Chapter - 2

Gauss' Law and Its Applications

In previous chapter we have studied about point charge and system of point charges at rest, and concept of electric field. We have also seen, how the principle of superposition is of help in calculating the electric field due to a system of discrete charges. In this chapter our aim is to determine electric field due to a continuous charge distrbution . For such cases, concept of charge density is utilized along with Couloumb's law. What we have to do is to divide the charge distribution into infintesimal elements of charges which may be considered to be a point charge. Electric field due to such an element can then be calculated using Coulomb's law. Accoding to principle of super position the total field is the sum (integral) of all such contributions over the charge distribution.

In principle this method is possible for any continuous charge distribution, however in many cases either it is cumbersome to perform the integration or impossible to solve it exaclty. In situations related to continuous charge distributions where charge distribution is uniform the Gauss's law is very helpful which makes the determination of electric field mathematically very simple. To work with Gauss's law it is essential for us to understand the concept of electric flux, so we start this chapter with the study of electric flux.

2.1 Electric Flux

Electric flux through a flat surface of area S lying in a uniform electric field E is defined by

$$\phi_E = Es\cos\theta \qquad \ldots (2.1)$$

Where θ is the angle between $\vec{E}$ and normal to the surface (Fig 2.1) $\vec{S}$ is area vector

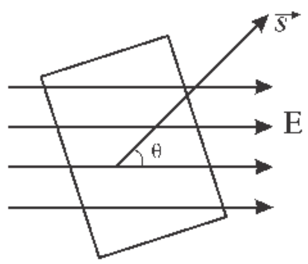


Fig 2.1 : Electric through a flat surface

The area of a flat surface can be represented by vector $\vec{S}$ (called area vector) whose magnitude is equal to the area S and whose direction is normal to the plane of the area. Accordingly we can write

$$\phi_E = \vec{E}\cdot\vec{s} \qquad \ldots (2.2)$$

Thus, electric flux is a scalar quanity value of which depends on electric field, area of surface under consideration and the angle between area vector and the electric field. The electric flux ϕ through a surface is proportional to the net number of electric field lines passing through that surface. ϕ_E is positive (when 90°> θ> 0°), negative 180° > θ> 90° and zero (when θ = 90°). When electric field lines are coming out from the area flux is regarded positive while field lines entering the area corresponds to negative flux. When field lines are parallel to the flat area the flux is considered to be zero.

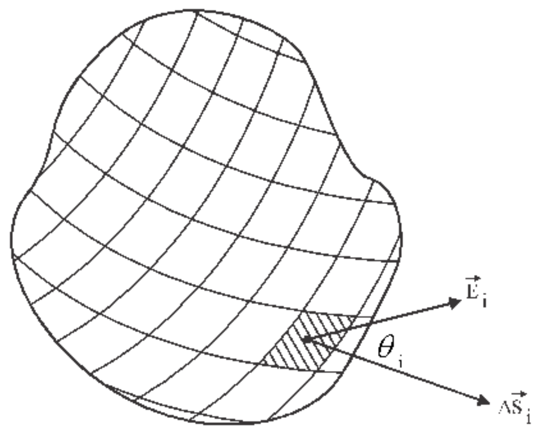


Fig 2.2 : Flux through a closed surface

In a general case where $\vec{E}$ is non uniform and surface is not flat, to calculate electric flux we consider the surface to be divided into a large number 'n' of small area elements Δs_1, $\Delta s_2 \ldots \Delta s_n$ (fig 2.2) where each area element Δs_i is small enough so that

(i) It can be assumed to be flat (planer)

(ii) The variation of electric field over this area element is so small that electric field E_i can be regarded as constant, then using the equation (2.2) the flux linked with such an area element is

$$\Delta\phi_{E_i} = \vec{E}_i \cdot \Delta\vec{s}_i \qquad \ldots(2.3)$$

Summing the contributions of all elements gives an approximation to the total flux through the surface.

$$\phi_E \simeq \sum \vec{E}_i \cdot \Delta\vec{s}_i$$

If the area of each element approaches zero the number of elements approaches infinity and the sum $\sum$ is replaced by an intergral. Therefore, the general expression of electric flux is

$$\phi_E = \int \vec{E} \cdot d\vec{s} \qquad \ldots(2.4)$$

The integral in equation (2.4) must be evaluated over the entire surface under question.

We are often interested in evaluating the flux through a **closed surface,** defined as a surface that divides space into an inside and an outside region so that one cannot enter in one region to the other without crossing the surface. For example, the surface of a sphere is a closed surface. Using the symbol $\oint$ to represent an integral over a closed surface the net flux through a closed surface can be written as

$$\phi_E = \oint \vec{E} \cdot d\vec{s} \qquad \ldots(2.5)$$

As we have described above, the vector area element is directed normal to the surface, however, normal can be in two directions. By convention for an area element of a closed surface, area vector ΔS_i always points outward. This convention is used in Fig 2.2. For this fig note that for different area elements, corresponding vector area elements ΔS_i will be pointing in different directions but each such vector will be along outward normal to its corresponding surface element. Also the flux leaving the surface is considered to be positive while that entering into it, is considered to be negative. If the number of field lines leaving the surface is more than those entering the net flux is position. If vice versa flux is negative.

The SI unit of electric flux is N m^{2}C^{-1} or Vm is and it has dimensions of $\left[M^1L^3T^{-3}A^{-1}\right]$

Example 2.1 : Find the electric flux through a vector area $\vec{S} = 5\times10^{-3}\,\hat{j}\,\text{m}^2$ placed in electric field $\vec{E} = 200\hat{i} + 300\hat{j}\,\text{Vm}^{-1}$

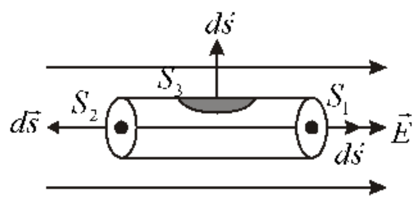
Solution : Electric flux

$$\phi = \vec{E}\cdot\vec{S} = \left(200\hat{i} + 300\hat{j}\right)\cdot\left(5\times10^{-3}\hat{j}\right)$$

$$\phi = 0 + 1500\times10^{-3} = 1.5\,\text{Vm}$$

Example 2.2 : A cylinder is lying in a uniform electric field such that its axis is along the electric field. Show that the net electric flux through the cylinder is zero.

Solution :



As shown in figure we can consider the cylinder to be consisting of three surfaces, two circular faces S_1 and S_2 and curved surfaces S_3, thus the net flux through cylinder.

$$\phi = \int_{S_1} \vec{E}\cdot d\vec{s} + \int_{S_2} \vec{E}\cdot d\vec{s} + \int_{S_3} \vec{E}\cdot d\vec{s}$$

From figure it is clear that for face S_1, $\vec{E}$ and $d\vec{s}$ are parallel $(\theta = 0)$ for face S_2, $\vec{E}$ and $d\vec{s}$ are antiparallel $(\theta = 180^\circ)$ and everywhere on curved surface $\vec{E}$ and $d\vec{s}$ are mutually perpendicular $(\theta = 90^\circ)$, hence

$$\phi = \int_{S_1} Eds\cos0^\circ + \int_{S_2} Eds\cos180^\circ + \int_{S_3} Eds\cos90^\circ$$

or $\quad \phi = ES_1 - ES_2 + 0$

As area $S_1 = S_2 =$ S (say)

$$\phi = ES - ES = 0$$

This result is expected since electric field is uniform so number of field lines entering the cylinder is equal to the number of field lines leaving.

Example 2.3 : A circular sheet of 5 cm radius is situated in a uniform electric field of $5\times10^{+5}\ \text{Vm}^{-1}$ such that its plane makes an angle of 30° with field. Determine electric flux through the sheet.

Solution : The angle made by area vector (normal to the plane of sheet) with electric field is $\theta = 90^\circ - 30^\circ$ or $\theta = 60^\circ$

So, electric flux $\phi = ES\cos\theta = E\left(\pi r^2\right)\cos 60^\circ$

$$\phi = 5\times10^{-5}\times3.14\times\left(5\times10^{-2}\right)^2\times\frac{1}{2}$$

$$\phi = 125\times3.14\times\frac{1}{2}\times10 = 1.96\times10^3\ \text{Vm}$$

2.2 Continuous Charge Distribution

On a microscopic scale, electric charge is quantised. However, there are often situations in which many charges are so close together that they can be conisdered to be continuously distributed. If we consider such a charge distribution to be consiting of point charges the number of such charges is enormously high e.g. a rod containing a small charge of only 1nC it contains 10^{10} point charges. Thus, though it is possible to imagine a charge distribution to be covered by point charges and to calculate the electric field at the desired point using Coulomb's law and then vector sum of electric fields due to all the point charges to give the net electric field, but presence of a very large number of point charges makes such an approach hoplessly complicated. Instead we regard the charge distribution to be continuous, use the concept of charge density and the method of calculus to calculate the electric field. The use of a continuous charge density to describe a large number of closely spaced charges is similar to the use of a continuous mass density to describe air which actually consists of a large number of discrete molecules.

If the net charge on some object is q we divide the charge distribution into many infinitesimal elements dq. Each such element has a length, area or volume dependingon whether considering charges that are respectively distributed in one, two ot three dimensions. We express dq in terms of the size of element and the charge density. Depending upon the number of dimensions over which the charge is distributed we define three types of charge densities as follows -

(i) Linear Charge density

In some situation charges are distributed along a line in space (or along the length of an object) such as charge on a thin rod or wire or on the circumference of a ring. In such cases we express dq in terms of linear charge density (charge per unit length) λ whose SI unit is C/m . If the length of charge element dq is dx by definition.

$$\lambda = \frac{dq}{dx} \qquad \text{. . . (2.6 a)}$$

or $dq = \lambda dx$ . . . (2.6 b)

If a charge q is spreaded uniformly on a rod of length L then we can write $\lambda = q/L$ and it is a constant.

(ii) Surface Charge Density

In some situations charge might be distributed over a two dimensional area such as the surface of a thin disc or sheet or surface of a conductor. In such cases elemental charge dq is expressed in terms of the surface charge density (charge per unit area) σ measured in SI units of C/m^2. If a charge dq is present in an elemental area dS then.

$$\sigma = dq/ds \qquad \text{. . . (2.7 a)}$$

or $dq = \sigma ds$ . . . (2.7 b)

If a charge q is spreaded uniformly over a surface of area S then $\sigma = q/S$ and is a constant.

(iii) Volume Charge density

The charge also might be spread throughout the volume of a three dimensional object. In this situation, we use volume charge density (charge per unit volume) ρ measured in the SI unit of C/m^3. If charge in a volument dV is dq, then

$$\rho = dq/dV \qquad \text{. . . (2.8 a)}$$

or $dq = \rho dV$ . . . (2.8 b)

If the charge q is distributed uniformly throughout the volume V then $\rho = q/V$ and is a constant.

