Solution of Triangle

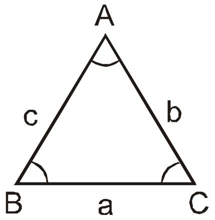
According to most accounts, geometry was first discovered among the Egyptians, taking its origin from the measurement of areas. For they found it necessary by reason of the flooding of the Nile, which wiped out everybody's proper boundaries. Nor is there anything surprising in that the discovery both of this and of the other sciences should have had its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the perfect.

..................Proclus

Sine Rule :

In any triangle ABC, the sines of the angles are proportional to the opposite sides

i.e. $\frac{a}{\sin\ A} = \frac{b}{\sin\ B} = \frac{c}{\sin\ C}$.



Example # 1 : How many triangles can be constructed with the data : a = 5, b = 7, sin A = 3/4

Solution : Since $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{5}{3/4} = \frac{7}{\sin B}$

$\Rightarrow$ $\sin B = \frac{21}{20} > 1$ not possible

$\therefore$ no triangle can be constructed.

Example # 2 : If in a triangle ABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then show that a^2, b^2, c^2 are in A.P.

Solution : We have $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$

$\Rightarrow \sin (B + C) \sin (B–C) = \sin (A + B) \sin (A–B)$ $\Rightarrow \sin^2B - \sin^2 C = \sin^2A - \sin^2B$

$\Rightarrow b^2 - c^2 = a^2 - b^2$ $\Rightarrow a^2, b^2, c^2$ are in A.P.

Self Practice Problems :

(1) In a ΔABC, the sides a, b and c are in A.P. , then prove that $\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right) : \cot\frac{B}{2} = 2 : 3$

(2) If the angles of ΔABC are in the ratio 1 : 2 : 3, then find the ratio of their corresponding sides

(3) In a ΔABC prove that $\frac{c}{a-b} = \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$.

Ans. (2) $1 : \sqrt{3} : 2$

Cosine Formula :

In any ΔABC

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc \cos A = b^2 + c^2 + 2bc \cos (B + C)$s

(ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ **(iii)** $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Example # 3 : In a triangle ABC, A, B, C are in A.P. Show that $2\cos\left(\frac{A-C}{2}\right) = \frac{a+c}{\sqrt{a^2-ac+c^2}}$.

Solution : $A + C = 2B \Rightarrow A + B + C = 3B \Rightarrow B = 60º$

$\therefore \cos60º = \frac{a^2+c^2-b^2}{2ac} \Rightarrow a^2 - ac + c^2 = b^2$

$$\Rightarrow \frac{a+c}{\sqrt{a^2-ac+c^2}} = \frac{a+c}{b} = \left[\frac{\sin A + \sin C}{\sin B}\right] = \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin B}$$

$= 2\cos\frac{A-C}{2}$ $\qquad$ ($\because$ A + C = 2B)

Example # 4 : In a ΔABC , prove that a (bcos C – c cosB) = $b^2 - c^2$

Solution : Since $\cos C = \frac{a^2+b^2-c^2}{2ab}$ & $\cos B = \frac{a^2+c^2-b^2}{2ac}$

$$\therefore \quad \text{L.H.S.} = a\left\{b\left(\frac{a^2+b^2-c^2}{2ab}\right) - c\left(\frac{a^2+c^2-b^2}{2ac}\right)\right\}$$

$$= \frac{a^2+b^2-c^2}{2} - \frac{(a^2+c^2-b^2)}{2} = (b^2 - c^2) = \text{R.H.S.}$$

Hence L.H.S. = R.H.S. **Proved**

Example # 5 : The sides of ΔABC are AB = $\sqrt{13}$ cm, BC = $4\sqrt{3}$ cm and CA = 7 cm. Then find the value of $\sin\theta$ where θ is the smallest angle of the triangle.

Solution : Angle opposite to AB is smallest . Therefore,

$\cos\theta = \frac{49+48-13}{2.7.4\sqrt{3}} = \frac{\sqrt{3}}{2}$ $\qquad \Rightarrow \sin\theta = \frac{1}{2}$

Self Practice Problems :

(4) If in a triangle ABC, 3 sinA = 6 sinB = $2\sqrt{3}$ sinC, Then find the angle A.

(5) If two sides a, b and angle A be such that two triangles are formed, then find the sum of two values of the third side.

Ans. (4) 90º (5) 2b cosA

Projection Formula :

In any ΔABC

(i) a = b cosC + c cosB **(ii)** b = c cosA + a cosC **(iii)** c = a cosB + b cosA

Example # 6 : If in a ΔABC, $c\cos^2\frac{A}{2} + a\cos^2\frac{C}{2} = \frac{3b}{2}$, then show that a, b, c are in A.P.

Solution : c (1 + cosA) + a (1 + cosC) = 3b

$\Rightarrow$ a + c + (c cosA + acosC) = 3b

$\Rightarrow$ a + c + b = 3b

$\Rightarrow$ a + c = 2b

Example # 7 : In a ΔABC, prove that (b + c) cos A + (c + a) cos B + (a + b) cos C = a + b + c.

Solution : $\because$ L.H.S. = (b + c) cos A + (c + a) cos B + (a + b) cos C

= b cos A + c cos A + c cos B + a cos B + a cos C + b cos C

= (b cos A + a cos B) + (c cos A + a cos C) + (c cos B + b cos C)

= a + b + c

= R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Self Practice Problems :

(6) The roots of $x^2 - 2\sqrt{3}x + 2 = 0$ represent two sides of a triangle. If the angle between them is $\frac{\pi}{3}$, then find the perimeter of triangle.

(7) In a triangle ABC, if cos A + cosB + cos C = 3/2, then show that the triangle is an equilateral triangle.

(8) In a ΔABC, prove that $\frac{\cos A}{c\cos B + b\cos C} + \frac{\cos B}{a\cos C + c\cos A} + \frac{\cos C}{a\cos B + b\cos A} = \frac{a^2 + b^2 + c^2}{2abc}$.

Ans. (6) $2\sqrt{3} + \sqrt{6}$

Napier's Analogy - tangent rule :

In any ΔABC

(i) $\tan\frac{B-C}{2} = \frac{b - c}{b + c}\cot\frac{A}{2}$ **(ii)** $\tan\frac{C - A}{2} = \frac{c - a}{c + a}\cot\frac{B}{2}$

(iii) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

Example # 8 : Find the unknown elements of the ΔABC in which $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, C = 90°.

Solution : $\because$ $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, C = 90°

$\because$ A + B + C = 180°

$\therefore$ A + B = 90° ....(i)

$\because$ From law of tangent, we know that $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

$$= \frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 45^\circ = \frac{2}{2\sqrt{3}}\cot 45^\circ \Rightarrow \tan\left(\frac{A-B}{2}\right) = \frac{1}{\sqrt{3}}$$

$\therefore$ $\frac{A-B}{2} = \frac{\pi}{6}$

$\Rightarrow$ $A - B = \frac{\pi}{3}$ ....(ii)

From equation (i) and (ii), we get $A = \frac{5\pi}{12}$ and $B = \frac{\pi}{12}$

Now, $c = \sqrt{a^2 + b^2} = 2\sqrt{2}$

$\therefore$ $c = 2\sqrt{2}$, $A = \frac{5\pi}{12}$, $B = \frac{\pi}{12}$ Ans.

Self Practice Problems :

(9) In a ΔABC if b = 3, c = 5 and $\cos(B - C) = \frac{7}{25}$, then find the value of $\sin\frac{A}{2}$.

(10) If in a ΔABC, we define $x = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$, $y = \tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$ and $z = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$, then show that $x + y + z = - xyz$.

Ans. (9) $\frac{1}{\sqrt{10}}$

Trigonometric Functions of Half Angles :

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{ca}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s\,(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s\,(s-b)}{ca}}$, $\cos\frac{C}{2} = \sqrt{\frac{s\,(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)} = \frac{(s-b)(s-c)}{\Delta}$, where $s = \frac{a+b+c}{2}$ is semi perimeter and Δ is the area of triangle.

(iv) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

Area of Triangle (Δ)

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)\,(s-b)\,(s-c)}$$

Example # 9 : If p_1, p_2, p_3 are the altitudes of a triangle ABC from the vertices A, B, C and Δ is the area of the triangle, then show that $p_1^{-1} + p_2^{-1} - p_3^{-1} = \frac{s-c}{\Delta}$

Solution : We have

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta}$$

$$= \frac{a+b-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta}$$

Example # 10 : In a ΔABC if b sinC(b cosC + c cosB) = 64, then find the area of the ΔABC.

Solution : $\because$ b sinC (b cosC + c cosB) = 64 ....(i) given

$\because$ From **projection rule**, we know that

a = b cosC + c cosB put in (i), we get

ab sinC = 64 ....(ii)

$\because$ $\Delta = \frac{1}{2}$ab sinC $\therefore$ from equation (ii), we get

$\therefore$ Δ = 32 sq. unit

Example # 11 : If A,B,C are the angle of a triangle, then prove that $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s^2}{\Delta}$

Solution : $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

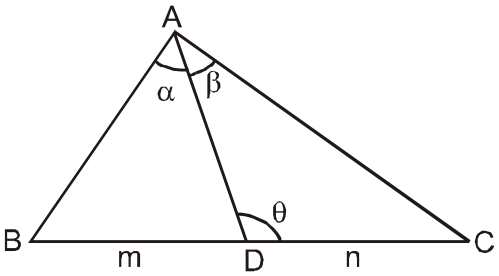
$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{\sqrt{s}(s-a+s-b+s-c)}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s}{\Delta}\,(3s - 2s) = \frac{s^2}{\Delta}$$

m - n Rule : In any triangle ABC if D be any point on the base BC, such that BD : DC :: m : n and if $\angle$BAD= α, $\angle$DAC = β, $\angle$ CDA = θ , then

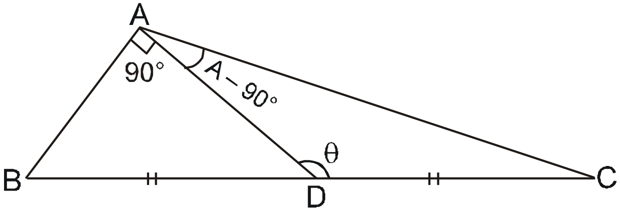
$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$

$$n\cot B - m\cot C$$



Example # 12 : In a ΔABC . AD divides BC in the ratio 2 : 1 such that at ∠BAD = 90º then prove that tanA + 3tanB = 0

Solution : From the figure , we see that θ = 90º + B (as θ is external angle of ΔABD)



Now if we apply **m-n rule** in ΔABC, we get

(2 + 1) cot (90º + B) = 2. cot 90° – 1.cot (A – 90°)

⇒ – 3 tan B = cot (90° – A)

⇒ – 3 tan B = tan A

⇒ tan A + 3 tan B = 0 **Hence proved.**

Example # 13 : The base of a Δ is divided into three equal parts . If α, β, γ be the angles subtended by these parts at the vertex, prove that :

$(\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma) = 4\text{cosec}^2\beta$

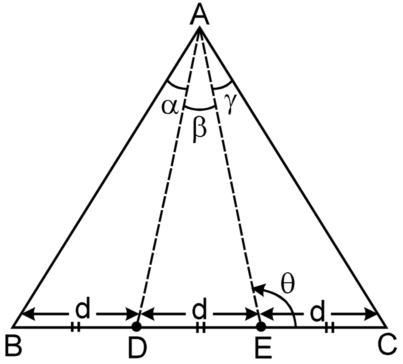
Solution : Let point D and E divides the base BC into three equal parts i.e. BD = DE = EC = d (Let) and let α, β and γ be the angles subtended by BD, DE and EC respectively at their opposite vertex. Now in ΔABC

∵ BE : EC = 2d : d = 2 : 1

∴ from **m-n rule**, we get

$(2 + 1)\cot\theta = 2\cot(\alpha + \beta) - \cot\gamma$

⇒ $3\cot\theta = 2\cot(\alpha + \beta) - \cot\gamma$ ....(i)



again

∵ in ΔADC

∵ DE : EC = d : d = 1 : 1

∴ if we apply **m-n rule** in ΔADC, we get

$(1 + 1)\cot\theta = 1.\cot\beta - 1\cot\gamma$

$2\cot\theta = \cot\beta - \cot\gamma$ ....(ii)

from (i) and (ii), we get $\frac{3\cot\theta}{2\cot\theta} = \frac{2\cot(\alpha+\beta) - \cot\gamma}{\cot\beta - \cot\gamma}$

⇒ $3\cot\beta - 3\cot\gamma = 4\cot(\alpha + \beta) - 2\cot\gamma$

$\Rightarrow$ 3cotβ – cotγ = 4 cot (α + β)

$\Rightarrow$ $3\cot\beta - \cot\gamma = 4\left\{\dfrac{\cot\alpha.\cot\beta - 1}{\cot\beta + \cot\alpha}\right\}$

$\Rightarrow$ $3\cot^2\beta + 3\cot\alpha\cot\beta - \cot\beta\cot\gamma - \cot\alpha\cot\gamma = 4\cot\alpha\cot\beta - 4$

$\Rightarrow$ $4 + 3\cot^2\beta = \cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\alpha\cot\gamma$

$\Rightarrow$ $4 + 4\cot^2\beta = \cot\alpha\cot\beta + \cot\alpha\cot\gamma + \cot\beta\cot\gamma + \cot^2\beta$

$\Rightarrow$ $4(1 + \cot^2\beta) = (\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma)$

$\Rightarrow$ $(\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma) = 4\text{cosec}^2\beta$

Self Practice Problems :

(11) In a ΔABC, the median to the side BC is of length $\dfrac{1}{\sqrt{11-6\sqrt{3}}}$ unit and it divides angle A into the angles of 30° and 45°. Prove that the side BC is of length 2 unit.

Radius of Circumcirlce :

If R be the circumradius of ΔABC, then $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} = \dfrac{abc}{4\Delta}$

Example # 14 : In a ΔABC , prove that sin2A + sin2B + sin2C = $2\Delta/R^2$

Solution : In a ΔABC, we know that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

and sin2A + sin2B + sin2C = 4sinAsinBsinC

$= \dfrac{4abc}{8R^3} = \dfrac{16\Delta R}{8R^3} = \dfrac{2\Delta}{R^2}$

Example # 15 : In a ΔABC if a = 22 cm, b = 28 cm and c = 36 cm, then find its circumradius.

Solution : $\because$ $R = \dfrac{abc}{4\Delta}$ ....(i)

$\because$ $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\because$ $s = \dfrac{a+b+c}{2} = 43$ cm

$\therefore$ $\Delta = \sqrt{43\times21\times15\times7} = 21\sqrt{215}$

$\therefore$ $R = \dfrac{22\times28\times36}{4\times21\sqrt{215}} = \dfrac{264}{\sqrt{215}}$ cm

Example # 16 : In a ΔABC, if $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right angled.

Solution : We have : $8R^2 = a^2 + b^2 + c^2$

$\Rightarrow 8R^2 = [4R^2\sin^2A + 4R^2\sin^2B + 4R^2\sin C]$ [$\because$ a = 2R sin A etc.]

$\Rightarrow 2 = \sin^2A + \sin^2B + \sin^2C \Rightarrow (1 - \sin^2A) - \sin^2B + (1 - \sin^2C) = 0$

$\Rightarrow (\cos^2A - \sin^2B) + \cos^2C = 0 \Rightarrow \cos(A + B)\cos(A - B) + \cos^2C = 0$

$\Rightarrow -\cos C\cos(A - B) + \cos^2C = 0 \Rightarrow -\cos C\{\cos(A - B) - \cos C\} = 0$

$\Rightarrow -\cos C[\cos(A - B) + \cos(A + B)] = 0 \Rightarrow -2\cos A\cos B\cos C = 0$

$\Rightarrow \cos A = 0$ or $\cos B = 0$ or $\cos C = 0$

$\Rightarrow A = \dfrac{\pi}{2}$ or $B = \dfrac{\pi}{2}$ or $C = \dfrac{\pi}{2}$

$\Rightarrow$ ΔABC is a right angled triangle.

Example # 17 : $\dfrac{b^2 - c^2}{2a} = R\sin(B - C)$

Solution : $\dfrac{b^2 - c^2}{2a} = \dfrac{4R^2(\sin^2B - \sin^2C)}{4R\sin A} = \dfrac{R\sin(B+C)\sin(B-C)}{\sin A} = R\sin(B - C)$

Self Practice Problems :

(12) In a ΔABC, prove that (a + b) = 4R $\cos\left(\frac{A-B}{2}\right)\cos\frac{C}{2}$

(13) In a ΔABC , if b = 15 cm and cos B = $\frac{4}{5}$, find R.

(14) In a triangle ABC if α, β, γ are the distances of the vertices of triangle from the corresponding points of contact with the incircle, then prove that $\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma} = r^2$

Ans. (13) 12.5

Radius of The Incircle :

If 'r' be the inradius of ΔABC, then

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(iii) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Radius of The Ex-Circles :

If r_1, r_2 , r_3 are the radii of the ex-circles of ΔABC opposite to the vertex A, B, C respectively, then

(i) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$;

(ii) $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r_1 = 4R\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$

Example # 18 : cos A + cosB + cos C = $\left(1+\frac{r}{R}\right)$

Solution : LHS = cosA + cosB + cosC

$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2}$

$= 2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right)-\sin\frac{C}{2}\right\}+1 = 2\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}+1$

$= 2\sin\frac{C}{2}\left\{2\sin\frac{A}{2}\sin\frac{B}{2}\right\} + 1 = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$= 1 + \frac{1}{R}\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 1 + \frac{r}{R}$ = RHS

Example # 19 : In a triangle ABC, find the value of $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$.

Solution : $\frac{b-c}{\left(\frac{\Delta}{s-a}\right)}+\frac{c-a}{\left(\frac{\Delta}{s-b}\right)}+\frac{a-b}{\left(\frac{\Delta}{s-c}\right)}$

$= \frac{1}{\Delta}$ [(b – c) (s – a) + (c – a) (s – b) + (a – b) (s – c)]

$= \frac{1}{\Delta}$ [s(b – c + c – a + a – b) –a (b – c) – b(c – a) – c (a – b)] = 0

Self Practice Problems :

(15) In a triangle ABC, r_1 , r_2, r_3 are in HP. If its area is 24 cm^2 and its perimeter is 24 cm. then find lengths of its sides.

(16) In a triangle ABC, a : b : c = 4 : 5 : 6 . Find the ratio of the radius of the circumcircle to that of the incircle.

(17) In a ΔABC, prove that $\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}$.

(18) If A, A_1, A_2 and A_3 are the areas of the inscribed and escribed circles respectively of a ΔABC, then prove that $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$.

Ans. (15) 6, 8 , 10 (16) 16 : 7

Length of Angle Bisectors, Medians & Altitudes :

(i) Length of an angle bisector from the angle A = $\beta_a = \frac{2bc \cos \frac{A}{2}}{b + c}$;

(ii) Length of median from the angle A = $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

& (iii) Length of altitude from the angle A = $A_a = \frac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

Example#20 : In ΔABC, AD & BE are its two median . If AD = 4 , $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$ then find the length of BE and area of ΔABC.

Solution : $AP = \frac{2}{3}$; $AD = \frac{8}{3}$; $PD = \frac{4}{3}$; Let PB = x

$\tan 60^\circ = \frac{8/3}{x}$ or $x = \frac{8}{3\sqrt{3}}$

Area of $\Delta ABP = \frac{1}{2} \times \frac{8}{3} \times \frac{8}{3\sqrt{3}} = \frac{32}{9\sqrt{3}}$

$\therefore$ Area of $\Delta ABC = 3 \times \frac{32}{9\sqrt{3}} = \frac{32}{3\sqrt{3}}$

Also, $BE = \frac{3}{2}x = \frac{4}{\sqrt{3}}$

Self Practice Problem :

(19) In a ΔABC if ∠A = 90°, b = 5 cm, c = 12 cm. If ‘G’ is the centroid of triangle, then find circumradius of ΔGAB.

Ans. (19) $\frac{13\sqrt{601}}{30}$ cm

The Distances of The Special Points from Vertices and Sides of Triangle :

(i) Circumcentre (O) : OA = R and O_a = R cos A

(ii) Incentre (I) : IA = r cosec $\frac{A}{2}$ and I_a = r

(iii) Excentre (I_1) : I_1 A = r_1 cosec $\frac{A}{2}$ and I_{1a} = r_1

(iv) Orthocentre (H) : HA = 2R cos A and H_a = 2R cos B cos C

(v) Centroid (G) : GA = $\frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ and G_a = $\frac{2\Delta}{3a}$

Example # 21 : If p_1,p_2, p_3 are respectively the lengths of perpendiculars from the vertices of a triangle ABC to the opposite sides, prove that :

(i) $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}=\frac{1}{R}$ (ii) $\frac{bp_1}{c}+\frac{cp_2}{a}+\frac{ap_3}{b} = \frac{a^2+b^2+c^2}{2R}$

Solution : (i) use $\frac{1}{p_1}=\frac{a}{2\Delta},\frac{1}{p_2}=\frac{b}{2\Delta},\frac{1}{p_3}=\frac{c}{2\Delta}$

$\therefore$ LHS = $\frac{1}{2\Delta}$ (a cosA + b cosB + c cosC)

= $\frac{R}{2\Delta}$ (sin 2A + sin 2B + sin 2C) = $\frac{4R\sin A\ \ \sin B\ \ \sin C}{2\Delta}$

= $\frac{4R}{2\Delta}\cdot\frac{a}{2R}.\frac{b}{2R}.\frac{c}{2R}=\frac{1}{4\Delta R^2}abc=\frac{1}{4\Delta R^2}.(4R\Delta)=\frac{1}{R}$ = RHS

(ii) LHS = $\frac{bp_1}{c}+\frac{cp_2}{a}+\frac{ap_3}{b} = \frac{a^2+b^2+c^2}{2R} = \frac{2b\Delta}{ac}+\frac{2c\Delta}{ab}+\frac{2a\Delta}{bc} = \frac{2\Delta(a^2+b^2+c^2)}{abc}$

$= \frac{2\Delta(a^2+b^2+c^2)}{4\Delta R} = \frac{a^2+b^2+c^2}{2R}$

Self Practice Problems :

(20) If I be the incentre of ΔABC, then prove that IA . IB . IC = abc tan $\frac{A}{2}$ tan $\frac{B}{2}$ tan $\frac{C}{2}$.

(21) If x, y, z are respectively be the perpendiculars from the circumcentre to the sides of ΔABC, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$.

Exercise-1

✍ **Marked questions are recommended for Revision.**

SUBJECTIVE QUESTIONS

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

A-1. In a ΔABC, prove that :

(i) $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$

(ii) $\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0$

(iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

(iv)✍ $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$

(v) $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$

(vi) $\frac{\sin B}{\sin C} = \frac{c - a\cos B}{b - a\cos C}$

A-2. Find the real value of x such that $x^2 + 2x$, $2x + 3$ and $x^2 + 3x + 8$ are lengths of the sides of a triangle.

A-3. The angles of a ΔABC are in A.P. (order being A, B, C) and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, then find $\angle$A.

A-4.✍ If $\cos A + \cos B = 4 \sin^2\left(\frac{C}{2}\right)$, prove that sides a, c, b of the triangle ABC are in A.P.

A-5. If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then prove that a^2, b^2, c^2 are in A.P.

A-6. In a triangle ABC, prove that for any angle θ, $b \cos (A - \theta) + a \cos (B + \theta) = c \cos \theta$.

A-7.✍ With usual notations, if in a Δ ABC, $\frac{b + c}{11} = \frac{c + a}{12} = \frac{a + b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

A-8. Let a, b and c be the sides of a ΔABC. If a^2, b^2 and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$, where P, Q & R are constants, then find the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ in terms of P, Q and R.

A-9.✍ If in a triangle ABC, the altitude AM be the bisector of $\angle$BAD, where D is the mid point of side BC, then prove that $(b^2 - c^2) = a^2/2$.

A-10. If in a triangle ABC, $\angle C = 60º$, then prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

A-11.✍ In a triangle ABC, $\angle C = 60º$ and $\angle A = 75º$. If D is a point on AC such that the area of the ΔABD is $\sqrt{3}$ times the area of the ΔBCD, find the $\angle$ABD.

A-12.✍ In a scalene triangle ABC, D is a point on the side AB such that $CD^2 = AD. DB$, if $\sin A. \sin B = \sin^2 \frac{C}{2}$ then prove that CD is internal bisector of $\angle$C.

A-13.✍ In triangle ABC,D is on AC such that AD = BC , BD = DC, $\angle DBC = 2x$, and $\angle BAD = 3x$, all angles are in degrees, then find the value of x.

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

B-1. In a ΔABC, prove that

(i) $2\left[a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right] = c + a - b.$

(ii) $\dfrac{\cos^2\frac{A}{2}}{a} + \dfrac{\cos^2\frac{B}{2}}{b} + \dfrac{\cos^2\frac{C}{2}}{c} = \dfrac{s^2}{abc}$

(iii) $4\left(bc.\cos^2\frac{A}{2}+ca.\cos^2\frac{B}{2}+ab.\cos^2\frac{C}{2}\right) = (a + b + c)^2$

(iv)✍ $(b - c)\cot\frac{A}{2} + (c - a)\cot\frac{B}{2} + (a - b)\cot\frac{C}{2} = 0$

(v) $4\Delta\,(\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$

(vi) $\left(\dfrac{2abc}{a+b+c}\right).\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2} = \Delta$

B-2. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan\frac{A}{2} + \tan\frac{C}{2}$ in terms of cot (B/2).

B-3. If in a Δ ABC, a = 6, b = 3 and cos(A – B) = 4/5, then find its area.

B-4.✍ If in a triangle ABC, $\angle A = 30^\circ$ and the area of triangle is $\dfrac{\sqrt{3}\,a^2}{4}$, then prove that either

B = 4 C or C = 4 B.

Section (C) Inradius and Exradius

C-1. In any ΔABC, prove that

(i) $Rr\,(\sin A + \sin B + \sin C) = \Delta$

(ii) $a\cos B\cos C + b\cos C\cos A + c\cos A\cos B = \dfrac{\Delta}{R}$

(iii) $\dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca} = \dfrac{1}{2Rr}.$

(iv)✍ $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + \dfrac{r}{2R}$

(v)✍ $a\cot A + b\cot B + c\cot C = 2(R + r)$

C-2. In any ΔABC, prove that

(i) $r.\,r_1.r_2.r_3 = \Delta^2$

(ii)✍ $r_1 + r_2 - r_3 + r = 4R\cos C.$

(iii) $\dfrac{1}{r^2}+\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}+\dfrac{1}{r_3^2}=\dfrac{a^2+b^2+c^2}{\Delta^2}$

(iv) $\left(\dfrac{1}{r}+\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)^2 = \dfrac{4}{r}\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)$

(v)✍ $\dfrac{bc-r_2r_3}{r_1} = \dfrac{ca-r_3r_1}{r_2} = \dfrac{ab-r_1r_2}{r_3} = r$

C-3. Show that the radii of the three escribed circles of a triangle are roots of the equation $x^3 - x^2(4R + r) + x\,s^2 - r\,s^2 = 0$.

C-4. The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.

C-5. If the area of a triangle is 100 sq.cm, $r_1 = 10$ cm and $r_2 = 50$ cm, then find the value of $(b - a)$.

Section (D) Miscellaneous

D-1. If α, β, γ are the respective altitudes of a triangle ABC, prove that

(i) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} = \dfrac{\cot A + \cot B + \cot C}{\Delta}$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} - \dfrac{1}{\gamma} = \dfrac{2ab}{(a + b + c)\ \Delta}\ \cos^2\dfrac{C}{2}$

D-2.✍ If in an acute angled ΔABC, line joining the circumcentre and orthocentre is parallel to side AC, then find the value of tan A.tan C.

D-3.✍ A regular hexagon & a regular dodecagon are inscribed in the same circle. If the side of the dodecagon is $\left(\sqrt{3} - 1\right)$, if the side of the hexagon is $\sqrt[4]{k}$, then find value of k.

D-4.✍ If D is the mid point of CA in triangle ABC and Δ is the area of triangle, then show that $\tan(\angle ADB) = \dfrac{4\Delta}{a^2 - c^2}$.

Exercise-2

✍ **Marked questions are recommended for Revision.**

PART-I (OBJECTIVE QUESTIONS)

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

A-1. In a ΔABC, A : B : C = 3 : 5 : 4. Then $a + b + c\sqrt{2}$ is equal to

(A) 2b (B) 2c (C) 3b (D) 3a

A-2*. In a triangle ABC, the altitude from A is not less than BC and the altitude from B is not less than AC. The triangle is

(A) right angled (B) isosceles (C) obtuse angled (D) equilateral

A-3. If in a Δ ABC, $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$, then the triangle is :

(A) right angled (B) isosceles (C) equilateral (D) obtuse angled

A-4. In a ΔABC $\dfrac{bc\sin^2 A}{\cos A + \cos B\cos C}$ is equal to

(A) $b^2 + c^2$ (B) bc (C) a^2 (D) $a^2 + bc$

A-5✍ Given a triangle ΔABC such that $\sin^2 A + \sin^2 C = 1001.\sin^2 B$. Then the value of $\dfrac{2(\tan A + \tan C).\tan^2 B}{\tan A + \tan B + \tan C}$ is

(A) $\dfrac{1}{2000}$ (B) $\dfrac{1}{1000}$ (C) $\dfrac{1}{500}$ (D) $\dfrac{1}{250}$

A-6. If in a triangle ABC, $(a + b + c)(b + c - a) = k.\ bc$, then :

(A) $k < 0$ (B) $k > 6$ (C) $0 < k < 4$ (D) $k > 4$

.

