y = \dfrac{x}{1+x^2}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(1+x^2).1-x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$

Let $\dfrac{dy}{dx}$ = g(x) (i.e. slope of tangent)

$\therefore g(x) = \dfrac{1-x^2}{(1+x^2)^2}$

$\Rightarrow g'(x) = \dfrac{(1+x^2)^2.(-2x)-(1-x^2).2(1+x^2).2x}{(1+x^2)^4}$

$= \dfrac{-2x(1+x^2)[(1-x^2)+2(1-x^2)]}{(1+x^2)^4} = \dfrac{-2x(3-x^2)}{(1+x^2)^3}$

For greatest or least values of m we should have

$g'(x) = 0 \quad \Rightarrow x = 0, x = \pm\sqrt{3}$

Now,

$g''(x) = \dfrac{(1+x^2)^3(6x^2-6)-(2x^3-6x).3(1+x^2)^2.2x}{(1+x^2)^6}$

Atx = 0, g″(x) = – 6 < 0

$\therefore$ g′(x) has maximum value at x = 0

$\Rightarrow$ (x = 0, y = 0) is the required point at which tangent to the curve has the greatest slope.

Sol 6: Given, y = cos (x + y)

$\Rightarrow \left(\dfrac{dy}{dx}\right) = -\sin(x+y).\left(1+\dfrac{dy}{dx}\right)$ ...(i)

Since, tangent is parallel to x + 2y = 0,

Then slope $\dfrac{dy}{dx} = -\dfrac{1}{2}$

$\therefore$ From Eq.(i),

$-\dfrac{1}{2} = -\sin(x+y).\left(1-\dfrac{1}{2}\right)$

$\Rightarrow$sin(x + y) = 1

Which shows cos (x + y) = 0

$\therefore$y = 0

$\Rightarrow x + y = \dfrac{\pi}{2}$ or $-\dfrac{3\pi}{2}$

$\therefore x = \dfrac{\pi}{2}$ or $-\dfrac{3\pi}{2}$

Thus, required points are

$\left(\dfrac{\pi}{2},0\right)$ and $\left(-\dfrac{3\pi}{2},0\right)$

$\therefore$ Equation of tangents are

$\dfrac{y-0}{x-\pi/2} = -\dfrac{1}{2}$ and $\dfrac{y-0}{x+3\pi/2} = -\dfrac{1}{2}$

$\Rightarrow 2y = -x + \dfrac{\pi}{2}$ and $2y = -x - \dfrac{3\pi}{2}$

$\Rightarrow x + 2y = \dfrac{\pi}{2}$ and $x + 2y = -\dfrac{3\pi}{2}$

are the required equation of tangents

Sol 7: Let P(a cos θ, 2 sin θ) be a point on the ellipse

$4x^2 + a^2y^2 = 4a^2$ ie, $\dfrac{x^2}{a^2} + \dfrac{y^2}{4} = 1$

Let A(0, – 2) be the given point. Then,

$(AP)^2 = a^2\cos^2\theta + 4(1 + \sin\theta)^2$

$\Rightarrow \dfrac{d}{d\theta}(AP)^2 = -a^2\sin^2 2\theta + 8(1+\sin\theta).\cos\theta$

$\Rightarrow \dfrac{d}{d\theta}(AP)^2 = [(8-2a^2)\sin\theta + 8]\cos\theta$

For maximum or minimum, we put $\dfrac{d}{d\theta}(AP)^2 = 0$

$\Rightarrow [(8 - 2a^2) \sin \theta + 8] \cos \theta = 0$

$\Rightarrow \cos \theta = 0$ or $\sin \theta = \dfrac{4}{a^2 - 4}$

$(\because 4 < a^2 < 8 \Rightarrow \dfrac{4}{a^2 - 4} > 1 \Rightarrow \sin \theta > 1$, which is impossible)

Now, $\dfrac{d^2}{d\theta^2}(AP)^2 = -\{(8-2a^2)\sin\theta + 8\}\sin\theta + (8-2a^2) . \cos^2\theta$

For $\theta = \dfrac{\pi}{2}$, we have

$\dfrac{d^2}{d\theta^2}(AP)^2 = -(16 - 2a^2) < 0$

Thus, AP^2 ie, AP is maximum when $\theta = \dfrac{\pi}{2}$. The point on the curve $4x^2 + a^2y^2 = 4a^2$ that is farthest from the point A (0, – 2) is

$\left(a\cos\dfrac{\pi}{2}, 2\sin\dfrac{\pi}{2}\right) = (0, 2)$

Sol 8: Since, equation of normal to $y^2 = 4ax$ is

$y = mx - 2am - am^3$

Equation of normal for $y^2 = x$ is

$y = mx - \dfrac{m}{2} - \dfrac{1}{4}m^3$ which passes through (c, 0)