2.2.1 Electric Field due to a continuous Charge Distribution

In this subsection we discuss the determination of electric field due to a continuous charge distribution, the general method for which is as follows -

1. Consider the charge distribution to be consist of a large number of infintesimal elements.
2. Choose an arbitrary charge element and express its charge dq in terms of relevent charge density given by equations 2.6, 2.7 and 2.8 depending on whether the charge is distributed over a line, surface or volume.
3. Treating this charge element dq as a point charge the intensity of electric field at the observation point p is given by

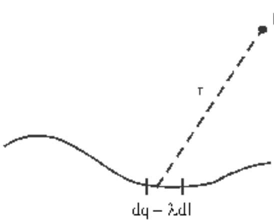
$$d\vec{E} = \frac{1}{4\pi \in_0} \frac{dq}{r^2} \hat{r}$$

here r is the distance between element dq and point P. The direction of vector $d\vec{E}$ is determined by the sign of charge dq according to the force that dq would exert on a unit test charge at P.

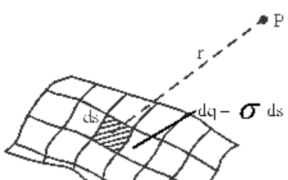
4. The total electric field at P for the entire charge distribution is obtained by taking vector sum of the contributions from all the elements. In the limiting case when the size of element tends to zero

$$\vec{E} = \int d\vec{E} = \frac{1}{4\pi \in_0} \int \frac{dq}{r^2} \hat{r} \qquad \ldots (2.10)$$

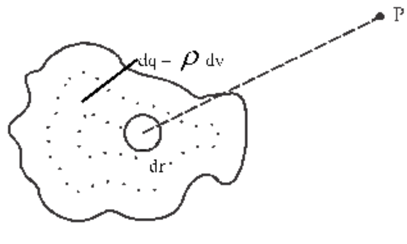
Fig (2.3) shows situations corresponding to linear, surface and volume charge distributions, the corresponding expressions for electric fields are as follows -



(a) linear charge distribution



(b) surface charge distribution



(c) volume charge distribution

Fig 2.3 : Determination of electric field due to (a) linear charge distribution (b) surface charge distribution (c) volume charge distribution

(i) Linear charge distribution : Here $dq = \lambda d\ell$

$$\therefore \quad \vec{E} = \frac{1}{4\pi \in_0} \oint_L \frac{\lambda d\ell}{r^2} \hat{r} \qquad \ldots (2.11)$$

Here the symbol L on integral represents a line integral. If λ is uniform

$$\vec{E} = \frac{1}{4\pi \in_0} \lambda \int_L \frac{d\ell}{r^2} \hat{r} \qquad \ldots (2.12)$$

(ii) Surface Charge Distribution : Here $dq = \sigma ds$

$$\therefore \quad \vec{E} = \frac{1}{4\pi \in_0} \int_S \frac{\sigma ds}{r^2} \hat{r} \qquad \ldots (2.13)$$

here the symbol s on integral suggest that it is a surface integral. If σ is uniform

$$\vec{E} = \frac{1}{4\pi \in_0} \sigma \int_S \frac{ds}{r^2} \hat{r} \qquad \ldots (2.14)$$

(iii) Volume Charge distribution : Here $dq = \rho dV$

$$\therefore \quad \vec{E} = \frac{1}{4\pi \in_0} \int_V \frac{\rho dV}{r^2} \hat{r} \qquad \ldots (2.15)$$

Here the symbol V on integral sign represents a volume integral. If ρ is a constant (uniform)

$$\vec{E} = \frac{1}{4\pi \in_0} \rho \int_V \frac{dV}{r^2} \hat{r} \qquad \ldots (2.16)$$

While solving various integrals it should be taken care of that direction of $d\vec{E}$ due to different elements may be different. Equation (2.10) infact represents a three dimensional vector equation. It can be written is its cartesion components as -

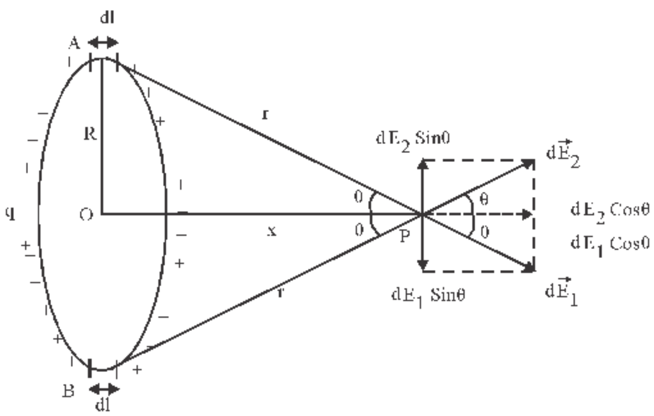
$$E_x = \int dE_x,\ E_y = \int dE_y,\ E_z = \int dE_z \ldots (2.17)$$

In many situations one or more of the above integrals may vanish or have identical values owing to symmetry in charge distribution.

If we wish to determine the force on a point charge q due to some continuous charge distribution, we can do so by first determining $\vec{E}$ using equation (2.10) (according to the dimensional situation of the charge distribution) and then use $\vec{F} = q\vec{E}$. In this chapter we shall limit our study to charge distributions for which corresponding charge density (λ, σ or ρ as the case may be) is uniform.

Example 2.4 A thin ring of radius R has a positive charque q uniformly distributed over it. Determine the electric field at a point on the axis of ring at a distance x from the centre of the ring. Discuss the behavior of the result for condition $x >> R$.

Solution : As the ring is uniformly charged, the linear charge density is constant and is given



$$\lambda = \frac{q}{L} = \frac{q}{2\pi R}$$

Now we consider two diametrically opposite elements A and B each of length $d\ell$ on the ring as shown in Fig.

Charge on each element

$$dq = \lambda d\ell$$

If r is the distance of each element from point P then the electric field at P due to element A.

$$d\vec{E}_1 = \frac{1}{4\pi \in_0} \frac{dq}{r^2} \text{ (In direction AP)}$$

and electric field at P due to element B

$$d\vec{E}_2 = \frac{1}{4\pi \in_0} \frac{dq}{r^2} \text{ (In direction BP)}$$

Clearly $|d\vec{E}_1| = |d\vec{E}_2|$

On resolving $d\vec{E}_1$ and $d\vec{E}_2$ as shown in fig, the perpendicular component $dE_1 \sin\theta$ and $dE_2 \sin\theta$ cancel each other being equal and opposite. While component along axis $dE_1 \cos\theta$ and $dE_2 \cos\theta$ add up being in same direction. We can divide the entire ring into pairs of such diametrically opposite elements. For each such pair the axial component is along OP, so electric field due to complete ring, at P.

$$E = \int_L dE \cos\theta = \frac{1}{4\pi \in_0} \int_L \frac{dq}{r^2} \cos\theta$$

From Fig. $\cos\theta = \frac{x}{r}$, and as $dq = \lambda d\ell$ and are R, λ constant and for a given point P, x, > and r are is also treated as constant.

$$E = \frac{\lambda x}{4\pi \in_0 r^3} \int_L d\ell$$

So $\int_L d\ell$ = length of complete ring $= 2\pi R$

So $$E = \frac{\lambda x}{4\pi \in_0 r^3} \cdot 2\pi R$$

$\because$ $\lambda \times 2\pi R = q$ and from Fig. $r = \left(R^2 + x^2\right)^{1/2}$

So $$E = \frac{qx}{4\pi \in_0 \left(R^2 + x^2\right)^{3/2}} = \frac{kqx}{\left(R^2 + x^2\right)^{3/2}}$$

Under condition $x >> R$ the above expression reduces to

$$E = \frac{kqx}{x^3} = \frac{kq}{x^2}$$

Which is identical to expression for the field produced by a point charge q at a point at a diffence x from it. Thus, for distant axial point the ring behaves as if its entire charge is concentrated at its centre.

Example 2.5 A uniformly charged ring and a uniformly charged sphere are both of equal radius R and each has a charge q. The centre of the sphere lies on the axis of ring at a distance of $R\sqrt{3}$ from the centre of ring. Find the electric force acting between sphere and the ring.

Solution : Electric field at an axial point at a distance x from the centre of a uniformly charge ring is given by

$$E = \frac{1}{4\pi \in_0} \frac{qx}{\left(R^2 + x^2\right)^{3/2}}$$

given $x = R\sqrt{3}$, hence the electric field at the location of the centre of sphere, due to ring is

$$E = \frac{1}{4\pi \in_0} \frac{qR\sqrt{3}}{\left(4R^2\right)^{3/2}} = \frac{1}{4\pi \in_0} \frac{\sqrt{3}q}{8R^2} = \frac{\sqrt{3}q}{32\pi \in_0 R^2}$$

From symmerty the total charge q uniformly distributed on the sphere can be considered to be concentrated at its centre, hence the force between sphere and the ring is

$$F = \frac{\sqrt{3}q^2}{32\pi \in_0 R^2}$$

2.3 Gauss's Law

Gauss's law states that the net flux of an electric field through an imaginary closed surface is $1/\in_0$ times the net charge enclosed by the closed surace.

$$\phi = \oint_S \vec{E} \cdot d\vec{s} = \frac{\Sigma q}{\varepsilon_0} \quad \ldots (2.18)$$

If the closed surface is in some medium other than free space or air, then

$$\phi = \oint_S \vec{E} \cdot d\vec{s} = \frac{\Sigma q}{\varepsilon_0} \quad \ldots (2.19)$$

The above equation gives Gauss's law for dielectric media. Regarding Gauss's law following points are worth noting

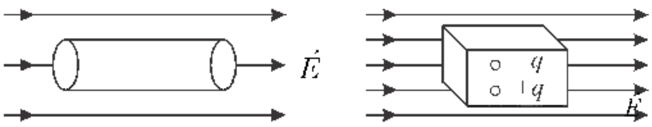
(i) Here Σq, represents the algebraic sum of charges enclosed by the surface.

(ii) The flux entering the surface is considered negative and the flux leaving it is considered positive. The net flux is the algebraic sum of the fluxleaving and flux entering the system.

(iii) The closed surface considered for applying Gauss's law is called Gaussian surface. It is an arbitrary imaginary closed surface i.e. it can have sphereical, cylindrical or any other arbitrary shape. It is usually chosen so that the symmetry of charge distribution (if any) gives, on at least part of the surface an electric field of constant magnitude which can then be factored out of the integral of equation (2.18) making calculations easier.