A-7. In a triangle ABC, a: b: c = 4: 5: 6. Then $3A + B$ equals to :

(A) 4C (B) 2π (C) $\pi - C$ (D) π

A-8.✍ The distance between the middle point of BC and the foot of the perpendicular from A is $(b > c)$:

(A) $\frac{-a^2 + b^2 + c^2}{2a}$ (B) $\frac{b^2 - c^2}{2\,a}$ (C) $\frac{b^2 + c^2}{\sqrt{bc}}$ (D) $\frac{b^2 + c^2}{2\,a}$

A-9*. If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the triangle is

(A) isosceles (B) right angled (C) equilateral (D) None of these

A-10.✍Triangle ABC is right angle at A. The points P and Q are on hypotenuse BC such that BP = PQ = QC. If AP = 3 and AQ = 4, then length BC is equal to

(A) $3\sqrt{5}$ (B) $5\sqrt{3}$ (C) $4\sqrt{5}$ (D) 7

A-11. In ΔABC, $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$, then ΔABC is

(A) isosceles but not necessarily equilaterial

(B) equilateral

(C) right angled but not neccessarily isosceles

(D) right angled isosceles

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

B-1. If in a triangle ABC, right angle at B, $s - a = 3$ and $s - c = 2$, then

(A) a = 2, c = 3 (B) a = 3, c = 4 (C) a = 4, c = 3 (D) a = 6, c = 8

B-2. If in a triangle ABC, $b \cos^2\frac{A}{2} + a \cos^2\frac{B}{2} = \frac{3}{2}c$, then a, c, b are :

(A) in A.P. (B) in G.P. (C) in H.P. (D) None

B-3.✍ If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to :

(A) R, R, R (B) $\sqrt{2}R$, $\sqrt{2}R$, $\sqrt{2}R$ (C) 2R, 2R, 2R (D) $\frac{R}{2}, \frac{R}{2}, \frac{R}{2}$

B-4. In a Δ ABC if $b + c = 3a$, then $\cot\frac{B}{2} \cdot \cot\frac{C}{2}$ has the value equal to:

(A) 4 (B) 3 (C) 2 (D) 1

B-5. In a ΔABC, $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and area (ΔABC) = $\frac{9\sqrt{3}}{2}$ cm^2. Then 'a' is

(A) $6\sqrt{3}$ cm (B) 9 cm (C) 18 cm (D) 7 cm

B-6.* The diagonals of a parallelogram are inclined to each other at an angle of 45º, while its sides a and b (a > b) are inclined to each other at an angle of 30º, then the value of $\frac{a}{b}$ is

(A) 2cos36º (B) $\sqrt{\frac{3 + \sqrt{5}}{4}}$ (C) $\frac{3 + \sqrt{5}}{4}$ (D) $\frac{\sqrt{5} + 1}{2}$

B-7. If in a ΔABC, $\Delta = a^2 - (b-c)^2$, then tan A is equal to

(A) 15/16 (B) 8/15 (C) 8/17 (D) 1/2

B-8*. If in a ΔABC, a = 5, b = 4 and cos (A – B) = $\frac{31}{32}$, then

(A) c = 6 (B) $\sin A = \left(\frac{5\sqrt{7}}{16}\right)$

(C) area of ΔABC = $\frac{15\sqrt{7}}{4}$ (D) c = 8

B-9. If R denotes circumradius, then in ΔABC, $\frac{b^2-c^2}{2a\ R}$ is equal to

(A) cos (B – C) (B) sin (B – C) (C) cos B – cos C (D) sin(B + C)

B-10*. Which of the following holds good for any triangle ABC?

(A) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$ (B) $\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{3}{2R}$

(C) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ (D) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$

B-11. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to:

(A) $\frac{9\sqrt{3}\ (1+\sqrt{3})}{\pi^2}$ (B) $\frac{9\sqrt{3}\ (\sqrt{3}-1)}{\pi^2}$ (C) $\frac{9\ \sqrt{3}\ (1+\sqrt{3})}{2\ \pi^2}$ (D) $\frac{9\ \sqrt{3}\ (\sqrt{3}-1)}{2\ \pi^2}$

B-12. In a ΔABC, a = 1 and the perimeter is six times the arithmetic mean of the sines of the angles. Then measure of $\angle$ A is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

B-13*. Three equal circles of radius unity touches one another. Radius of the circle touching all the three circles is :

(A) $\frac{2-\sqrt{3}}{\sqrt{3}}$ (B) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$ (C) $\frac{2+\sqrt{3}}{\sqrt{3}}$ (D) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$

B-14.✍ Triangle ABC is isosceles with AB = AC and BC = 65 cm. P is a point on BC such that the perpendicular distances from P to AB and AC are 24 cm and 36 cm, respectively. The area of triangle ABC (in sq. cm is)

(A) 1254 (B) 1950 (C) 2535 (D) 5070

Section (C) Inradius and Exradius

C-1. In a Δ ABC, the value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to:

(A) $\frac{r}{R}$ (B) $\frac{R}{2r}$ (C) $\frac{R}{r}$ (D) $\frac{2r}{R}$

C-2. In a triangle ABC, if a : b : c = 3 : 7 : 8, then R : r is equal to

(A) 2 : 7 (B) 7 : 2 (C) 3 : 7 (D) 7 : 3

C-3*. If $r_1 = 2r_2 = 3r_3$, then

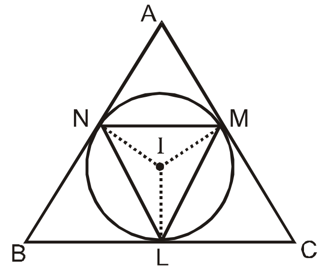
(A) $\frac{a}{b} = \frac{4}{5}$ (B) $\frac{a}{b} = \frac{5}{4}$ (C) $\frac{a}{c} = \frac{3}{5}$ (D) $\frac{a}{c} = \frac{5}{3}$

C-4*. In a ΔABC, following relations hold good. In which case(s) the triangle is a right angled triangle?
(A) $r_2 + r_3 = r_1 - r$ (B) $a^2 + b^2 + c^2 = 8R^2$ (C) $r_1 = s$ (D) $2R = r_1 - r$

C-5. The perimeter of a triangle ABC right angled at C is 70, and the inradius is 6, then $|a - b|$ equals
(A) 1 (B) 2 (C) 8 (D) 9

C-6. In a triangle ABC, if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in:
(A) A.P. (B) G.P. (C) H.P. (D) none of these

C-7. If the incircle of the Δ ABC touches its sides at L, M and N as shown in the figure and if x, y, z be the circumradii of the triangles MIN, NIL and LIM respectively, where I is the incentre, then the product xyz is equal to :



(A) Rr^2 (B) rR^2 (C) $\frac{1}{2}Rr^2$ (D) $\frac{1}{2}rR^2$

C-8. If in a ΔABC, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan\frac{A}{2}\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)$ is equal to :
(A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 3

C-9. If in a ΔABC, $\angle A = \frac{\pi}{2}$, then $\tan\frac{C}{2}$ is equal to
(A) $\frac{a-c}{2b}$ (B) $\frac{a-b}{2c}$ (C) $\frac{a-c}{b}$ (D) $\frac{a-b}{c}$

C-10. In any ΔABC, $\frac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{Rs^2}$ is always equal to
(A) 8 (B) 27 (C) 16 (D) 4

C-11*. In a triangle ABC, right angled at B, then
(A) $r = \frac{AB + BC - AC}{2}$ (B) $r = \frac{AB + AC - BC}{2}$
(C) $r = \frac{AB + BC + AC}{2}$ (D) $R = \frac{s-r}{2}$

C-.12*. With usual notations, in a Δ ABC the value of $\Pi\,(r_1 - r)$ can be simplified as:
(A) $abc\,\Pi\tan\frac{A}{2}$ (B) $4rR^2$ (C) $\frac{(abc)^2}{R(a+b+c)^2}$ (D) $4Rr^2$

C-13. **STATEMENT-1 :** In a triangle ABC, the harmonic mean of the three exradii is three times the inradius.
STATEMENT-2 : In any triangle ABC, $r_1 + r_2 + r_3 = 4R$.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

Section (D) Miscellaneous

D-1.✍ If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then cos B + cos C is equal to :

(A) 0 (B) 1 (C) 2 (D) 1/2

D-2.✍ In a ΔABC, if AB = 5 cm, BC = 13 cm and CA = 12 cm, then the distance of vertex 'A' from the side BC is (in cm)

(A) $\frac{25}{13}$ (B) $\frac{60}{13}$ (C) $\frac{65}{12}$ (D) $\frac{144}{13}$

D-3.✍ If AD, BE and CF are the medians of a ΔABC, then $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ is equal to

(A) 4 : 3 (B) 3 : 2 (C) 3 : 4 (D) 2 : 3

D-4*.✍ In a triangle ABC, with usual notations the length of the bisector of internal angle A is :

(A) $\frac{2bc \cos\frac{A}{2}}{b+c}$ (B) $\frac{2bc \sin\frac{A}{2}}{b+c}$ (C) $\frac{abc \ \ \text{cosec}\frac{A}{2}}{2R \ \ (b+c)}$ (D) $\frac{2\Delta}{b+c}\cdot\text{cosec}\frac{A}{2}$

D-5.✍ Let f, g, h be the lengths of the perpendiculars from the circumcentre of the Δ ABC on the sides BC, CA and AB respectively. If $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \lambda\frac{a\,b\,c}{f\,g\,h}$, then the value of '$\lambda$' is:

(A) 1/4 (B) 1/2 (C) 1 (D) 2

D-6.✍ In an acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to

(A) $\frac{\Delta}{2R}$ (B) $\frac{\Delta}{3R}$ (C) $\frac{\Delta}{4R}$ (D) $\frac{\Delta}{R}$

D-7.✍ AA_1, BB_1 and CC_1 are the medians of triangle ABC whose centroid is G. If points A, C_1, G and B_1 are concyclic, then

(A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$

D-8. If 'ℓ' is the length of median from the vertex A to the side BC of a ΔABC, then

(A) $4\ell^2 = b^2 + 4ac \cos B$ (B) $4\ell^2 = a^2 + 4bc \cos A$

(C) $4\ell^2 = c^2 + 4ab \cos C$ (D) $4\ell^2 = b^2 + 2c^2 - 2a^2$

D-9*. The product of the distances of the incentre from the angular points of a Δ ABC is:

(A) $4 R^2 r$ (B) $4 Rr^2$ (C) $\frac{(a\,b\,c)\,R}{s}$ (D) $\frac{(a\,b\,c)\,r}{s}$

D-10.✍ In a triangle ABC, B = 60° and C = 45°. Let D divides BC internally in the ratio 1 : 3, then value of $\frac{\sin\angle BAD}{\sin\angle CAD}$ is

(A) $\sqrt{\frac{2}{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{3}$

D-11*.✍ In a triangle ABC, points D and E are taken on side BC such that BD = DE = EC. If angle ADE = angle AED = θ, then:

(A) $\tan\theta = 3 \tan B$ (B) $3 \tan\theta = \tan C$

(C) $\frac{6\tan\theta}{\tan^2\theta - 9} = \tan A$ (D) angle B = angle C

D-12. **STATEMENT-1 :** If R be the circumradius of a ΔABC, then circumradius of its excentral $\Delta I_1I_2I_3$ is 2R.

STATEMENT-2 : If circumradius of a triangle be R, then circumradius of its pedal triangle is $\frac{R}{2}$.

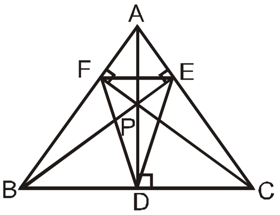
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

PART-II (COMPREHENSION)

Comprehension # 1 (Q. No. 1 to 4)

The triangle DEF which is formed by joining the feet of the altitudes of triangle ABC is called the Pedal Triangle.

Answer The Following Questions :



1.✍ Angle of triangle DEF are

(A) $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$ (B) $\pi + 2A$, $\pi + 2B$ and $\pi + 2C$
(C) $\pi - A$, $\pi - B$ and $\pi - C$ (D) $2\pi - A$, $2\pi - B$ and $2\pi - C$

2*.✍ Sides of triangle DEF are

(A) b cosA, a cosB, c cosC (B) a cosA, b cosB, c cosC
(C) R sin 2A, R sin 2B, R sin 2C (D) a cotA, b cotB, c cotC

3.✍ Circumraii of the triangle PBC, PCA and PAB are respectively

(A) R, R, R (B) 2R, 2R, 2R (C) R/2, R/2, R/2 (D) 3R, 3R, 3R

4*.✍ Which of the following is/are correct

(A) $\frac{\text{Perimeter of } \Delta DEF}{\text{Perimeter of } \Delta ABC} = \frac{r}{R}$ (B) Area of ΔDEF = 2 Δ cosA cosB cosC

(C) Area of ΔAEF = $\Delta \cos^2 A$ (D) Circum-radius of ΔDEF =

Comprehension # 2 (Q. 5 to 8)

The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle and in this case internal angle bisector of triangle ABC are the altitudes of triangles $I_1I_2I_3$

5.✍ Incentre I of Δ ABC is the .... of the excentral $\Delta\, I_1 I_2 I_3$.

(A) Circumcentre (B) Orthocentre (C) Centroid (D) None of these

6.✍ Angles of the $\Delta\, I_1 I_2 I_3$ are

(A) $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$ (B) $\frac{\pi}{2} + \frac{A}{2}$, $\frac{\pi}{2} + \frac{B}{2}$ and $\frac{\pi}{2} + \frac{C}{2}$

(C) $\frac{\pi}{2} - A$, $\frac{\pi}{2} - B$ and $\frac{\pi}{2} - C$ (D) None of these

7.✍ Sides of the $\Delta\, I_1 I_2 I_3$ are

(A) $R\cos\frac{A}{2}$, $R\cos\frac{B}{2}$ $R\cos\frac{C}{2}$ (B) $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ $4R\cos\frac{C}{2}$

(C) $2R\cos\frac{A}{2}$, $2R\cos\frac{B}{2}$ $2R\cos\frac{C}{2}$ (D) None of these

8.✍ Value of $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 =$

(A) $4R^2$ (B) $16R^2$ (C) $32R^2$ (D) $64R^2$

PART-III (MATCH THE COLUMN)

1. Match the column

Column– I		**Column–II**	
(A)	In a ΔABC, $2B = A + C$ and $b^2 = ac$. Then the value of $\frac{a^2(a+b+c)}{3abc}$ is equal to	(p)	8
(B)	In any right angled triangle ABC, the value of $\frac{a^2+b^2+c^2}{R^2}$ is always equal to (where R is the circumradius of ΔABC)	(q)	1
(C)	In a ΔABC if $a = 2$, $bc = 9$, then the value of $2R\Delta$ is equal to	(r)	5
(D)	In a ΔABC, $a = 5$, $b = 3$ and $c = 7$, then the value of $3\cos C + 7\cos B$ is equal to	(s)	9

2. **Match the column**

Column – I		**Column – II**	
(A)	In a ΔABC, $a = 4$, $b = 3$ and the medians AA_1 and BB_1 are mutually perpendicular, then square of area of the ΔABC is equal to	(p)	27
(B)✍	In any ΔABC, minimum value of $\frac{r_1\ r_2\ r_3}{r^3}$ is equal to	(q)	7
(C)	In a ΔABC, $a = 5$, $b = 4$ and $\tan\frac{C}{2} = \sqrt{\frac{7}{9}}$, then side 'c' is equal to	(r)	6
(D)✍	In a ΔABC, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of $(8\cos B)$ is equal to	(s)	11

Exercise-3

✍ Marked Questions may have for Revision Questions.

*** Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is

[IIT-JEE 2010, Paper-1, (3, –1), 84]

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

2.✍ Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

[IIT-JEE 2010, Paper-1, (3, 0), 84]

(A) $-\left(2+\sqrt{3}\right)$ (B) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$

3.✍ Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to

[IIT-JEE 2010, Paper-2, (3, 0), 79]

4. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

[IIT-JEE 2012, Paper-2, (3, –1), 66]

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

5.*✍ In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) 16 (B) 18 (C) 24 (D) 22

6. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

7*. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

[JEE (Advanced) 2016, Paper-1, (4, –2)/62]

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

8*. In a triangle PQR, let $\angle PQR = 30^{\circ}$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?

(A) $\angle QPR = 45^{\circ}$ **[JEE(Advanced) 2018, Paper-1,(4, –2)/60]**

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^{\circ}$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(D) The area of the circumcircle of the triangle PQR is 100π

9. In a non-right-angled triangle ΔPQR, Let p, q, r denote the lengths of the sies opposite to the angles at P, Q, R respectively. The median form R meets the side PQ at S, the perpendicualr4 from P meets the side QR at E, and RS and PE intersect at O. If p = $\sqrt{3}$, q = 1, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

[JEE(Advanced) 2019, Paper-1,(4, –1)/62]

(A) Length of RS = $\frac{\sqrt{7}}{2}$

(B) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

(C) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2 - \sqrt{3})$

(D) Length of OE = $\frac{1}{6}$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

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1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A **false** statement among the following is **[AIEEE - 2010 (4, –1), 144]**

(1) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$.

(2) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$.

(3) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$.

(4) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$.

2.✍ ABCD is a trapezium such that AB and CD are parallel and BC $\perp$ CD. If $\angle ADB = \theta$, BC = p and CD = q, then AB is equal to : **[AIEEE - 2013, (4, –1),120]**

(1) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$

(2) $\frac{p^2+q^2\cos\theta}{p\cos\theta + q\sin\theta}$

(3) $\frac{p^2+q^2}{p^2\cos\theta + q^2\sin\theta}$

(4) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta + q\sin\theta)^2}$

3. With the usual notation, in ΔABC, if $\angle A + \angle B = 120°$, a = $\sqrt{3}$ + 1 and b = $\sqrt{3}$ –1, then the ratio $\angle A : \angle B$, is: **[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]**

(1) 9 : 7

(2) 7 : 1

(3) 3 : 1

(4) 5 : 3

4. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is **[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]**

(1) $\frac{c}{\sqrt{3}}$

(2) $\frac{3}{2}y$

(3) $\frac{c}{3}$

(4) $\frac{y}{\sqrt{3}}$

Answers

EXERCISE - 1

Section (A) :

A-2.	$x > 5$	**A-3.**	75°	**A-8.**	$\frac{P}{2\sqrt{R}}$	**A-11.**	30°	**A-13.**	10°

Section (B) :

B-2.	$\frac{2}{3}\cot\frac{B}{2}$	**B-3.**	9 sq. unit

Section (C)

C-4.	6, 8, 10 cm	**C-5.**	8

Section (D)

D-2.	3	**D-3.**	$\sqrt{2}$

EXERCISE - 2

PART -I

Section (A) :

A-1.	(C)	**A-2.**	(AB)	**A-3.**	(C)	**A-4.**	(C)	**A-5**	(D)	**A-6.**	(C)	**A-7.**	(D)
A-8.	(B)	**A-9.**	(AB)	**A-10.**	(A)	**A-11.**	(A)						

Section (B) :

B-1.	(B)	**B-2.**	(A)	**B-3.**	(A)	**B-4.**	(C)	**B-5.**	(B)	**B-6.**	(AD)	**B-7.**	(B)
B-8.	(ABC)	**B-9.**	(B)	**B-10.**	(AB)	**B-11.**	(A)	**B-12.**	(C)	**B-13.**	(AC)	**B-14.**	(C)

Section (C) :

C-1.	(A)	**C-2.**	(B)	**C-3.**	(BD)	**C-4.**	(ABCD)	**C-5.**	(A)	**C-6.**	(A)	**C-7.**	(C)
C-8.	(B)	**C-9.**	(D)	**C-10.**	(D)	**C-11*.**	(AD)	**C-.12*.**	(ACD)	**C-13.**	**(C)**		

Section (D)

D-1.	(B)	**D-2.**	(B)	**D-3.**	(C)	**D-4.**	(ACD)	**D-5.**	(A)	**D-6.**	(D)	**D-7.**	(C)
D-8.	(B)	**D-9.**	(BD)	**D-10.**	(C)	**D-11.**	(ACD)	**D-12.**	(A)				

PART -II

1.	(A)	**2.**	(BC)	**3.**	(A)	**4.**	(ABCD)	**5.**	(B)	**6.**	(A)	**7.**	(B)
8.	(B)												

PART -III

1. (A) → (q), (B) → (p), (C) → (s), (D) → (r)

2. (A) → (s), (B) → (p), (C) → (r), (D) → (q)

EXERCISE - 3

PART -I

1.	(D)	**2.**	(B)	**3.**	3	**4.**	(C)	**5.**	(BD)	**6.**	(B)
7.	(A,C,D)	**8.**	(BCD)	**9.**	(ACD)						

PART -II

1.	(2)	**2.**	(1)	**3.**	(2)	**4.**	(1)

HLP Answers

SUBJECTIVE QUESTIONS

This questions paste Staright Line sheets

1. In ΔABC , P is an interior point such that $\angle PAB = 10^{\circ}$ $\angle PBA = 20^{\circ}$, $\angle PCA = 30^{\circ}$, $\angle PAC = 40^{\circ}$ then prove that ΔABC is isosceles

2. In a triangle ABC, if $a \tan A + b \tan B = (a + b) \tan\left(\frac{A+B}{2}\right)$, prove that triangle is isosceles.

3. In any triangle ABC, if $2\Delta a - b^2c = c^3$, (where Δ is is the area of triangle), then prove that $\angle A$ is obtuse

4. If in a triangle ABC, $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$ prove that the triangle ABC is either isosceles or right angled.

5. In a Δ ABC, $\angle$ C = 60° and $\angle$ A = 75°. If D is a point on AC such that the area of the Δ BAD is $\sqrt{3}$ times the area of the Δ BCD, find the $\angle$ ABD.

6. In a ΔABC, if a, b and c are in A.P., prove that $\cos A.\cot\frac{A}{2}$, $\cos B.\cot\frac{B}{2}$, and $\cos C.\cot\frac{C}{2}$ are in A.P.

7. In a triangle ABC, prove that the area of the incircle is to the area of triangle itself is,

$\pi : \cot\left(\frac{A}{2}\right) . \cot\left(\frac{B}{2}\right) . \cot\left(\frac{C}{2}\right)$.

8. In ΔABC, prove that $a^2 (s - a) + b^2 (s - b) + c^2 (s - c) = 4R\Delta \left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

9. In any ΔABC, prove that

(i) $(r_3+ r_1)(r_3+ r_2)\sin C = 2 r_3 \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$

(ii) $\frac{\tan\frac{A}{2}}{(a - b)(a - c)} + \frac{\tan\frac{B}{2}}{(b - a)(b - c)} + \frac{\tan\frac{C}{2}}{(c - a)(c - b)} = \frac{1}{\Delta}$

(iii) $(r+r_1)\tan\frac{B - C}{2} + (r + r_2)\tan\frac{C-A}{2} + (r + r_3)\tan\frac{A-B}{2} = 0$

(iv) $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.

10. In an acute angled triangle ABC, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then prove that $b + 3c < 3a < 3b + 3c$

11. If the inradius in a right angled triangle with integer sides is r. Prove that

(i) If r = 4, the greatest perimeter (in units) is 90

(ii) If r = 5, the greatest area (in sq. units) is 330

12. If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2$, then prove that the triangle is right angled.

13. DEF is the triangle formed by joning the points of contact of the incircle with the sides of the triangle ABC; prove that

(i) its sides are $2r\cos\frac{A}{2}$, $2r\cos\frac{B}{2}$ and $2r\cos\frac{C}{2}$,

(ii) its angles are $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$

and

(iii) its area is $\frac{2\Delta^3}{(abc)s}$, i.e. $\frac{1}{2}\frac{r}{R}\Delta$.

14. Three circles, whose radii are a, b and c, touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is $\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$.

15. OA and OB are the equal sides of an isoscles triangle lying in the first quadrant making angles θ and ϕ respectively with x-axis. Show that the gradient of the bisector of acute angle AOB is cosec β – cot β where $\beta = \phi + \theta$. (Where O is origin)

16. The hypotenuse BC = a of a right-angled triangle ABC is divided into n equal segments where n is odd. The segment containing the midpoint of BC subtends angle α at A. Also h is the altitude of the triangle through A. Prove that $\tan\alpha = \frac{4nh}{a\left(n^2-1\right)}$.

Answers

5. $\angle$ ABD = 30°

Exercise-1

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

SUBJECTIVE QUESTIONS

विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule
खण्ड (A) : ज्या नियम, कोज्या नियम, स्पर्शज्या नियम या नेपियर एनालोजि, प्रक्षेप नियम

A-1. In a ΔABC, prove that :
त्रिभुज ABC में सिद्ध कीजिए कि :

(i) $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$

(ii) $\dfrac{a^2 \sin(B-C)}{\sin A} + \dfrac{b^2 \sin(C-A)}{\sin B} + \dfrac{c^2 \sin(A-B)}{\sin C} = 0$

(iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

(iv)✍ $(a - b)^2 \cos^2\dfrac{C}{2} + (a + b)^2 \sin^2\dfrac{C}{2} = c^2$

(v) $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$

(vi) $\dfrac{\sin B}{\sin C} = \dfrac{c - a\cos B}{b - a\cos C}$

Sol. (i) L.H.S. = $a \sin (B - C) + b \sin (C - A) + c \sin (A - B)$
$= k \sin A \sin (B - C) + k \sin B \sin (C - A) + k \sin C \sin (A - B)$
$= k (\sin^2 B - \sin^2 C) + k (\sin^2 C - \sin^2 A) + k (\sin^2 A - \sin^2 B)$
$= 0 =$ R.H.S.

(ii) L.H.S. = $\dfrac{a^2 \sin(B-C)}{\sin\ A} + \dfrac{b^2 \sin(C-A)}{\sin\ B} + \dfrac{c^2 \sin(A-B)}{\sin\ C}$

first term = $\dfrac{a^2 \sin(B-C)}{\sin\ A} = \dfrac{k^2 \sin^2 A\ \ \sin(B-C)}{\sin\ A}$
$= k^2 \sin (B + C)\ \sin (B - C)$
$= k^2 (\sin^2 B - \sin^2 C)$

Similarly $\dfrac{b^2 \sin(C-A)}{\sin B} = k^2 (\sin^2 C - \sin^2 A)$

and $\dfrac{c^2 \sin(A-B)}{\sin C} = k^2 (\sin^2 A - \sin^2 B)$

$\therefore$ L.H.S. = $k^2 (\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B)$
$= 0 =$ R.H.S.

(iii) L.H.S. $= 2bc \cos A + 2ca \cos B + 2ab \cos C$
$= b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2$
$= a^2 + b^2 + c^2$
= R.H.S

(iv) L.H.S. = $a^2 \left(\cos^2\dfrac{C}{2} + \sin^2\dfrac{C}{2}\right) + b^2\left(\cos^2\dfrac{C}{2} + \sin^2\dfrac{C}{2}\right) - 2ab\left(\cos^2\dfrac{C}{2} - \sin^2\dfrac{C}{2}\right)$
$= a^2 + b^2 - 2ab \cos C$
$= a^2 + b^2 - (a^2 + b^2 - c^2)$
$= c^2 =$ R.H.S.

(v) $\because$ L.H.S. $= b^2 \sin 2C + c^2 \sin 2B$
$= 2b^2 \sin C \cos C + 2c^2 \sin B \cos B$
$= 2k^2 \sin^2 B \cos C \sin C + 2k^2 \sin^2 C \sin B \cos B$ ($\because b = k\sin B, c = k\sin C$)
$= 2k^2 \sin B \sin C [\sin B \cos C + \cos B \sin C]$
$= 2(k \sin B) (k \sin C) \sin (B + C)$

$= 2bc \sin A$

(vi) $\because$ R.H.S $= \dfrac{c - a\cos B}{b - a\cos C}$ $\qquad \because$ $c = a \cos B + b \cos A$,

$b = c \cos A + a \cos C$

$= \dfrac{b\cos A}{c\cos A} = \dfrac{b}{c}$

$= \dfrac{\sin B}{\sin C}$ = L.H.S.

Hindi. (i) बायां पक्ष (L.H.S.) $= a \sin (B - C) + b \sin (C - A) + c \sin (A - B)$

$= k \sin A \sin (B - C) + k \sin B \sin (C - A) + k \sin C \sin (A - B)$

$= k (\sin^2 B - \sin^2 C) + k (\sin^2 C - \sin^2 A) + k (\sin^2 A - \sin^2 B)$

$= 0$ = दायां पक्ष (R.H.S.)

(ii) बायां पक्ष (L.H.S.) $= \dfrac{a^2 \sin(B - C)}{\sin A} + \dfrac{b^2 \sin(C - A)}{\sin B} + \dfrac{c^2 \sin(A - B)}{\sin C}$

प्रथम पद $= \dfrac{a^2 \sin(B - C)}{\sin A} = \dfrac{k^2 \sin^2 A \sin(B - C)}{\sin A}$

$= k^2 \sin (B + C) \sin (B - C)$

$= k^2 (\sin^2 B - \sin^2 C)$

इसी प्रकार, $\dfrac{b^2 \sin(C - A)}{\sin B} = k^2 (\sin^2 C - \sin^2 A)$

और $\dfrac{c^2 \sin(A - B)}{\sin C} = k^2 (\sin^2 A - \sin^2 B)$

$\therefore$ बायां पक्ष (L.H.S.) $= k^2 (\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B)$

$= 0$ = दायां पक्ष (R.H.S.)

(iii) बायां पक्ष (L.H.S.) $= 2bc \cos A + 2ca \cos B + 2ab \cos C$

$= b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2$

$= a^2 + b^2 + c^2$

= R.H.S. दायां पक्ष (R.H.S.)

(iv) बायां पक्ष (L.H.S.) $= a^2 \left(\cos^2 \dfrac{C}{2} + \sin^2 \dfrac{C}{2}\right) + b^2 \left(\cos^2 \dfrac{C}{2} + \sin^2 \dfrac{C}{2}\right) - 2ab\left(\cos^2 \dfrac{C}{2} - \sin^2 \dfrac{C}{2}\right)$

$= a^2 + b^2 - 2ab \cos C$

$= a^2 + b^2 - (a^2 + b^2 - c^2)$

$= c^2$ = दायां पक्ष (R.H.S.)

(v) $\because$ बायां पक्ष (L.H.S.) $= b^2 \sin 2C + c^2 \sin 2B$

$= 2b^2 \sin C \cos C + 2c^2 \sin B \cos B$

$= 2k^2 \sin^2 B \cos C \sin C + 2k^2 \sin^2 C \sin B \cos B$ ($\because$ $b = k\sin B$, $c = k\sin C$)

$= 2k^2 \sin B \sin C [\sin B \cos C + \cos B \sin C]$

$= 2(k \sin B) (k \sin C) \sin (B + C)$

$= 2bc \sin A$

(vi) $\because$ दायां पक्ष (R.H.S) $= \dfrac{c - a\cos B}{b - a\cos C}$ $\qquad \because$ $c = a \cos B + b \cos A$,

$b = c \cos A + a \cos C$

$= \dfrac{b\cos A}{c\cos A} = \dfrac{b}{c}$

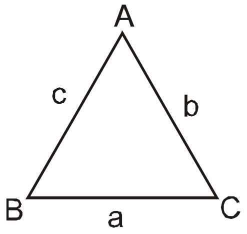
$= \dfrac{\sin B}{\sin C}$ = बायां पक्ष (L.H.S.)

A-2. Find the real value of x such that $x^2 + 2x$, $2x + 3$ and $x^2 + 3x + 8$ are lengths of the sides of a triangle.

x के वह वास्तविक मान ज्ञात कीजिए जबकि $x^2 + 2x$, $2x + 3$ तथा $x^2 + 3x + 8$ एक त्रिभुज की भुजायें हो।

Ans. $x > 5$

Sol.



For ΔABC $\quad a + b > c, \quad b + c > a, \quad c + a > b$

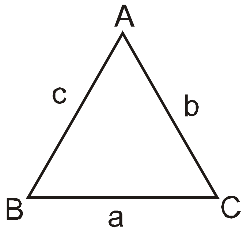
$x^2 + 4x + 3 > x^2 + 3x + 8 \Rightarrow \quad x > 5$

$x^2 + 5x + 11 > x^2 + 2x \quad \Rightarrow \quad x > \frac{-11}{3}$

$2x^2 + 5x + 8 > 2x + 3 \quad \Rightarrow \quad 2x^2 + 3x + 5 > 0 \Rightarrow \quad x \in R$

Common to all is $x > 5$.

Hindi.