$\therefore 0 = m\left(c - \dfrac{1}{2} - \dfrac{m^2}{4}\right) \Rightarrow m = 0$

and $\dfrac{m^2}{4} = c - \dfrac{1}{2} \Rightarrow m = \pm 2\sqrt{c - \dfrac{1}{2}}$

Which gives a normal as x-axis and for other two normals

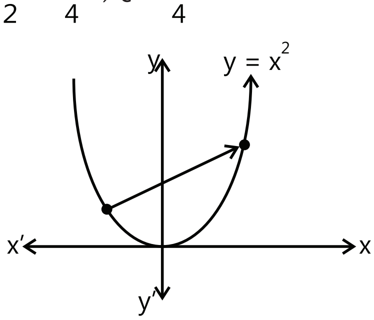
$c - \dfrac{1}{2} > 0 \Rightarrow c > \dfrac{1}{2}$

Now, if normals are perpendicular

$\Rightarrow \left(2\sqrt{c - \dfrac{1}{2}}\right) . \left(-2\sqrt{c - \dfrac{1}{2}}\right) = -1$

$\Rightarrow c - \dfrac{1}{2} = \dfrac{1}{4} \Rightarrow c = \dfrac{3}{4}$

Sol 9:



Any point on the parabola $y = x^2$ is of the form (t, t^2).

Now, $\dfrac{dy}{dx} = 2x \Rightarrow \left[\dfrac{dy}{dx}\right]_{x=t} = 2t$

Which is the slope of the tangent. So, the slope of the normal to $y = x^2$ at $A(t, t^2)$ is $-1/2t$.

Therefore, the equation of the normal to $y = x^2$ at $A(t, t^2)$ is

$Y - t^2 = \left(-\dfrac{1}{2t}\right)(x - t)$ .... (i)

Suppose eq. (1) meets the curve again at $B(t_1, t_1^2)$

Then, $t_1^2 - t^2 = -\dfrac{1}{2t}(t_1 - t)$

$\Rightarrow (t_1 - t)(t_1 + t) = -\dfrac{1}{2t}(t_1 - t)$

$\Rightarrow (t_1 + t) = -\dfrac{1}{2t}$

$\Rightarrow t_1 = -t - \dfrac{1}{2t}$

Therefore, length of chord,

$L = AB^2 = (t - t_1)^2 + (t^2 - t_1^2)^2$

$= (t - t_1)^2 + (t - t_1)^2(t + t_1)^2$

$= (t - t_1)^2 [1 + (t + t_1)^2]$

$= \left(t + t + \dfrac{1}{2t}\right)^2 \left[1 + \left(t - t - \dfrac{1}{2t}\right)^2\right]$

$\Rightarrow L = \left(2t + \dfrac{1}{2t}\right)^2 \left(1 + \dfrac{1}{4t^2}\right) = 4t^2\left(1 + \dfrac{1}{4t^2}\right)^3$

$\therefore$ On differentiating w.r.t., we get

$\dfrac{dL}{dt} = 8t\left(1 + \dfrac{1}{4t^2}\right)^3 + 12t^2\left(1 + \dfrac{1}{4t^2}\right)^2\left(-\dfrac{2}{4t^3}\right)$

$= 2\left(1 + \dfrac{1}{4t^2}\right)^2 \left[4t\left(1 + \dfrac{1}{4t^2}\right) - \dfrac{3}{t}\right]$

$= 2\left(1 + \dfrac{1}{4t^2}\right)^2\left(4t - \dfrac{2}{t}\right) = 4\left(1 + \dfrac{1}{4t^2}\right)^2\left(2t - \dfrac{1}{t}\right)$

For maxima or minima, we must have $\dfrac{dL}{dt} = 0$

$\Rightarrow 2t - \dfrac{1}{t} = 0 \Rightarrow t^2 = \dfrac{1}{2} \Rightarrow t = \pm\dfrac{1}{\sqrt{2}}$

Next,

$\dfrac{d^2L}{dt^2} = 8\left(1 + \dfrac{1}{4t^2}\right)\left(-\dfrac{1}{2t^3}\right)\left(2t - \dfrac{1}{t}\right) + 4\left(1 + \dfrac{1}{4t^2}\right)^2\left(2 + \dfrac{1}{t^2}\right)$

$$\Rightarrow \left[\frac{d^2L}{dt^2}\right]_{t=\pm 1/\sqrt{2}} = 0 + 4\left(1+\frac{1}{2}\right)^2 (2+2) > 0$$

Therefore, L is minimum, when $t = \pm\frac{1}{\sqrt{2}}$, point A is $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and point B is $(-\sqrt{2}, 2)$ when $t = \frac{-1}{\sqrt{2}}$, A is $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, B is $(+\sqrt{2}, 2)$