(iv) Gauss's law considers only on the net charge enclosed in the closed surface. The value of flux does not depend on shape and size of the Gaussian surface. It deos not depend on the location or ditribution of charges inside Gaussian surface. It depends on amount of enclosed charges, their nature and medium. For static charge distribution Gauss's law and Coulomb's law are equivalent. However Gauss's law is more general in that it is always valid whether or not the charges are static.

(v) If the net change enclosed by a surface is zero the flux linked with it is always zero whether it is placed in a uniform or nonuniform electric field. For such a surface, flux entering is equal to the flux leaving the surface. [see fig 2.4 (a) and (b)]



(a) (b)

i.e. $\phi_{net} = \phi_{in} + \phi_{out} = 0$

(vi) Gauss's law is valid only for those vector fields which obeys inverse square law.

(vii) Electric field $\vec{E}$ at any point on the Gaussian surface is the net electric field at that point. This $\vec{E}$ result from all charges both those inside and those outside the Gaussian surface, however term q on the right hand side represent only the net charge enclosed in the Gaussian surface. In some specific cases though the net charge enclosed in the Gaussian surface may be zero but electric field may not. For example if a dipole is enclosed by a Gaussian surface than charge enclosed is zero and ϕ is zero but at some point on the surface $\vec{E}$ is non zero.

(viii) Charges present outside the Gaussian surface do not contribute toward the net flux through the surface.

2.3.1 Gauss's law Derived from Coulomb's law

Considere a positive point charge q at point O enclosed by an arbitrarily shaped Gaussian surface.

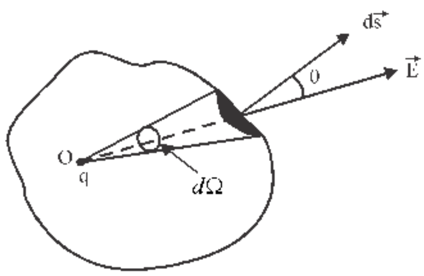


Fig. 2.5 : Solid Angle

Consider an area element $d\vec{s}$, at a distance r from the point charge (Fig 2.5), the electric field at this location is

$$\vec{E} = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} \hat{r} \qquad \ldots (2.21)$$

So flux linked with this area element is

$$d\phi = E ds \cos\theta = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} ds \cos\theta$$

hence the total flux linked with entire closed surface

$$\phi = \int_S \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} ds \cos\theta = \frac{q}{4\pi \epsilon_0} \int_S \frac{ds \cos\theta}{r^2}$$

By definition $\frac{ds \cos\theta}{r^2} = d\Omega$ solid angle subtended by the area element ds at point O.

$$\phi = \frac{q}{4\pi \epsilon_0} \int_S d\Omega = \frac{4\pi q}{4\pi \epsilon_0} \quad \left[\because \int_S d\Omega = 4\pi\right]$$

or $\phi = \frac{q}{\epsilon_0}$

which is the mathematical statement of Gauss's law.

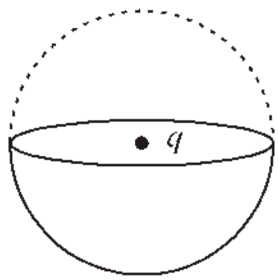
Example 2.6 A point charge of 7.6 μC is situated at the centre of a spherical surface of 0.03 m radius. Find the flux linked with the spherical surface. What will be the change in flux if the radius of surface be doubled.

Solution : As

$$\phi = \frac{q}{\epsilon_0} = \frac{7.6 \times 10^{-6}}{8.85 \times 10^{-12}} = 8.6 \times 10^5 \text{ Nm}^2\text{C}^{-1}$$

As Gauss law does not depend on the size of surface it will not change on doubling the radius of spherical surface.

Example 2.7 A charge q is placed at the centre of a hemispherical surface. Determine the flux of electric field through the surface of hemisphere.



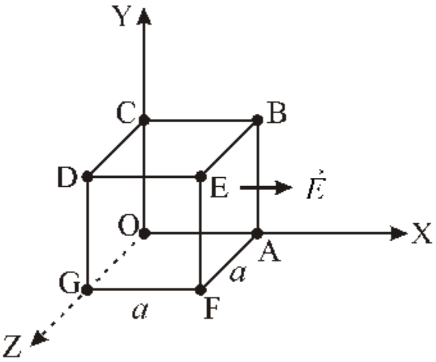
Solution : As Gauss's law deals with the electric flux depends on the charge enclosed by a closed surface, so to enclose the charge and keeping symmetry in view we imagine a complete spherical surface centred at location of q. Flux through this surface.

$$\phi' = \frac{q}{\epsilon_0}$$

Since the charge is placed at the centre, from symmetry considerations we expect that flux through the surface of hemisphere

$$\phi = \frac{\phi'}{2} = \frac{q}{2\epsilon_0}$$

Example 2.8 Figure shows a closed Gaussian surface in the shape of a cube placed in a region where the electric field is given by $\vec{E} = E_0 x\hat{i}$. Each edge of the cube has length a = 1 cm and constant $E_0 = 2.5\times10^5\ \text{NC}^{-1}\text{m}^{-1}$. Find the net electric flux linked with cube and the net charge enclosed by the cube.



Solution : Area of each face of the cube

$$S = a^2$$

Total flux through the cube

$$\phi = \left(\vec{E}\cdot\vec{S}\right)_{ABEF} + \left(\vec{E}\cdot\vec{S}\right)_{OCDG} + \left(\vec{E}\cdot\vec{S}\right)_{BCDE}$$

$$+\left(\vec{E}\cdot\vec{S}\right)_{OAFG} + \left(\vec{E}\cdot\vec{S}\right)_{OABC} + \left(\vec{E}\cdot\vec{S}\right)_{DEFG}$$

As for each of the faces BCDE, OAFG, OABC, and DEFG, $\vec{E}$ is perpendicular to so corresponding flux terms are zero.

$$\phi = \left(E_0 x\hat{i}\cdot a^2\hat{i}\right)_{ABEF} + \left(E_0 x\hat{i}\cdot a^2\left(-\hat{i}\right)\right)_{OCDG}$$

and as for the face OCDG x=0 the contribution of this term toward flux is also zero. As for face ABEF x=a

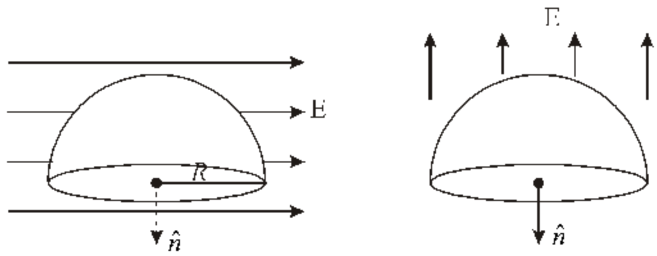
$$\phi = E_0 a^3 - 0 = E_0 a^3$$

$$= \left(2.5\times10^5\right)\times\left(1\times10^{-2}\right)^3 = 0.25\ \text{Nm}^2\text{C}^{-1}$$

and from Gauss's law, the enclosed charge is

$$q = \epsilon_0\ \phi = 8.85\times10^{-12}\times0.25 = 2.21\times10^{-12}\ \text{C}$$

Example 2.9 A hemispherical body is placed in a uniform electric field E. What is the electric flux linked with curved surface if the electric field is (a) parallel to its base Fig (a) (b) perpendicular to its base.(Fig. b)



(a) (b)

Solution : We can consider the hemispherical body as a closed body with a curved surface and a flat base (cross section), the flux linked with this body will be zero as it does not enclose any charge. So if ϕ_C and ϕ_b are flux be linked with curved surface and base respectively

$$\phi = \phi_c + \phi_s = 0$$

or $\quad \phi_c = -\phi_b$

(a) For the situation of Fig (a), as the electric field is parallel to the base, vector area of base is perpendicular to $\vec{E}$ thus $\phi_b = 0$ and so $\phi_c = 0$.

(b) For the situation of Fig (b) area vector of base is antiparallel to $\vec{E}$

So $\quad \phi_b = E\cos180^o = -E\pi R^2$

$$\phi_C = -\phi_b = E\cdot\pi R^2$$

(In this case the flux linked with curved surface depends on the radius of cross section (base) and not on the shape of curved surface)

2.4 Application of Gauss's Law

Using Gauss's law $\phi = \oint_S \vec{E} \cdot d\vec{s} = \frac{\Sigma q}{\epsilon_0}$

The electric field due to a highly symmetrical charge distribution can often be easily calculated. The aim in this type of calculation is to determine a surface (Gaussian surface) which satisties one or more of the following conditions -

1. The value of electric field can be argued by symmetry to be constant over the portion of surface.

2. The area vectors of the portions of surface are either parallel or pendicular to $\vec{E}$.

In following subsections we will discuss selection of Gaussian surfaces for some symmetrical charge distributions.

2.4.1 Electric Field Intensity due to an Infinite Line Charge

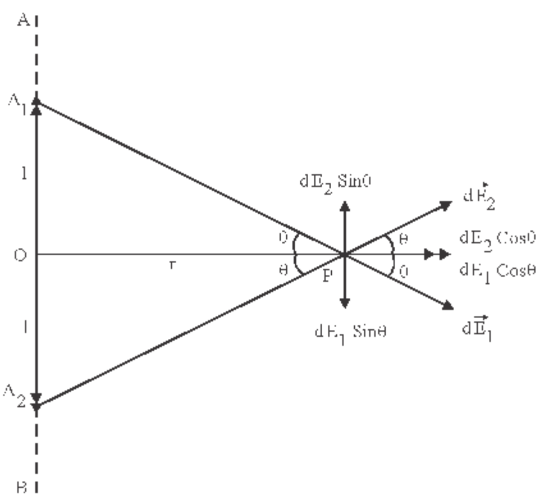


Fig 2.4 : Electric Field due to an infinite line charge

Consider an infinite line charge of constant linear charge density λ. We wish to determine electric field at a point P at as perpendicular distance OP= r from this line charge.

Now consider two small elements A_1 and A_2 of equal lengths situated symmetrically with respect to O on this line charge (Fig 2.6). The electric field inensities at P due to these elements are $d\vec{E}_1$ and $d\vec{E}_2$ respectively with $d\vec{E}_1$ and $d\vec{E}_2$ directed along A_1P and A_2P . On resolving these electric field along OP and perpendicular to OP we note that their perpendicular componenets $dE_1 \sin\theta$ and $dE_2 \sin\theta$ cancel while components along OP, $dE_1 \cos\theta$ and $dE_2 \cos\theta$ add. In this manner the infinite line charge can be divided into such symmetrical pairs. Resulting electric field from each such pair is along OP. Thus we conclude that the electric field due to infinite line charge is directed perpendicular to the line charge i.e. it is in redial direction (which is redially outward or inward depending upon whether the charge is positive or negative). This result is expected on the basis of symmetry. Imagine that while you are watching some one rotates the line charge about its perpendicular axis. When you look again you will not be able to detect any change. From this symetry it can be concluded that the only uniquely specified direction in this situation is along a radial line.