ΔABC के लिए $\quad a + b > c, \quad b + c > a, \quad c + a > b$

$x^2 + 4x + 3 > x^2 + 3x + 8 \Rightarrow \quad x > 5$

$x^2 + 5x + 11 > x^2 + 2x \quad \Rightarrow \quad x > \frac{-11}{3}$

$2x^2 + 5x + 8 > 2x + 3$

$\Rightarrow \quad 2x^2 + 3x + 5 > 0 \Rightarrow \quad x \in R$

सभी के लिए उभयनिष्ठ $\quad x > 5$.

A-3. The angles of a ΔABC are in A.P. (order being A, B, C) and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, then find $\angle A$.

त्रिभुज ABC के कोण A, B, C इसी क्रम में समान्तर श्रेढ़ी में है तथा $b : c = \sqrt{3} : \sqrt{2}$ है, तो $\angle A$ ज्ञात कीजिए।

Ans. 75°

Sol. $\because \quad 2B = A + C,$

$\Rightarrow \quad B = 60^{\circ}$

from Sine-rule ज्या नियम के अनुसार

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad \frac{b}{c} = \frac{\sin B}{\sin C} = \frac{\sqrt{3}}{\sqrt{2}}$$

$\therefore \quad \sin C = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad C = 45^{\circ}$

$\therefore \quad A = 75^{\circ}$

A-4. If $\cos A + \cos B = 4 \sin^2\left(\frac{C}{2}\right)$, prove that sides a, c, b of the triangle ABC are in A.P.

यदि $\cos A + \cos B = 4 \sin^2\left(\frac{C}{2}\right)$ हो, तो सिद्ध कीजिए कि त्रिभुज ABC की भुजाएँ a, c, b समान्तर श्रेढ़ी में हैं।

Sol. $\because \quad \cos A + \cos B = 4 \sin^2\frac{C}{2} \quad \Rightarrow \quad 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 4\sin^2\frac{C}{2}$

$\Rightarrow \quad 2\cos\left(\frac{A-B}{2}\right) = 4\sin\frac{C}{2} \quad \Rightarrow \quad 2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) = 4\sin\frac{C}{2}\cos\frac{C}{2}$

$\Rightarrow \quad 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2\sin C \quad \Rightarrow \quad \sin A + \sin B = 2\sin C$

$\Rightarrow$ $a + b = 2c$ $\Rightarrow$ a, c, b are in A.P. a, c, b समान्तर श्रेढी में है।

A-5. If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then prove that a^2, b^2, c^2 are in A.P.

यदि त्रिभुज ABC में, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$ हो, तो सिद्ध कीजिए कि a^2, b^2, c^2 समान्तर श्रेढ़ी में है।

Sol. $\because$ $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$ $\Rightarrow \sin(B + C) \sin(B - C) = \sin(A + B) \sin(A - B)$

$\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$ $\Rightarrow 2 \sin^2 B = \sin^2 A + \sin^2 C$

$\Rightarrow 2b^2 = a^2 + c^2$ $\Rightarrow$ a^2, b^2, c^2 are in A.P. $\Rightarrow$ a^2, b^2, c^2 समान्तर श्रेणी में होगें।

A-6. In a triangle ABC, prove that for any angle θ, $b \cos (A - \theta) + a \cos (B + \theta) = c \cos \theta$.

त्रिभुज ABC में किसी कोण θ के लिए सिद्ध कीजिए कि $b \cos (A - \theta) + a \cos (B + \theta) = c \cos \theta$.

Sol. $\because$ L.H.S. बायां पक्ष $= b(\cos A \cos \theta + \sin A \sin \theta) + a(\cos B \cos \theta - \sin B \sin \theta)$

$= \cos \theta (b \cos A + a \cos B) + \sin \theta (b \sin A - a \sin B)$

$= c \cos \theta + 0$ $\{\because b \sin A - a \sin B = 0\}$

$= c \cos \theta =$ R.H.S. दायां पक्ष

A-7.✍ With usual notations, if in a Δ ABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

यदि त्रिभुज ABC में, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ हो, तो सिद्ध कीजिए कि $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

Sol. $\because$ $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = k$ $\therefore$ $\left.\begin{array}{l} b+c=11\ k \\ c+a=12\ k \\ a+b=13\ k \end{array}\right\} \Rightarrow$ $a = 7k, b = 6k, c = 5k$

$$\because \quad \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36+25-49}{2\times6\times5} = \frac{1}{5}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{25+49-36}{2\times5\times7} = \frac{19}{35}$$

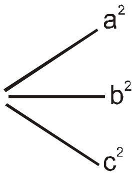
$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49+36-25}{2\times7\times6} = \frac{5}{7} \qquad \therefore \quad \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

A-8. Let a, b and c be the sides of a ΔABC. If a^2, b^2 and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$, where P, Q & R are constants, then find the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ in terms of P, Q and R.

माना a, b तथा c त्रिभुज ABC की भुजाएँ है। यदि a^2, b^2 एवं c^2 समीकरण $x^3 - Px^2 + Qx - R = 0$, जहाँ P, Q तथा R अचर है, के मूल हो तो $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ का मान P, Q एवं R के पदो में ज्ञात कीजिए।

Ans. $\frac{P}{2\sqrt{R}}$

Sol. $\because$ $x^3 - Px^2 + Qx - R = 0$



$\therefore$ $a^2 + b^2 + c^2 = P$

$a^2b^2 + b^2c^2 + c^2a^2 = Q$

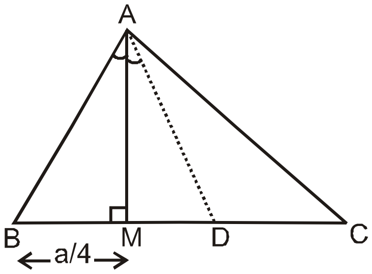
$a^2b^2c^2 = R \Rightarrow abc = \sqrt{R}$

$$\because \quad \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{1}{2abc}[a^2 + b^2 + c^2] = \frac{P}{2\sqrt{R}}$$

A-9. If in a triangle ABC, the altitude AM be the bisector of ∠BAD, where D is the mid point of side BC, then prove that $(b^2 - c^2) = a^2/2$.

यदि त्रिभुज ABC में शीर्षलम्ब AM कोण BAD का अर्धक हो, जहाँ D भुजा BC का मध्य बिन्दु है, तो सिद्ध कीजिए कि $(b^2 - c^2) = a^2/2$.

Sol.



$\because \ \frac{a}{4} = c \cos B$

$\frac{a}{4} = c\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$

$\therefore \ \frac{a^2}{2} = a^2 + c^2 - b^2$

$\therefore \ b^2 - c^2 = \frac{a^2}{2}$.

A-10. If in a triangle ABC, ∠C = 60º , then prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

यदि ABC में ∠C = 60º तब सिद्ध कीजिए कि $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

Sol. By the cosine formula, we ahve

$c^2 = a^2 + b^2 - 2ab \cos C$

or $c^2 = a^2 + b^2 - 2ab \cos 60^\circ = a^2 + b^2 - ab$ .....(i)

Now, $\frac{1}{a+c} + \frac{1}{b+c} - \frac{3}{a+b+c} = \left[\frac{(b+c)(a+b+c) + (a+c)(a+b+c) - 3(a+c)(b+c)}{(a+b)(b+c)(a+b+c)}\right]$

$= \frac{(a^2 + b^2 - ab) - c^2}{(a+b)(b+c)(a+b+c)} = 0$ [from eq. (i)]

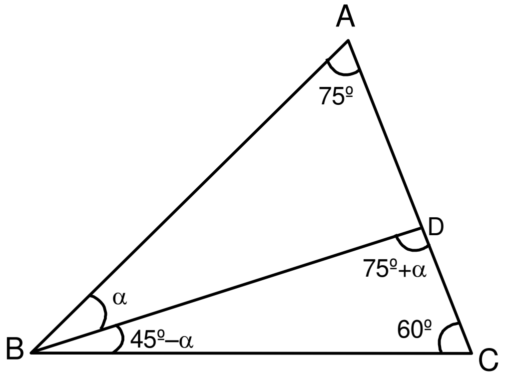
or $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

A-11. In a triangle ABC, ∠C = 60º and ∠A = 75º. If D is a point on AC such that the area of the ΔABD is $\sqrt{3}$ times the area of the ΔBCD, find the ∠ABD.

त्रिभुज ABC में ∠C = 60º तथा ∠A = 75º. यदि D, AC पर बिन्दु इस प्रकार है कि ΔABD का क्षेत्रफल, ΔBCD के क्षेत्रफल का $\sqrt{3}$ गुना है तब ∠ABD ज्ञात कीजिए।

Ans. 30°

Sol. Let h be the length of perpendicular from B on AC



Given that $\frac{\Delta BAD}{\Delta BCD} = \sqrt{3}$

$\Rightarrow \frac{\frac{1}{2}h.AD}{\frac{1}{2}h.DC} = \frac{AD}{CD} = \sqrt{3}$ .....(i)

In ΔBAD, taking $\angle ABD = \alpha$, we have

$\frac{AD}{\sin\alpha} = \frac{BD}{\sin 75º}$ .....(ii)

And in ΔBCD, we have $\frac{CD}{\sin(45°-\alpha)} = \frac{BD}{\sin 60º}$ .....(iii)

$\therefore$ From (2) and (3), we get

$$\frac{AD\sin(45°-\alpha)}{CD\sin\alpha} = \frac{\sin 60º}{\sin 75º}$$

$\Rightarrow \sqrt{3}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\left(\frac{\cos\alpha-\sin\alpha}{\sqrt{2}}\right) = \frac{\sqrt{3}}{2}\sin\alpha$

$\Rightarrow \sqrt{\sqrt{3}+1}\cos\alpha = (3+\sqrt{3})\sin\alpha$

$\Rightarrow \tan\alpha = 1/\sqrt{3}$

$\Rightarrow \alpha = \pi/6 = 30º$

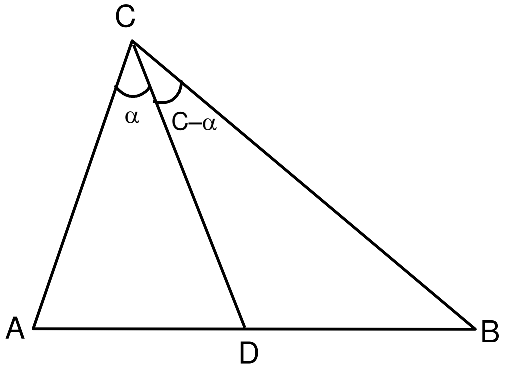
Hence $\angle ABD = 30º$

A-12. In a scalene triangle ABC, D is a point on the side AB such that $CD^2 = AD.\ DB$, if $\sin A.\ \sin B = \sin^2\frac{C}{2}$ then prove that CD is internal bisector of $\angle C$.

एक विषमबाहु त्रिभुज ABC में D भुजा AB पर एक बिन्दु D इस प्रकार है $CD^2 = AD.\ DB$, यदि $\sin A.\ \sin B = \sin^2\frac{C}{2}$ तब सिद्ध कीजिए कि CD, कोण C का आन्तरिक अर्द्धक है।

Sol. Let $\angle ACD = \alpha \Rightarrow \angle DCB = (C - \alpha)$

Figuire



Applying the sine rule in ΔACD and in ΔDCB respectively, we get

$$\Rightarrow \frac{AD}{\sin\alpha} = \frac{CD}{\sin A} \text{ and } \frac{BD}{\sin(C-\alpha)} = \frac{CD}{\sin B} \Rightarrow \frac{AD.BD}{\sin\alpha\sin(C-\alpha)} = \frac{CD^2}{\sin A.\sin B}$$

$$\Rightarrow \frac{1}{2}\,[\cos(2\alpha - C) - \cos C] = \sin^2\frac{C}{2}$$

$$\Rightarrow \frac{1}{2}\,[\cos(2\alpha - C) - 1 + 2\sin^2\frac{C}{2}] = \sin^2\frac{C}{2}$$

$$\Rightarrow \cos(2\alpha - C) = 1$$

$$\Rightarrow \alpha = \frac{C}{2}$$

Thus, CD is the internal angle bisector of angle C.

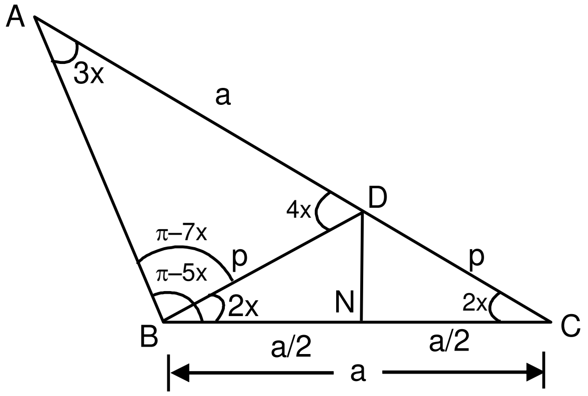
A-13.✍In triangle ABC,D is on AC such that AD = BC , BD = DC, $\angle$DBC = 2x, and $\angle$BAD = 3x, all angles are in degrees, then find the value of x.

ΔABC में D ,AC पर इस प्रकार है कि AD = BC , BD = DC, $\angle$DBC = 2x, और $\angle$BAD = 3x, सभी कोण, डिग्री में है तब x का मान ज्ञात कीजिए।

Ans. 10°

Sol. In ΔABC ,

$$\frac{AC}{\sin 5x} = \frac{BC}{\sin 3x} \Rightarrow \frac{a+p}{\sin 5x} = \frac{a}{\sin 3x}$$



In ΔBDN, $\cos 2x = \frac{a}{2p}$

or a = 2p cos2x

From eq. (i), $\frac{2p\cos 2x + p}{\sin 5x} = \frac{2p\cos 2x}{\sin 3x}$

or 2sin3x cos2x + sin3x = 2sin5x cos2x

or sin5x + sinx + sin3x = sin7x + sin3x

or sin7x – sin5x = sinx

or 2cos6x sinx = sinx

or $\cos 6x = \frac{1}{2} \quad \Rightarrow x = 10^\circ$

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

खण्ड (B) अर्द्धकोण, त्रिभुज का क्षेत्रफल और परित्रिज्या में त्रिकोणमितिय सम्बन्ध

B-1. In a ΔABC, prove that

त्रिभुज ABC में सिद्ध कीजिए कि –

(i) $2\left[a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right]$ = c + a – b.   (ii) $\frac{\cos^2\frac{A}{2}}{a}+\frac{\cos^2\frac{B}{2}}{b}+\frac{\cos^2\frac{C}{2}}{c}=\frac{s^2}{abc}$

(iii) $4\left(bc.\cos^2\frac{A}{2}+ca.\cos^2\frac{B}{2}+ab.\cos^2\frac{C}{2}\right)$ = (a + b + c)^2

(iv)✍ (b – c) cot $\frac{A}{2}$ + (c – a) cot $\frac{B}{2}$ + (a – b) cot $\frac{C}{2}$ = 0

(v) 4Δ (cot A + cot B + cot C) = a^2 + b^2 + c^2

(vi) $\left(\frac{2abc}{a+b+c}\right)$. cos $\frac{A}{2}$ .cos $\frac{B}{2}$. cos $\frac{C}{2}$ = Δ

Sol. (i) L.H.S. (बायां पक्ष) = 2a sin^2 $\frac{C}{2}$ + 2 c sin^2 $\frac{A}{2}$

= a(1 – cos c) + c(1 – cos A)
= a + c – (a cos C + c cos A)
= a + c – b
= R.H.S. (दायां पक्ष)

(ii) ∵ L.H.S. (बायां पक्ष) = $\frac{\cos^2\frac{A}{2}}{a}+\frac{\cos^2\frac{B}{2}}{b}+\frac{\cos^2\frac{C}{2}}{c}$

$$=\frac{1}{a}.\frac{s(s-a)}{bc}+\frac{1}{b}.\frac{s(s-b)}{ca}+\frac{1}{c}.\frac{s(s-c)}{ab}=\frac{s(3s-(a+b+c))}{abc}=\frac{s^2}{abc}.$$

(iii) L.H.S. (बायां पक्ष) = 2bc(1 + cos A) + 2ca(1 + cos B) + 2ab(1 + cos C)

= 2bc + 2ca + 2ab + 2bc cos A + 2ca cos B + 2 ab cos C
= 2 $\sum$ab + a^2 + b^2 + c^2 = (a + b + c)^2 = R.H.S. (दायां पक्ष)

(iv) ∵ L.H.S. (बायां पक्ष) = (b – c) cot $\frac{A}{2}$ + (c – a) cot $\frac{B}{2}$ + (a – b) cot $\frac{C}{2}$

$$\because\ (b-c)\cot\frac{A}{2}=k(\sin B-\sin C)\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=2k\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$$

$$=2k\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)=k\,[\cos C-\cos B]$$

similarly (इसी प्रकार) (c – a) cot $\frac{B}{2}$ = k[cos A – cos C]

and (और) (a – b) cot $\frac{C}{2}$ = k[cos B – cos A]

∴ L.H.S. (दायां पक्ष) = k[cos C – cos B + cos A – cos C + cos B – cos A]
= 0
= R.H.S. (दायां पक्ष)

(v) L.H.S. बायां पक्ष = 4Δ (cot A + cot B + cot C)

$$=4\Delta\left(\frac{\cos\ A}{\sin\ A}+\frac{\cos B}{\sin\ B}+\frac{\cos\ C}{\sin\ C}\right)\qquad\left\{\therefore\ \Delta=\frac{1}{2}\ bc\ \sin A\right\}$$

= 2bc cos A + 2 ca cos B + 2ab cos C
= a^2 + b^2 + c^2 = R.H.S. दायां पक्ष

(vi) L.H.S. बायां पक्ष = $\frac{2abc}{a+b+c}\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$

$$= \frac{2\ abc}{2s}\sqrt{\frac{s(s-a)}{bc}\ .\ \frac{s(s-b)}{ca}\ .\ \frac{s(s-c)}{ab}} = \sqrt{s(s-a)(s-b)\ \ (s-c)} = \Delta$$ = R.H.S. दायां पक्ष .

B-2. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan\frac{A}{2} + \tan\frac{C}{2}$ in terms of cot (B/2).

यदि किसी त्रिभुज की भुजाएँ a, b, c समान्तर श्रेढ़ी में हो, तो $\tan\frac{A}{2} + \tan\frac{C}{2}$ का मान cot (B/2) के पदों में ज्ञात कीजिए।

Ans. $\frac{2}{3}\cot\frac{B}{2}$

Sol. $\because$ 2b = a + c ...(i)

$$\because\ \tan\frac{A}{2} + \tan\frac{C}{2} = \sqrt{\frac{(s-b)\ \ (s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)\ \ (s-b)}{s(s-c)}}$$

$$= \sqrt{\frac{s-b}{s}}\left[\frac{s-c+s-a}{\sqrt{(s-a)(s-c)}}\right] = \frac{b}{s}\sqrt{\frac{s(s-b)}{(s-a)\ \ (s-c)}}$$

$$= \frac{2b}{2s}\sqrt{\frac{s(s-b)}{(s-a)\ \ (s-c)}} = \frac{2b}{3b}\cot\frac{B}{2} = \frac{2}{3}\cot\frac{B}{2}.$$

B-3. If in a Δ ABC, a = 6, b = 3 and cos(A – B) = 4/5, then find its area.
यदि त्रिभुज ABC में, a = 6, b = 3 एवं cos(A – B) = 4/5 हो, तो इसका क्षेत्रफल ज्ञात कीजिए।

Ans. 9 sq. unit

Sol. $\because$ a = 6, b = 3 and तथा cos(A – B) = 4/5

$\because$ $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$ .... (i)

$\because$ $\tan^2\left(\frac{A-B}{2}\right) = \frac{1-\cos(A-B)}{1+\cos(A-B)} = \frac{1}{9}$

$\therefore$ $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$

$\therefore$ from (i) से, we get $\frac{1}{3} = \frac{1}{3}\cot\frac{C}{2} \Rightarrow$ C = 90°

$\therefore$ Area क्षेत्रफल = $\frac{1}{2}$ab = 9 sq. unit वर्ग इकाई

B-4.✍ If in a triangle ABC, $\angle$A = 30º and the area of triangle is $\frac{\sqrt{3}\ a^2}{4}$, then prove that either B = 4 C or C = 4 B.

यदि त्रिभुज ABC में, $\angle$A = 30º तथा त्रिभुज का क्षेत्रफल $\frac{\sqrt{3}\ a^2}{4}$ हो, तो सिद्ध कीजिए कि या तो B = 4 C या C = 4 B.

Sol. $\because$ $\angle$A = 30° and तथा $\Delta = \frac{\sqrt{3}\ \ a^2}{4}$

$\because$ $\frac{1}{2}bc\sin A = \frac{\sqrt{3}}{4}a^2 \Rightarrow \frac{1}{2}bc\sin 30° = \frac{\sqrt{3}}{4}a^2 \Rightarrow bc = \sqrt{3}\,a^2$

$\Rightarrow$ $\sin B\sin C = \sqrt{3}\sin^2 A \Rightarrow \sin B\sin C = \frac{\sqrt{3}}{4}$ as जैसा कि $\sin A = \frac{1}{2}$

$\Rightarrow$ $\cos(B-C) - \cos(B+C) = \frac{\sqrt{3}}{2} \Rightarrow \cos(B-C) + \cos A = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\cos(B - C) = 0$ (as जैसा कि $\angle A = 30^{\circ} \Rightarrow \cos A = \frac{\sqrt{3}}{2}$)

$\Rightarrow$ $B - C = 90^{\circ}$ or या $B - C = -90^{\circ}$

But लेकिन $B + C = 150^{\circ}$ as $A = 30^{\circ}$ (क्योंकि दिया है $A = 30^{\circ}$)

case (i) : if यदि $B - C = 90^{\circ}$

and तथा $B + C = 150^{\circ}$ $\Rightarrow$ $B = 120^{\circ}$ and तथा $C = 30^{\circ}$ $\Rightarrow$ $B = 4C$

case (ii) : if यदि $B - C = -90^{\circ}$

and तथा $B + C = 150^{\circ}$ $\Rightarrow$ $B = 30^{\circ}$ and तथा $C = 120^{\circ}$ $\therefore$ $C = 4B$.

Section (C) Inradius and Exradius
खण्ड(C) अन्तत्रिज्या परित्रिज्या

C-1. In any ΔABC, prove that
त्रिभुज ABC में सिद्ध कीजिए कि –

(i) $Rr(\sin A + \sin B + \sin C) = \Delta$ (ii) $a\cos B\cos C + b\cos C\cos A + c\cos A\cos B = \frac{\Delta}{R}$

(iii) $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$. (iv)✍ $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + \frac{r}{2R}$

(v)✍ $a\cot A + b\cot B + c\cot C = 2(R + r)$

Sol. (i) $\because$ L.H.S. (बायां पक्ष) $= Rr(\sin A + \sin B + \sin C)$

$$= Rr\left(\frac{a+b+c}{2R}\right) \qquad \because \quad r = \frac{\Delta}{s}$$

$$= \frac{r(2s)}{2} = rs = \Delta$$

(ii) $\because$ L.H.S. (बायां पक्ष) $= a\cos B\cos C + \cos A(b\cos C + c\cos B)$

$$= a[\cos B\cos C + \cos A]$$
$$= a[\cos B\cos C - \cos(B + C)]$$
$$= a\sin B\sin C$$
$$= a.\frac{b}{2R}.\frac{c}{2R} = \frac{abc}{4R^2} = \frac{4R\Delta}{4R^2} = \frac{\Delta}{R}$$

(iii) $\because$ L.H.S. (बायां पक्ष) $= \frac{c+a+b}{abc} = \frac{2s}{4R\Delta} = \frac{1}{2R\frac{\Delta}{s}} = \frac{1}{2Rr}$

(iv) L.H.S. बायां पक्ष $= \frac{1}{2}(1 + \cos A + 1 + \cos B + 1 + \cos C)$

$$= \frac{1}{2}\left(3+1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 2 + \frac{1}{2}\frac{r}{R} = 2 + \frac{r}{2R}$$

(v) L.H.S. बायां पक्ष $= a\frac{\cos A}{\sin A} + b\frac{\cos B}{\sin B} + c\frac{\cos C}{\sin C}$

$$= 2R\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 2R + 2r = 2(R + r)$$

C-2. In any ΔABC, prove that
किसी त्रिभुज ABC में सिद्ध कीजिए कि :–

(i) $r.\,r_1.r_2.r_3 = \Delta^2$

(ii)✍ $r_1 + r_2 - r_3 + r = 4R\cos C$. (iii) $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$

(iv) $\left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2 = \frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$ (v)✍ $\frac{bc-r_2r_3}{r_1} = \frac{ca-r_3r_1}{r_2} = \frac{ab-r_1r_2}{r_3} = r$

Sol. (i) $r.\,r_1\,.r_2\,.r_3 = \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = D^2$

(ii) $r_1 + r_2 - r_3 + r = 4R\cos C$ ∵ L.H.S. $= \frac{\Delta}{s-a}+\frac{\Delta}{s-b}-\frac{\Delta}{s-c}+\frac{\Delta}{s}$

$$= \Delta\left[\frac{s-b+s-a}{(s-a)(s-b)}\right]-\Delta\left[\frac{1}{s-c}-\frac{1}{s}\right] = \Delta\left[\frac{c}{(s-a)\ (s-b)}-\frac{c}{s(s-c)}\right] = \Delta c\left[\frac{s(s-c)-(s-a)(s-b)}{s\ (s-a)\ (s-b)\ (s-c)}\right]$$

$$= \frac{\Delta c\ \ [s^2-sc-s^2+s(a+b)-ab]}{\Delta^2} = \frac{c\ \ [s(a+b-c)-ab]}{\Delta} = \frac{c[(a+b+c)\ \ (a+b-c)-2ab]}{2\Delta}$$

$$= c\left[\frac{(a+b)^2-c^2-2ab}{2\Delta}\right] = \frac{c\ \ (a^2+b^2-c^2)}{2\Delta}$$

∵ $\cos C = \frac{a^2+b^2-c^2}{2ab}$ ∴ L.H.S. $= \frac{c\ \ (2ab\cos C)}{2\Delta}$

$$= \frac{abc\cos C}{\Delta} = \frac{4\ \ R\ \ \Delta\cos C}{\Delta} = 4R\cos C$$

(iii) ∵ L.H.S. $= \frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} = \frac{1}{\Delta^2}\ [s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2]$

$$= \frac{1}{\Delta^2}\ [4s^2 - 2s(a+b+c) + \Sigma a^2] = \frac{\Sigma\ a^2}{\Delta^2} = \text{R.H.S.}$$

(iv) ∵ L.H.S. $= \left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2 = \frac{1}{\Delta^2}\ (s + s - a + s - b + s - c)^2 = 4\ \frac{s^2}{\Delta^2} = \frac{4}{r^2}$

∵ R.H.S. $= \frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right) = \frac{4}{r}\cdot\frac{1}{\Delta}(s-a+s-b+s-c) = \frac{4}{r}\cdot\frac{s}{\Delta} = \frac{4}{r^2}$

(v) ∵ $\frac{bc-r_2r_3}{r_1} = \frac{bc\left(1-\frac{r_2r_3}{bc}\right)}{r_1} = \frac{bc\left(1-\frac{\Delta^2}{bc\ \ (s-b)(s-c)}\right)}{r_1}$

$$= \frac{bc\left(1-\frac{s(s-a)}{bc}\right)}{r_1} = \frac{bc\left(1-\cos^2\frac{A}{2}\right)}{s\tan\frac{A}{2}}$$

$$= \frac{bc\sin^2\frac{A}{2}\cos\frac{A}{2}}{s\cdot\sin\frac{A}{2}} = \frac{bc.(2\sin\frac{A}{2}\cos\frac{A}{2})}{2\ \ s} = \frac{bc\ \ \sin A}{2\ \ s} = \frac{2\Delta}{2s} = r$$

similarly we can show that इसी प्रकार दर्शाया जा सकता है कि $\frac{ca-r_3r_1}{r_2} = \frac{ab-r_1r_2}{r_3} = r$

C-3. Show that the radii of the three escribed circles of a triangle are roots of the equation $x^3 - x^2(4R + r) + x\,s^2 - r\,s^2 = 0$.

प्रदर्शित कीजिए कि त्रिभुज के तीनों बर्हिवृत्तों की त्रिज्याएँ, समीकरण $x^3 - x^2(4R + r) + x\,s^2 - r\,s^2 = 0$ के मूल है।

Sol. ∵ $\sum r_1 = r_1 + r_2 + r_3$

∵ $r_1 + r_2 + r_3 - r = 4R$ ∴ $\sum r_1 = 4R + r$

∵ $\sum r_1r_2 = s^2$

$\because \quad \Pi r_1 = \dfrac{\Delta^3}{(s-a)(s-b)(s-c)} = \dfrac{s\Delta^3}{\Delta^2} = s\Delta = rs^2$

$\therefore$ equation having root r_1, r_2, r_3 is $\qquad$ $\therefore$ r_1, r_2, r_3 मूलों वाली समीकरण होगी

$x^3 - (4R + r)x^2 + (s^2)x - rs^2 = 0.$

C-4. The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.

यदि त्रिभुज ABC का क्षेत्रफल 24 वर्ग सेमी. तथा परिमाप 24 सेमी हो तथा साथ ही बर्हिवृत्तों की त्रिज्याऐं r_1, r_2, r_3 ह. श्रे. में हो, तो इसकी भुजाओं की लम्बाई ज्ञात कीजिए।

Ans. 6, 8, 10 cm सेमी

Sol. $\because \quad \Delta = 24$ sq. cm वर्ग सेमी .... (i)

$2s = 24 \quad \Rightarrow \quad s = 12$ .... (ii)

$\because \quad r_1, r_2, r_3$ are in H.P. $\qquad$ r_1, r_2, r_3 हरात्मक श्रेढी में है।

$\therefore \quad \dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{r_3}$ are in A.P. $\qquad$ $\dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{r_3}$ समान्तर श्रेढी में है।

$\therefore \quad \dfrac{s-a}{\Delta}, \dfrac{s-b}{\Delta}, \dfrac{s-c}{\Delta}$ are in A.P. $\qquad$ $\dfrac{s-a}{\Delta}, \dfrac{s-b}{\Delta}, \dfrac{s-c}{\Delta}$ समान्तर श्रेढी में है।

$\therefore$ a, b, c are in A.P. $\qquad$ a, b, c समान्तर श्रेढी में है $\quad \Rightarrow \quad 2b = a + c$

$\because \quad 2s = 24$

$\therefore \quad a + b + c = 24$

$3b = 24$

$\therefore \quad b = 8 \quad \Rightarrow \quad a + c = 16$

But लेकिन $\Delta = \sqrt{s(s-a)\ (s-b)(s-c)}$

$\Rightarrow \Delta = \sqrt{12(12-a)(12-8)(12-c)} \quad \Rightarrow \quad 24 \times 24 = 12 \times (12 - a) \times 4 \times (12 - c)$

$\Rightarrow 2 \times 6 = 144 - 12\ (a + c) + ac \quad \Rightarrow \quad 12 = 144 - 192 + ac$

$\therefore$ ac = 60 and और a + c = 16 $\quad \therefore \quad$ a = 10, c = 6 or या a = 6, c = 10 and और b = 8

C-5. If the area of a triangle is 100 sq.cm, $r_1 = 10$ cm and $r_2 = 50$ cm, then find the value of (b – a).