Again, when $t = \frac{1}{\sqrt{2}}$, the equation of AB is

$$\frac{y-2}{\frac{1}{2}-2} = \frac{x+\sqrt{2}}{\frac{1}{\sqrt{2}}+\sqrt{2}}$$

$$\Rightarrow (y-2)\left\{\left(\frac{1}{\sqrt{2}}+\sqrt{2}\right)\right\} = (x+\sqrt{2})\left(\frac{1}{2}-2\right)$$

$$\Rightarrow -2y + 4 = \sqrt{2}x + 2$$

$$\Rightarrow \sqrt{2}x + 2y - 2 = 0$$

And when $t = -\frac{1}{\sqrt{2}}$, the equation of AB is $\frac{y-2}{\frac{1}{2}-2}$

$$= \frac{x-\sqrt{2}}{\left(-\frac{1}{\sqrt{2}}\right)-\sqrt{2}}$$

$$\Rightarrow (y-2)\left(-\frac{1}{\sqrt{2}}-\sqrt{2}\right) = (x-\sqrt{2})\left(\frac{1}{2}-2\right)$$

$$\Rightarrow 2y - 4 = \sqrt{2}(x-\sqrt{2})$$

$$\Rightarrow \sqrt{2}x - 2y + 2 = 0$$

Sol 10: Given, $y = (1+x)^y + \sin^{-1}(\sin^2)x)$

Let $y = u + v$,

where $u = (1+x)^y$, $v = \sin^{-1}(\sin^2 x)$

On differentiating, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{... (i)}$$

Now, $u = (1+x)^y$

Take logarithm on both sides, we get

$\text{Log}_e u = y \log_e(1+x)$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{1+x} + \frac{dy}{dx}\{\log_e(1+x)\}$$

$$\Rightarrow \frac{du}{dx} = (1+x)^y$$

$$\left[\frac{y}{1+x} + \frac{dy}{dx}\log_e(1+x)\right] \quad \text{... (ii)}$$

Again, $v = \sin^{-1}(\sin^2 x)$

$\Rightarrow \sin v = \sin^2 x$

$$\Rightarrow \cos v \frac{dv}{dx} = 2\sin x \cos x$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\cos v}(2\sin x \cos x)$$

$$\Rightarrow \frac{dv}{dx} = \frac{2\sin x\cos x}{\sqrt{1-\sin^2 v}} = \frac{2\sin x\cos x}{\sqrt{1-\sin^4 x}} \quad \text{... (iii)}$$

$\therefore$ From Eq. (i)

$$\frac{dy}{dx} = (1+x)^y\left[\frac{y}{1+x} + \frac{dy}{dx}\log_e(1+x)\right] + \frac{2\sin x\cos x}{\sqrt{1-\sin^4 x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(1+x)^{y-1} + 2\sin x\cos x/\sqrt{1-\sin^4 x}}{1-(1+x)^y\log_e(1+x)}$$

At x = 0,

$y = (1+0)^y + \sin^{-1}\sin(0) = 1$

$$\therefore \frac{dy}{dx} = \frac{1(1+0)^{1-1} + 2\sin 0.\cos 0/\sqrt{(1-\sin^4 0)}}{1-(1+0)^1\log_e(1+0)}$$

$$\Rightarrow \frac{dy}{dx} = 1$$

Again, the slope of the normal is

$$m = -\frac{1}{dy/dx} = -1$$

Hence, the required equation of the normal is

$y - 1 = (-1)(x - 0)$

ie, $y + x - 1 = 0$

Sol 11: Let any point P_1 on $y = x^3$ be (h, h^3)

Then tangent at P_1 is

$y - h^3 = 3h^2(x - h)$....(i)

It meets $y = x^3$ at P_2.

On putting the value of y in Eq. (i)