Now consider a Gaussian surface in the form of a closed cylinder of length l coaxial with the line charge such that the point P lies on the its curved surface. (Fig 2.7)

The charge enclosed in this Gaussian surface

$$\Sigma q = \lambda \ell \qquad \ldots (2.24)$$

so from Gauss's law

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{\Sigma q}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0} \qquad \ldots (2.25)$$

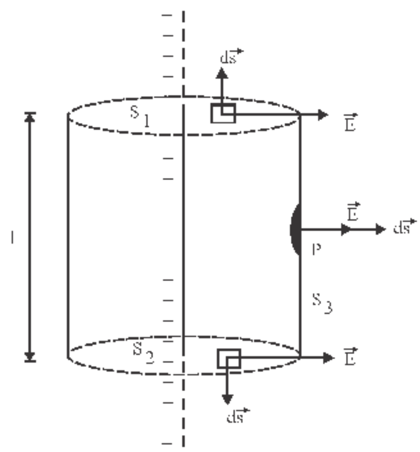


Fig 2.7 : Cylindrical Gaussian surface for a line charge

This closed cylindrical surface can be subdivided into three parts

(i) Upper circular face (cap) S_1

(ii) Lower circular face (cap) S_2

(iii) Curved surface S_3

So the equation (2.25) can be rewritten as

$$\int_{S_1} \vec{E}\cdot d\vec{s} + \int_{S_2} \vec{E}\cdot d\vec{s} + \int_{S_3} \vec{E}\cdot d\vec{s} = \frac{\lambda\ell}{\epsilon_0}$$

or

$$\int_{S_1} Eds\cos 90^\circ + \int_{S_2} Eds\cos 90^\circ + \int_{S_3} Eds\cos 0^\circ = \frac{\lambda\ell}{\epsilon_0}$$

or $$0+0+\int_{S_3} Eds = \frac{\lambda\ell}{\epsilon_0} \qquad \ldots(2.26)$$

Since the magnitude of electric field E is same everywhere on the curved surface so E can be taken out of the integral in equation 2.26, giving

$$E\int_{S} ds = \frac{\lambda\ell}{\epsilon_0}$$

or $$E\times 2\pi r\ell = \frac{\lambda\ell}{\epsilon_0} \qquad \because \int_{S_3} ds = 2\pi r\ell$$

or $$E = \frac{\lambda}{2\pi r \epsilon_0} \qquad \ldots(2.27)$$

In vector form $$\vec{E} = \frac{\lambda}{2\pi \epsilon_0 r}\hat{r} \qquad \ldots(2.28)$$

Clearly the magnitude of electric field due to an infinite line charge is inversely proportional to distance and directly proportional to the charge density and corresponding graphs are as shown in fig 2.8

i.e. $$E \propto \frac{1}{r} \text{ and } E \propto \lambda \qquad \ldots(2.29)$$

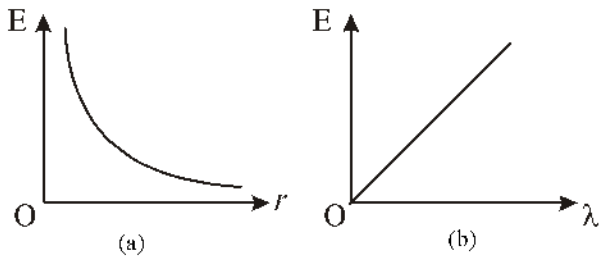


Fig 2.8 : Variation of electric field due to a line charge with (a) distance (b) charge density

Example 2.10 : The linear charge density of a straight infinite wire is 2 μC/m. Find the magnitude of electric field at a point 20 cm from the wire in air.

Solution :

$$\because \quad E = \frac{\lambda}{2\pi \epsilon_0 r} = \frac{2\lambda}{4\pi \epsilon_0 r}$$

$$= 9\times10^9 \times \frac{2\times2\times10^{-6}}{20\times10^{-2}}$$

or $$E = 1.8\times10^5 \text{ NC}^{-1}$$

Example 2.11 : An electron is circulating on a path of radius 0.1 m around a infinite line charge. If the linear charge density is 10^{-6} cm^{-1} then find the magnitude of the velocity of electron. [Given $m_e = 9.0\times10^{-31}$ kg, $e = 1.6\times10^{-19}$ C]

Solution : Force on electron due to infinite line charge

$$F = qE = eE = \frac{1}{4\pi \epsilon_0}\frac{2e\lambda}{r}$$

This force provides the electron necessary centripetal force, so $\frac{m_e v^2}{r} = \frac{2e\lambda}{4\pi \in_0 r}$

or $\quad v = \sqrt{\frac{2e\lambda}{4\pi \in_0 m_e}}$

$$= \sqrt{\frac{9\times10^9 \times 2\times1.6\times10^{-19} \times10^{-6}}{9.0\times10^{-31}}}$$

$$v = \sqrt{2\times16\times10^{14}} = 4\sqrt{2}\times10^7$$

$$= 5.65\times10^7 \text{ ms}^{-1}$$

2.4.2 Electric Field due to an Infinite Uniformly Charged Non Conducting Sheet

Let ABCD is some portion of a uniformly charged non conducting sheet of infinite extension. The surface charge density σ for this surface is uniform. We wish to determine electric field at a point P at a normal distance OP = *r* from the sheet. (Fig 2.9)

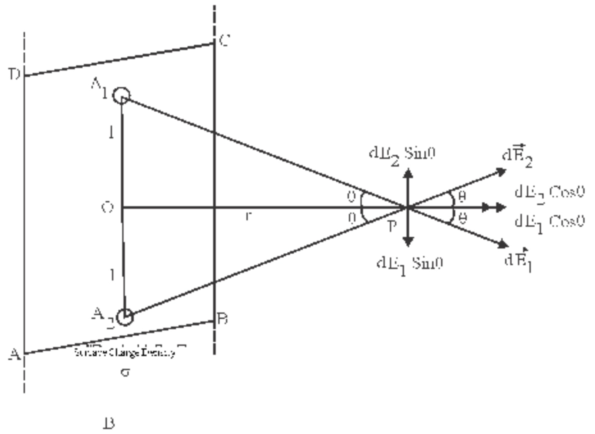


Fig 2.9 : Electric field due to an infinite non conducting charges sheet

Consider two small area elements A_1 and A_2 equidistant from O on its sides at a distance l then electric field $d\vec{E}_1$ and $d\vec{E}_2$ due to these elements have same magnitude and these are directed along A_1P and A_2P respectively. On resolving these electric fields along OP and perpendicular to it, the perpendicular components $dE_1 \sin\theta$ and $dE_2 \sin\theta$ cancel while parallel components $dE_1 \cos\theta$ and $dE_2 \cos\theta$ add.

In this manner we can divide the entire charge sheet into pair of symmetrically placed area elements. For each such pair the net electric field is along OP. Thus we can say that the net electric field due to complete sheet at point P is along the normal joining P to the sheet.

Now imagine a cylindrical closed surface of cross sectional area *S* and length 2*r* with point P at one of its circular face (cap). The sheet divides this cylinder into two equal parts (Fig 2.10).

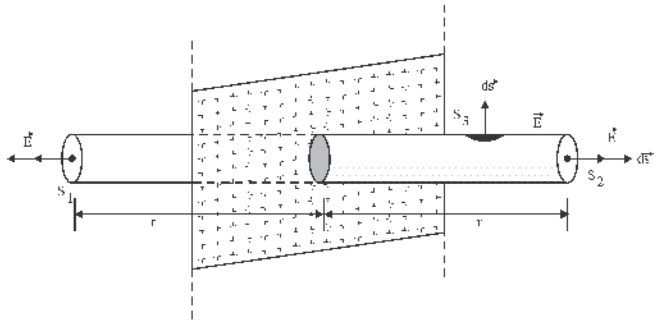


Fig 2.10 : Construction of Gaussion surface for a non conducting uniformly charged infinite sheet.

The net charge enclosed by this surface is

$$\Sigma q = \sigma S \qquad \ldots (2.30)$$

therefore the net electric flux lined with it

$$\phi = \oint_S \vec{E}\cdot d\vec{s} = \frac{\Sigma q}{\in_0} = \frac{\sigma S}{\in_0} \qquad \ldots (2.31)$$

We can consider this closed cylindrical surface to be consisting of three parts (i) circular cap S_1 (ii) circular cap S_2 and (iii) curved surface S_3 . Thus the equation (2.31) can be written as.

$$\int_{S_1} \vec{E}\cdot d\vec{s} + \int_{S_2} \vec{E}\cdot d\vec{s} + \int_{S_3} \vec{E}\cdot d\vec{s} = \frac{\sigma S}{\in_0} \qquad \ldots (2.32)$$

For both surfaces S_1 and S_2 $\vec{E}$ and $d\vec{s}$ are parallel so at these surfaces $\vec{E}\cdot d\vec{s} = Eds$, while for surface S_3, $\vec{E}$ is perpendicular to $d\vec{s}$ so $\vec{E}\cdot d\vec{s}=0$.

So from equation (2.23)

$$\int_{S_1} Eds + \int_{S_2} Eds + 0 = \frac{\sigma S}{\epsilon_0}$$

As for surfaces S_1 and S_2 E is same at every point, so

$$E\int_{S_1} ds + E\int_{S_2} ds = \frac{\sigma S}{\epsilon_0}$$

$$\because \qquad \int_{S_1} ds = \int_{S_2} ds = S$$

so $$ES + ES = \frac{\sigma S}{\epsilon_0}$$

or $$2ES = \frac{\sigma S}{\epsilon_0}$$

or $$E = \frac{\sigma}{2\epsilon_0} \qquad \ldots (2.33)$$

In vector form $$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n} \qquad \ldots (2.34)$$

Where $\hat{n}$, is a unit vector normal to the sheet. It is obvious that the electric field due to an infinite sheet of charge does not depend on distance i.e. such a charge distribution produces a uniform electric field. This result can be extended for points in the close vicinity if a large but finite sheet of charge provided the points are not near its edges.