यदि एक त्रिभुज का क्षेत्रफल 100 वर्ग सेमी, $r_1 = 10$ सेमी एवं $r_2 = 50$ सेमी हो, तो (b – a) का मान ज्ञात कीजिए।

Ans. 8

Sol. $\because \quad D = 100\ cm^2$

$r_1 = \dfrac{\Delta}{s-a} = 10 \quad \Rightarrow \quad s - a = 10$ ....(i)

$r_2 = \dfrac{\Delta}{s-b} = 50 \quad \Rightarrow \quad s - b = 2$ ....(ii)

(i) – (ii)

$\therefore \quad b - a = 8$

Section (D) Miscellaneous
खण्ड (D) विविध

D-1. If α, β, γ are the respective altitudes of a triangle ABC, prove that

यदि त्रिभुज ABC के शीर्षलम्ब क्रमशः α, β, γ हो, तो सिद्ध कीजिए कि –

(i) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} = \dfrac{\cot A + \cot B + \cot\ C}{\Delta}$ $\qquad$ (ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} - \dfrac{1}{\gamma} = \dfrac{2ab}{(a\ +\ b\ +\ c)\ \Delta} \cos^2 \dfrac{C}{2}$

Sol. (i) $\because \quad \alpha = \frac{2\Delta}{a}, \beta = \frac{2\Delta}{b}, \gamma = \frac{2\Delta}{c} \quad \Rightarrow \quad \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{a^2+b^2+c^2}{4\ \Delta^2}$

R.H.S. दायां पक्ष = $\frac{\cot\ A\ +\ \cot\ B\ +\ \cot\ C}{\Delta} = \frac{1}{\Delta}\left(\frac{bc\ \cos\ A}{2\Delta}+\frac{ca\ \cos\ B}{2\Delta}+\frac{ab\ \cos\ C}{2\Delta}\right)$

$= \frac{a^2+b^2+c^2}{4\Delta^2}$

$\therefore$ L.H.S. बायां पक्ष = R.H.S. दायां पक्ष

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\gamma} = \frac{a+b-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta}$

R.H.S. दायां पक्ष = $\frac{2ab}{(a+b+c)\Delta}.\cos^2\frac{C}{2} = \frac{2ab}{(2\ \ s).\Delta}.\ \frac{s(s-c)}{ab}$

$= \frac{s-c}{\Delta}$

L.H.S. बायां पक्ष = R.H.S. दायां पक्ष

D-2. If in an acute angled ΔABC, line joining the circumcentre and orthocentre is parallel to side AC, then find the value of tan A.tan C.

यदि किसी न्यूनकोण ΔABC में, परिकेन्द्र तथा लम्बकेन्द्र को मिलाने वाली रेखा AC के समान्तर हो तो tan A.tan C का मान ज्ञात कीजिए।

Ans. 3

Sol. $\because$ line joining the circumcentre and orthocentre is parallel to side AC

$\because$ परिकेन्द्र तथा लम्ब केन्द्र को मिलाने वाली रेखा, भुजा AC के समान्तर है,

$\Rightarrow$ R cos B = 2R cos A cos C $\quad\Rightarrow\quad$ –cos (A+C) = 2 cos A cos C

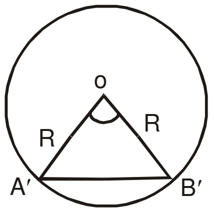
$\Rightarrow$ sin A sin C = 3 cos A cos C $\quad\Rightarrow\quad$ tan A tan C = 3

D-3. A regular hexagon & a regular dodecagon are inscribed in the same circle. If the side of the dodecagon is $\left(\sqrt{3}-1\right)$, if the side of the hexagon is $\sqrt[4]{k}$, then find value of k.

एक समषट्भुज और एक समबारह भुज एक ही वृत्त के अन्दर निर्मित है, यदि बारहभुज की एक भुजा की लम्बाई $\left(\sqrt{3}-1\right)$ हो, तो षट्भुज की भुजा की लम्बाई $\sqrt[4]{k}$ है, तो k का मान है–

Ans. $\sqrt{2}$

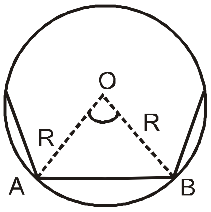
Sol.



For dodecagon $\angle$ A 'OB' = $\frac{2\pi}{12}$ = 30°

$\Rightarrow \quad \angle$ OA 'B' = $\angle$ OB 'A' = 75° $\Rightarrow \quad \frac{R}{\sin 75°} = \frac{\sqrt{3}-1}{\sin 30°}$

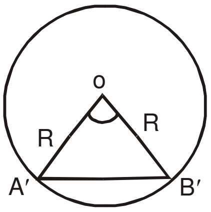
$\Rightarrow \quad R = \frac{(\sqrt{3}-1)\ \ (\sqrt{3}+1)}{2\sqrt{2}\times\frac{1}{2}} \quad \Rightarrow \quad R = \sqrt{2}$



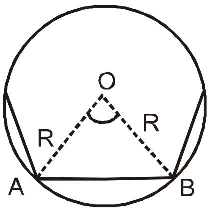
For hexagon $\angle$ AOB = $\frac{2\pi}{6}$ = 60°

$\Rightarrow$ Δ AOB is equilatecal $\Rightarrow$ AB = R = $\sqrt{2}$

Hindi समबारह भुज हेतु $\angle$ A 'OB' = $\frac{2\pi}{12}$ = 30° $\Rightarrow\angle$ OA 'B' = $\angle$ OB 'A' = 75°



$$\Rightarrow \quad \frac{R}{\sin 75^\circ} = \frac{\sqrt{3}-1}{\sin 30^\circ} \Rightarrow \quad R = \frac{(\sqrt{3}-1)\ \ (\sqrt{3}+1)}{2\sqrt{2}\times\frac{1}{2}} \quad \Rightarrow \quad R = \sqrt{2}$$

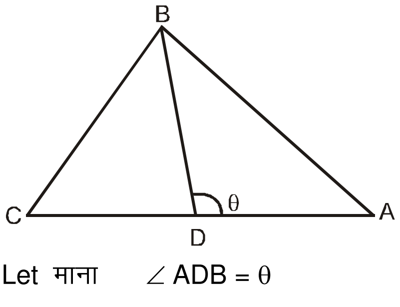


षट्भुज के लिए $\angle$ AOB = $\frac{2\pi}{6}$ = 60°

$\Rightarrow$ Δ AOB समबाहु त्रिभुज है $\Rightarrow$ AB = R = $\sqrt{2}$

D-4. If D is the mid point of CA in triangle ABC and Δ is the area of triangle, then show that tan ($\angle$ADB) = $\frac{4\Delta}{a^2-c^2}$.

यदि त्रिभुज ABC में CA का मध्य बिन्दु D तथा त्रिभुज का क्षेत्रफल Δ हो, तो प्रदर्शित कीजिए कि tan ($\angle$ADB) = $\frac{4\Delta}{a^2-c^2}$.



Sol.

Let माना $\angle$ ADB = θ

$\therefore$ we have to prove that हमें सिद्ध करना है कि tanθ = $\frac{4\Delta}{a^2-c^2}$

if we aply m – n rule, then m – n प्रमेय का प्रयोग करने पर

(1 + 1) cotθ = 1.cot C – 1.cotA.

$$= \frac{\cos C}{\sin C} - \frac{\cos A}{\sin A} = \frac{ab\cos C}{2\Delta} - \frac{bc\cos A}{2\Delta} = \frac{1}{2\Delta}\left[\frac{b^2+a^2-c^2}{2} - \frac{b^2+c^2-a^2}{2}\right]$$

$$= \frac{1}{4\Delta}\,[2(a^2-c^2)]$$

$\therefore \quad 2\cot\theta = \frac{a^2-c^2}{2\Delta} \qquad \therefore \quad \tan\theta = \frac{4\Delta}{a^2-c^2}$

Exercise-2

✍ **Marked questions are recommended for Revision.**

✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

PART-I (OBJECTIVE QUESTIONS)

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule
खण्ड (A) : ज्या नियम, कोज्या नियम, स्पर्शज्या नियम या नेपियर एनालोजि, प्रक्षेप नियम

A-1. In a ΔABC, A : B : C = 3 : 5 : 4. Then a + b + c $\sqrt{2}$ is equal to

त्रिभुज ABC में, A : B : C = 3 : 5 : 4 हो, तो a + b + c $\sqrt{2}$ का मान है –

(A) 2b (B) 2c (C*) 3b (D) 3a

Sol. $\because$ A : B : C = 3 : 5 : 4 $\therefore$ A = 45° , B = 75° , C = 60°

$\because$ from Sine - rule ज्या नियम के अनुसार

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \quad \Rightarrow \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{c}{\frac{\sqrt{3}}{2}} = k \qquad (\because \sin 75° = \sin(45° + 30°))$$

$\therefore \quad a = \frac{k}{\sqrt{2}}, \; b = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ k and और $c = \frac{k\sqrt{3}}{2}$

$$\therefore \quad a + b + c\sqrt{2} = \frac{k}{\sqrt{2}} + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k + \left(\frac{k\sqrt{3}}{2}\right)\sqrt{2} = \frac{k}{2\sqrt{2}}\left[2+(\sqrt{3}+1)+2\sqrt{3}\right] = \frac{3k(\sqrt{3}+1)}{2\sqrt{2}} = 3b$$

A-2*. In a triangle ABC, the altitude from A is not less than BC and the altitude from B is not less than AC. The triangle is

(A*) right angled (B*) isosceles (C) obtuse angled (D) equilateral

ABC में A से शीर्षलम्ब, BC से कम नहीं है तथा B से शीर्षलम्ब AC से कम नहीं है, तब त्रिभुज है –

(A) समकोण (B) समद्विबाहु (C) अधिक कोण (D) समबाहु

Sol. Given c sin B $\geq$ a

$\Rightarrow$ sinC sinB $\geq$ sinA

and a sinC $\geq$ b $\Rightarrow$ sinC sinA $\geq$ sinB

$\Rightarrow$ sinC. sinA $\geq$ sinB $\geq \frac{\sin A}{\sin C}$

$\therefore$ $\sin^2 C \geq 1$ $\Rightarrow$ sinC = 1 $\Rightarrow$ $\angle$C is $\frac{\pi}{2}$

A-3. If in a Δ ABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is :

(A) right angled (B) isosceles (C*) equilateral (D) obtuse angled

यदि त्रिभुज ABC में, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ हो, तो त्रिभुज है –

(A) समकोण त्रिभुज (B) समद्विबाहु त्रिभुज (C) समबाहु त्रिभुज (D) अधिक कोण त्रिभुज

Sol. given दिया है $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ ..... (i)

$\because$ a = k sin A, b = k sin B, c = k sin C $\therefore$ (i) becomes (i) में प्रयोग करने पर

$\frac{\cot A}{k} = \frac{\cot B}{k} = \frac{\cot C}{k}$ $\therefore$ A = B = C

$\therefore$ ΔABC is an equilateral triangle (त्रिभुज ABC एक समबाहु त्रिभुज होगा।)

A-4. In a ΔABC $\frac{bc\sin^2 A}{\cos A + \cos B\cos C}$ is equal to

(A) $b^2 + c^2$ (B) bc (C*) a^2 (D) $a^2 + bc$

त्रिभुज ABC में, $\frac{bc\sin^2 A}{\cos A + \cos B\cos C}$ बराबर है–

(A) $b^2 + c^2$ के (B) bc के (C*) a^2 के (D) $a^2 + bc$ के

Sol. $\because$ $\frac{bc\sin^2 A}{\cos A + \cos B\cos C} = \frac{k^2 \sin B\sin C\sin^2 A}{-\cos(B+C) + \cos B\cos C} = \frac{k^2 \sin B\sin C\sin^2 A}{\sin B\sin C} = k^2 \sin^2 A = a^2$.

A-5✍ Given a triangle ΔABC such that $\sin^2A + \sin^2C = 1001.\sin^2B$. Then the value of $\frac{2(\tan A + \tan C).\tan^2 B}{\tan A + \tan B + \tan C}$ is

दिए गए त्रिभुज ABC में $\sin^2A + \sin^2C = 1001.\sin^2B$. तब $\frac{2(\tan A + \tan C).\tan^2 B}{\tan A + \tan B + \tan C}$ बराबर है –

(A) $\frac{1}{2000}$ (B) $\frac{1}{1000}$ (C) $\frac{1}{500}$ (D*) $\frac{1}{250}$

Sol. $\sin^2A + \sin^2C = 1001 \sin^2B$

$\Rightarrow a^2 + c^2 = 1001\, b^2$ (using sine rule)

Now, $\frac{(2\tan A + \tan C).\tan^2 B}{\tan A + \tan B + \tan C} = \frac{2(\tan A + \tan C).\tan^2 B}{\tan A.\tan B\tan C} = 2\left(\frac{\cot A + \cot C}{\cot B}\right)$

- $\sin B \frac{2(\cos A\sin C + \sin A\cos C)}{\sin A.\sin C.\cos B} = \frac{2\sin(\pi - B).\sin B}{\sin A\sin C\cos B} = \frac{2\sin^2 B}{\sin A\sin C\cos B}$

$= \frac{2 \times 2b^2}{2ac.\cos B} = \frac{2 \times 2b^2}{a^2 + c^2 - b^2} = \frac{2 \times 2b^2}{1000b^2} = \frac{1}{250}$

A-6. If in a triangle ABC, (a + b + c) (b + c – a) = k. bc, then :

(A) k < 0 (B) k > 6 (C*) 0 < k < 4 (D) k > 4

यदि त्रिभुज ABC में (a + b + c) (b + c – a) = k. bc हो, तो –

(A) k < 0 (B) k > 6 (C*) 0 < k < 4 (D) k > 4

Sol. $\because$ (a + b + c) (b + c – a) = kbc $\Rightarrow$ $(b + c)^2 - a^2 = kbc$

$b^2 + c^2 - a^2 = (k - 2)\, bc$ $\Rightarrow$ $\frac{b^2 + c^2 - a^2}{2bc} = \frac{k-2}{2} = \cos A$

$\because$ In a ΔABC में $-1 < \cos A < 1$ $\therefore$ $-1 < \frac{k-2}{2} < 1$

0 < k < 4

.

A-7. In a triangle ABC, a: b: c = 4: 5: 6. Then 3A + B equals to :

त्रिभुज ABC में, a: b: c = 4: 5: 6 हो, तो 3A + B का मान है–

(A) 4C (B) 2π (C) $\pi - C$ (D*) π

Sol. $\because$ a : b : c = 4 : 5 : 6 $\therefore$ a = 4k, b = 5k , c = 6k

$\because$ $\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{9}{16}$ $\because$ $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25 + 36 - 16}{2 \times 5 \times 6} = \frac{3}{4}$

$\because$ cos 3A = 4 $\cos^3$ A – 3 cos A

$$= 4 \times \frac{27}{64} - 3 \times \frac{3}{4} = \frac{27}{16} - \frac{9}{4} = \frac{27-36}{16} = \frac{-9}{16}$$

cos 3A = –cos B = cos (π – B)

$\therefore$ 3A + B = π

A-8.✍ The distance between the middle point of BC and the foot of the perpendicular from A is (b > c) :

(A) $\frac{-a^2+b^2+c^2}{2a}$ (B*) $\frac{b^2-c^2}{2\,a}$ (C) $\frac{b^2+c^2}{\sqrt{bc}}$ (D) $\frac{b^2+c^2}{2\,a}$

भुजा BC के मध्य बिन्दु एवं शीर्ष A से डाले गए लम्ब के पाद के मध्य दूरी है (b > c)

(A) $\frac{-a^2+b^2+c^2}{2a}$ (B*) $\frac{b^2-c^2}{2\,a}$ (C) $\frac{b^2+c^2}{\sqrt{bc}}$ (D) $\frac{b^2+c^2}{2\,a}$

Sol. $\because$ ED = $\frac{a}{2}$ – c cos B

$$= \frac{a}{2} - c\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$= \frac{a}{2} - \left(\frac{a^2+c^2-b^2}{2a}\right) = \frac{a^2-a^2-c^2+b^2}{2a} = \frac{b^2-c^2}{2a}$$

A-9*. If in a triangle ABC, cos A cos B + sin A sin B sin C = 1, then the triangle is

(A*) isosceles (B*) right angled (C) equilateral (D) None of these

यदि त्रिभुज ABC में cos A cos B + sin A sin B sin C = 1 हो, तो त्रिभुज है –

(A) समद्विबाहु त्रिभुज (B) समकोण त्रिभुज (C) समबाहु त्रिभुज (D) इनमें से कोई नहीं

Sol. $\because$ sin C = $\frac{1-\cos A\cos B}{\sin A\sin B} \le 1$ $\Rightarrow$ cos (A – B) $\ge$ 1 $\Rightarrow$ cos (A – B) = 1

$\Rightarrow$ A – B = 0 $\Rightarrow$ A = B $\therefore$ sin C = $\frac{1-\cos^2 A}{\sin^2 A}$ = 1 $\Rightarrow$ C = 90º

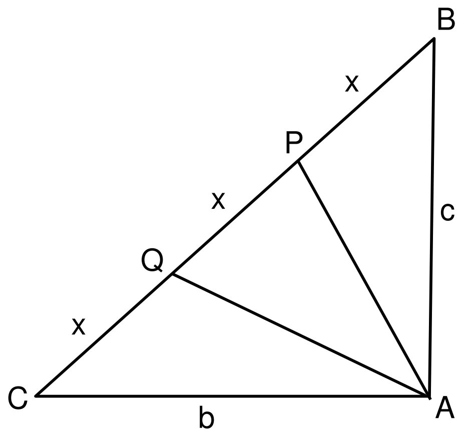
A-10.✍Triangle ABC is right angle at A. The points P and Q are on hypotenuse BC such that BP = PQ = QC. If AP = 3 and AQ = 4, then length BC is equal to

त्रिभुज ABC में A पर समकोण है। बिन्दु P और Q कर्ण BC पर इस प्रकार है कि BP = PQ = QC.

यदि AP = 3 और AQ = 4 तब BC की लम्बाई है –

(A*) $3\sqrt{5}$ (B) $5\sqrt{3}$ (C) $4\sqrt{5}$ (D) 7

Sol.



Let BP = PQ = QC = x

In ΔABP, using cosine rule

9 = $c^2 + x^2$ – 2cx cosB

But $\cos B = \frac{c}{3x}$ ....(1)

$\Rightarrow 9 = x^2 + \frac{c^2}{3}$

similarly using cosine rule in ΔACQ , we get

$16 = x^2 + \frac{b^2}{3}$ ....(2)

Adding (1) and (2), we get $25 = 2x^2 + \left(\frac{b^2+c^2}{3}\right)$

$\therefore 25 = 2x^2 + \frac{(3x)^2}{3}$

$\therefore 25 = 2x^2 + 3x^2$

$\therefore x^2 = 5$

$\therefore BC = 3x = 3\sqrt{5}$

A-11. In ΔABC, $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$, then ΔABC is
(A*) isosceles but not necessarily equilaterial
(B) equilateral
(C) right angled but not neccessarily isosceles
(D) right angled isosceles

ΔABC में $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$, तब ΔABC है –
(A) समद्विबाहु परन्तु आवश्यक नहीं कि समबाहु (B) समबाहु
(C) समकोण त्रिभुज परन्तु आवश्यक नहीं कि समद्विबाहु (D) समद्विबाहु समकोण त्रिभुज

Sol. $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$

$\Rightarrow bc = 2 \cos A (b^2 + c^2 - 2bc \cos A)$

$\Rightarrow bc = (2\cos A)a^2$

$\Rightarrow \frac{bc}{2a^2} = \cos A = \frac{b^2+c^2-a^2}{2bc}$

$\Rightarrow b^2c^2 = a^2(b^2 + c^2 - a^2)$

$\Rightarrow (a^2 - b^2)(a^2 - c^2) = 0$

Thus, triangle is isosceles

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

खण्ड (B) अर्द्धकोण, त्रिभुज का क्षेत्रफल और परित्रिज्या में त्रिकोणमितिय सम्बन्ध

B-1. If in a triangle ABC, right angle at B, $s - a = 3$ and $s - c = 2$, then
त्रिभुज ABC जो B पर समकोण है, में $s - a = 3$ एवं $s - c = 2$ हो, तो –
(A) a = 2, c = 3 (B*) a = 3, c = 4 (C) a = 4, c = 3 (D) a = 6, c = 8

Sol. $\because$ $s - a = 3$ ...(1) and $s - c = 2$ ...(2)

by (1) – (2), we get
$c - a = 1$

(1) + (2), we get $2s - a - c = 5$ $\Rightarrow$ $b = 5$

$\because$ ΔABC is right angled at B

$\therefore$ $a^2 + c^2 = 25$ ...(3)

$\therefore$ $(c - a)^2 + 2ac = 25$

$ac = 12$ ...(4)

$\therefore$ $a(1 + a) = 12 \Rightarrow a^2 + a - 12 = 0$

$\Rightarrow (a + 4)(a - 3) = 0$

$\Rightarrow a = 3$ and $c = 4$.

Hindi $\because$ $s - a = 3$ ...(1) तथा $s - c = 2$ ...(2)

समीकरण (1) में से (2) को घटाने पर

$c - a = 1$

समीकरण (1) में (2) को जोड़ने पर

$2s - a - c = 5 \Rightarrow b = 5$

$\because$ ΔABC शीर्ष B पर समकोण है, $\therefore$ $a^2 + c^2 = 25$ ...(3)

$\therefore$ $(c - a)^2 + 2ac = 25$

$ac = 12$ ...(4)

$\therefore$ $a(1 + a) = 12 \Rightarrow a^2 + a - 12 = 0 \Rightarrow (a + 4)(a - 3) = 0 \Rightarrow a = 3$ और $c = 4$.

B-2. If in a triangle ABC, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c$, then a, c, b are :

(A*) in A.P. (B) in G.P. (C) in H.P. (D) None

यदि किसी त्रिभुज ABC में, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c$, हो, तो a, c, b हैं –

(A) समान्तर श्रेढ़ी में (B) गुणोत्तर श्रेढ़ी में (C) हरात्मक श्रेढ़ी में (D) इनमें से कोई नहीं।

Sol. $\because$ $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c. \Rightarrow b\,\frac{s(s-a)}{bc} + a\,\frac{s(s-b)}{ac} = \frac{3}{2}c.$

$\Rightarrow \frac{s}{c}[s - a + s - b] = \frac{3}{2}c \Rightarrow \frac{s}{c} \times c = \frac{3}{2}c \Rightarrow \frac{a+b+c}{2} = \frac{3c}{2} \Rightarrow a + b = 2c$

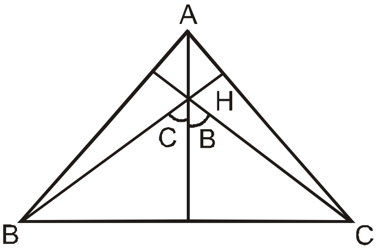
$\Rightarrow$ a, c, b are in A.P. a, c, b समान्तर श्रेढी में है।

B-3. If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to :

यदि त्रिभुज ABC का लम्बकेन्द्र H हो, तो त्रिभुजों BHC, CHA एवं AHB के परिगत वृत्तों की त्रिज्याएँ क्रमशः हैं –

(A*) R, R, R (B) $\sqrt{2}R, \sqrt{2}R, \sqrt{2}R$ (C) 2R, 2R, 2R (D) $\frac{R}{2}, \frac{R}{2}, \frac{R}{2}$

Sol.



In ΔHBC is we apply Sine-rule, then we get

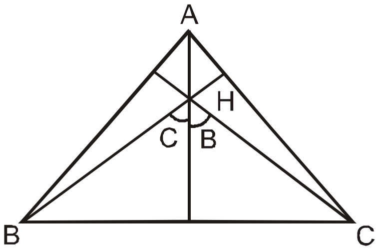
$$\frac{BC}{\sin(B+C)} = 2R'$$

$$\frac{a}{\sin\ A} = 2R' \Rightarrow 2R = 2R' \Rightarrow R = R'$$

$\therefore$ circumradius of ΔHBC (i.e. R') = R

Similarly we can prove for ΔHCA and ΔHAB.

Hindi



त्रिभुज HBC में ज्या नियम के प्रयोग से

$\frac{BC}{\sin(B+C)} = 2R'$

$\frac{a}{\sin\ A} = 2R' \Rightarrow 2R = 2R' \Rightarrow R = R'$

$\therefore$ ΔHBC की परित्रिज्या (अर्थात् R′) = R

इसी प्रकार ΔHCA और ΔHAB के लिये भी सिद्ध किया जा सकता है।

B-4. In a Δ ABC if b + c = 3a, then $\cot\frac{B}{2}\cdot\cot\frac{C}{2}$ has the value equal to:

त्रिभुज ABC में यदि b + c = 3a हो, तो $\cot\frac{B}{2}\cdot\cot\frac{C}{2}$ का मान है –

(A) 4 (B) 3 (C*) 2 (D) 1

Sol. $\because\ \cot\frac{B}{2}\ \cot\frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\ \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{s}{s-a} = \frac{2s}{2s-2a}$

$= \frac{a+b+c}{b+c-a} = \frac{4a}{2a} = 2$ $(\because\ b + c = 3a)$

B-5. In a ΔABC, A = $\frac{2\pi}{3}$, b – c = 3 $\sqrt{3}$ cm and area (ΔABC) = $\frac{9\sqrt{3}}{2}$cm^2. Then 'a' is

(A) 6 $\sqrt{3}$ cm (B*) 9 cm (C) 18 cm (D) 7 cm

त्रिभुज ABC में A = $\frac{2\pi}{3}$, b – c = 3 $\sqrt{3}$ सेमी एवं क्षेत्रफल (ΔABC) = $\frac{9\sqrt{3}}{2}$ वर्ग सेमी हो, तो 'a' का मान है–

(A) 6$\sqrt{3}$ सेमी (B) 9 सेमी (C) 18 सेमी (D) 7 सेमी

Sol. $\because\ A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ and तथा Area (क्षेत्रफल) = $\frac{9\sqrt{3}}{2}$ cm^2

$\because\ \Delta = \frac{1}{2}bc\ \ \sin A \Rightarrow \frac{9\sqrt{3}}{2} = \frac{1}{2}bc\sin\frac{2\pi}{3} \Rightarrow bc = 18$

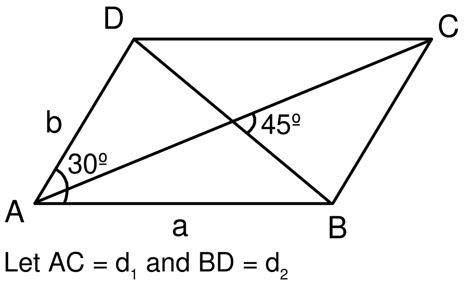
$\because\ \cos\frac{2\pi}{3} = \frac{b^2+c^2-a^2}{2bc} = -\frac{1}{2} \Rightarrow \frac{(b-c)^2+2bc-a^2}{2bc} = -\frac{1}{2} \Rightarrow a = 9$

B-6.* The diagonals of a parallelogram are inclined to each other at an angle of 45º, while its sides a and b (a > b) are inclined to each other at an angle of 30º, then the value of $\frac{a}{b}$ is

समान्तर चतुर्भुज के विकर्ण एक दूसरे को 45º पर झुके हुए है जबकि इसकी भुजाऐं a और b (a > b) एक दूसरे से 30º, कोण पर झुकी है तब $\frac{a}{b}$ का मान है –

(A*) 2cos36º (B) $\sqrt{\frac{3+\sqrt{5}}{4}}$ (C) $\frac{3+\sqrt{5}}{4}$ (D*) $\frac{\sqrt{5}+1}{2}$

Sol.



Let AC = d_1 and BD = d_2

Area of parallelogram is $\frac{1}{2}d_1d_2 \sin 45º = 2\left(\frac{1}{2}ab\sin 30º\right) \Rightarrow d_1d_2 = \sqrt{2}\,ab$ ...(i)

where $d_1^2 = a^2 + b^2 - 2ab\cos 150º = a^2 + b^2 + \sqrt{3}\,ab$

and $d_2^2 = a^2 + b^2 - 2ab\cos 30º = a^2 + b^2 - \sqrt{3}\,ab$

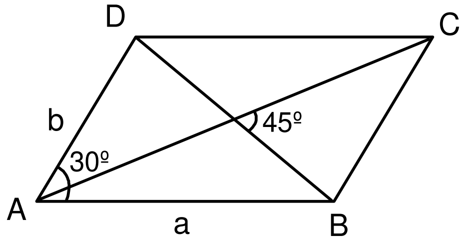
$\Rightarrow d_1^2d_2^2 = (a^2 + b^2)^2 - 3a^2b^2$

$\Rightarrow 2a^2b^2 = (a^2 + b^2)^2 - 3a^2b^2$

$\Rightarrow a^4 + b^4 - 3a^2b^2 = 0$

$\Rightarrow \left(\frac{a}{b}\right)^4 - 3\left(\frac{a}{b}\right)^2 + 1 = 0 \Rightarrow \left(\frac{a}{b}\right)^2 = \frac{3+\sqrt{5}}{2}$ (as a > b)

$\Rightarrow \left(\frac{a}{b}\right)^2 = \frac{(\sqrt{5}+1)^2}{4} \quad \Rightarrow \frac{a}{b} = \frac{\sqrt{5}+1}{2}$



Hindi.