$x^3 - h^3 = 3h^2(x - h)$

$\Rightarrow (x - h)(x^2 + xh + h^2) = 3h^2(x - h)$

$\Rightarrow x^2 + xh + h^2 = 3h^2$

or x = h

$\Rightarrow x^2 + xh + 2h^2 = 0 \Rightarrow (x - h)(x + 2h) = 0$

$\Rightarrow x = h$ or $x = -2h$

Therefore, x = – 2h is the point P_2,

Which implies $y = -8h^3$

Hence, point $P_2 \equiv (-2h, -8h^3)$

Again, tangent at P_2 is

$y + 8h^3 = 3(-2h)^2(x + 2h)$

It meets $y = x^3$ at P_3

$\Rightarrow x^3 + 8h^3 = 12h^2(x + 2h)$

$\Rightarrow x^3 - 2hx - 8h^2 = 0$

$\Rightarrow (x + 2h)(x - 4h) = 0$

$\Rightarrow x = 4h \Rightarrow y = 64h^3$

Therefore, $P_3 \equiv (4h, 64h^3)$

Similarly, we get $P_4 \equiv (-8h, -8^3 h^3)$

Hence, the abscissae are

h, –2h, 4h, –8h, ... which form a GP

Let $D' = \Delta P_1P_2P_3$ and $D'' = \Delta P_2P_3P_4$

$$\frac{D'}{D''} = \frac{\Delta P_1P_2P_3}{\Delta P_2P_3P_4} = \frac{\frac{1}{2}\begin{vmatrix} h & h^3 & 1 \\ -2h & -8h^3 & 1 \\ 4h & 64h^3 & 1 \end{vmatrix}}{\frac{1}{2}\begin{vmatrix} -2h & -8h^3 & 1 \\ 4h & 64h^3 & 1 \\ -8h & -512h^3 & 1 \end{vmatrix}}$$

$$= \frac{\frac{1}{2}\begin{vmatrix} h & h^3 & 1 \\ -2h & -8h^3 & 1 \\ 4h & 64h^3 & 1 \end{vmatrix}}{\frac{1}{2}\times(-2)\times(-8)\begin{vmatrix} h & h^3 & 1 \\ -2h & -8h^3 & 1 \\ 4h & 64h^3 & 1 \end{vmatrix}}$$

$$= \frac{1}{16}$$

which is the required ratio.

Sol 12: (A, C)

Let the sides of rectangle be 15k and 8k and side of square be x then (15k – 2x)(8k – 2x)x is volume.

$$v = 2(2x^3 - 23kx^2 + 60k^2x)$$

$$\left.\frac{dv}{dx}\right|_{x=5} = 0$$

$$6x2 - 46kx + 60k2\,|_{x=5} = 0$$

$$6k^2 - 23k + 15 = 0$$

$k = 3,\ k = \frac{5}{6}$. Only k = 3 is permissible

So, the sides are 45 and 24.

Sol 13: (B)

$$P \to \frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}$$

$$= \frac{\frac{1}{\sqrt{1+y^2}} + \frac{y^2}{\sqrt{1+y^2}}}{\frac{\sqrt{1-y^2}}{y} + \frac{y}{\sqrt{1-y^2}}} = \frac{\sqrt{1+y^2}}{\frac{1}{y\sqrt{1-y^2}}} = y\sqrt{1-y^4}$$

$$\Rightarrow \frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}\right)^2 + y^4$$

$$= \frac{1}{y^2}\left(y^2\left(1-y^4\right)\right) + y^4 = 1 - y^4 + y^4 = 1$$

$Q \to \cos x + \cos y + \cos z = 0$

$\sin x + \sin y + \sin z = 0$

$\cos x + \cos y = -\cos z$ ... (i)

$\sin x + \sin y = -\sin z$ ... (ii)

$(1)^2 + (2)^2$

$1 + 1 + 2(\cos x \cos y + \sin x \sin y) = 1$

$2 + 2\cos(x - y) = 1$

$2\cos(x - y) = -1$

$$\cos(x - y) = -\frac{1}{2}$$

$$\Rightarrow 2\cos^2\left(\frac{x-y}{2}\right) - 1 = -\frac{1}{2} \Rightarrow 2\cos^2\left(\frac{x-y}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \cos^2\left(\frac{x-y}{2}\right) = \frac{1}{4} \Rightarrow \cos\left(\frac{x-y}{2}\right) = \frac{1}{2}$$

$$R \to \cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x$$

$$= \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$$

$$\left[\cos\left(\frac{\pi}{4} - x\right) - \cos\left(\frac{\pi}{4} + x\right)\right]\cos 2x$$

$= (\cos x \sin 2x - \sin x \sin 2x)\sec x$

$\frac{2}{\sqrt{2}}$ sin x cos 2x = (cos x – sin x) sin 2x sec x

$\sqrt{2}$ sin x cos 2x = (cos x – sin x) 2 sin x

$$\frac{1}{\sqrt{2}} = \frac{1}{\cos x + \sin x} \Rightarrow x = \frac{\pi}{4}$$

$$\sec x = \sec\frac{\pi}{4} = \sqrt{2}$$

$$S \to \cot\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\cot\alpha = \frac{x}{\sqrt{1-x^2}}$$

$$\tan^{-1}(x\sqrt{6}) = \phi$$

$$\sin\phi = \frac{x\sqrt{6}}{\sqrt{6x^2+1}}$$

$$\Rightarrow \quad \frac{x}{\sqrt{1-x^2}} = \frac{x\sqrt{6}}{\sqrt{6x^2+1}}$$

$$6x^2 + 1 = 6 - 6x^2$$

$$12x^2 = 5$$

$$x = \sqrt{\frac{5}{12}} = \frac{1}{2}\sqrt{\frac{5}{3}}$$