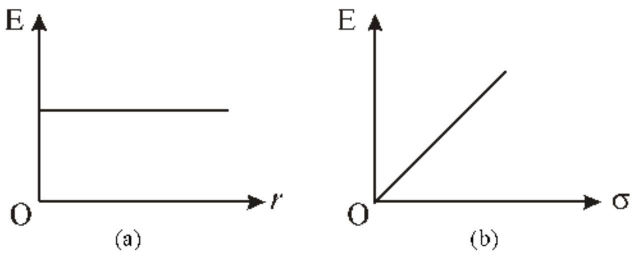


Fig 2.11 : Dependence of electric field for a Uniformly charged infinite sheet

Example 2.12 : For a uniformly charged infinite non conducting sheet, area of 1 cm^2 anywhere on the sheet contains 1 µC charge. Calculate the electric field near the sheet in air.

Solution : Surface charge density

$$\sigma = \frac{q}{A} = \frac{17.70\times10^{-6}C}{10^{-4}m^2}$$

$$\sigma = 17.70\times10^{-2}\ \text{C/m}^2$$

So electric field

$$E = \frac{\sigma}{2\epsilon_0} = \frac{17.70\times10^{-2}}{2\times8.85\times10^{-12}} = 10^{10}\ \text{NC}^{-1}$$

2.4.3 Electric Field due to an Uniformly Charged Infinite Conducting Plate

For a uniformly charged infinite conducting plate the electric field is directed normal to the plate as in case of an uniformly charged infinite non conducting sheet. This can be shown by arguments based on symmetry similar to those used in earlier subsection.

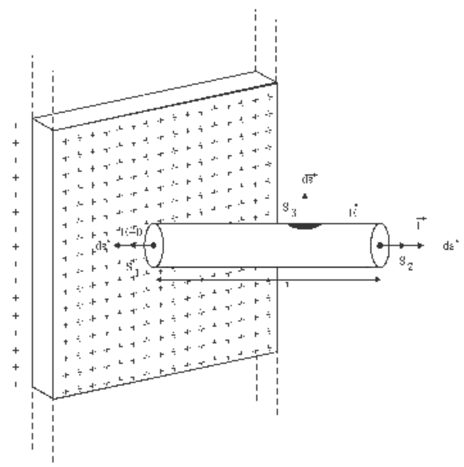


Fig 2.12 : Electric field due to uniformly charged conducting plate

Let the surface charge density for the conductor plate is σ. We wish to determine the magnitude of electric field at a perpendicular distance r from the plate. Imagine a cylindrical Gaussian surface of cross sectional area S and length r as shown in Fig (2.12)

Charged enclosed by this surface $\Sigma q = \sigma S$ ... (2.35) Form Gauss's law the flux linked with this cylindrical surface

$$\phi = \oint \vec{E} \cdot d\vec{s} = \frac{\Sigma q}{\epsilon_0} = \frac{\sigma S}{\epsilon_0} \quad \ldots (2.36)$$

This Gaussian surface can be considered to be consisting of three parts. (i) Left circular cap S_1 (ii) right circular cap S_2 (iii) curved surface S_3. So the equation (2.36) can be rewritten as

$$\int_{S_1} \vec{E} \cdot d\vec{s} + \int_{S_2} \vec{E} \cdot d\vec{s} + \int_{S_3} \vec{E} \cdot d\vec{s} = \frac{\sigma S}{\epsilon_0} \quad \ldots (2.37)$$

As circular surface S_1 is inside conductor so E=0 for it while for S_3 $\vec{E}$ and $d\vec{s}$ mutually perpendicular, thus $\vec{E} \cdot d\vec{s} = 0$. Also for surface S_2 $\vec{E}$ and $d\vec{s}$ are parallel and E is same for every point on S_2, from these considerations equation (2.37) yields

$$0 + E\int_{S_2} ds + 0 = \frac{\sigma S}{\epsilon}$$

or $$ES = \frac{\sigma s}{\epsilon}$$

or $$E = \frac{\sigma}{\epsilon} \quad \ldots (2.38)$$

In vector form

$$\vec{E} = \frac{\sigma}{\epsilon}\hat{n} \quad \ldots (2.39)$$

Where $\hat{n}$, is a unit vector normal to the surace of conductor. Thus, the electric field due to a uniformly charged infinite conductor plate does not depend on distance i.e. electric field is uniform. This result is approximately true for a uniformly charged conducting plate of finite dimensions for points in close vicinity of it.

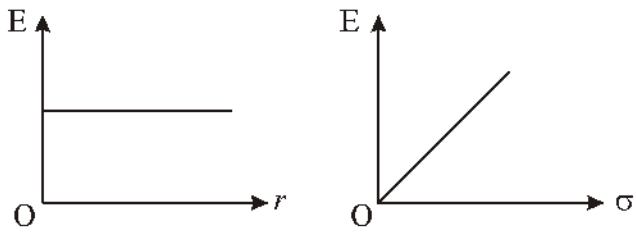


Fig 2.13 : Dependence of electric field for a uniformly charged conducting plate

Example 2.13 : The surface charge density for a uniformly chargedinfinite conductor plate is $4\times10^{-6}\,\mathrm{Cm}^{-2}$. Find the magnitude of force on a charge -2×10^{-6} C placed near it.

Solution : Electric field for the conducting plate

$$E = \frac{\sigma}{\epsilon_0}$$

Magnitude of force on q

$$F = qE = \frac{q\sigma}{\epsilon_0} = \frac{2\times10^{-6}\times4\times10^{-6}}{8.85\times10^{-12}} = \frac{8}{8.85}$$

$$F = 0.903\,\mathrm{N}$$

2.4.4 Electric Field Intensity due to a Uniformly Charged Spherical Shell

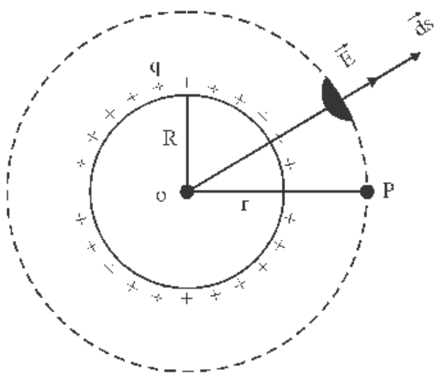


Fig 2.14 : electric field outside the charged spherical shell

Suppose a charge Q is distributed uniformly on the surface of a spherical shell of radius R. The surface charge density for this shell is then.

$$\sigma = \frac{Q}{A} = \frac{Q}{4\pi R^2} \quad \ldots (2.40)$$

We wish to determine electric field at a point P from the centre of the shell. The Gaussian surface for such a charge distribution must be spherical. Depending upon location of P three situations are possible.

(a) When P is outside of the sphere ($r > R$)

For this case consider a spherical Gaussian surface of radius r ($r > R$) concentric with spherical shell as shown in fig 2.14. The net charge enclosed by the Gaussian surface is then

$$\Sigma q = Q \quad \ldots (2.41)$$

So the net flux linked with Gaussian surface

$$\phi = \oint_S \vec{E} \cdot d\vec{s} = \frac{\Sigma q}{\in_0} = \frac{Q}{\in_0} \quad \ldots (2.42)$$

The intensity of electric field $\vec{E}$ and area element $d\vec{s}$ are both act along the radial line for every point on this Gaussian surface. Also note that as each point on this surface is equidistant from centre the magnitude of $\vec{E}$ is same so from equation (2.42)

$$\phi = \oint_S \vec{E} \cdot d\vec{s} = \oint_S E ds = E \oint_S ds = \frac{Q}{\in} \ldots (2.43)$$

$\because \quad \oint_S ds = 4\pi r^2 =$ Area of spherical Gaussian surface

so $\quad \phi = E \times 4\pi r^2 = \dfrac{Q}{\in_0}$

so $\quad E_{out} = \dfrac{Q}{4\pi \in_0 r^2} = \dfrac{\sigma R^2}{\in_0 r^2} \quad \ldots (2.44)$

$\{\because$ From equation (2.40) $\dfrac{Q}{4\pi} = \sigma R^2 \}$

From equatin (2.44) it is clear that for external points a uniformly charged spherical shell behaves as if its entire charge is concentrated at its centre. Thus the force due to a uniformly charged sphere having a charge Q on another charge placed outside the shell is same as the force on this charge due to a point charge Q placed at the centre of shell.

(b) When point P is on the surface of shell ($r = R$) : For this case on substituting r = R in equation (2.44) we obtain

$$E_S = \frac{Q}{4\pi \in R^2} = \frac{\sigma}{\in_0} \quad \ldots (2.45)$$

(c) When point P is inside the Shell ($r < R$) In this case Gaussian surface is well within the spherical shell (Fig 2.15) and as the charge is on the outer surface of shell the net charge enclosed by the is Gaussian surface is zero.

$$\Sigma q = 0$$

So form Gauss's law

$$\phi = \oint_S \vec{E} \cdot d\vec{s} = \frac{\Sigma q}{\varepsilon_0} = 0$$

$\because \quad ds \neq 0$ and $\vec{E}$ and $d\vec{s}$ are not mutually perpendicular hence at every point inside the spherical shell

$$E_{in} = 0 \quad \ldots (2.46)$$

Also if some charged particle is situated inside a charged spherical shell no force is exerted on it by the charge on the shell.

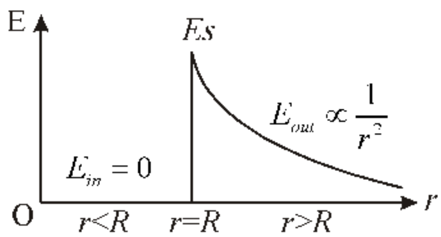


Fig 2.16 Variation with distance of electric field due to a uniformly charged spherical shell

The variation of electric field with distance for a charged sphere is shown in Fig 2.16

2.4.5 Electric Field Intensity due to a Uniformly Charged Conducting Sphere

If an excess charge is placed on an isolated conductor that amount of charge will move entirely to the surface of conductor. None of the excess charge can reside within the body of conductor. This is logical considering that charges of same sign repel each other. We may consider that by moving to the surface the added charges are getting as far away from each other as they can. Under electrostatic condition the electric field inside the conductor must be zero. For metallic conductors this is easy to explain. If it were not so the field would exert forces on free electrons and thus current would always exists with in a conductor of course, there is no such perpetual current in side a conductor and so interal electric field must be zero. An electric field does appear when excess charges is given to the conductor but the added charge distributes very quickly to the outer surface (in a time of the order of nano seconds) so the interal electric field due to all charges both inside and outside is zero. Then the movement of charges cases. As the net electric force on each charge is zero now the conductor is said to be in electronstatic equilibrium.

As the charge resides on the surface a spherical uniformly charged conductor behaves like a uniformly charged spherical shell and expressions for electric fields are exactly same as have been derived in subesection 2.4.4.