माना AC = d_1 तथा BD = d_2

समान्तर चतुर्भुज क्षेत्रफल = $\frac{1}{2}d_1d_2 \sin 45º = 2\left(\frac{1}{2}ab\sin 30º\right) \Rightarrow d_1d_2 = \sqrt{2}\,ab$ ...(i)

जहाँ $d_1^2 = a^2 + b^2 - 2ab\cos 150º = a^2 + b^2 + \sqrt{3}\,ab$

और $d_2^2 = a^2 + b^2 - 2ab\cos 30º = a^2 + b^2 - \sqrt{3}\,ab$

$\Rightarrow d_1^2d_2^2 = (a^2 + b^2)^2 - 3a^2b^2$

$\Rightarrow 2a^2b^2 = (a^2 + b^2)^2 - 3a^2b^2 \quad \Rightarrow a^4 + b^4 - 3a^2b^2 = 0$

$\Rightarrow \left(\frac{a}{b}\right)^4 - 3\left(\frac{a}{b}\right)^2 + 1 = 0 \Rightarrow \left(\frac{a}{b}\right)^2 = \frac{3+\sqrt{5}}{2}$ (चूंकि a > b)

$\Rightarrow \left(\frac{a}{b}\right)^2 = \frac{(\sqrt{5}+1)^2}{4} \quad \Rightarrow \frac{a}{b} = \frac{\sqrt{5}+1}{2}$

B-7. If in a ΔABC, $\Delta = a^2 - (b - c)^2$, then tan A is equal to

यदि किसी त्रिभुज ABC में, $\Delta = a^2 - (b - c)^2$ हो, तो tan A का मान है –

(A) 15/16 (B*) 8/15 (C) 8/17 (D) 1/2

Sol. $\Delta = (a + b - c)(a - b + c)$

$\Delta = 4(s - c)(s - b) \quad \Rightarrow \quad \frac{\Delta}{(s-b)\ (s\ -c)} = \frac{1}{4} \quad \therefore \quad \tan\frac{A}{2} = \frac{1}{4}$

$\because\ \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \quad \Rightarrow \quad \tan A = \frac{8}{15}$

B-8*. If in a ΔABC, a = 5, b = 4 and $\cos(A - B) = \frac{31}{32}$, then

(A*) c = 6 (B*) $\sin A = \left(\frac{5\sqrt{7}}{16}\right)$ (C*) area of $\Delta ABC = \frac{15\sqrt{7}}{4}$ (D) c = 8

यदि किसी ΔABC के लिए, a = 5, b = 4 और $\cos(A - B) = \frac{31}{32}$ हो, तो

(A*) $c = 6$ (B*) $\sin A = \left(\frac{5\sqrt{7}}{16}\right)$ (C*) ΔABC का क्षेत्रफल $= \frac{15\sqrt{7}}{4}$ (D) $c = 8$

Sol. (A) $\because$ $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$ ....(i)

$$\because \tan^2\left(\frac{A-B}{2}\right) = \frac{1-\cos(A-B)}{1+\cos(A-B)} = \frac{1-\frac{31}{32}}{1+\frac{31}{32}} = \frac{1}{63}$$

$$\therefore \tan\left(\frac{A-B}{2}\right) = \frac{1}{3\sqrt{7}} \qquad \because a = 5 \text{ and तथा } b = 4$$

$\therefore$ from equation (i), we get समीकरण (i) से

$$\frac{1}{3\sqrt{7}} = \left(\frac{5-4}{5+4}\right)\cot\frac{C}{2} \Rightarrow \frac{1}{3\sqrt{7}} = \frac{1}{9}\cot\frac{C}{2} \Rightarrow \cot\frac{C}{2} = \frac{3}{\sqrt{7}}$$

$$\because \cos C = \frac{1-\tan^2 C/2}{1+\tan^2 C/2} = \frac{1-7/9}{1+7/9} = \frac{2}{16} = \frac{1}{8}$$

$$\because \cos C = \frac{b^2+a^2-c^2}{2ab} \Rightarrow c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow c = 6$$

(B), (C) $\because$ Area क्षेत्रफल $= \frac{1}{2}ab\sin C$ $\because$ $\cos C = \frac{1}{8} \Rightarrow \sin C = \sqrt{1-\frac{1}{64}} = \frac{3\sqrt{7}}{8}$

$$\text{Area} = \frac{1}{2} \times 5 \times 4 \times \frac{3\sqrt{7}}{8}$$

Area क्षेत्रफल = sq. unit. वर्ग इकाई $\because$ From Sine rule ज्या नियम के प्रयोग से

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \sin A = \frac{a\sin C}{c} = \frac{5\times 3\sqrt{7}}{6\times 8} \quad \therefore \quad \sin A = \frac{5\sqrt{7}}{16}$$

B-9. If R denotes circumradius, then in ΔABC, $\frac{b^2-c^2}{2a\ R}$ is equal to

(A) $\cos(B-C)$ (B*) $\sin(B-C)$ (C) $\cos B - \cos C$ (D) $\sin(B+C)$

यदि R परित्रिज्या को प्रदर्शित करता हो, तो त्रिभुज ABC में $\frac{b^2-c^2}{2a\ R}$ का मान है –

(A) $\cos(B-C)$ (B) $\sin(B-C)$ (C) $\cos B - \cos C$ (D) $\sin(B+C)$

Sol. $$\because \frac{b^2-c^2}{2aR} = \frac{4R^2\left(\sin^2 B - \sin^2 C\right)}{2.2R\sin A.R} = \frac{\sin(B+C).\sin(B-C)}{\sin A} = \sin(B-C)$$

B-10*. Which of the following holds good for any triangle ABC?
किसी भी ΔABC के लिये, निम्न में से सत्य है :–

(A*) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$ (B*) $\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{3}{2R}$

(C) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ (D) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$

Sol. (A) $$\because \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

(B) $$\because \frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{a}{2R.a} + \frac{b}{2R.b} + \frac{c}{2R.c} = \frac{3}{2R}$$

(C) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c} \Rightarrow \cot A = \cot B = \cot C \Rightarrow A = B = C$

true for equilateral triangle only केवल समबाहु त्रिभुज के लिए सत्य है

(D) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$

$$\Rightarrow \frac{2\sin A\cos A}{k^2\sin^2 A} = \frac{2\sin B\cos B}{k^2\sin^2 B} = \frac{2\sin C\cos C}{k^2\sin^2 C}$$

$\Rightarrow$ cot A = cot B = cot C $\Rightarrow$ A = B = C $\Rightarrow$ true for equilateral triangle only

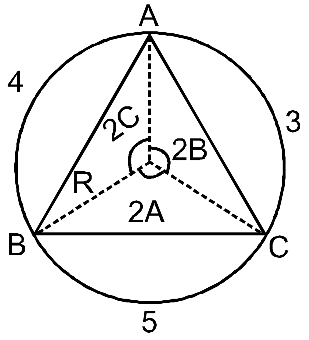
केवल समबाहु त्रिभुज के लिए सत्य है

B-11. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to:

किसी वृत्त में एक त्रिभुज बनाया जाता है। त्रिभुज के शीर्ष वृत्त को 3 इकाई, 4 इकाई एवं 5 इकाई के तीन चापों में विभाजित करते हो, तो त्रिभुज का क्षेत्रफल है –

(A*) $\frac{9\sqrt{3}\,(1+\sqrt{3})}{\pi^2}$ (B) $\frac{9\sqrt{3}\,(\sqrt{3}-1)}{\pi^2}$ (C) $\frac{9\ \sqrt{3}\ (1+\sqrt{3})}{2\,\pi^2}$ (D) $\frac{9\ \sqrt{3}\ (\sqrt{3}-1)}{2\,\pi^2}$

Sol.



angle = $\frac{\text{arc}}{\text{radius}}$ ....(1) कोण = $\frac{\text{चाय}}{\text{त्रिज्या}}$

$\because$ 4 + 5 + 3 = 2πR $\Rightarrow$ R = 6/π

$\because\ 2A = \frac{5}{R} = \frac{5\pi}{6}$,

$2B = \frac{3}{R} = \frac{\pi}{2}$ and और

$2C = \frac{4}{R} = \frac{2\pi}{3}$

Area of Δ ABC का क्षेत्रफल = $\frac{1}{2}R^2\left[\sin\frac{2\pi}{3} + \sin\frac{5\pi}{6} + \sin\frac{\pi}{2}\right]$

$$= \frac{R^2}{2}\left[\frac{\sqrt{3}}{2} + \frac{1}{2} + 1\right] = \frac{R^2}{2}\left[\frac{\sqrt{3}+3}{2}\right] = \frac{\sqrt{3}(\sqrt{3}+1)}{4} \times \frac{36}{\pi^2} = \frac{9\sqrt{3}(\sqrt{3}+1)}{\pi^2}$$

B-12. In a ΔABC, a = 1 and the perimeter is six times the arithmetic mean of the sines of the angles. Then measure of ∠ A is

त्रिभुज ABC में, a = 1 और परिमाप, कोणों की ज्याओं के समान्तर माध्य का छः गुना हो, तो कोण A का मान है –

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C*) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

Sol. a = 1

$\because\ 2s = 6\left(\frac{\sin A + \sin B + \sin C}{3}\right)$

$2s = 2\left(\frac{a+b+c}{2R}\right)$

R = 1

$\because\ \frac{a}{\sin\ A} = 2R \Rightarrow \sin A = \frac{1}{2}$

$A = \frac{\pi}{6}$

B-13*. Three equal circles of radius unity touches one another. Radius of the circle touching all the three circles is :
इकाई त्रिज्या के तीन समान वृत्त एक दूसरे को स्पर्श करते है। तीनों वृत्तों को स्पर्श करने वाले वृत्त की त्रिज्या है –

(A*) $\frac{2-\sqrt{3}}{\sqrt{3}}$ (B) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$ (C*) $\frac{2+\sqrt{3}}{\sqrt{3}}$ (D) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$

Sol. Let the radius of the inner circle be x
माना दिये गये तीनों वृत्तों को स्पर्श करने वाले छोटे वृत्त (चित्रानुसार) की त्रिज्या x है।

$\because\ \cos 30° = \frac{1}{x+1} = \frac{\sqrt{3}}{2}$

$x + 1 = \frac{2}{\sqrt{3}}$

$x = \frac{2-\sqrt{3}}{\sqrt{3}}$

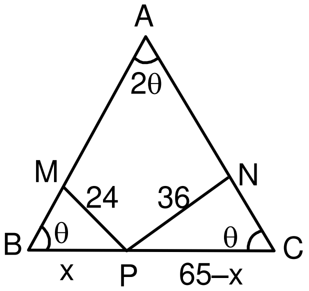
$\therefore$ radius of other (shaded) circle
दिये गये तीनों वृत्तों को स्पर्श करने वाले बडे वृत्त (चित्रानुसार) की त्रिज्या

$= 2 + x = 2 + \frac{2-\sqrt{3}}{\sqrt{3}} = \frac{2+\sqrt{3}}{\sqrt{3}}$

B-14. Triangle ABC is isosceles with AB = AC and BC = 65 cm. P is a point on BC such that the perpendicular distances from P to AB and AC are 24 cm and 36 cm, respectively. The area of triangle ABC (in sq. cm is)
त्रिभुज ABC समद्विबाहु त्रिभुज है जिसमें AB = AC तथा BC = 65 cm है। बिन्दु P, BC पर इस प्रकार है कि P से AB व AC पर क्रमशः लम्बवत् दूरियाँ 24 सेमी और 36 सेमी है। त्रिभुज ABC का क्षेत्रफल (वर्ग सेमी. में)

(A) 1254 (B) 1950 (C*) 2535 (D) 5070

Sol.



$A = \frac{1}{2}b^2 \sin 2\theta = b^2 \sin\theta \cos\theta$

Now, $\frac{x}{24} = \frac{65-x}{36}$ $(\because \Delta PMB \cong \Delta PNC)$

or 60x = (24) (65) or x = 26

$\therefore \sin\theta = \dfrac{12}{13}$ and $\cos\theta = \dfrac{5}{13}$

Again, $\dfrac{b}{\sin\theta} = \dfrac{65}{\sin 2\theta}$

or $b = \dfrac{65}{2\cos\theta} = \dfrac{(65)(13)}{(2)(5)} = \dfrac{13^2}{2}$

From eq. (i) we get

$A = \dfrac{13^4}{4} \times \dfrac{12}{13} \times \dfrac{5}{13} = (169)\ (15) = 2535$

Section (C) Inradius and Exradius
खण्ड(C) अन्तत्रिज्या परित्रिज्या

C-1. In a Δ ABC, the value of $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to:

त्रिभुज ABC में $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c}$ का मान है –

(A*) $\dfrac{r}{R}$ (B) $\dfrac{R}{2r}$ (C) $\dfrac{R}{r}$ (D) $\dfrac{2r}{R}$

Sol. $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c} = \dfrac{R(\sin 2A + \sin 2B + \sin 2C)}{2R(\sin A + \sin B + \sin C)}$

$= \dfrac{4\sin A \sin B \sin\ \ C}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$= \dfrac{r}{R}$.

C-2. In a triangle ABC, if a : b : c = 3 : 7 : 8, then R : r is equal to

यदि किसी त्रिभुज ABC में a : b : c = 3 : 7 : 8 हो, तो R : r होगा –

(A) 2 : 7 (B*) 7 : 2 (C) 3 : 7 (D) 7 : 3

Sol. a = 3k ; b = 7k ; c = 8k

$\because$ s = 9 k.

$\because \quad \Delta = \sqrt{9k\cdot 6k\cdot 2k\cdot k} = k^2\ 6\sqrt{3} \qquad \therefore \quad R = \dfrac{abc}{4\Delta} = \dfrac{(3\ \ k)\ \ (7\ \ k)\ \ (8\ \ k)}{4\times k^2\times\ \ 6\sqrt{3}} = \dfrac{7\ \ k}{\sqrt{3}}$

$\because \quad r = \dfrac{\Delta}{s} = \dfrac{k^2\ \ 6\sqrt{3}}{9\ \ k} = \dfrac{2k}{\sqrt{3}} \qquad \therefore \quad R : r = 7 : 2$

C-3*. If $r_1 = 2r_2 = 3r_3$, then

यदि $r_1 = 2r_2 = 3r_3$, तो

(A) $\dfrac{a}{b} = \dfrac{4}{5}$ (B*) $\dfrac{a}{b} = \dfrac{5}{4}$ (C) $\dfrac{a}{c} = \dfrac{3}{5}$ (D*) $\dfrac{a}{c} = \dfrac{5}{3}$

Sol. $r_1 = 2r_2 = 3r_3$

$\Rightarrow \quad \dfrac{\Delta}{s-a} = \dfrac{2\Delta}{s-b} = \dfrac{3\Delta}{s-c} \qquad \Rightarrow \quad \underset{(i)}{\dfrac{1}{s-a}} = \underset{(ii)}{\dfrac{2}{s-b}} = \underset{(iii)}{\dfrac{3}{s-c}}$

From (i) and (ii) we get a – b = c/3 ...(1)

From (i) and (iii), we get 2a – b = 2c ...(2)

From (ii) and (iii), we get a – 5b = –5c ...(3)

let c = k, then from (1) and (2), we get

$a = \dfrac{5k}{3}$ and $b = \dfrac{4k}{3} \qquad \therefore \dfrac{a}{b} = \dfrac{5}{4}\ ;\ \dfrac{a}{c} = \dfrac{5}{3}$.

Hindi. $r_1 = 2r_2 = 3r_3$

$$\Rightarrow \quad \frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{3\Delta}{s-c} \qquad \Rightarrow \quad \underset{(i)}{\frac{1}{s-a}} = \underset{(ii)}{\frac{2}{s-b}} = \underset{(iii)}{\frac{3}{s-c}}$$

(i) व (ii) से $a - b = c/3$ ...(1)

(i) व (iii) से $2a - b = 2c$ ...(2)

(ii) व (iii) से $a - 5b = -5c$ ...(3)

माना $c = k$ तब (1) तथा (2) को हल करने पर

$a = \frac{5k}{3}$ और $b = \frac{4k}{3}$ $\quad \therefore \frac{a}{b} = \frac{5}{4}; \frac{a}{c} = \frac{5}{3}$.

C-4*. In a ΔABC, following relations hold good. In which case(s) the triangle is a right angled triangle?
किसी त्रिभुज ABC में निम्नलिखित सम्बन्ध सत्य है। किस स्थिति में त्रिभुज समकोण त्रिभुज है ?

(A*) $r_2 + r_3 = r_1 - r$ (B*) $a^2 + b^2 + c^2 = 8 R^2$

(C*) $r_1 = s$ (D*) $2 R = r_1 - r$

Sol. (A) $\frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{\Delta}{s-a} - \frac{\Delta}{s} \Rightarrow \frac{1}{(s-b)(s-c)} = \frac{1}{s(s-a)}$

$\Rightarrow \tan^2\frac{A}{2} = 1 \Rightarrow A = 90°$

(B) $4R^2 (\sin^2A + \sin^2B + \sin^2C) = 8R^2$

$1 - (\cos^2A - \sin^2B) + \sin^2C = 2$

$1 - \cos(A + B) \cos(A - B) + 1 - \cos^2C = 2.$

$\cos C \cos(A - B) - \cos^2C = 0$

$\cos C[\cos(A - B) - \cos C] = 0$

$\cos C[\cos(A - B) + \cos (A + B)] = 0$

$2 \cos A \cos B \cos C = 0$

$\Rightarrow$ $A = 90°$ or या $B = 90°$ or या $C = 90°$

(C) $r_1 = s$.

$s \tan A/2 = s \quad \Rightarrow \quad \tan A/2 = 1 \quad \Rightarrow \quad A = 90°$

(D) $\frac{a}{\sin A} = \frac{a\Delta}{s(s-a)} \qquad \Rightarrow \frac{1}{\sin A} = \tan A/2$

$\Rightarrow$ $2 \sin^2A/2 = 1$

$\Rightarrow$ $1 - \cos A = 1 \quad \Rightarrow \cos A = 0 \quad \Rightarrow A = 90°$

C-5. The perimeter of a triangle ABC right angled at C is 70, and the inradius is 6, then $|a - b|$ equals
कोण C पर समकोण त्रिभुज ABC का परिमाप 70 है तथा अन्तत्रिज्या 6 है तब $|a - b|$ बराबर है –

(A*) 1 (B) 2 (C) 8 (D) 9

Sol. We known $\Delta = sr = (70/2) \times 6 = 210$

$\Rightarrow (1/2)\, ab = 210 \Rightarrow ab = 420$

Now $(a + b)^2 = (70 - c)^2$

$\Rightarrow a^2 + b^2 + 2ab = 4900 - 140c + c^2$

$\Rightarrow c = \frac{4900 - 2 \times 420}{140} = 29 \qquad [\because a^2 + b^2 = c^2]$

$\Rightarrow (a - b)^2 = a^2 + b^2 - 2ab = c^2 - 2ab = 841 - 840 = 1$

C-6. In a triangle ABC, if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in:

(A*) A.P. (B) G.P. (C) H.P. (D) none of these

त्रिभुज ABC में, यदि $\frac{a-b}{b-c} = \frac{s-a}{s-c}$ हो, तो r_1, r_2, r_3 है–

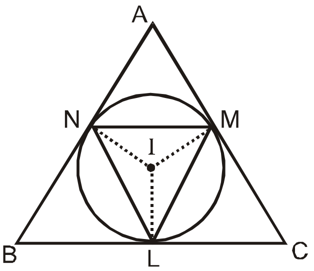
(A) समान्तर श्रेढ़ी में (B) गुणोत्तर श्रेढ़ी में (C) हरात्मक श्रेढ़ी में (D) इनमें से कोई नहीं।

Sol. $\frac{(s-b)-(s-a)}{(s-c)-(s-b)} = \frac{s-a}{s-c}$ $\Rightarrow \frac{\frac{\Delta}{r_2}-\frac{\Delta}{r_1}}{\frac{\Delta}{r_3}-\frac{\Delta}{r_2}} = \frac{\frac{\Delta}{r_1}}{\frac{\Delta}{r_3}}$ $\Rightarrow \frac{(r_1-r_2)\ \ r_2r_3}{r_1r_2(r_2-r_3)} = \frac{r_3}{r_1}$

$\Rightarrow 2r_2 = r_1 + r_3$

$\Rightarrow r_1, r_2, r_3$ are in A.P. $\Rightarrow r_1, r_2, r_3$ समान्तर श्रेढ़ी में है।

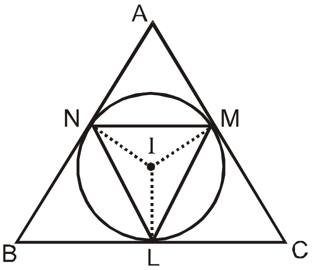
C-7. If the incircle of the Δ ABC touches its sides at L, M and N as shown in the figure and if x, y, z be the circumradii of the triangles MIN, NIL and LIM respectively, where I is the incentre, then the product xyz is equal to :



(A) $R r^2$ (B) $r R^2$

(C*) $\frac{1}{2} R r^2$ (D) $\frac{1}{2} r R^2$

यदि त्रिभुज ABC का अन्तःवृत्त इसकी भुजाओं को L, M एवं N पर चित्रानुसार स्पर्श करता है तथा यदि त्रिभुज MIN, NIL एवं LIM की परित्रिज्याएँ क्रमशः x, y एवं z हो, जहाँ I अन्तःकेन्द्र है, तो गुणनफल xyz का मान है –



(A) $R r^2$ (B) $r R^2$

(C*) $\frac{1}{2} R r^2$ (D) $\frac{1}{2} r R^2$

Sol. MINA is a cyclic quadrilatral MINA एक चक्रीय चतुर्भुज है

$\therefore \frac{MN}{\sin A} = AI \Rightarrow MN = r \operatorname{cosec}\frac{A}{2} \sin A = 2r\cos\frac{A}{2}$

IM = IN = r

$\therefore x = \frac{\left(2r\cos\frac{A}{2}\right)(r)(r)}{4\times\frac{1}{2}r\times r\sin A} = \frac{2r^3\cos\frac{A}{2}}{2r^2\sin A}$

$\frac{r\ \ \cos\ \frac{A}{2}}{\sin\ \ A} = \frac{r}{2\sin\frac{A}{2}}$

similarly इसी प्रकार $y = \frac{r}{2\sin\frac{B}{2}}$ and और $z = \frac{r}{2\sin\frac{C}{2}}$

$$\therefore\ xyz = \frac{r^3}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r^3}{2\frac{r}{R}} = \frac{1}{2}r^2 R$$

C-8. If in a ΔABC, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ is equal to :

(A) 2 (B*) $\frac{1}{2}$ (C) 1 (D) 3

यदि त्रिभुज ABC में, $\frac{r}{r_1} = \frac{1}{2}$, हो, तो $\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ का मान है –

(A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 3

Sol. $\because\ \frac{r}{r_1} = \frac{1}{2} \quad \Rightarrow \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{1}{2} \quad \Rightarrow \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{2} \quad \because\ \sum\tan\frac{A}{2}\tan\frac{B}{2} = 1$

$\Rightarrow \quad \tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) + \tan\frac{B}{2}\ \tan\frac{C}{2} = 1 \qquad \therefore\ \tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) = \frac{1}{2}$

C-9. If in a ΔABC, $\angle A = \frac{\pi}{2}$, then $\tan\frac{C}{2}$ is equal to

यदि त्रिभुज ABC में, $\angle A = \frac{\pi}{2}$ हो, तो $\tan\frac{C}{2}$ का मान है –

(A) $\frac{a-c}{2b}$ (B) $\frac{a-b}{2c}$ (C) $\frac{a-c}{b}$ (D*) $\frac{a-b}{c}$

Sol. $\because\ a^2 = b^2 + c^2$

$\because\ \tan C = \frac{c}{b}$

$\because\ \tan C = \frac{2\tan\frac{C}{2}}{1-\tan^2\frac{C}{2}} = \frac{c}{b}$

$\frac{2t}{1-t^2} = \frac{c}{b}$ where (जहाँ) $t = \tan\frac{C}{2}$

$t^2(c) + (2b)t - c = 0$

$\therefore\ t = \frac{-2b \pm \sqrt{4b^2 + 4\times c^2}}{2c}$

$t = \frac{-b \pm a}{c}$

$\Rightarrow t = \frac{a-b}{c} = \tan\frac{C}{2}$

C-10. In any ΔABC, $\frac{(r_1+\ r_2)\ (r_2\ +r_3)(r_3\ +r_1)}{Rs^2}$ is always equal to

(A) 8 (B) 27 (C) 16 (D*) 4

किसी त्रिभुज ABC में, $\frac{(r_1+\ r_2)\ (r_2\ +r_3)(r_3\ +r_1)}{Rs^2}$ का मान सदैव किसके बराबर होता है ?

(A) 8 के (B) 27 के (C) 16 के (D) 4

Sol. $\because\ r_1 + r_2 = \frac{\Delta c}{(s-a)(s-b)}$

$$\therefore \ \Pi (r_1 + r_2) = \frac{\Delta^3 abc}{(s-a)^2(s-b)^2(s-c)^2} = \frac{\Delta^3 (abc)s^2}{\Delta^4}$$

$$= \frac{(abc)s^2}{\Delta} = \frac{4R\Delta s^2}{\Delta} = 4Rs^2$$

$$\therefore \qquad \frac{\Pi \ (r_1 + r_2)}{Rs^2} = 4$$

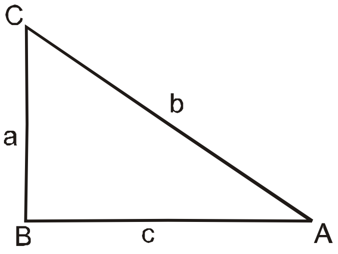
C-11*. In a triangle ABC, right angled at B, then

(A*) $r = \frac{AB + BC - AC}{2}$ (B) $r = \frac{AB + AC - BC}{2}$ (C) $r = \frac{AB + BC + AC}{2}$ (D*) $R = \frac{s - r}{2}$

त्रिभुज ABC में $\angle$ B समकोण है, तो

(A*) $r = \frac{AB + BC - AC}{2}$ (B) $r = \frac{AB + AC - BC}{2}$ (C) $r = \frac{AB + BC + AC}{2}$ (D*) $R = \frac{s - r}{2}$

Sol.



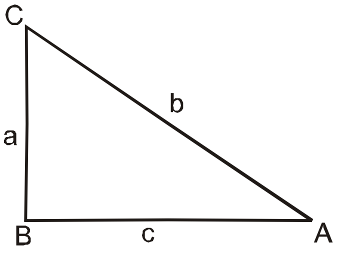
$\because \ r = (s - b) \tan \frac{B}{2}$

$r = s - b$ $(\because \ B = 90º)$

$\therefore \ r = \frac{2s - 2b}{2} = \frac{AB + BC + CA - 2CA}{2}$

$\therefore \ r = \frac{AB + BC - CA}{2}$.

Again.



$\because \ R = \frac{b}{2}$

$\because \ r = (s - b) \tan \frac{B}{2}$

$\Rightarrow r = (s - b) \Rightarrow r = s - 2R \Rightarrow R = \frac{s - r}{2}$

C-.12*. With usual notations, in a Δ ABC the value of Π $(r_1 - r)$ can be simplified as:

सामान्य संकेतानुसार त्रिभुज ABC में Π $(r_1 - r)$ का सरलीकृत मान है –

(A*) $abc \ \Pi \tan \frac{A}{2}$ (B) $4 \ r \ R^2$ (C*) $\frac{(abc)^2}{R\,(a+b+c)^2}$ (D*) $4 \ R \ r^2$

Sol. $r_1 - r = \frac{\Delta}{s - a} - \frac{\Delta}{s} = \frac{\Delta a}{s(s - a)} = a \tan \frac{A}{2}$

$$\therefore \Pi (r_1 - r) = abc \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2}$$

$$= abc\, \Pi \tan\frac{A}{2}$$

$$= abc \frac{\sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}}{\cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}} = \frac{(abc)r}{4R.\frac{(\sin A + \sin B + \sin C)}{4}} = \frac{(abc)\ r}{R\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right)}$$

$$= \frac{2(abc)r}{2s} = \frac{4R\Delta r}{s} = 4Rr^2$$

C-13. **STATEMENT-1 :** In a triangle ABC, the harmonic mean of the three exradii is three times the inradius.
STATEMENT-2 : In any triangle ABC, $r_1 + r_2 + r_3 = 4R$.

कथन -1 : एक त्रिभुज ABC में, बाह्य त्रिज्याओं का हरात्मक माध्य, अन्तः त्रिज्या का तीन गुना होता है।

कथन -2 : किसी ΔABC में, $r_1 + r_2 + r_3 = 4R$

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C*) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false

(A) कथन–1 सत्य है, कथन–2 सत्य है ; कथन–2, कथन–1 का सही स्पष्टीकरण है।
(B) कथन–1 सत्य है, कथन–2 सत्य है ; कथन–2, कथन–1 का सही स्पष्टीकरण नहीं है।
(C*) कथन–1 सत्य है, कथन–2 असत्य है।
(D) कथन–1 असत्य है, कथन–2 सत्य है।
(E) सभी कथन असत्य है।

Sol. **Statement-1 :**

$\because$ H.M. of the three ex-radii = $\frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} = \frac{3\Delta}{s - a + s - b + s - c} = \frac{3\Delta}{s} = 3r$

= 3 times the inradius

$\therefore$ statement-1 is true

Statement-2 : $\because$ L.H.S. = $r_1 + r_2 + r_3$

$$= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$= \Delta\left[\frac{(s-b)(s-c) + (s-a)(s-c) + (s-a)(s-b)}{(s-a)(s-b)(s-c)}\right]$$

$$= s\Delta\left[\frac{3s^2 - 2s(a+b+c) + ab + bc + ca}{\Delta^2}\right]$$

$$= \frac{s\Delta\ \left[ab + bc + ca - s^2\right]}{\Delta^2}$$

$$= \frac{s\left(ab + bc + ca - s^2\right)}{\Delta}$$

$\because$ R.H.S. = 4R = $\frac{abc}{\Delta}$

$\therefore$ L.H.S. $\neq$ R.H.S.

$\therefore$ Statement 2 is false.

Hindi **कथन-1 :**

$\because$ तीन बाह्य त्रिज्याओं का हरात्मक माध्य $= \dfrac{3}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}} = \dfrac{3\Delta}{s-a+s-b+s-c} = \dfrac{3\Delta}{s} = 3r$

= अन्तःत्रिज्या का तीन गुना

$\therefore$ कथन-1 सत्य है।

कथन-2 : $\because$ बायां पक्ष $= r_1 + r_2 + r_3$

$$= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$= \Delta\left[\frac{(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)}{(s-a)(s-b)(s-c)}\right]$$

$$= s\Delta\left[\frac{3s^2-2s(a+b+c)+ab+bc+ca}{\Delta^2}\right]$$

$$= \frac{s\Delta\ \left[ab+bc+ca-s^2\right]}{\Delta^2}$$

$$= \frac{s\left(ab+bc+ca-s^2\right)}{\Delta}$$

$\because$ दायां पक्ष $= 4R = \dfrac{abc}{\Delta}$

$\therefore$ बायां पक्ष $\neq$ दायां पक्ष

$\therefore$ कथन -2 असत्य है।

Section (D) Miscellaneous

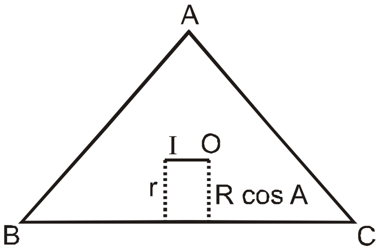
D-1. If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then cos B + cos C is equal to :

(A) 0 (B*) 1 (C) 2 (D) 1/2

यदि त्रिभुज ABC में परिकेन्द्र एवं अन्तःकेन्द्र को मिलाने वाली रेखा भुजा BC के समान्तर हो, तो cos B + cos C का मान है–

(A) 0 (B) 1 (C) 2 (D) 1/2

Sol.