2.4.6 Intensity of Electric Field due to a Uniformly Charged Non Conducting Sphere

Consider a non conducting sphere of radius R which is given a charge Q which is distributed uniformly over its entire volume. Therefore the volume charge density is given by

$$\rho = \frac{Q}{V} = \frac{Q}{(4/3)\pi R^3} \qquad \ldots (2.47)$$

We wish to determine electric field at a point P at a distance r from the centre O of the sphere. The Gaussian surface to be considered here is spherical with radius r centred at O. Depending upon the location of P three situations are possible.

(A) When point P lies outside the charged sphere ($r > R$) : In this situation the charged enclosed by the Gaussian surface is same as the charge on the sphere under consideration (Fig 2.17)

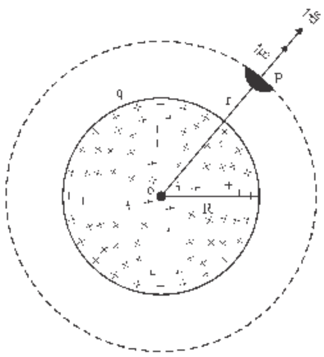


Fig 2.17: Gaussian surface for determination of electric field outside a uniformly charged non conducting sphere

Thus $\Sigma q = Q$ ... (2.48)

Hence, from Gauss's law the flux linked with this Gaussian surface is

$$\phi = \oint_S \vec{E} \cdot d\vec{s} = \frac{\Sigma q}{\epsilon_0} = \frac{Q}{\epsilon_0} \qquad \ldots (2.49)$$

or $$\phi = \oint_S E ds = \frac{Q}{\epsilon_0}$$ {∵ since for spherical surface $\vec{E}$ and $d\vec{s}$ are parallel}

As from symmetry magnitude of $\vec{E}$ is same at every point on Gaussian surface we heve

or $$\phi = E\oint_S ds = \frac{Q}{\epsilon_0}$$

or $$\phi = E \times 4\pi r^2 = \frac{Q}{\epsilon_0} \qquad \ldots (2.50)$$

or $$E = \frac{Q}{4\pi \epsilon_0 r^2} \qquad \ldots (2.51)$$

or $$E = \frac{\rho}{3\epsilon_0}\left(\frac{R^3}{r^2}\right) \qquad \ldots (2.52)$$

{∵ From equation 2.47 $Q = \frac{4}{3}\pi R^3 \rho$ }

In vector form

$$\vec{E} = \frac{Q}{4\pi \in_0 r^2}\hat{r} = \frac{\rho}{3\in_0}\left(\frac{R^3}{r^2}\right)\hat{r} \quad \ldots (2.53)$$

Thus we can say that for points outer to the surface the charged sphere behaves as if the entire charge is concentrated at the centre.

(b) When the point P is on the surface (r=R). In this case on substituting r=R in equation (2.53) yields

$$E = \frac{Q}{4\pi \in_0 r^2} = \frac{\rho R}{3\in_0} \quad \ldots (2.54)$$

and in vector form

$$\vec{E} = \frac{Q}{4\pi \in_0 r^2}\hat{r} = \frac{\rho R}{3\in_0}\hat{r} \quad \ldots (2.55)$$

(c) When point P lies inside the sphere $(r < R)$. In this case spherical Gaussian surface is inside the charged sphere (Fig 2.18) and charge enclosed by it say Q^1 is given by

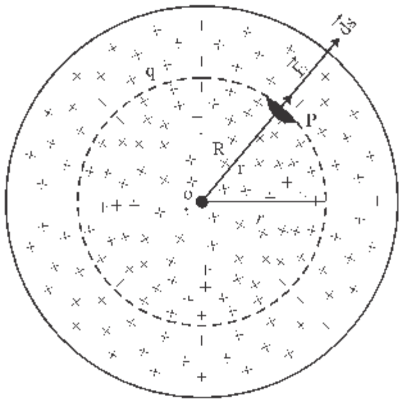


Fig 2.18 : Gaussian surface for Determining E at internal point

$$Q' = \rho \times \frac{4}{3}\pi r^3$$

$$Q' = \frac{Q}{\frac{4}{3}\pi R^3}\cdot\frac{4}{3}\pi r^3 = \frac{Qr^3}{R^3} \quad \ldots (2.56)$$

So the flux linked with Gaussian surface

$$\phi = \oint_S \vec{E}\cdot d\vec{s} = \frac{Q'}{\varepsilon_0} = \frac{Qr^3}{\in_0 R^3}$$

or
$$\phi = \oint_S E ds = \frac{Qr^3}{\in_0 R^3}$$

{As at each point on Gaussian surface $\vec{E} \| d\vec{s}$ }

From symmetry of charge distribution magnitude of E is same for every point on this surface.

So $\phi = \oint_S E ds = \frac{Qr^3}{\in_0 R^3}$

So
$$E = \frac{q}{4\pi \in_0 R^3} r = \frac{\rho}{3\in_0} r \quad \ldots (2.57)$$

In vector form

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^3}\vec{r} = \frac{\rho}{3\varepsilon_0}\vec{r} \quad \ldots (2.58)$$

as at centre of sphere r = 0, so from equation (2.57)

E centre = O

From above discussion for a spherical charge distribution it is clear that

(i) At the centre E = O

(ii) For the interior of the sphere, electric field is directly proportional to distance (r) from centre $E_{in} \propto r$

(iii) Electric field is maximum at the surface of sphere

(iv) For points outside the sphere electric field is inversely propotional to the square of distance

$$E_{out} \propto \frac{1}{r^2}$$

So for a uniformly charged non conducting sphere the variation of electric field with distance from the centre is as shown in Fig 2.19

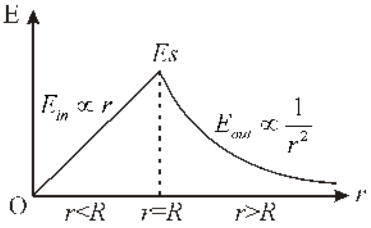


Fig 2.19 : Variation of Electric Field due to a uniformly charged non conducting sphere with distance r

Example 2.14 A conducting sphere of 10m radius is given 1 μC charge. Determine electric field at (a) its centre (b) a point 5 cm from centre (c) at a point 10 cm from centre (d) at a point 15 cm from centre in air.

Solution : (a) The electric field at the centre of a conducting sphere is zero.

(b) r = 5 cm wheres radius of sphere R = 14 cm, so the point is inside the conducting sphere. Electric field inside conductor is zero (c) r = 10 cm, point is on the surface of sphere.

$$E = \frac{Q}{4\pi \in_0 R^2} = 9\times10^9 \times \frac{1\times10^{-6}}{\left(10\times10^{-2}\right)^2}$$

$$= 9\times10^5 \text{ NC}^{-1}$$

(d) r = 15 cm, point is outside the sphere, so

$$E = \frac{Q}{4\pi \in_0 r^2} = \frac{9\times10^9\times1\times10^{-6}}{\left(15\times10^{-2}\right)^2}$$

$$= 4\times10^5 \text{ NC}^{-1}$$

Example 2. 15 : A sphere of diameter 10 cm is uniformly charged so that electric field at its surface is $5\times10^5 \text{ Vm}^{-1}$. Calculate the force on a $5\times10^{-2}\ \mu C$ charge situated at a distance of 25 cm from the centre of sphere.

Solution : Let q be the charge given to the sphere, then at surface

$$E_{सतह} = \frac{q}{4\pi \in_0 R^2}$$

Electric field outside the sphere at a distance r from centre

$$E_{surface} = \frac{q}{4\pi \in_0 r^2}$$

so $\frac{E}{E_{surface}} = \frac{R^2}{r^2}$

or $E = \frac{R^2}{r^2}\times E_{surface} = \frac{(5)^2}{(25)^2}\times5\times10^5 = \frac{125\times10^5}{625}$

$$E = 2\times10^4 \text{ Vm}^{-1}$$

Force on charge $F = q_0 E = 5\times10^{-8}\times2\times10^4$

$$= 10\times10^{-4} = 10^{-3} \text{ N}$$

Example 2.16 : A charge of 0.5 μC is distributed uniformly over a non conducting sphere of radius 10 cm. Determine the electric field at a point (a) at the centre of sphere (b) 8 cm from the centre (c) 10 cm from the centre (d) 20 cm from the centre in air.

Solution : At the centre of sphere E = 0

(b) When r = 8 $= 8\times10^{-2}$ m cm the point is internal to sphere so

$$E = \frac{Q}{4\pi \in_0 R^3} r$$

$$= \frac{9\times10^9\times0.5\times10^{-6}\times8\times10^{-2}}{\left(10\times10^{-2}\right)^3} = \frac{360}{10^{-3}}$$

$$E = 3.6\times10^5 \text{ V/m}$$

(c) When r = 10 cm point is on the surface of sphere

$$E = \frac{Q}{4\pi \in_0 R^2}$$

$$= \frac{9\times10^9\times0.5\times10^{-6}}{\left(10\times10^{-2}\right)^2} = 4.5\times10^5 \text{ V/m}$$

(d) When r = 20 cm, point is outside sphere so

$$E = \frac{Q}{4\pi \in_0 r^2}$$

$$= \frac{9\times10^9\times0.5\times10^{-6}}{\left(20\times10^{-2}\right)^2}$$

$$= 1.125\times10^5 \text{ V/m}$$

2.5 Force on the surface of a charged conductor :

Excess charge given to a surface gets distributed over its surface. The charge present in any small portion of the conductor is under repulsion from the charge present in the remaining portion of the conductor. Thus a force of repulsion acts on every surface element of the conductor and the net force on the surface is the vector sum of forces acting on all such elements. Thus a charge conductor surface experiences an outward pressure.