$\because$ R cos A = r

$R \cos A = 4R \sin\dfrac{A}{2}\sin\dfrac{B}{2}\ \sin\dfrac{C}{2}$

cos A = cos A + cos B + cos C – 1

cos B + cos C = 1

D-2. In a ΔABC, if AB = 5 cm, BC = 13 cm and CA = 12 cm, then the distance of vertex ‘A’ from the side BC is (in cm)

त्रिभुज ABC में यदि AB = 5 सेमी., BC = 13 सेमी. एवं CA = 12 सेमी. हो, तो शीर्ष ‘A’ की भुजा BC से लम्बाई है (सेमी. में) –

(A) $\frac{25}{13}$ (B*) $\frac{60}{13}$ (C) $\frac{65}{12}$ (D) $\frac{144}{13}$

Sol. $\because$ required distance अभीष्ट दूरी = $\frac{2\Delta}{a}$

$\because \ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$a = 13; b = 12; c = 5 \quad \Rightarrow \quad s = 15$

$\therefore \ \Delta = \sqrt{15\times2\times3\times10} = 5 \times 3 \times 2 = 30$

$\therefore$ required distance अभीष्ट दूरी = $\frac{2\times30}{13} = \frac{60}{13}$

D-3.✍ If AD, BE and CF are the medians of a ΔABC, then $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ is equal to

यदि किसी त्रिभुज ABC की माध्यिकाएँ AD, BE एवं CF हो, तो $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ का मान है –

(A) 4 : 3 (B) 3 : 2 (C*) 3 : 4 (D) 2 : 3

Sol. $\because \ AD^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2)$,

$BE^2 = \frac{1}{4}(2c^2 + 2a^2 - b^2)$ and तथा

$CF^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2) \qquad \therefore \ \frac{AD^2+BE^2+CF^2}{BC^2+CA^2+AB^2} = \frac{3}{4}$

D-4*.✍ In a triangle ABC, with usual notations the length of the bisector of internal angle A is :

त्रिभुज ABC में संकेतों के सामान्य प्रचलित अर्थ है, तो अन्तः कोण A के अर्धक की लम्बाई है –

(A*) $\frac{2bc\cos\frac{A}{2}}{b+c}$ (B) $\frac{2bc\sin\frac{A}{2}}{b+c}$ (C*) $\frac{abc \ \ \text{cosec}\frac{A}{2}}{2R \ \ (b+c)}$ (D*) $\frac{2\Delta}{b+c}\cdot\text{cosec}\frac{A}{2}$

Sol. $\because \ \beta_a = \frac{2bc}{b+c} \ \cos\frac{A}{2}$

(A) correct

(B) incorrect

(C) $\frac{abc\,\text{cosec}\frac{A}{2}}{2R(b+c)} = \frac{abc\,\text{cosec}\frac{A}{2}}{\frac{a}{\sin A}.(b+c)} = \frac{bc \ \ 2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}.(b+c)} = \frac{2bc}{(b+c)} \ \cos\frac{A}{2}$

(D) $\because \ \frac{2\Delta}{(b+c)}\text{cosec} \ \frac{A}{2} = \frac{bc\sin A}{(b+c)} . \frac{1}{\sin\frac{A}{2}} = \frac{2bc\sin\frac{A}{2}\cos\frac{A}{2}}{(b+c)} . \frac{1}{\sin\frac{A}{2}} = \frac{2bc}{b+c}\cos\frac{A}{2}$

D-5.✍ Let f, g, h be the lengths of the perpendiculars from the circumcentre of the Δ ABC on the sides BC, CA and AB respectively. If $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \lambda\frac{a\,b\,c}{f\,g\,h}$, then the value of '$\lambda$' is:

मानाकि त्रिभुज ABC के परिकेन्द्र से भुजाओं BC, CA एवं AB पर डाले गये लम्बों की लम्बाईयाँ क्रमशः f, g, h हैं। यदि $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \lambda\frac{a\,b\,c}{f\,g\,h}$ हो, तो 'λ' का मान है–

(A*) 1/4 (B) 1/2 (C) 1 (D) 2

Sol. f = RcosA , g = R cos B, h = R cosC.

$\because \quad \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \frac{2R\sin A}{R\cos A} + \frac{2R\sin B}{R\cos B} + \frac{2R\sin C}{R\cos C}$

$$= 2\left(\sum \tan A\right)$$

$$\because \quad \frac{abc}{fgh} = 8\left(\prod \tan \ A\right)$$

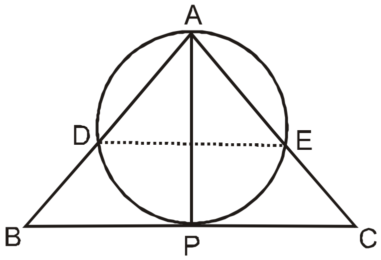
$$\because \quad \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh} \quad \Rightarrow \quad 2 \sum \tan A = \lambda.8\left(\prod \tan A\right) \quad \Rightarrow \quad \lambda = \frac{1}{4}$$

D-6.✍ In an acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to

एक न्यूनकोण त्रिभुज ABC में AP शीर्षलम्ब है। AP को व्यास मानकर खींचा गया वृत्त भुजाओं AB एवं AC को क्रमशः D एवं E पर काटता है, तो DE की लम्बाई है –

(A) $\frac{\Delta}{2R}$ (B) $\frac{\Delta}{3R}$ (C) $\frac{\Delta}{4R}$ (D*) $\frac{\Delta}{R}$

Sol.



$$\because \quad \frac{DE}{\sin A} = AP \quad \Rightarrow \quad DE = \frac{2\Delta}{a} \sin A$$

$$= \frac{2\Delta \sin A}{2R \sin A} = \frac{\Delta}{R}$$

D-7.✍ AA_1, BB_1 and CC_1 are the medians of triangle ABC whose centroid is G. If points A, C_1, G and B_1 are concyclic, then

(A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C*) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$

त्रिभुज ABC जिसका केन्द्रक G है, की माध्यिकाएँ AA_1, BB_1 एवं CC_1 है। यदि बिन्दु A, C_1, G एवं B_1 समचक्रीय हो, तो

(A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$

Sol. $\because$ A, C_1, G and (और) B_1 are cyclic (चक्रीय है)

$\therefore$ $BC_1 . BA = BG . BB_1$

$$\frac{c}{2} . c = \left(\frac{2}{3}BB_1\right).BB_1$$

$$\frac{c^2}{2} = \frac{2}{3} \times \frac{1}{4} (2c^2 + 2a^2 - b^2)$$

$\Rightarrow$ $c^2 + b^2 = 2a^2$

D-8. If 'ℓ' is the length of median from the vertex A to the side BC of a ΔABC, then

किसी ΔABC में शीर्ष A से गुजरने वाली माध्यिका की लम्बाई 'ℓ' हो, तो –

(A) $4\ell^2 = b^2 + 4ac \cos B$ (B*) $4\ell^2 = a^2 + 4bc \cos A$

(C) $4\ell^2 = c^2 + 4ab \cos C$ (D) $4\ell^2 = b^2 + 2c^2 - 2a^2$

Sol. $\because$ $\ell = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$\therefore$ $4\ell^2 = 2b^2 + 2c^2 - a^2$

$= a^2 + 2(b^2 + c^2 - a^2)$

$= a^2 + 2(2bc \cos A)$

$4\ell^2 = a^2 + 4bc \cos A$

D-9*. The product of the distances of the incentre from the angular points of a Δ ABC is:
त्रिभुज ABC के शीर्षों से अन्तःकेन्द्र की दूरियों का गुणनफल है

(A) $4R^2r$ (B*) $4Rr^2$ (C) $\frac{(a\,b\,c)\,R}{s}$ (D*) $\frac{(a\,b\,c)\,r}{s}$

Sol. Product of distances of incenter from angular points
शीर्षों से अन्तःकेन्द्र की दूरियों का गुणनफल

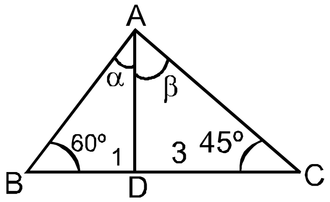
$$= \frac{r^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r^3}{r/4R} = 4Rr^2 = \frac{abc}{\Delta}r^2 = \frac{(abc)(r)}{\frac{\Delta}{r}} = \frac{(abc)(r)}{s}.$$

D-10.✍ In a triangle ABC, B = 60° and C = 45°. Let D divides BC internally in the ratio 1 : 3, then value of $\frac{\sin\angle BAD}{\sin\angle CAD}$ is

त्रिभुज ABC में, $\angle B = 60°$ एवं $\angle C = 45°$ है। यदि बिन्दु D भुजा BC को 1 : 3 के अनुपात में अन्तः विभाजित करता हो, तो $\frac{\sin\angle BAD}{\sin\angle CAD}$ का मान है –

(A) $\sqrt{\frac{2}{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C*) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{3}$

Sol.



if we apply Sine-Rule in Δ BAD, we get Δ BAD में ज्या नियम का प्रयोग करने पर

$$\frac{BD}{\sin\alpha} = \frac{AD}{\text{Sin } 60°} \quad ...(1)$$

if we apply Sine-Rule in Δ CAD, we get. Δ CAD में ज्या नियम का प्रयोग करने पर

$$\frac{CD}{\sin\ \beta} = \frac{AD}{\sin\ 45^{o}} \quad ...(2)$$

divide (2) by (1) (2) में (1) का भाग देने पर

$$\frac{\sin\ \alpha}{\sin\ \beta} \times \frac{CD}{BD} = \frac{\sin\ 60^{o}}{\sin\ 45^{o}}$$

$$\frac{\sin\ \alpha}{\sin\ \beta} \times \frac{3}{1} = \frac{\sqrt{3}}{2\times\frac{1}{\sqrt{2}}}$$

$$\frac{\sin\ \alpha}{\sin\ \beta} = \frac{1}{\sqrt{6}}$$

D-11*.✍ In a triangle ABC, points D and E are taken on side BC such that BD = DE = EC. If angle ADE = angle AED = θ, then:

(A*) $\tan\theta = 3\tan B$ (B) $3\tan\theta = \tan C$

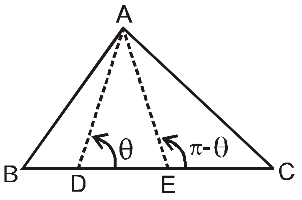
(C*) $\frac{6\tan\theta}{\tan^2\theta - 9} = \tan A$ (D*) angle B = angle C

त्रिभुज ABC में, भुजा BC पर बिन्दु D एवं E इस प्रकार लिए जाते हैं कि BD = DE = EC. यदि $\angle ADE = \angle AED = \theta$ हो, तो –

(A) $\tan\theta = 3\tan B$ (B) $3\tan\theta = \tan C$

(C) $\frac{6\tan\theta}{\tan^2\theta - 9} = \tan A$ (D) कोण B = कोण C

Sol.



if we apply m-n Rule in Δ ABE, we get Δ ABE में m-n प्रमेय का प्रयोग करने पर

$(1+1) \cot \theta = 1.\cot B - 1.\cot \theta$

$2 \cot\theta = \cot B - \cot\theta$

$3 \cot\theta = \cot B$

$\tan \theta = 3 \tan B$ ....(1)

Similarly, if we apply m-n Rule in Δ ACD, we get इसी प्रकार, ΔACD में m-n प्रमेय का प्रयोग करने पर

$(1+1) \cot (\pi - \theta) = 1.\cot\theta - 1.\cot C.$

$\cot C = 3 \cot\theta \Rightarrow \tan \theta = 3 \tan C$ ....(2)

form (1) and (2) we can say that समीकरण (1) तथा (2) से

$\tan B = \tan C \Rightarrow B=C$

$\because \quad A + B + C = \pi$

$\therefore A = \pi - (B + C)$

$= \pi - 2B \quad \because \quad B = C$

$\therefore \tan A = - \tan2B$

$$= -\left(\frac{2\tan B}{1-\tan^2 B}\right) = -\frac{\frac{2\tan\theta}{3}}{1-\frac{\tan^2\theta}{9}} \Rightarrow \tan A = \frac{6\tan\theta}{\tan^2\theta-9}$$

D-12. **STATEMENT-1 :** If R be the circumradius of a ΔABC, then circumradius of its excentral $\Delta I_1I_2I_3$ is 2R.

STATEMENT-2 : If circumradius of a triangle be R, then circumradius of its pedal triangle is $\frac{R}{2}$.

(A*) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

(E) Both STATEMENTS are false

कथन-1 : यदि एक ΔABC की परित्रिज्या R है, तो बहिष्केन्द्रीय $\Delta I_1I_2I_3$ के बाहय केन्द्र की परित्रिज्या 2R होगी।

कथन-2 : यदि एक त्रिभुज की परित्रिज्या R है, तो इसके पदिक त्रिभुज की परित्रिज्या $\frac{R}{2}$ होगी।

(A*) कथन–1 सत्य है, कथन–2 सत्य है ; कथन–2, कथन–1 का सही स्पष्टीकरण है।

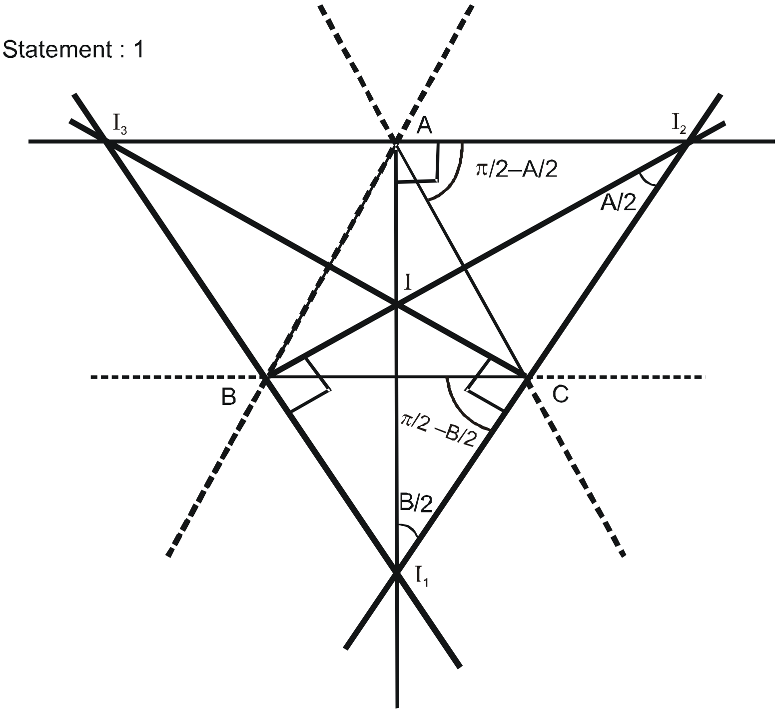
(B) कथन–1 सत्य है, कथन–2 सत्य है ; कथन–2, कथन–1 का सही स्पष्टीकरण नहीं है।

(C) कथन–1 सत्य है, कथन–2 असत्य है।

(D) कथन–1 असत्य है, कथन–2 सत्य है।

(E) सभी कथन असत्य है।

Sol.



$I_1 I_2 = 4R \cos\frac{C}{2}$ if we apply Sine-Rule in $\Delta I_1 I_2 I_3$, then $\Delta I_1 I_2 I_3$ में ज्या नियम का प्रयोग करने पर–

$$2 R_{ex} = \frac{I_1 \ \ I_2}{\sin\left(\frac{A}{2}+\frac{B}{2}\right)} = \frac{4R\cos\frac{C}{2}}{\sin\left(\frac{A+B}{2}\right)}$$

$$= \frac{4R\cos\frac{C}{2}}{\cos\frac{C}{2}}$$

$2R_{ex} = 4R \qquad R_{ex} = 2R$

$\therefore$ Δ ABC is pedal triangle of $\Delta I_1 I_2 I_3$

Δ ABC, त्रिभुज $\Delta I_1 I_2 I_3$ का पदिक त्रिभुज है।

$\therefore$ statement - 1 and statement - 2 both are correct and statement -2 also explains Statement - 1

कथन - 1 तथा कथन - 2 दोनों सत्य है तथा कथन -2, कथन- 1 सही व्याख्या भी करता है।

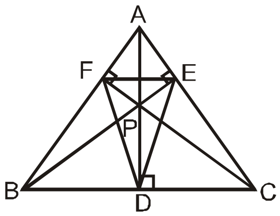
PART-II (COMPREHENSION)

अनुच्छेद (COMPREHENSION)

Comprehension # 1 (Q. No. 1 to 4)

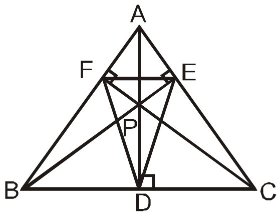
The triangle DEF which is formed by joining the feet of the altitudes of triangle ABC is called the Pedal Triangle.

Answer The Following Questions :



अनुच्छेद # 1

त्रिभुज ABC शीर्ष लम्बों पादों को मिलाने से बना त्रिभुज DEF दिए गए त्रिभुज का पदिक त्रिभुज कहलाता है, निम्न प्रश्नों के उत्तर दीजिए



1.✍ Angle of triangle DEF are

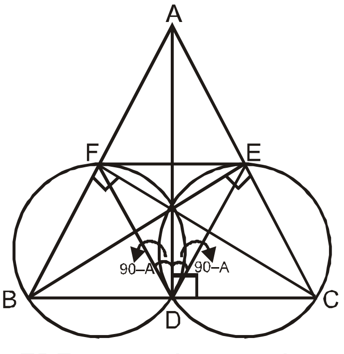
(A*) $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$ (B) $\pi + 2A$, $\pi + 2B$ and $\pi + 2C$

(C) $\pi - A$, $\pi - B$ and $\pi - C$ (D) $2\pi - A$, $2\pi - B$ and $2\pi - C$

त्रिभुज DEF के कोण है

(A) $\pi - 2A$, $\pi - 2B$ और $\pi - 2C$ (B) $\pi + 2A$, $\pi + 2B$ और $\pi + 2C$

(C) $\pi - A$, $\pi - B$ और $\pi - C$ (D) $2\pi - A$, $2\pi - B$ और $2\pi - C$



Sol.

$\angle EDF = 90 - A + 90 - A$

$= 180 - A$

2*.✍ Sides of triangle DEF are

त्रिभुज DEF की भुजाऐं है–

(A) b cosA, a cosB, c cosC (B*) a cosA, b cosB, c cosC

(C*) R sin 2A, R sin 2B, R sin 2C (D) a cotA, b cotB, c cotC

Sol. ΔAEF : AF = bcosA, AE = ccosA

$$\therefore \quad \cos A = \frac{b^2\cos^2 A + C^2\cos^2 A - EF^2}{2b\cos A.c\cos A}$$

$$\Rightarrow \quad (EF)^2 = (b^2 + c^2 - 2bc\cos A)\cos^2 A$$

$$(EF)^2 = a^2\cos^2 A$$

$$EF = a.\cos A$$

3.✍ Circumraii of the triangle PBC, PCA and PAB are respectively

त्रिभुज PBC, PCA तथा PAB की परित्रिज्याऐं क्रमशः है–

(A*) R, R, R (B) 2R, 2R, 2R (C) R/2, R/2, R/2 (D) 3R, 3R, 3R

Sol. Circumraii of the triangle PBC = $\frac{BC}{2.\sin(B+C)}$

$$= \frac{a}{2\sin(\pi-A)} = \frac{a}{2\sin A} = R$$

4*.✍ Which of the following is/are correct

(A*) $\frac{\text{Perimeter of } \Delta DEF}{\text{Perimeter of } \Delta ABC} = \frac{r}{R}$ (B*) Area of $\Delta DEF = 2\,\Delta \cos A \cos B \cos C$

(C*) Area of $\Delta AEF = \Delta \cos^2 A$ (D*) Circum-radius of $\Delta DEF =$

निम्न में से कौनसा सही है–

(A) $\frac{\text{त्रिभुज DEF का परिमाप}}{\text{त्रिभुज ABC का परिमाप}} = \frac{r}{R}$ (B) त्रिभुज DEF का क्षेत्रफल $= 2\,\Delta \cos A \cos B \cos C$

(C) त्रिभुज AEF का क्षेत्रफल $= \Delta \cos^2 A$ (D) त्रिभुज DEF की परित्रिज्या $= \frac{R}{2}$

Sol. $\because$ FE = a cos A = R sin 2A

DE = c cos C = R sin 2C

FD = b cos B = R sin 2B

(A) $= \frac{R\left(\sum \sin 2A\right)}{a+b+c}$

$$= \frac{R(4\sin A \sin B \sin C)}{2R\left(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)} = \frac{8\left(\prod \sin\frac{A}{2}\right)\left(\prod \cos\frac{A}{2}\right)}{2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{r}{R}$$

(B) $\because$ Area of ΔDEF का क्षेत्रफल

$$= \frac{1}{2} FD \times DE \sin(\pi - 2A) = \frac{1}{2} b \cos B . c \cos C . \sin 2A$$

$$= \frac{1}{2} bc \cos B . \cos C . 2 \sin A . \cos A = 2\left(\frac{1}{2} bc \sin A\right) \cos A . \cos B . \cos C$$

$$= 2\Delta \cos A . \cos B . \cos C$$

(C) Area of ΔAEF का क्षेत्रफल $= \frac{1}{2} AE \times AF \sin A$

$$= \frac{1}{2} (c \cos A)(b \cos A) \sin A = \left(\frac{1}{2} bc \sin A\right)\cos^2 A = \Delta \cos^2 A$$

(D) $R_{DEF} = \frac{FE \times DE \times FD}{4\Delta_{DEF}} = \frac{abc \cos A \cos B \cos C}{4 \times 2\Delta \cos A \cos B \cos C} = \frac{abc}{8\Delta} = \frac{4R\Delta}{8\Delta} = \frac{R}{2}$

Comprehension # 2 (Q. 5 to 8)

अनुच्छेद # 2

The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle and in this case internal angle bisector of triangle ABC are the altitudes of triangles $I_1I_2I_3$

त्रिभुज ABC के तीन बहिष्केन्द्रो I_1, I_2 तथा I_3 को मिलाने से बना त्रिभुज बहिष्केन्द्रीय त्रिभुज कहलाता है तथा इस स्थिति में Δ ABC के आन्तरिक कोण अर्द्धक त्रिभुज $I_1I_2I_3$ के शीर्ष लम्ब है।

5.✍ Incentre I of Δ ABC is the .... of the excentral $\Delta\, I_1 I_2 I_3$.

(A) Circumcentre (B*) Orthocentre (C) Centroid (D) None of these

त्रिभुज ABC का अन्तःकेन्द्र I , बहिष्केन्द्र त्रिभुज $I_1 I_2 I_3$ का ...., है–

(A) परिकेन्द्र (B) लम्बकेन्द्रक (C) केन्द्रक (D) इनमें से कोई नहीं

Sol. Clearly

6.✍ Angles of the $\Delta\, I_1 I_2 I_3$ are

(A*) $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$ (B) $\frac{\pi}{2}+\frac{A}{2}, \frac{\pi}{2}+\frac{B}{2}$ and $\frac{\pi}{2}+\frac{C}{2}$

(C) $\frac{\pi}{2}-A, \frac{\pi}{2}-B$ and $\frac{\pi}{2}-C$ (D) None of these

त्रिभुज $I_1 I_2 I_3$ के कोण है–

(A*) $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ तथा $\frac{\pi}{2}-\frac{C}{2}$ (B) $\frac{\pi}{2}+\frac{A}{2}, \frac{\pi}{2}+\frac{B}{2}$ तथा $\frac{\pi}{2}+\frac{C}{2}$

(C) $\frac{\pi}{2}-A, \frac{\pi}{2}-B$ तथा $\frac{\pi}{2}-C$ (D) इनमें से कोई नहीं

Sol. Let $\angle I_3 I_1 I_2 = \theta$

Then angle of pedal trinagle पदिक त्रिभुज के कोण= $\pi - 2\theta = A$

$$\theta = \frac{\pi}{2}-\frac{A}{2}$$

7.✍ Sides of the $\Delta\, I_1 I_2 I_3$ are

(A) $R\cos\frac{A}{2}, R\cos\frac{B}{2}$ तथा $R\cos\frac{C}{2}$ (B*) $4R\cos\frac{A}{2}, 4R\cos\frac{B}{2}$ तथा $4R\cos\frac{C}{2}$

(C) $2R\cos\frac{A}{2}, 2R\cos\frac{B}{2}$ तथा $2R\cos\frac{C}{2}$ (D) None of these

त्रिभुज $I_1 I_2 I_3$ की भुजाऐं है–

(A) $R\cos\frac{A}{2}, R\cos\frac{B}{2}$ and $R\cos\frac{C}{2}$ (B*) $4R\cos\frac{A}{2}, 4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$

(C) $2R\cos\frac{A}{2}, 2R\cos\frac{B}{2}$ and $2R\cos\frac{C}{2}$ (D) इनमें से कोई नहीं

Sol. Side of pedal triangle पदिक त्रिभुज की भुजाएं = $I_2I_3\cos\theta = BC$

$$I_2I_3 = \frac{a}{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)}$$

$$I_2I_3 = 4R\cos\left(\frac{A}{2}\right)$$

8.✍ Value of $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 =$

$II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 =$ का मान है–

(A) $4R^2$ (B*) $16R^2$ (C) $32R^2$ (D) $64R^2$

Sol $I I_1 = 4 R \sin\frac{A}{2}$

$I_2I_3 = 4 R \cos\frac{A}{2}$

$\therefore \quad II_1^2 + I_2I_3^2 = 16R^2$

PART-III (MATCH THE COLUMN)

भाग– III (कॉलम को सुमेलित कीजिए (MATCH THE COLUMN)

1. Match the column

Column– I **Column–II**

(A) In a ΔABC, $2B = A + C$ and $b^2 = ac$. Then the value of $\frac{a^2(a+b+c)}{3abc}$ is equal to (p) 8

(B) In any right angled triangle ABC, the value of $\frac{a^2+b^2+c^2}{R^2}$ is always equal to (where R is the circumradius of ΔABC) (q) 1

(C) In a ΔABC if $a = 2$, $bc = 9$, then the value of $2R\Delta$ is equal to (r) 5

(D) In a ΔABC, $a = 5$, $b = 3$ and $c = 7$, then the value of $3 \cos C + 7 \cos B$ is equal to (s) 9

Ans. (A) → (q), (B) → (p), (C) → (s), (D) → (r)

मिलान कीजिए

स्तम्भ – I **स्तम्भ – II**

(A) किसी त्रिभुज ABC में, $2B = A + C$ और $b^2 = ac$ हो, तो $\frac{a^2(a+b+c)}{3abc}$ का मान होगा– (p) 8

(B) किसी समकोण त्रिभुज ABC में, $\frac{a^2+b^2+c^2}{R^2}$ का मान हमेशा होता है– (जहाँ R, DABC की परित्रिज्या है) (q) 1

(C) किसी ΔABC में, यदि $a = 2$, $bc = 9$ हो, तो $2R\Delta$ का मान है– (r) 5

(D) किसी ΔABC में, $a = 5$, $b = 3$ और $c = 7$ है, तब $3 \cos C + 7 \cos B$ का मान होगा– (s) 9

Ans. (A) → (q), (B) → (p), (C) → (s), (D) → (r)

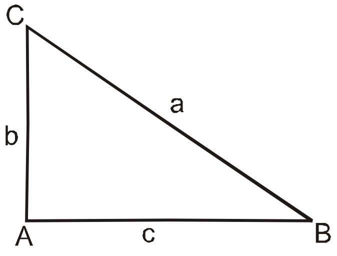
Sol. (A) $\because\ 2B = A + C \Rightarrow B = \frac{\pi}{3}$ and $A + C = \frac{2\pi}{3}$

$\because\ b^2 = ac \quad \Rightarrow \sin^2 B = \sin A.\sin C \quad \Rightarrow \sin A \sin C = \frac{3}{4}$

$\Rightarrow \cos(A - C) - \cos(A + C) = \frac{3}{2} \qquad \because\ A + C = \frac{2\pi}{3}$

$\Rightarrow \cos(A - C) = 1 \qquad \Rightarrow A = C = \frac{\pi}{3} = B \qquad \Rightarrow a = b = c \qquad \therefore\ \frac{a^2(a+b+c)}{3abc} = 1$

(B) $\because\ a^2 = b^2 + c^2$ and और $2R = a \qquad \therefore\ \frac{a^2+b^2+c^2}{R^2} = \frac{2a^2}{R^2} = 8$



(C) $\because\ \Delta = \frac{1}{2}bc \sin A \Rightarrow \Delta = \frac{1}{2}.9. \sin A = \frac{9}{2} \times \frac{a}{2R} \qquad \because\ a = 2$

$\therefore\ 2R\Delta = 9$

(D) $\because\ a = 5, b = 3$ and और $c = 7$

and because we know that और हम जानते है कि

$b \cos C + c \cos B = a$

$\therefore\ 3 \cos C + 7 \cos B = 5$

2. **Match the column**

Column – I		**Column – II**	
(A)	In a ΔABC, a = 4, b = 3 and the medians AA_1 and BB_1 are mutually perpendicular, then square of area of the ΔABC is equal to	(p)	27
(B)✍	In any ΔABC, minimum value of $\frac{r_1\ r_2\ r_3}{r^3}$ is equal to	(q)	7
(C)	In a ΔABC, a = 5, b = 4 and $\tan \frac{C}{2} = \sqrt{\frac{7}{9}}$, then side 'c' is equal to	(r)	6
(D)✍	In a ΔABC, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of $(8 \cos B)$ is equal to	(s)	11

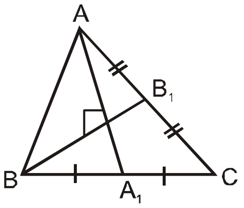
Ans. (A) → (s), (B) → (p), (C) → (r), (D) → (q)

मिलान कीजिए

स्तम्भ – I		**स्तम्भ – II**	
(A)	किसी त्रिभुज ΔABC में, a = 4, b = 3 तथा माध्यिकाएं AA_1 और BB_1 परस्पर लम्बवत् है तब ΔABC के क्षेत्रफल के वर्ग का मान होगा –	(p)	27
(B)	किसी त्रिभुज ABC में, $\frac{r_1\ r_2\ r_3}{r^3}$ का न्यूनतम मान है–	(q)	7
(C)	यदि त्रिभुज ΔABC में, a = 5, b = 4 और $\tan \frac{C}{2} = \sqrt{\frac{7}{9}}$ हो, तो भुजा c का मान होगा –	(r)	6
(D)	त्रिभुज ΔABC में, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, तब 8 cos B का मान होगा–	(s)	11

Ans. (A) → (s), (B) → (p), (C) → (r), (D) → (q)

Sol.



Match the column स्तंम्भ मिलान कीजिए

(A) AA_1 and BB_1 are perpendicular

AA_1 तथा BB_1 परस्पर लम्बवत् है,

$\therefore\quad a^2 + b^2 = 5c^2$

$\therefore\quad c^2 = \frac{a^2+b^2}{5} = 5 \quad \Rightarrow \quad c = \sqrt{5}$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{16+9-5}{2\times4\times3} = \frac{5}{6}$$

$$\sin C = \frac{\sqrt{11}}{6}$$

$\therefore \quad \Delta = \frac{1}{2}$ ab sin C = $\sqrt{11}$

$\therefore \quad \Delta^2 = 11$

(B) $\because$ G.M. $\geq$ H.M. $\quad \because$ गुणोत्तर माध्य $\geq$ समान्तर माध्य

$$(r_1\, r_2\, r_3)^{1/3} \geq \frac{3}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}}$$

$$\Rightarrow (r_1\, r_2\, r_3)^{1/3} \geq 3r$$

$$\Rightarrow \frac{r_1 \quad r_2 \quad r_3}{r^3} \geq 27$$

(C) $\tan^2 \frac{C}{2} = \frac{(s-a)(s-b)}{s(s-c)} \qquad \because \qquad a = 5, b = 4 \qquad 2s = 9 + c$

$$= \frac{(9+c-10)(9+c-8)}{(9+c)(9-c)} = \frac{c^2-1}{81-c^2}$$

$\Rightarrow \quad \frac{7}{9} = \frac{c^2-1}{81-c^2} \qquad \Rightarrow c^2 = 36 \qquad \Rightarrow c = 6$

(D) $\because \quad 2a^2 + 4b^2 + c^2 = 4ab + 2ac.$

$\Rightarrow \quad (a - 2b)^2 + (a - c)^2 = 0$

$\Rightarrow \quad a = 2b = c$

$\because \quad \cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{7}{8}$

$\therefore \quad 8 \cos B = 7$

Exercise-3

✍ Marked Questions may have for Revision Questions.
✍ चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

भाग - I : JEE (ADVANCED) / IIT-JEE (पिछले वर्षो) के प्रश्न

*** Marked Questions may have more than one correct option.**
*** चिन्हित प्रश्न एक से अधिक सही विकल्प वाले प्रश्न है -**

1. If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is

यदि किसी त्रिभुज के कोण A, B एवं C समान्तर श्रेढ़ी में हैं तथा कोणों A, B एवं C की सम्मुख भुजाओं की लम्बाईयाँ क्रमशः a, b तथा c हैं, तो व्यंजक $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ का मान है– **[IIT-JEE 2010, Paper-1, (3, –1), 84]**

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D*) $\sqrt{3}$

Sol. $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = \frac{2}{2R}(a \cos C + c \cos A) = \frac{b}{R} = 2 \sin B = 2 \sin 60^{\circ} = \sqrt{3}$

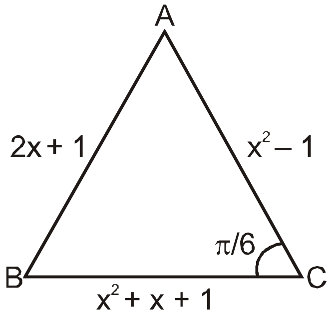
2.✍ Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

मानाकि ABC एक त्रिभुज है जिसमें $\angle ACB = \frac{\pi}{6}$ तथा A, B तथा C की सम्मुख भुजाओं की लम्बाईयाँ क्रमशः a, b तथा c हैं। x के वह मान जिनके लिए $a = x^2 + x + 1$, $b = x^2 - 1$ एवं $c = 2x + 1$ हों, निम्न हैं

(A) $-\left(2+\sqrt{3}\right)$ (B*) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$

[IIT-JEE 2010, Paper-1, (3, 0), 84]

Sol.



$$\cos\frac{\pi}{6} = \frac{(x^2-1)^2+(x^2+x+1)^2-(2x+1)^2}{2(x^2+x+1)(x^2-1)}$$

$$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2+(x^2+3x+2)(x^2-x)}{2(x^2+x+1)(x^2-1)}$$

$$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2+(x+1)(x+2)x(x-1)}{2(x^2+x+1)(x^2-1)}$$

$\Rightarrow \quad \sqrt{3} = \frac{x^2-1+x(x+2)}{x^2+x+1} \quad \Rightarrow \quad \sqrt{3}\,(x^2 + x + 1) = 2x^2 + 2x - 1$

$\Rightarrow \quad (\sqrt{3} - 2)\, x^2 + (\sqrt{3} - 2)\, x + (\sqrt{3} + 1) = 0$

on solving हल करने पर

$x^2 + x - (3\sqrt{3}+5) = 0$ we get

$x = \sqrt{3} + 1, -(2+\sqrt{3})$

$\because$ At $x = -(2+\sqrt{3})$, Side c becomes negative. पर भुजा c ऋणात्मक होती है

$\therefore$ $x = \sqrt{3}+1$

3.✍ Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to **[IIT-JEE 2010, Paper-2, (3, 0), 79]**

दिया है कि किसी त्रिभुज ABC के शीर्षो A, B एवं C की सम्मुख भुजाओं की लम्बाइयाँ क्रमशः a, b एवं c हैं। माना कि a = 6, b = 10 तथा त्रिभुज का क्षेत्रफल $15\sqrt{3}$ है। यदि $\angle ACB$ अधिक कोण (obtuse angle) है तथा त्रिभुज के अन्तःवृत्त की त्रिज्या r है, तो r^2 का मान ज्ञात कीजिए।

Ans. 3

Sol. Area of triangle त्रिभुज का क्षेत्रफल $= \frac{1}{2}$ ab sin C $= 15\sqrt{3}$

$\Rightarrow \quad \frac{1}{2} \,.\, 6 \,.\, 10 \sin C = 15\sqrt{3} \quad \Rightarrow \quad \sin C = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad C = \frac{2\pi}{3}$ (C is obtuse angle अधिक कोण)

Nowतथा $\cos C = \frac{a^2+b^2-c^2}{2ab} \quad \Rightarrow \quad -\frac{1}{2} = \frac{36+100-c^2}{2.6.10} \quad \Rightarrow \quad c = 14$

∴ $r = \dfrac{\Delta}{s} = \dfrac{15\sqrt{3}}{\dfrac{6+10+14}{2}} = \sqrt{3}$ ⇒ $r^2 = 3$

4. Let PQR be a triangle of area Δ with $a = 2$, $b = \dfrac{7}{2}$ and $c = \dfrac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\dfrac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals **(SOT)**

त्रिभुज PQR का क्षेत्रफल Δ है जिसके लिए $a = 2$, $b = \dfrac{7}{2}$ और $c = \dfrac{5}{2}$ है, जहाँ a, b और c क्रमशः कोण P, Q और R की सम्मुख भुजाओं की लम्बाईयाँ हैं। तब $\dfrac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ का मान निम्न है– **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) $\dfrac{3}{4\Delta}$ (B) $\dfrac{45}{4\Delta}$ (C*) $\left(\dfrac{3}{4\Delta}\right)^2$ (D) $\left(\dfrac{45}{4\Delta}\right)^2$

Sol. **Ans. (C)**

$a = 2 = QR$

$b = \dfrac{7}{2} = PR$

$c = \dfrac{5}{2} = PQ$

$s = \dfrac{a+b+c}{2} = \dfrac{8}{4} = 4$

$$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{2\sin P(1-\cos P)}{2\sin P(1+\cos P)} = \frac{1-\cos P}{1+\cos P} = \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2} = \frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

Hindi.

$a = 2 = QR$

$b = \dfrac{7}{2} = PR$

$c = \dfrac{5}{2} = PQ$

$s = \dfrac{a+b+c}{2} = \dfrac{8}{4} = 4$

$$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{2\sin P(1-\cos P)}{2\sin P(1+\cos P)} = \frac{1-\cos P}{1+\cos P} = \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$$

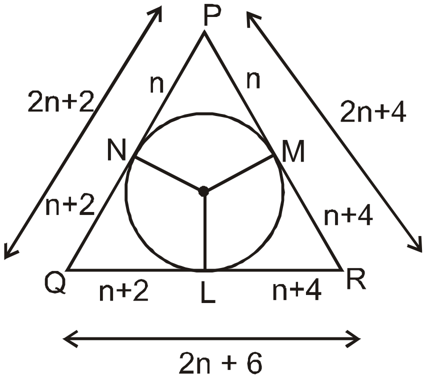
$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2} = \frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

5.*✍ In a triangle PQR, P is the largest angle and $\cos P = \dfrac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

त्रिभुज PQR में, P वृहत्तम कोण है तथा cosP = $\frac{1}{3}$ । इसके अतिरिक्त त्रिभुज का अन्तःवृत्त भुजाओं PQ, QR तथा RP को क्रमशः N, L तथा M पर इस तरह स्पर्श करता है कि PN, QL तथा RM की लम्बाईयाँ क्रमागत सम पूर्ण संख्याएं है। तब त्रिभुज की भुजा (भुजाओं) की सम्भावित लम्बाई (लम्बाईयाँ) है (हैं) **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) 16 (B*) 18 (C) 24 (D*) 22

Sol. **(B, D)**



$$\cos P = \frac{(2n+2)^2 + (2n+4)^2 - (2n+6)^2}{2\ (2n+2)\ (2n+4)} = \frac{1}{3}$$

$$\Rightarrow \qquad \frac{4n^2 - 16}{8\ (n+1)\ (n+2)} = \frac{1}{3}$$

$$= \frac{n^2 - 4}{2\ (n+1)\ (n+2)} = \frac{1}{3} \qquad \Rightarrow \qquad \frac{n-2}{2\ (n+1)} = \frac{1}{3}$$

$= 3n - 6 = 2n + 2$

$\Rightarrow n = 8$

$\Rightarrow 2n + 2 = 18$

$\Rightarrow 2n + 4 = 720$

$\Rightarrow 2n + 6 = 22$

6. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

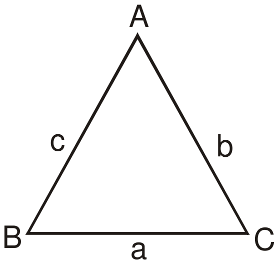
(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

एक त्रिभुज की दो भुजाओं का योग x है तथा उन्हीं भुजाओं का गुणनफलन y है। यदि $x^2 - c^2 = y$, जहाँ c त्रिभुज की तीसरी भुजा है, तब त्रिभुज की अंतःत्रिज्या (in-radius) एवं परिवृत–त्रिज्या (circum-radius) का अनुपात (ratio) है– **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

Ans. **(B)**

Sol.



$a + b = x$

$ab = y$

$x^2 - c^2 = y$

$(a + b)^2 - c^2 = ab$

$a^2 + b^2 + ab = c^2$

$a^2 + b^2 - c^2 = -ab$

$$\frac{a^2+b^2-c^2}{2ab} = \frac{7}{2}$$

$$\cos C = \frac{-1}{2}$$

$$C = \frac{2\pi}{3}$$

$$\frac{r}{R} = \frac{\Delta \times 4\Delta}{s \times abc} = \frac{4 \times \frac{1}{4}a^2b^2\sin^2 C}{(a+b+c)abc} = \frac{3ab}{4c(x+c)}$$

$$= \frac{3y}{4c(x+c)}$$

7*. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

[JEE (Advanced) 2016, Paper-1, (4, –2)/62]

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

माना कि त्रिभुज XYZ में कोणों X, Y, Z के सामने की भुजाओं की लम्बाइयाँ क्रमाः x, y, z है और $2s = x + y + z$ है। यदि $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$, और त्रिभुज XYZ के अंतर्वत्त (incircle) का क्षेत्रफल $\frac{8\pi}{3}$ है, तब

(A) त्रिभुज XYZ का क्षेत्रफल $6\sqrt{6}$ है

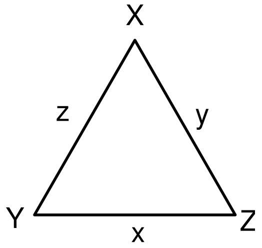
(B) त्रिभुज XYZ के परिवत्त (circumcircle) की त्रिज्या $\frac{35}{6}\sqrt{6}$ है

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Ans. **(A,C,D)**

Sol.



$2S = x + y + z \quad \Rightarrow \quad \frac{S-x}{4} = \frac{S-y}{3} = \frac{S-z}{2} = \lambda$

$S - x = 4\lambda$

$S - y = 3\lambda$

$S - z = 2\lambda$

$S = 9\lambda$

Adding all we get सभी को जोड़नें पर, हम पाते है–

$S = 9\lambda,\ x = 5\lambda,\ y = 6\lambda,\ = z = 7\lambda$

$\pi r^2 = \frac{8\pi}{3} \quad \Rightarrow \quad r^2 = \frac{8}{3}$

$\Delta = \sqrt{S(S-x)(S-y)(S-z)} \quad \Rightarrow \quad \Delta = \sqrt{9\lambda.4\lambda.3\lambda.2\lambda.} = 6\sqrt{6}\ \lambda^2$

$R = \frac{xyz}{4\Delta} = \frac{5\lambda.6\lambda.7\lambda}{4.6\sqrt{6}\lambda^2} = \frac{35}{4\sqrt{6}}\lambda \quad \Rightarrow \quad r^2 = \frac{8}{3} = \frac{\Delta^2}{S^2} = \frac{216\lambda^4}{81\lambda^2} = \frac{24}{9}\lambda^2 = \frac{8}{3}\lambda^2 = \frac{8}{3}$

we get हम पाते है $\lambda = 1$

(A) $\Delta = 6\sqrt{6}$ (B) $R = \frac{35}{4\sqrt{6}}\lambda = \frac{35}{4\sqrt{6}}$

(C) $r = 4R \sin\frac{X}{2}\ \sin\frac{Y}{2}\ \sin\frac{Z}{2} \quad \Rightarrow \quad \frac{2\sqrt{2}}{\sqrt{3}} = 4.\ \frac{35}{4\sqrt{6}}\ .\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$

$\frac{4}{35} = \sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\frac{Z}{2} = \frac{S(S-z)}{xy} = \frac{9.2}{5.6} = \frac{3}{5}$

8*. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?

(A) $\angle QPR = 45^\circ$ **[JEE(Advanced) 2018, Paper-1,(4, –2)/60]**

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(D) The area of the circumcircle of the triangle PQR is 100π

एक त्रिभुज (triangle) PQR में, मानाकि $\angle PQR = 30^\circ$ और भुजाओं PQ और QR की लम्बाईयाँ क्रमशः $10\sqrt{3}$ और 10 है। तब निम्नलिखित में से कौनसा (से) कथन सत्य है (हैं) ?

(A) $\angle QPR = 45^\circ$

(B) त्रिभुज PQR का क्षेत्रफल (area) $25\sqrt{3}$ है और $\angle QRP = 120^\circ$

(C) त्रिभुज PQR के अंतर्वृत्त (incircle) की त्रिज्या (radius) $10\sqrt{3} - 15$ है।

(D) त्रिभुज PQR के परिवृत्त (circumcircle) का क्षेत्रफल 100π है।

Ans. **(BCD)**

Sol. $\cos Q = \frac{100 + 300 - (PR)^2}{2.10.10\sqrt{3}} \quad \Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{100 + 300 - (PR)^2}{2.10.10\sqrt{3}}$

$300 = 400 - (PR)^2 \quad \Rightarrow \quad PR = 10$

$\Delta = \frac{1}{2}(PQ)(QR)\sin Q = \frac{1}{2}10.10\sqrt{3} \times \frac{1}{2} = 25\sqrt{3}$

$r = \frac{\Delta}{s} = \frac{25\sqrt{3} \times 2}{(20 + 10\sqrt{3})} = \frac{50\sqrt{3}}{20 + 10\sqrt{3}} = \frac{5\sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = 5(2\sqrt{3} - 3) = 10\sqrt{3} - 15$

by sine rule ज्या नियम $\frac{10\sqrt{3}}{\sin R} = \frac{10}{\sin Q} \quad \Rightarrow \quad \angle R = 30^\circ$

2(circumradiusपरित्रिज्या) = $\frac{PR}{\sin Q} = \frac{10}{1/2} \quad \Rightarrow$ circumradiusपरित्रिज्या = 10

Hence area of circumcircle = $\pi R^2 = 100\pi$

अतः परिवृत्त का क्षेत्रफल = $\pi R^2 = 100\pi$

9. In a non-right-angled triangle ΔPQR, Let p, q, r denote the lengths of the sies opposite to the angles at P, Q, R respectively. The median form R meets the side PQ at S, the perpendicualr4 from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, q = 1, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

{Solution of Triangle [ST-RA]-T-305}

[JEE(Advanced) 2019, Paper-1,(4, –1)/62]

(A) Length of RS = $\frac{\sqrt{7}}{2}$ (B) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

(C) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2-\sqrt{3})$ (D) Length of OE = $\frac{1}{6}$

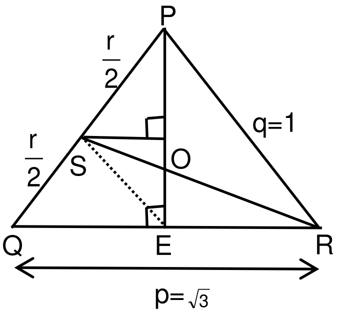
एक असमकोणीय त्रिभुज (non-right-angled) ΔPQR के लिए, माना कि p, q, r क्रमशः कोण P, Q, R के सामने वाली भुजाओं की लम्बाइयाँ दर्शायी है। R से खींची गयी माध्यिका (median) भुजा PQ से S पर मिलती है, P से खींचा गया अभिलम्ब (perpendicular) भुजा QR से E पर मिलता है तथा RS और PE एक दूसरे को O पर काटती है। यदि $p = \sqrt{3}$, q = 1 और ΔPQR के परिवृत्त (circumcircle) की त्रिज्या (radius) 1 है, तब निम्न में से कौन सा (से) विकल्प सही है (हैं) ?

(A) RS की लम्बाई = $\frac{\sqrt{7}}{2}$ (B) ΔSOE का क्षेत्रफल (area) = $\frac{\sqrt{3}}{12}$

(C) ΔPQR के अंतर्वृत (incircle) की त्रिज्या = $\frac{\sqrt{3}}{2}(2-\sqrt{3})$ (D) OE की लम्बाई = $\frac{1}{6}$

Ans. **(ACD)**

Sol.



$\frac{p}{\sin P} = \frac{q}{\sin Q} = 2(1) \quad \Rightarrow \quad \sin P = \frac{\sqrt{3}}{2}, \sin Q = \frac{1}{2}$

$\Rightarrow \quad \angle P = 60° \text{ or } 120°$ and और $\angle Q = 30° \text{ or } 150°$

because $\angle P + \angle Q$ must be less than 180° but not equal to 90°

क्योकि $\angle P + \angle Q$ का मान 180° से कम होगा परन्तु 90° के बराबर नहीं है।

$\angle P = 120°$ and और $\angle Q = 30°$ and और $\angle R = 30°$ $\quad \frac{r}{\sin R} = 2 \Rightarrow r = 1$

Now length of median माध्यिका की लम्बाई RS = $\frac{1}{2}\sqrt{2p^2+2q^2-r^2}$ = $\frac{1}{2}\sqrt{6+2-1}=\frac{\sqrt{7}}{2}$

$\Rightarrow$ option विकल्प (A) is correct सही है।

Inradius अंत:त्रिज्या = $\frac{2\Delta}{p+q+r}=\frac{\frac{2pqr}{4\times(1)}}{p+q+r}=\frac{1}{2}\left(\frac{1\times1\times\sqrt{3}}{1+1+\sqrt{3}}\right)$ = $\frac{\sqrt{3}}{2}\left(\frac{2-\sqrt{3}}{1}\right)$ $\Rightarrow$ option (C) is correct

$\Rightarrow$ $\frac{1}{2}\times\sqrt{3}\times PE=\frac{pqr}{4(1)}$ (equal area of Δ के क्षेत्रफल के बराबर)

$\Rightarrow$ PE = $\frac{1\times1\times\sqrt{3}}{4}\times\frac{2}{\sqrt{3}}=\frac{1}{2}$

$\Rightarrow$ OE = $\frac{2(\text{Area of }\Delta OQR)}{QR}$ = $\frac{2\times\frac{1}{3}\left(\frac{1}{2}.1.\sqrt{3}\sin 30^\circ\right)}{\sqrt{3}}=\frac{1}{6}$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

भाग - II : JEE (MAIN) / AIEEE (पिछले वर्षो) के प्रश्न

1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A **false** statement among the following is **[AIEEE - 2010 (4, –1), 144]**

(1) There is a regular polygon with $\frac{r}{R}=\frac{1}{\sqrt{2}}$. (2*) There is a regular polygon with $\frac{r}{R}=\frac{2}{3}$.

(3) There is a regular polygon with $\frac{r}{R}=\frac{\sqrt{3}}{2}$. (4) There is a regular polygon with $\frac{r}{R}=\frac{1}{2}$.

एक समबहुभुज के लिए, माना अन्त:वृत्त तथा परिवृत्त की त्रिज्याएँ क्रमशः r तथा R है। निम्नलिखित में से कौनसा प्रकथन मिथ्या है ?

(1) एक समबहुभुज ऐसा है जिसके लिए $\frac{r}{R}=\frac{1}{\sqrt{2}}$. (2*) एक समबहुभुज ऐसा है जिसके लिए $\frac{r}{R}=\frac{2}{3}$.

(3) एक समबहुभुज ऐसा है जिसके लिए $\frac{r}{R}=\frac{\sqrt{3}}{2}$. (4) एक समबहुभुज ऐसा है जिसके लिए $\frac{r}{R}=\frac{1}{2}$.

Ans. **(2)**

Sol. $\frac{r}{R}$ = cos $\left(\frac{\pi}{n}\right)$

Let cos $\frac{\pi}{n}$ = $\frac{2}{3}$ for some $n \geq 3$, $n \in N$

As $\frac{1}{2}<\frac{2}{3}<\frac{1}{\sqrt{2}}$ $\Rightarrow$ $\cos\frac{\pi}{3}<\cos\frac{\pi}{n}<\cos\frac{\pi}{n}$ $\Rightarrow$ $\frac{\pi}{3}>\frac{\pi}{n}>\frac{\pi}{n}$

$\Rightarrow$ $3 < n < 4$, which is not possible
so option (2) is the false statement
so it will be the right choice
Hence correct option is (2)

Hindi. $\frac{r}{R}$ = cos $\left(\frac{\pi}{n}\right)$

माना cos $\frac{\pi}{n}$ = $\frac{2}{3}$ किसी $n \geq 3$, $n \in N$ के लिए

चूँकि $\frac{1}{2} < \frac{2}{3} < \frac{1}{\sqrt{2}}$ ⇒ $\cos\frac{\pi}{3} < \cos\frac{\pi}{n} < \cos\frac{\pi}{n}$ ⇒ $\frac{\pi}{3} > \frac{\pi}{n} > \frac{\pi}{n}$

⇒ $3 < n < 4$, जो संभव नहीं है।

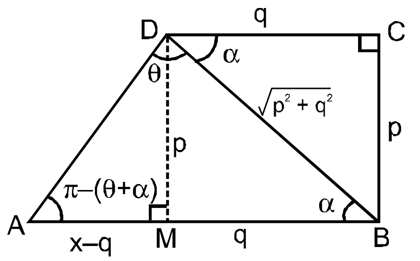
विकल्प (2), असत्य कथन है। अतः सही विकल्प (2) है।

2.✍ ABCD is a trapezium such that AB and CD are parallel and BC ⊥ CD. If $\angle ADB = \theta$, BC = p and CD = q, then AB is equal to : **[AIEEE - 2013, (4, –1),120]**

ABCD एक ऐसा समलंब चतुर्भुज है जिसमें AB तथा CD समांतर हैं तथा BC ⊥ CD है। यदि $\angle ADB = \theta$, BC = p तथा CD = q, है, तो AB बराबर है : **[AIEEE - 2013, (4, –¼),360]**

(1*) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (2) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$ (3) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (4) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

Sol. **(1)**



Let (माना) AB = x

$\tan(\pi-\theta-\alpha) = \frac{p}{x-q} \Rightarrow \tan(\theta+\alpha) = \frac{p}{q-x}$

⇒ $q - x = p\cot(\theta+\alpha)$

⇒ $x = q - p\cot(\theta+\alpha)$

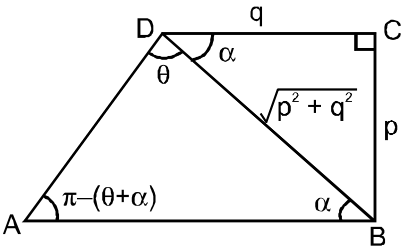
$= q - p\left(\frac{\cot\theta\ \cot\alpha - 1}{\cot\alpha + \cot\theta}\right)$

$= q - p\left(\frac{\frac{q}{p}\ \cot\theta\ -1}{\frac{q}{p}\ +\cot\theta}\right) = q - p\left(\frac{q\cot\theta\ -p}{q+p\cot\theta}\right) = q - p\left(\frac{q\cos\theta\ -p\sin\theta}{q\sin\theta+p\cos\theta}\right)$

⇒ $x = \frac{q^2\ \sin\theta + pq\cos\theta - pq\cos\theta + p^2\sin\theta}{p\cos\theta+q\sin\theta} \Rightarrow AB = \frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$.

Alternative

From Sine Rule ज्या नियम से



$$\frac{AB}{\sin\theta} = \frac{\sqrt{p^2+q^2}}{\sin(\pi-(\theta+\alpha))}$$

$$AB = \frac{\sqrt{p^2+q^2}\ \sin\theta}{\sin\theta\ \cos\alpha + \cos\theta\ \sin\alpha}$$

$$= \frac{(p^2+q^2)\ \sin\theta}{q\sin\ \theta + p\cos\theta} \qquad \left(\because \cos\alpha = \frac{q}{\sqrt{p^2+q^2}}\right)$$

$$= \frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}.$$

3. With the usual notation, in ΔABC, if $\angle A + \angle B = 120°$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is:

सामान्य संकेतों में ΔABC में यदि $\angle A + \angle B = 120°$, $a = \sqrt{3} + 1$ तथा $b = \sqrt{3} - 1$ है, तो अनुपात $\angle A : \angle B$ बराबर है:

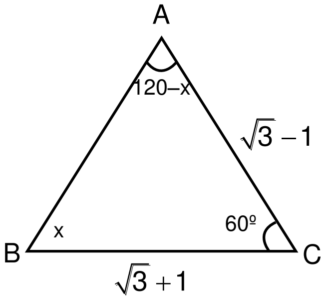
[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]

(1) 9 : 7 (2) 7 : 1 (3) 3 : 1 (4) 5 : 3

Ans. **(2)**

Sol. $\frac{\sqrt{3}+1}{\sin(120-x)} = \frac{\sqrt{3}-1}{\sin x}$

$\frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{\sin(120^0 - x)}{\sin x}$



$\frac{\sqrt{3}+1}{\sqrt{3}-1} - \frac{1}{2} = \frac{\sqrt{3}}{2}$ cotx

$\frac{3+2\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$ cotx

cotx = $\sqrt{3}$ + 2

tanx = $2 - \sqrt{3}$

x = 15º

120 – x = 105º

$\therefore \frac{\angle A}{\angle B} = \frac{7}{1}$ (7 : 1)

4. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is

एक त्रिभुज की दो भुजाओं की लम्बाई का योग x है और इन्ही दो भुजाओं की लम्बाई का गुणनफल y है। यदि $x^2 - c^2 = y$ जहाँ c त्रिभुज की तीसरी भुजा की लम्बाई है, तब त्रिभुज के परिवृत्त की त्रिज्या है –

[JEE(Main) 2019, Online (11-01-19),P-1 (4, – 1), 120]

(1) $\frac{c}{\sqrt{3}}$ (2) $\frac{3}{2}y$ (3) $\frac{c}{3}$ (4) $\frac{y}{\sqrt{3}}$

Ans. **(1)**

Sol. Let a, b, c be the three sides, given

त्रिभुज की भुजाएँ a, b, c

$a + b = x$, $ab = y$, $(a + b)^2 - c^2 = ab$

here यहाँ $\frac{a^2+b^2-c^2}{2ab} = -\frac{1}{2} \Rightarrow \cos C = -\frac{1}{2}$

$\frac{c}{\sin C} = 2R \Rightarrow \frac{2c}{\sqrt{3}} = 2R \Rightarrow R = \frac{c}{\sqrt{3}}$

HLP Answers

SUBJECTIVE QUESTIONS

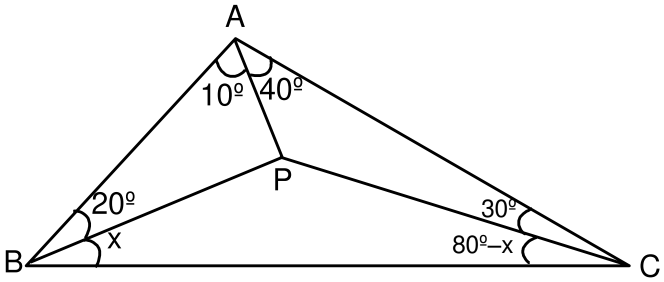
विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

This questions paste Staright Line sheets

1. In ΔABC , P is an interior point such that ∠PAB = 10º ∠PBA = 20º, ∠PCA = 30º, ∠PAC = 40º then prove that ΔABC is isosceles

ΔABC में P आन्तरिक बिन्दु P इस प्रकार है कि ∠PAB = 10º ∠PBA = 20º, ∠PCA = 30º, ∠PAC = 40º तब सिद्ध कीजिए कि ΔABC समद्विबाहु है।

Sol.



From ΔAPB, ΔPBC and ΔPCA , using sine rule

$$\frac{AP}{\sin 20} = \frac{BP}{\sin 10}$$

$$\frac{BP}{\sin(80-x)} = \frac{PC}{\sin x}$$

$$\frac{PC}{\sin 40} = \frac{AP}{\sin 30}$$

$$\frac{AP}{\sin\angle ABP}\frac{BP}{\sin\angle PCB}.\frac{CP}{\angle PAC} = \frac{AP}{\sin\angle PCA}\frac{BP}{\sin\angle PBC}.\frac{CP}{\sin\angle PAB}$$

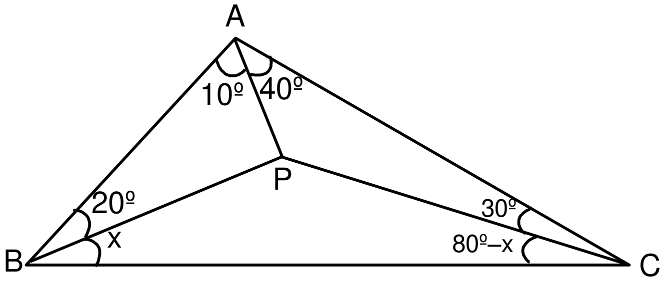
$\Rightarrow$ sin30º . sinx.sin10º = sin20º.sin(80º – x) sin40º

$\Rightarrow$ x = 60º

$\therefore$ ∠BCA = ∠CAB = 50º

So, ΔABC is an isosceles triangle.

Sol.



ΔAPB, ΔPBC और ΔPCA , ज्या नियम से

$$\frac{AP}{\sin 20} = \frac{BP}{\sin 10}$$

$$\frac{BP}{\sin(80-x)} = \frac{PC}{\sin x}$$

$$\frac{PC}{\sin 40} = \frac{AP}{\sin 30}$$

$$\frac{AP}{\sin\angle ABP}\frac{BP}{\sin\angle PCB}.\frac{CP}{\angle PAC} = \frac{AP}{\sin\angle PCA}\frac{BP}{\sin\angle PBC}.\frac{CP}{\sin\angle PAB}$$

$\Rightarrow \sin30^{\circ} . \sin x.\sin10^{\circ} = \sin20^{\circ}.\sin(80^{\circ} - x) \sin40^{\circ}$

$\Rightarrow x = 60^{\circ}$

$\therefore \angle BCA = \angle CAB = 50^{\circ}$

इसलिए ΔABC समद्विबाहु त्रिभुज है।

2. In a triangle ABC, if $a \tan A + b \tan B = (a + b) \tan\left(\frac{A+B}{2}\right)$, prove that triangle is isosceles.