Let the surface charge density on a conductor surface be σ. Now consider two points P_1 and P_2 placed symmetrically with respect to the conductor with P_1 just inside and P_2 just outside the conductor (Fig 2.20)

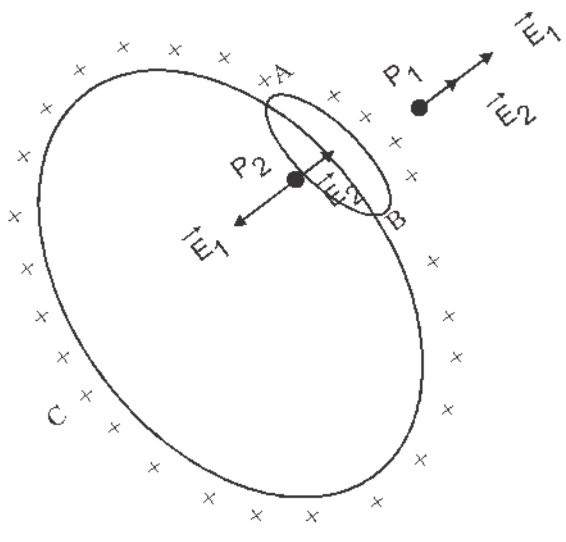


Fig 2.20 : Determination of force on the surface a charged conductor

As the electric field out side the conductor is $\sigma / \in_0$ so the electric field at point P_1

$$E_{P_1} = \frac{\sigma}{\in_0} \quad \ldots (2.60)$$

and since the electric field inside a conductor is zero so at point P_2

$$E_{P_2} = 0 \quad \ldots (2.61)$$

Next, we consider this conductor to be consisting of two parts (i) element AB having area ds and (ii) remainder of the conductor, ACB. It $\vec{E}_1$ and $\vec{E}_2$ are the electric field due to AB and ACB respectively at points in their near vicinity then from fig

$$E_{P_1} = E_1 + E_2 \quad \ldots (2.62)$$

(E_1 and E_2 are in same direction at Point P_1)

$$\text{and } E_{P_2} = E_1 - E_2 \quad \ldots (2.63)$$

(E_1 and E_2 are directed opposite at point P_2)

From equation (2.16) and (2.63)

$$E_1 - E_2 = 0$$

i.e. $$E_1 = E_2 \quad \ldots (2.64)$$

From equation (2.60) (2.62) and (2.64)

$$E_2 + E_2 = \frac{\sigma}{\in_0}$$

or $$E_2 = \frac{\sigma}{2\in_0} \quad \ldots (2.65)$$

So, the electric field due to part ACB at the location of area element AB can be considered as $\frac{\sigma}{2\in_0}$.

If the total charge on element AB is dq then force on it

$dF = E_2 dq = \frac{\sigma}{2\in_0} dq$: As $(\because dq = \sigma ds)$, so

$$dF = \frac{\sigma^2}{2\in_0} ds = \frac{1}{2}\in_0 E^2 ds \quad \ldots (2.66)$$

$$\left\{ \because E = \frac{\sigma}{\in_0} \text{ so } \sigma = \in_0 E \right\}$$

Force acting on the complete surface is then given by

$$F = \oint_s \frac{\sigma^2}{2\in_0} ds = \oint_s \frac{\in_0 E^2}{2} ds \quad \ldots (2.67)$$

and the force per unit area i.e. pressure

$$P = \frac{dF}{ds} = \frac{\sigma^2}{2 \in_0} = \frac{1}{2} \in_0 E^2 \qquad \ldots (2.68)$$

this pressure is called as electrostatic pressure

2.6 Energy per unit Volume for an electric Field

We have seen that an electric force acts along outward normal on the surface of a charged conductor. For increasing the amount of charge on the conductor or to increase the volume of region in which the electric field is present, the work is required to be done agains this force. This work gets stored in the form of energy in electric field.

For the sake of simplicity we consider a spherical shell of radius r for which surface charge density is σ (Fig 2.21)

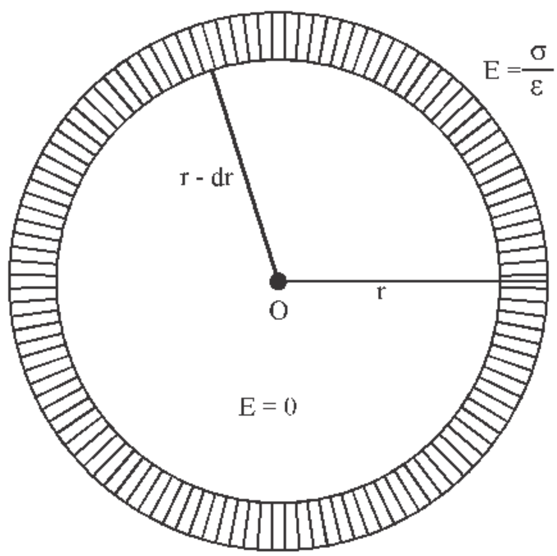


Fig 2.21 : Spherical Charge Distribution

Outward pressure on the surface of shell

$$P = \frac{\sigma^2}{2 \in_0} \qquad \ldots (2.69)$$

So the outward force on the surface

$$F = PA = \frac{\sigma^2}{2 \in_0} \times 4\pi r^2 \qquad \ldots (2.70)$$

Work done against this force in compressing the shell by a small amount dr is

$$dW = Fdr = \frac{\sigma^2}{2 \in_0} 4\pi r^2 dr$$

Reduction in volume of sphere (or increase in volume of the region where electric field is present)

$dV = 4\pi r^2 dr$

So $$dW = \frac{\sigma^2}{2 \in_0} dV \qquad \ldots (2.71)$$

So energy stored in electric field

$$W = U = \int \frac{\sigma^2}{2 \in_0} dV = \int \frac{1}{2} \in_0 E^2 dV \qquad \ldots (2.72)$$

and the energy stored per unit voume in electric field or energy density.

$$U_V = \frac{dW}{dV} = \frac{\sigma^2}{2 \in_0} = \frac{1}{2} \in_0 E^2 \qquad \ldots (2.73)$$

If some other medium other then free space or air is considered then

$$U_V = \frac{dW}{dV} = \frac{\sigma^2}{2 \in_0} = \frac{1}{2} \in_0 E^2 \qquad \ldots (2.73)$$

Although the above relations have been derived by considering a spherical shell but their validity is general.

2.7 Equilibrium of a Charged Soap Bubble

For a soap bubble, the pressure at its internal surface is more than the atmospheric pressure present at its outer surface. This excess pressure is balanced by pressure due to surface tension. If the radius of bubble is r and surface tension is T then excess pressure.

$$P_{ex} = \frac{4T}{r} \qquad \ldots (2.74)$$

If now the soap buble is charged with surface charge density σ then an outward electricstatic pressure

$\frac{\sigma^2}{2 \in_0}$ also acts on the surface of bubble. In this case

$$P_{ex} + \frac{\sigma^2}{2\epsilon_0} = \frac{4T}{r}$$

or $$P_{ex} = \frac{4T}{r} - \frac{\sigma^2}{2\epsilon_0} \qquad \ldots (2.75)$$

On charging bubble in such a manner a situation arise in which excess pressure becomes zero after which the bubble bursts, so for equilibrium,

$$P_{ex} = \frac{4T}{r} - \frac{\sigma^2}{2\epsilon_0} = 0$$

or $$\frac{4T}{r} = \frac{\sigma^2}{2\epsilon_0} \qquad \ldots (2.76)$$

Equilibrium radius of bubble $$r = \frac{8T\epsilon_0}{\sigma^2} \qquad \ldots (2.77)$$

and surface charge density $$\sigma = \sqrt{\frac{8T\epsilon_0}{r}} \qquad \ldots (2.78)$$

and charge $$q = \sigma \times 4\pi r^2 = 4\pi\sqrt{8T\epsilon_0 r^3} \qquad \ldots (2.79)$$

Example 2.17 : The surface charge density for a charged soap bubble is 2.96 $\mu C/m^2$. The surface tension of soap solution is 4×10^{-4} N/m . Find the radius of soap bubble so that excess pressure is zero and the bubble is in equilibrium.

Solution :

$$r = \frac{8T\epsilon_0}{\sigma^2} = \frac{8\times4\times10^{-4}\times8.85\times10^{-12}}{\left(2.96\times10^{-6}\right)^2}$$

$$= 3.2\times10^{-1}\ \text{m}$$

$$r = 0.32\ \text{m}$$

Important Points

1. Electricflux through a surface is proportional to the net number of electric field lines passing through that area. electric flux through a flat area in a uniform electric field is

   $\phi = ES\cos\theta$

   Where θ is the angle between direction of E and normal to the area. If electric field vector is denoted by $\vec{E}$ be and area vector by $\vec{S}$ then

   $\phi = \vec{E}\cdot\vec{S}$

2. If the electric field is non uniform and (or) the surface is not flat then electric flux

   $\phi = \int \vec{E}\cdot d\vec{s}$

   Where $d\vec{s}$ is vector area of some surface element and $\vec{E}$ is electric field at this element. If the surface is closed

   $\phi = \oint \vec{E}\cdot d\vec{s}$

3. Depending upon the relative orientation of $\vec{E}$ and $\vec{S}$ (or $d\vec{s}$) ϕ may be positive, negative or zero.

4. For determining electric field due to some continuous charge distribution, it is divided into small charge elements dq and electric field due to each such element is then integrated to obtain total electric field. Electric field at a distance r from charge element dq

$$dE = \frac{1}{4\pi \in_0} \frac{dq}{r^2}$$

For complete continuous distribution

$$E = \int dE = \frac{1}{4\pi \in_0} \int \frac{dq}{r^2}$$

If charge distribution is linear $dq = \lambda d\ell$ where λ is charge linear charge density and is length of element.

If charge distribution is superfecial $dq = \sigma ds$ σ is surface charge density and ds is area of element.

In case of a volume charge distribution $dq = \rho dV$ where p is volume charge density and dV is volume element.

5. Gauss's law is valid only for closed surfaces. According to it the the net flux through a closed surface in an electric field

$$\phi = \oint \vec{E} \cdot d\vec{s} = \frac{q}{\in_0}$$

here q is the net charge enclosed by the closed surface.

6. Gauss's law is very helpful for finding electric field when charge distributions have a high degree of symmetry.

7. For an infinite line charge of linear charge density λ

$$E = \frac{\lambda}{2\pi \in_0 r}$$

8. For a uniformly charged infinity non conducting sheet (surface charge density σ)

$$E = \frac{\sigma}{2\varepsilon_0}$$ (uniform electric Field)

9. For a uniformly charged infinite conducting (surface charge density) $E = \frac{\sigma}{\in_0}$ (uniform electric field)

10. Continuous charge distribution can be classified as

(i) linear charge distribution distribution (ii) surface cahrge distribution (iii) volume charge distribution

11. For a uniformly charged spherical shell or spherical conductor Esurface

$$E_{surface} = \frac{q}{4\pi \in_0 R^2}(r = R); \ E_{in} = 0$$

[Here Q is the charge on sphere and R is its radius]

12. For a uniformly charged non conducting sphere

$$E_{surface} = \frac{Q}{4\pi \in_0 R^2}(r = R)$$

$$E_{in} = \frac{Qr}{4\pi \in_0 R^3}(r < R)$$

13. Pressure on surface of a charged conductor

$P = \frac{\sigma^2}{2\in_0} = \frac{1}{2}\in_0 E^2$ this is due to mutual repulsion between charges residing on surface.