त्रिभुज ABC में, यदि $a \tan A + b \tan B = (a + b) \tan\left(\frac{A+B}{2}\right)$ हो, तो सिद्ध कीजिए कि त्रिभुज समद्विबाहु है।

Sol. $\because\ a \tan A + b \tan B = (a + b) \tan\left(\frac{A+B}{2}\right)$

$$\Rightarrow a\left[\tan A - \tan\ \left(\frac{A+B}{2}\right)\right] = b\left[\tan\ \left(\frac{A+B}{2}\right) - \tan B\right]$$

$$\Rightarrow a\left[\frac{\sin A\cos\left(\frac{A+B}{2}\right) - \sin\left(\frac{A+B}{2}\right)\cos A}{\cos A\cos\ \left(\frac{A+B}{2}\right)}\right] = b\,\frac{\sin\left(\frac{A+B}{2} - B\right)}{\cos B\cos\left(\frac{A+B}{2}\right)}$$

$$\Rightarrow \frac{a\quad \sin\left(A - \frac{A+B}{2}\right)}{\cos A\cos\left(\frac{A+B}{2}\right)} = \frac{b\quad \sin\left(\frac{A-B}{2}\right)}{\cos B\cos\left(\frac{A+B}{2}\right)}$$

$$\Rightarrow \sin\left(\frac{A-B}{2}\right)\left(\frac{a}{\cos A} - \frac{b}{\cos B}\right) = 0$$

$\Rightarrow \sin\left(\frac{A-B}{2}\right) = 0$ or या $\frac{a}{\cos A} - \frac{b}{\cos B} = 0$

$\Rightarrow A = B$ or या $2R (\tan A - \tan B) = 0$

$\Rightarrow \tan A = \tan B$

$\Rightarrow A = B$

3. In any triangle ABC, if $2\Delta a - b^2c = c^3$, (where Δ is is the area of triangle), then prove that $\angle A$ is obtuse

किसी त्रिभुज ABC में यदि $2\Delta a - b^2c = c^3$, (जहाँ Δ त्रिभुज का. क्षेत्रफल है) तब सिद्ध कीजिए $\angle A$ अधिक कोण है।

Sol. $2\Delta a - b^2c = c^3$

$\Rightarrow 2\Delta a^2b = abc (b^2+ c^2)$

$$\Rightarrow \frac{(a^2b)abc}{2R} = abc(b^2 + c^2)$$

$$\Rightarrow \frac{a^2b}{2R} = b^2 + c^2$$

$\Rightarrow a^2 \sin B = b^2 + c^2$

If $\sin B = 1$, then $a^2 = b^2 + c^2$, which is not possible

$\therefore \sin B \neq 1$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 \sin B - a^2}{2bc}$$

< 0

$\therefore$ A is obtuse

Sol. $2\Delta a - b^2c = c^3$

$\Rightarrow 2\Delta a^2b = abc\ (b^2 + c^2)$

$\Rightarrow \frac{(a^2b)abc}{2R} = abc(b^2 + c^2)$

$\Rightarrow \frac{a^2b}{2R} = b^2 + c^2$

$\Rightarrow a^2 \sin B = b^2 + c^2$

यदि $\sin B = 1$, तब $a^2 = b^2 + c^2$, जो कि संभव नहीं है।

$\therefore \sin B \neq 1$

$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 \sin B - a^2}{2bc}$

< 0

$\therefore$ A अधिक कोण है।

4. If in a triangle ABC, $\frac{\cos A\ +\ 2\cos C}{\cos A\ +\ 2\cos B} = \frac{\sin B}{\sin C}$ prove that the triangle ABC is either isosceles or right angled.

यदि त्रिभुज ABC में $\frac{\cos A\ +\ 2\cos C}{\cos A\ +\ 2\cos B} = \frac{\sin B}{\sin C}$ हो, तो सिद्ध कीजिए कि त्रिभुज ABC या तो समद्विबाहु त्रिभुज है या समकोण त्रिभुज है।

Sol. $\cos A(\sin B - \sin C) + (\sin 2B - \sin 2C) = 0$

$\Rightarrow \cos A.(\sin B - \sin C) + 2\cos(B + C)\sin(B - C) = 0$ $\quad \because\ B + C = \pi - A$

$\Rightarrow \cos A.(\sin B - \sin C) - 2\cos A.\sin(B - C) = 0$

$\Rightarrow \cos A[(\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C)] = 0$

$\Rightarrow$ either या तो $\cos A = 0 \Rightarrow A = 90°$ $\Rightarrow$ right angled समकोण त्रिभुज होगा

or या $(\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C) = 0$

$\Rightarrow (b - c) - 2\left(b.\frac{a^2 + b^2 - c^2}{2ab} - c.\frac{a^2 + c^2 - b^2}{2ac}\right) = 0$

$\Rightarrow a(b - c) - 2(b^2 - c^2) = 0$

$(b - c)\ [a - 2(b + c)] = 0$

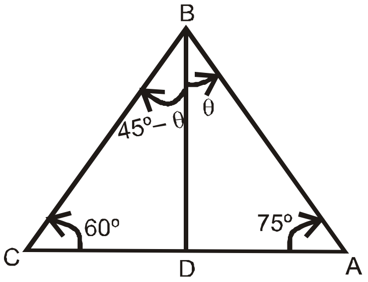
$\therefore\ b - c = 0 \Rightarrow b = c$ $\Rightarrow$ isosceles अतः त्रिभुज समद्विबाहु त्रिभुज होगा।

5. In a Δ ABC, $\angle$ C = 60° and $\angle$ A = 75°. If D is a point on AC such that the area of the Δ BAD is $\sqrt{3}$ times the area of the Δ BCD, find the $\angle$ ABD.

त्रिभुज ABC में $\angle C = 60°$ एवं $\angle A = 75°$ है। यदि भुजा AC पर एक बिन्दु D इस प्रकार है कि ΔBAD का क्षेत्रफल, त्रिभुज BCD के क्षेत्रफल का $\sqrt{3}$ गुना है। $\angle$ABD का मान ज्ञात कीजिए।

Ans. $\angle$ ABD = 30°

Sol.



Area of ΔBAD = $\sqrt{3}$ × Area of ΔBCD $\qquad$ ΔBAD का क्षेत्रफल = $\sqrt{3}$ × ΔBCD का क्षेत्रफल

$\Rightarrow \quad \frac{1}{2} BD \times BA \sin\theta = \sqrt{3} \times \frac{1}{2} BC \times BD \sin(45^{\circ} - \theta)$

$$\frac{BA}{BC} = \sqrt{3}\ \frac{\sin(45^\circ-\theta)}{\sin\theta} \qquad \text{.............(1)}$$

From Sine-Rule ज्या नियम का प्रयोग करने पर

$$\frac{BC}{\sin 75^\circ} = \frac{AB}{\sin 60^\circ}$$

$$\therefore \quad \frac{BA}{BC} = \frac{\sin 60^\circ}{\sin 75^\circ} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{3}+1}$$

$\therefore$ From equation (1) समीकरण (1) से

$$\frac{\sqrt{3}.\sqrt{2}}{(\sqrt{3}+1)} = \sqrt{3}\left[\frac{1}{\sqrt{2}}\cot\theta - \frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow \quad \frac{2}{(\sqrt{3}+1)} = \cot\theta - 1 \quad \Rightarrow \quad \frac{2\left(\sqrt{3}-1\right)}{2} = \cot\theta - 1$$

$$\Rightarrow \quad \cot\theta = \sqrt{3} \quad \Rightarrow \quad \theta = 30^\circ \quad \Rightarrow \angle ABD = 30^\circ$$

6. In a ΔABC, if a, b and c are in A.P., prove that $\cos A.\cot\frac{A}{2}$, $\cos B.\cot\frac{B}{2}$, and $\cos C.\cot\frac{C}{2}$ are in A.P.

यदि किसी त्रिभुज ABC में a, b एवं c समान्तर श्रेढ़ी में हो, तो सिद्ध कीजिए कि $\cos A\frac{A}{2}$. cot , $\cos B.\cot\frac{B}{2}$ एवं $\cos C.\cot\frac{C}{2}$ समान्तर श्रेढ़ी में है।

Sol. $\because \cos A \cot\frac{A}{2} = \left(1-2\sin^2\frac{A}{2}\right)\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} = \cot\frac{A}{2} - \sin A$

Similarly $\cos B \cot\frac{B}{2} = \cot\frac{B}{2} - \sin B$

and $\cos C \cot\frac{C}{2} = \cot\frac{C}{2} - \sin C$

$\because$ a, b, c are in A.P.

$\therefore$ sin A, sin B, sin C are also in A.P.

$\because$ a, b, c are in A.P.

$\therefore$ $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are also in A.P.

$\therefore$ $\cot\frac{A}{2} - \sin A, \cot\frac{B}{2} - \sin B, \cot\frac{C}{2} - \sin C$ are also in A.P.

Hindi. $\because \cos A \cot\frac{A}{2} = \left(1-2\sin^2\frac{A}{2}\right)\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} = \cot\frac{A}{2} - \sin A$

इसी प्रकार $\cos B \cot\frac{B}{2} = \cot\frac{B}{2} - \sin B$

और $\cos C \cot\frac{C}{2} = \cot\frac{C}{2} - \sin C$

$\because$ a, b, c समान्तर श्रेढी में है।

$\therefore$ sin A, sin B, sin C भी समान्तर श्रेढ़ी में होगें।

$\because$ a, b, c समान्तर श्रेढ़ी में है।

$\therefore\ \cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ भी समान्तर श्रेढ़ी में होगें।

$\therefore\ \cot\frac{A}{2} - \sin A, \cot\frac{B}{2} - \sin B, \cot\frac{C}{2} - \sin C$ भी समान्तर श्रेढ़ी में होगें।

7. In a triangle ABC, prove that the area of the incircle is to the area of triangle itself is, $\pi : \cot\left(\frac{A}{2}\right) . \cot\left(\frac{B}{2}\right) . \cot\left(\frac{C}{2}\right)$.

त्रिभुज ABC में सिद्ध कीजिए कि अन्तःवृत्त के क्षेत्रफल और स्वयं इस त्रिभुज के क्षेत्रफल का अनुपात $\pi : \cot\left(\frac{A}{2}\right) . \cot\left(\frac{B}{2}\right) . \cot\left(\frac{C}{2}\right)$ है।

Sol. $\because\ \dfrac{\text{Area of incircle}}{\text{Area of } \Delta ABC}\ \dfrac{\text{अन्तः वृत्त का क्षे०}}{\Delta ABC \text{ का क्षे०}} = \dfrac{\pi r^2}{\frac{1}{2}bc \quad \sin A}$

$$= \frac{\pi \times 16R^2 \times \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\frac{1}{2}(2R\sin B)\quad(2R\sin C)\quad\left(2\sin\frac{A}{2}\quad\cos\frac{A}{2}\right)} = \frac{4\pi\sin\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\quad\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)\quad\cos\frac{A}{2}}$$

$$= \frac{\pi\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{\pi}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}} = \pi : \cot\frac{A}{2}\ \cot\frac{B}{2}\ \cot\frac{C}{2}$$

8. In ΔABC, prove that $a^2 (s - a) + b^2 (s - b) + c^2 (s - c) = 4R\Delta\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

ΔABC, में सिद्ध कीजिए $a^2 (s - a) + b^2 (s - b) + c^2 (s - c) = 4R\Delta\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

Sol. L.H.S. $= \frac{1}{2}[a^2(b + c - a) + b^2 (c + a - b) + c^2 (a + b - c)]$

$= \frac{1}{2}[a(b^2 + c^2 - a^2) + b(c^2 + a^2 - b^2) + c (a^2 + b^2 - c^2)]$

$= \frac{1}{2}(2abc \cos A + 2abc \cos B + 2abc \cos C)$

$= abc\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

$= 4R\Delta\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

9. In any ΔABC, prove that
त्रिभुज ABC में सिद्ध कीजिए कि –

(i) $(r_3 + r_1)\ (r_3 + r_2) \sin C = 2\ r_3\ \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$

(ii) $\dfrac{\tan\frac{A}{2}}{(a - b)\ (a - c)} + \dfrac{\tan\frac{B}{2}}{(b - a)\ (b - c)} + \dfrac{\tan\frac{C}{2}}{(c - a)\ (c - b)} = \dfrac{1}{\Delta}$

(iii) $(r + r_1) \tan\dfrac{B - C}{2} + (r + r_2) \tan\dfrac{C - A}{2} + (r + r_3) \tan\dfrac{A - B}{2} = 0$

(iv) $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.

Sol. (i) L.H.S. बायां पक्ष $= (r_3 + r_1)(r_3 + r_2) \sin C$

$$= \frac{\Delta b}{(s-a)(s-c)} \; \frac{\Delta a}{(s-c)(s-b)} \sin C$$

$$= \frac{ab\Delta^2}{(s-a)(s-b)(s-c)(s-c)} \sin C$$

$$= \frac{ab \;\; s(s-a)(s-b)(s-c)}{(s-a)(s-b)(s-c)(s-c)} \sin C$$

$$= \frac{abs \sin C}{(s-c)}$$

$$= \frac{2\Delta.s}{(s-c)} = 2\, sr_3 \;\sqrt{r_2r_3 + r_3r_1 + r_1r_2}$$

R.H.S. दायां पक्ष $= 2r_3$

$$= 2r_3 \sqrt{s^2} = 2sr_3$$

$\therefore$ L.H.S. बायां पक्ष = R.H.S. दायां पक्ष

(ii) L.H.S. बायां पक्ष $= -\frac{1}{\Delta}\left[\frac{(s-b)(s-c)}{(a-b)(c-a)} + \frac{(s-a)(s-c)}{(a-b)(b-c)} + \frac{(s-a)(s-b)}{(c-a)(b-c)}\right]$

$$= -\frac{1}{\Delta}\left[\frac{(s-b)(s-c)(b-c) + (s-a)(s-c)(c-a) + (s-a)(s-b)(a-b)}{(a-b)(b-c)(c-a)}\right]$$

$= \frac{1}{\Delta}$ = R.H.S. दायां पक्ष

(iii) First term (प्रथम पद) $= (r + r_1) \tan\frac{B-C}{2}$

$$= \left(\frac{\Delta}{s} + \frac{\Delta}{s-a}\right)\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$$

$$= \frac{\Delta(2s-a)}{s(s-a)} \cdot \left(\frac{b-c}{b+c}\right) \cdot \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$= b - c$

similarly second term (इसी प्रकार द्वितीय पद)= $c - a$

& third term (और तृतीय पद) = $a - b$

$\therefore$ L.H.S. बायां पक्ष = $b - c + c - a + a - b = 0$ = R.H.S. दायां पक्ष

(iv) $\because\; r_1 + r_2 + r_3 - r = 4R$

$\therefore$ $(r_1 + r_2 + r_3 - r)^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2(r_1r_2 + r_2r_3 + r_3 r_1)$ ....(i)

$\because$ $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$

and (और) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

$\therefore$ from equation (i) समीकरण (i) से

$16R^2 = r^2 + r_1^2 + r_2^2 + r_3^2 - 2(ab + bc + ca - s^2) + 2s^2$

$\therefore$ $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - 4s^2 + 2(ab + bc + ca)$

$= 16R^2 - (a + b + c)^2 + 2(ab + bc + ca)$

$= 16R^2 - a^2 - b^2 - c^2$

10. In an acute angled triangle ABC, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then prove that $b + 3c < 3a < 3b + 3c$

न्यूनकोण त्रिभुज ABC में $r + r_1 = r_2 + r_3$ और $\angle B > \frac{\pi}{3}$, तब सिद्ध कीजिए $b + 3c < 3a < 3b + 3c$

Sol. $r - r_2 = r_3 - r_1 \Rightarrow \frac{\Delta}{s} - \frac{\Delta}{s-b} = \frac{\Delta}{s-c} - \frac{\Delta}{s-a}$

or या $\frac{-b}{s(s-b)} = \frac{c-a}{(s-c)(s-a)}$

or या $\frac{(s-a)(s-c)}{s(s-b)} = \frac{a-c}{b} \Rightarrow \tan^2\frac{B}{2} = \frac{a-c}{b}$

But परन्तु $\frac{B}{2} \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$. Therefore इसलिए,

$\tan^2 \frac{B}{2} \in \left(\frac{1}{3}, 1\right) \Rightarrow \frac{1}{3} < \frac{a-c}{b} < 1$

or या $b < 3a - 3c < 3b$

$b + 3c < 3a < 3b + 3c$

11. If the inradius in a right angled triangle with integer sides is r. Prove that

(i) If r = 4, the greatest perimeter (in units) is 90

(ii) If r = 5, the greatest area (in sq. units) is 330

यदि समकोण त्रिभुज जिसकी भुजाऐं पूर्णांक है अन्तः त्रिज्या r है तब सिद्ध कीजिए कि

(i) यदि r = 4, तब अधिकतम परिमाप 90 (ईकाई में) है।

(ii) यदि r = 5 हो तो अधिकतम क्षेत्रफल (ईकाई वर्ग में) 330 है।

Sol. (i and ii) Let a, b and c $(a < b < c)$ be the sides of given triangle.

Also, $2r = a + b - c$

When r = 4 then, (a, b) = (9, 40), (10, 24), (12,16)

$\therefore$ Greatest perimeter = 9 + 40 + 41 = 90 units

when r = 5 then (a, b) = (11, 60) (12,35) (15,20)

$\therefore$ Greatest area = $\frac{11 \times 60}{2}$ = 330 sq. unit

Sol. (i और ii) माना a, b और c $(a < b < c)$ त्रिभुज की भुजाएं

तथा, $2r = a + b - c$

जब r = 4 तब, (a, b) = (9, 40), (10, 24), (12,16)

$\therefore$ अधिकतम परिमाप = 9 + 40 + 41 = 90 units

जहा r = 5 तब (a, b) = (11, 60) (12,35) (15,20)

$\therefore$ अधिकतम क्षेत्रफल = $\frac{11 \times 60}{2}$ = 330 वर्ग इकाई

12. If $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, then prove that the triangle is right angled.

यदि $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$ हो, तो सिद्ध कीजिए कि त्रिभुज, समकोण त्रिभुज है।

Sol. $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2 \quad \Rightarrow \quad \left(1 - \frac{s-b}{s-a}\right)\left(1 - \frac{s-c}{s-a}\right) = 2$

$\Rightarrow \quad \frac{(b-a)\ \ (c-a)}{(s-a)^2} = 2 \quad \Rightarrow \quad 2(bc - ab - ac + a^2) = (2s - 2a)^2$

$\Rightarrow \quad 2bc - 2ab - 2ca + 2a^2 = (b^2 + c^2 + a^2 - 2ab + 2bc - 2ca)$

$a^2 = b^2 + c^2$

$\Rightarrow$ triangle is right angled. त्रिभुज, समकोण त्रिभुज होगा।

13. DEF is the triangle formed by joning the points of contact of the incircle with the sides of the triangle ABC; prove that

(i) its sides are $2r\cos\frac{A}{2}$, $2r\cos\frac{B}{2}$ and $2r\cos\frac{C}{2}$,

(ii) its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$

and

(iii) its area is $\frac{2\Delta^3}{(abc)s}$, i.e. $\frac{1}{2}\frac{r}{R}\Delta$.

त्रिभुज ABC की भुजाओं को अन्तःवृत्त जिन बिन्दुओं पर स्पर्श करता है उनको मिलाने से त्रिभुज DEF निर्मित होता है। सिद्ध कीजिए कि –

(i) इसकी भुजाएँ $2r\cos\frac{A}{2}$, $2r\cos\frac{B}{2}$ and $2r\cos\frac{C}{2}$ हैं।

(ii) इसके कोण $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$ हैं।

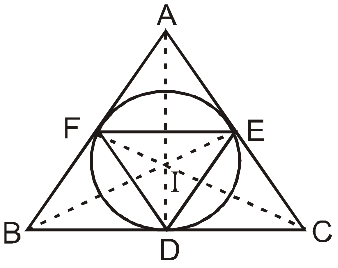
और

(iii) इसका क्षेत्रफल $\frac{2\Delta^3}{(abc)s}$ अर्थात् $\frac{1}{2}\frac{r}{R}\Delta$ है।

Sol. (i) EIFA is a cyclic quadrilateral

$\therefore$ $\frac{EF}{\sin A} = AI$

$\because$ AI = r cosec A/2



$\therefore$ EF = r cosec A/2.sin A
= 2 r cos A/2

similarly DF = 2 r cos B/2

and DE = 2r cos C/2.

(ii) IECD is a cyclic quadrilateral

$\therefore$ $\angle ICE = \angle IDE = \frac{C}{2}$

similarly $\angle IDF = \angle IBF = \frac{B}{2}$

$\therefore$ $\angle FDE = \frac{B}{2}+\frac{C}{2} = \frac{\pi-A}{2}$

$= \frac{\pi}{2}-\frac{A}{2}$

(iii) area of $\Delta DEF = \frac{1}{2}$ FD . DE sin $\angle FDE$

$= \frac{1}{2}\left(2r \cos \frac{B}{2}\right)\left(2r \cos \frac{C}{2}\right)\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)$

$$= 2r^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$= 2r^2\left(\frac{\sin A + \sin B + \sin C}{4}\right)$$

$$= \frac{r^2}{2}\left(\frac{2\ \ \Delta}{bc} + \frac{2\Delta}{ca} + \frac{2\Delta}{ab}\right)$$

$$= \frac{r^2}{2}\left[\frac{2\Delta\ \ (a+b+c)}{abc}\right] = \frac{r^2\Delta\ \ .\ \ 2s}{abc}$$

$$= \frac{2\ \ r^2.\ \ \Delta s^2}{(abc)s} = \frac{2\Delta\ \ (r\ \ s)^2}{(abc)s}$$

$$= \ \frac{2\Delta^3}{(abc)s} = \frac{1}{2}\ \frac{r\Delta}{R}.$$

Hindi (i) EIFA एक चक्रीय चतुर्भुज है

$\therefore$ $\frac{EF}{\sin A} = A\,I$

$\because$ AI = r cosec A/2

$\therefore$ EF = r cosec A/2 sin A

= 2 r cos A/2

इसी प्रकार, DF = 2 r cos B/2

और DE = 2r cos C/2.

(ii) IECD एक चक्रीय चतुर्भुज है

$\therefore$ $\angle ICE = \angle IDE = \frac{C}{2}$

इसी प्रकार $\angle IDF = \angle IBF = \frac{B}{2}$

$\therefore$ $\angle FDE = \frac{B}{2} + \frac{C}{2} = \frac{\pi - A}{2}$

$$= \frac{\pi}{2} - \frac{A}{2}$$

(iii) ΔDEF का क्षेत्रफल = $\frac{1}{2}$ FD . DE sin $\angle$FDE

$$= \frac{1}{2}\left(2r\ \ \cos\ \ \frac{B}{2}\right)\left(2r\ \ \cos\ \ \frac{C}{2}\right)\sin\ \ \left(\frac{\pi}{2} - \frac{A}{2}\right)$$

$$= 2r^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$= 2r^2\left(\frac{\sin A + \sin B + \sin C}{4}\right)$$

$$= \frac{r^2}{2}\left(\frac{2\ \ \Delta}{bc} + \frac{2\Delta}{ca} + \frac{2\Delta}{ab}\right)$$

$$= \frac{r^2}{2}\left[\frac{2\Delta\ \ (a+b+c)}{abc}\right] = \frac{r^2\Delta\ \ .\ \ 2s}{abc}$$

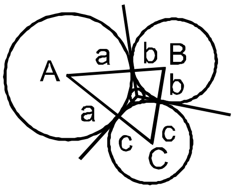
$$= \frac{2\ \ r^2.\ \ \Delta s^2}{(abc)s} = \frac{2\Delta\ \ (r\ \ s)^2}{(abc)s}$$

$$= \ \frac{2\Delta^3}{(abc)s} = \frac{1}{2}\ \frac{r\Delta}{R}.$$

14. Three circles, whose radii are a, b and c, touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is $\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$.

तीन वृत्त जिनकी त्रिज्याएँ a, b, c है, एक दूसरे को बाह्य स्पर्श करते है तथा उनके स्पर्श बिन्दुओं पर स्पर्श रेखाएँ एक बिन्दु पर मिलती है। सिद्ध कीजिए कि इस बिन्दु की उनके किसी स्पर्श बिन्दु से दूरी $\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$ हैं।

Sol.



required distance = inradius of Δ ABC   अभीष्ट दूरी = Δ ABC की अन्तःत्रिज्या

$\because\ 2s = a + b + b + c + c + a$

$= 2\,(a + b + c)$

$s = a + b + c$

$\Delta = \sqrt{s(s-(a+b))(s-(b+c))\quad(s-(c+a))} = \sqrt{(a+b+c)(abc)}$

$\therefore$ required distance अभीष्ट दूरी

$$= \frac{\Delta}{s} = \frac{\sqrt{(a+b+c)(abc)}}{(a+b+c)} = \sqrt{\frac{abc}{a+b+c}} = \left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$$

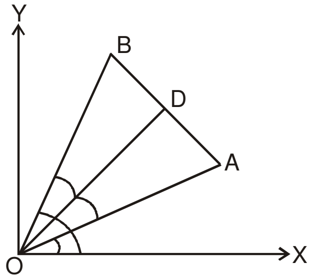
15. OA and OB are the equal sides of an isoscles triangle lying in the first quadrant making angles θ and φ respectively with x-axis. Show that the gradient of the bisector of acute angle AOB is cosec β – cot β where β = φ + θ. (Where O is origin)

OA और OB समद्विबाहु त्रिभुज की दो बराबर भुजाएं है जो प्रथम चतुर्थांश में क्रमशः x के साथ θ और φ कोण बनाती है। दर्शाइये कि न्यूनकोण AOB के अर्धक की प्रवणता cosec β – cot β है जहाँ β = φ + θ. (जहाँ O मूल बिन्दु है)

Sol. From the fig.

$\angle AOD = \alpha = \angle DOB$ $\phi = \theta = 2\alpha$

or, $a = \frac{\phi - \theta}{2}$ or, $\angle DOX = \theta + \alpha = \frac{\theta + \phi}{2}$



The gradient of OD = $\tan\frac{\theta + \phi}{2} = \frac{\sin\frac{\theta + \phi}{2}}{\cos\frac{\theta + \phi}{2}}$

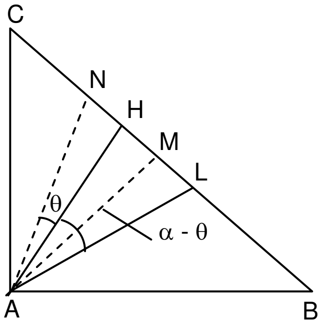
$$= \frac{2\sin^2\frac{\theta + \phi}{2}}{2\cos\frac{\theta + \phi}{2}\sin\frac{\theta + \phi}{2}} = \frac{1+\cos(\theta + \phi)}{\sin(\theta + \phi)} = \text{cosec B} - \text{cot B}.$$

16. The hypotenuse BC = a of a right-angled triangle ABC is divided into n equal segments where n is odd. The segment containing the midpoint of BC subtends angle α at A. Also h is the altitude of the triangle through A. Prove that $\tan\alpha = \dfrac{4nh}{a\left(n^2-1\right)}$.

समकोण त्रिभुज ABC के कर्ण BC = a को n बराबर खण्डों में विभाजित किया जाता है जहाँ n विषम है। BC के मध्य बिन्दु को रखने वाला खण्ड, A पर कोण α बनाती है तथा h, A से जाने वाला शीर्षलम्ब है तब सिद्ध कीजिए $\tan\alpha = \dfrac{4nh}{a\left(n^2-1\right)}$

Sol. Let LN be the segment of the side BC containing its midpoint M. We have $BM = MC = AM = \dfrac{a}{2}$. Let AH be the altitude from A on BC, with AH = h. Also $\angle LAN = \alpha$. Let $\angle NAH = \theta \Rightarrow \angle HAL = \alpha - \theta$.

From ΔAMH, we have $MH = \sqrt{\dfrac{a^2}{4} - h^2}$. Also $LM = MN = \dfrac{a}{2n}$



Now $\tan\alpha = \tan\left(\alpha - \theta + \theta\right) = \dfrac{\tan\left(\alpha-\theta\right) + \tan\theta}{1 - \tan\theta\tan\left(\alpha-\theta\right)} = \dfrac{\dfrac{LH}{h} + \dfrac{NH}{h}}{1 - \dfrac{LH}{h}\cdot\dfrac{NH}{h}}$

$$= \frac{h\left(LH + NH\right)}{h^2 - LH \cdot NH} = \frac{h \cdot a}{n\left[h^2 - \left(\dfrac{a}{2n} + \sqrt{\dfrac{a^2}{4} - h^2}\right)\left(\dfrac{a}{2n} - \sqrt{\dfrac{a^2}{4} - h^2}\right)\right]}$$

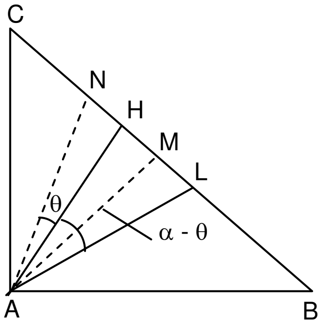
$$= \frac{ha}{n\left[h^2 - \dfrac{a^2}{4n^2} + \dfrac{a^2}{4} - h^2\right]} = \frac{4nh}{a\left(n^2-1\right)}.$$

Hindi माना LN भुजा BC का खण्ड है जो इसके मध्य बिन्दु M को रखता है।

यहाँ $BM = MC = AM = \dfrac{a}{2}$.

माना AH, A से BC पर शीर्षलम्ब, AH = h. तथा $\angle LAN = \alpha$. माना $\angle NAH = \theta \Rightarrow \angle HAL = \alpha - \theta$.

ΔAMH से यहाँ $MH = \sqrt{\dfrac{a^2}{4} - h^2}$. इसलिए $LM = MN = \dfrac{a}{2n}$



अब $\tan\alpha = \tan(\alpha-\theta+\theta) = \dfrac{\tan(\alpha-\theta)+\tan\theta}{1-\tan\theta\tan(\alpha-\theta)} = \dfrac{\dfrac{LH}{h}+\dfrac{NH}{h}}{1-\dfrac{LH}{h}\cdot\dfrac{NH}{h}}$

$$= \frac{h(LH+NH)}{h^2-LH\cdot NH} = \frac{h\cdot a}{n\left[h^2-\left(\dfrac{a}{2n}+\sqrt{\dfrac{a^2}{4}-h^2}\right)\left(\dfrac{a}{2n}-\sqrt{\dfrac{a^2}{4}-h^2}\right)\right]}$$

$$= \frac{ha}{n\left[h^2-\dfrac{a^2}{4n^2}+\dfrac{a^2}{4}-h^2\right]} = \frac{4nh}{a(n^2-1)}.$$