14. Energy per unit volume in an electric field

$$U_V = \frac{\sigma^2}{2\in_0} = \frac{1}{2}\in_0 E^2$$

15. For the equilibrium of a charged soap bubble.

$\frac{4T}{r} = \frac{\sigma^2}{2\in}$ (T is surface Tension, r is radius of soap bubble)

Questions For practice

Multiple Choice Questions

1. The electric field due to a uniformly charged solid non conducting sphere is maximum at -

(a) the centre

(b) the mid point between centre and surface

(c) its surface (d) infinity

2. For free space the energy density in some region where electric field is E, is given by

(a) $\frac{1}{2}\in_0 E$ (b) $\frac{E^2}{2\in_0}$

(c) $\frac{1}{2}E\in_0^2$ (d) $\frac{1}{2}\in_0 E^2$

3. 1 μC charge is present at the centre of a cube of edge a. The flux through each face of the cube will be (in VM)

(a) 1.12×10^4 (b) 2.2×10^4

(c) 1.88×10^4 (d) 3.14×10^4

4. Two electric dipoles having charges $\pm q$ are placed mutually perpendicular to each other. The net electric flu through the cube is

(a) $\frac{q}{\in_0}$ (b) $\frac{4q}{\in_0}$

(c) Zero (d) $\frac{2q}{\in_0}$

5. On charging a soap bubble with negative charge its radius

(a) decreases

(b) increases

(c) remains unchanged

(d) nothing can be said due to incomplete information

6. A charge q is in a sphere and flux through the sphere is $\frac{q}{\in_0}$. On reducing the radius of sphere by half the change in the flux is
(a) four times its intital value
(b) one fourth of its initial value
(c) half of its initial value
(d) unchanged

7. Complete flux due to a unit charge placed in air is

(a) $\in_0$ (b) $\in_0^{1}$

(c) $(4\pi \in_0)^{-1}$ (d) $4\pi \in_0$

8. The radii of two conducting spheres are a and b. When these are charged with same surface charge density the ratio of electric field intensities at their surfaces is

(a) $b^2 : a^2$ (b) 1 : 1

(c) $a^2 : b^2$ (d) b : a

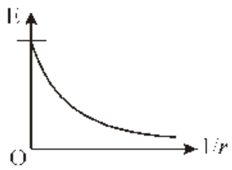
9. The radii of two conducting spheres are a and b. When these are given same charge the ratio of electric field intensities at their surfaces is

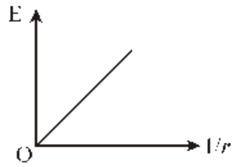
(a) $b^2 : a^2$ (b) 1 : 1

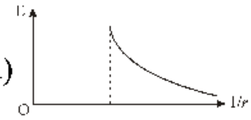
(c) $a^2 : b^2$ (d) $b : a$

10. Electric field intensity due to a long straight charged wire varies with $1/r$ as shown in

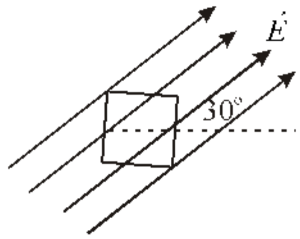
(a)


(b)


(c)


(d)


11. A square is situated in a uniform horizantal electric field such that a line drawn in the plane of square makes angle of 30° with electric field (Fig). If side of square is a the flux through the sphere will be



(a) $\frac{\sqrt{3}Ea^2}{2}$ (b) $\frac{Ea^2}{2}$

(c) Zero (d) none of these

Very Short Answer Questions :

1. When does the electric flux through an area element placed in an electric field $\vec{E}$ is zero?
2. At what positions the electric field intensity due to a uniformly charged sphere is zero?
3. Write the expression for force per unit area of a charged conductor and give its direction.
4. Where does the energy due to a charge is stored?
5. A charge Q is a given to a conducting sphere of diameter d what is the value of electric field inside the sphere?
6. Suppose the Coulomb's law has a $1 / r^3$ dependence instead of $1 / r^2$ dependence, is the Gauss's law still valid?
7. If the net charge enclosed by a Gaussian surface is positive then what is the nature of flux through the surface?

8. If the net flux through some closed surface in an electric field is zero what can be said about the surface?

9. If net charged enclosed by a Gaussian surface is zero does it mean that electric field at every point on the surface is zero.

10. Define linear charge density.

11. What will be the change in electric field in moving from one side to the other of a charged plane sheet having surface charge density σ.

12. Graph the variation of electric field with distance for a uniformly charged non conducting sheet.

13. What is the value of electric field at the centre of a uniformly charged non conducting sphere?

14. A charge q is at the centre of a sphere. If now this charge is placed at the centre of a cylinder of same volume then what will be the ratio of net flux in the tweo cases?

Short Answer Questions :

1. Explain the term electric flux. Write its SI unit and dimensions.

2. Explain the term linear charge density. Write its SI unit.

3. Explain the term surface charge density. Write its SI unit.

4. Explain the term volume charge density. Write its SI unit.

5. State Gauss's law for electrostatics.

6. The excess charge given to a conductor reside always on its outer surface? Why?

7. Establish expressions for electric force and electrostatic pressure on the surface of a charged conductor?

8. Establish expression for energy stored per unit volume in electric field.

9. Establish expression for maximum charged density for the equilibrium of a charged soap bubble.

10. Verify Gauss's law from Coulomb's law.

11. You are travelling in a car. Lighting is expected what should you do about your safety?

12. Consider two long straight line charges having linear charge densities λ_1 and λ_2 . Derive expression for the force per unit length acting between them.

13. Consider two infinite parallel planes having charge densities + and - respectively. What is the magnitude of electric field at some point in region between them.

Essay type Questions

1. For a spherical conductor of radius R having a charge q determine electric field for following situations

   (A) $r > R$ (B) $r < R$

   (C) at its surface (D) at its centre

   Graph the variation of electric field with distance.

2. Determine the electric field due to a uniformly charged sphere for following cases

   (A) Outside the sphere

   (B) At the surface of sphere

   (C) Inside the sphere

   (D) At the centre of sphere

3. Using Gauss's law determine the intensity of electric field at a point near a uniformly charged infinite wire. Graph the variation of electric field with distance.

4. Using Gauss's law determine the intensity of electric field at a point near a uniformly charged infinite non conducting plane. Explain the dependence of electric field.

5. Determine the direction of electric field due to a uniformly charged infinite conducting plate for points in its vicinity. Using Gauss's law determine expression for its electric field. Draw necessary diagrams.

Answers (Multiple Choice Questions)

1. (c) 2. (d)   3. (c)   4. (c)   5. (b)   6. (d)   7. (b)
8. (b) 9. (a)   10. (c) 11. (c)

Very Short Answer Questions :

1. When $\vec{E}$ and $d\vec{s}$ are mutually perpendicular.

2. At its centre and infinity

3. $\frac{\sigma^2}{2\in_0}$ and is directed normally outward.

4. In the region of elecric field.

5. Zero

6. No, Gauss's law holds only for fields which obeys inverse square law.

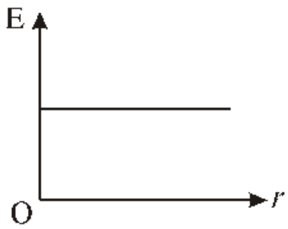
7. Positive and outwards

8. Net charged enclosed by the surface is zero and $\phi_{out} = \phi_{in}$

9. No, from $\phi = \oint_S \vec{E}\cdot d\vec{s} = \frac{q}{\in_0} = 0$ this situation is possible when $\vec{E}$ but $\vec{E}$ is perpendicular to $d\vec{s}$.

10. Amount of charge per unit length.

11. $\frac{\sigma}{2\in_0} - \left(-\frac{\sigma}{2\in_0}\right) = \frac{\sigma}{\in_0}$

12. 

13. Zero

14. 1 : 1

Numerical Problems

1. The flux entering and leaving a closed surface are400 Nm^2/C and 800 $N\,m^2/C$ respectively. What is the net charge enclosed by this surface. (Ans : 3.54 nC)

2. The surface charge density on a uniformly charged conducting sphere is $80\,\mu C/m^2$. Calculate the charge on sphere and net flux through surface.

   (Ans : 1.45 mC, 1.63×10^8 Nm^2/C)

3. Consider a cube of side a, Let a charge q be placed

   (i) at entre   (ii) at one corner of cube

   (iii) at one face of the cube

   For each of the above cases calculate the total flux linked with cube and flux linked with each face.

Ans : (i) $\frac{q}{6\varepsilon_0}\ \frac{q}{6\varepsilon_0}$ (ii) $\frac{q}{4\in_0}, \frac{q}{16\in_0}$ (iii) $\frac{q}{2\in_0}, \frac{q}{10\in_0}$

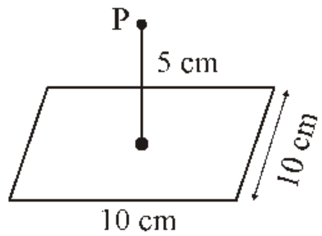
4. The intensity of electric field due to a charged sphere at a point at a distance of 20 cm from in centre is $10\,V/m$. The radius of sphere is 5 cm. Determine the intensity of electric field at a distance of 8 cm from the centre.

   (Ans : 62.5 V / m)

5. An infinite line charge produces on electric field of 9×10^4 N/C at 2 cm from it. Determine the linear charge density.

   (Ans : 10^{-7} C/m½)

6. A charge of 10 μC is placed directly above the centre of a square of 10 cm side at a height of 5 cm as shown in Fig. Determine the magnitude of electric flux through the square?



   (Ans : 1.88×10^5 Nm^2/C)

7. A charge of 10 μC is given to a metallic plate of area $10^{-2}\,m^2$. Determine the intensity of electric fields at points near by.

   (Ans : 5.65×10^7 V/ m)

8. Two metallic plates each of area are $1m^2$ placed parallel to each other at a separation of 0.05 m. Both have charges of equal magnitude but of opposite nature. If the magnitude of electric field in

space between them is 5.5 V/m then calculate the magnitude of charge on each plate.

(Ans : 4.87 × 10^{10} C)

9. A particle of mass 9×10^{-5} gm is placed at some height above a uniformly charged horizontal infinite non conducting plate having a surface charge density $5 \times 10^{-5}\ C/m^2$. What should be the charge on the particle so that on releasing it will not fall down.

(Ans : $3.12 \times 10^{-13}\ C$)

10. A large uniformly charged sheet having a surface charge density of $5 \times 10^{16}\ C/m^2$ lies in X – Y plane. Calculate the electric flux through a circular loop of radius 0.1 m, whose axis makes on angle of 60^0 with Z axis.

(Ans : $4.44 \times 10^{7}\ Nm^2/C$)

11. An electron of 10^3 eV energy is fired from a distance of 5mm perpendicularly towards an infinite charged conducting plate. What should be the minimum charge density on plate so that electron fails to strike the plate.

(Ans : $1.77 \times 10^{-6}\ C/m^2$)

12. The internal and external pressures for a soap bubble are same. The surface tension or soap soultion is 0.04 N/m and its diameter is 4 cm. Determine the charge on soap bubble.

(Ans : 59.8 nC)