3.5 Rigid body dynamics

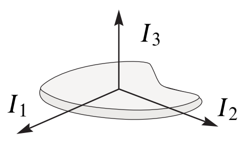
Moment of inertia tensor

Moment of inertia tensor[a]	$I_{ij}=\int (r^2\delta_{ij}-x_ix_j)\,\mathrm{d}m$	(3.136)	r δ_{ij}	$r^2=x^2+y^2+z^2$ Kronecker delta
	$\mathbf{I}=\begin{pmatrix} \int (y^2+z^2)\,\mathrm{d}m & -\int xy\,\mathrm{d}m & -\int xz\,\mathrm{d}m \\ -\int xy\,\mathrm{d}m & \int (x^2+z^2)\,\mathrm{d}m & -\int yz\,\mathrm{d}m \\ -\int xz\,\mathrm{d}m & -\int yz\,\mathrm{d}m & \int (x^2+y^2)\,\mathrm{d}m \end{pmatrix}$	(3.137)	$\mathbf{I}$ $\mathrm{d}m$ x_i I_{ij}	moment of inertia tensor mass element position vector of $\mathrm{d}m$ components of $\mathbf{I}$
Parallel axis theorem	$I_{12}=I^{\star}_{12}-ma_1a_2$ $I_{11}=I^{\star}_{11}+m(a_2^2+a_3^2)$ $I_{ij}=I^{\star}_{ij}+m(\lvert\boldsymbol{a}\rvert^2\delta_{ij}-a_ia_j)$	(3.138) (3.139) (3.140)	$I^{\star}_{ij}$ $a_i,\boldsymbol{a}$ m	tensor with respect to centre of mass position vector of centre of mass mass of body
Angular momentum	$\boldsymbol{J}=\mathbf{I}\boldsymbol{\omega}$	(3.141)	$\boldsymbol{J}$ $\boldsymbol{\omega}$	angular momentum angular velocity
Rotational kinetic energy	$T=\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{J}=\frac{1}{2}I_{ij}\omega_i\omega_j$	(3.142)	T	kinetic energy

[a] I_{ii} are the moments of inertia of the body. I_{ij} ($i\neq j$) are its products of inertia. The integrals are over the body volume.

Principal axes

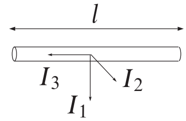
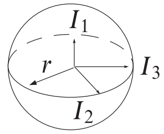
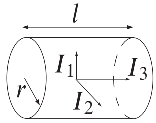
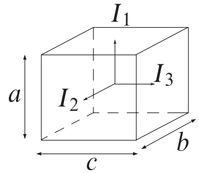
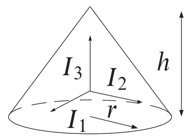
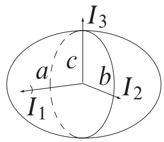
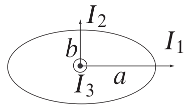
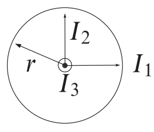
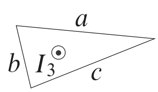
Principal moment of inertia tensor	$\mathbf{I}'=\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$	(3.143)	$\mathbf{I}'$ I_i	principal moment of inertia tensor principal moments of inertia
Angular momentum	$\boldsymbol{J}=(I_1\omega_1,I_2\omega_2,I_3\omega_3)$	(3.144)	$\boldsymbol{J}$ ω_i	angular momentum components of $\boldsymbol{\omega}$ along principal axes
Rotational kinetic energy	$T=\frac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)$	(3.145)	T	kinetic energy
Moment of inertia ellipsoid[a]	$T=T(\omega_1,\omega_2,\omega_3)$ $J_i=\frac{\partial T}{\partial \omega_i}$ ($\boldsymbol{J}$ is $\perp$ ellipsoid surface)	(3.146) (3.147)		
Perpendicular axis theorem	$I_1+I_2\begin{cases}\geq I_3 & \text{generally} \\ =I_3 & \text{flat lamina} \perp \text{to 3-axis}\end{cases}$	(3.148)		
Symmetries	$I_1\neq I_2\neq I_3$ asymmetric top $I_1=I_2\neq I_3$ symmetric top $I_1=I_2=I_3$ spherical top	(3.149)		



lamina

[a] The ellipsoid is defined by the surface of constant T.

Moments of inertia[a]

Thin rod, length l	$$I_1 = I_2 = \frac{ml^2}{12}$$ $$I_3 \simeq 0$$	(3.150) (3.151)	
Solid sphere, radius r	$$I_1 = I_2 = I_3 = \frac{2}{5} mr^2$$	(3.152)	
Spherical shell, radius r	$$I_1 = I_2 = I_3 = \frac{2}{3} mr^2$$	(3.153)	
Solid cylinder, radius r, length l	$$I_1 = I_2 = \frac{m}{4}\left(r^2 + \frac{l^2}{3}\right)$$ $$I_3 = \frac{1}{2} mr^2$$	(3.154) (3.155)	
Solid cuboid, sides a,b,c	$$I_1 = m(b^2 + c^2)/12$$ $$I_2 = m(c^2 + a^2)/12$$ $$I_3 = m(a^2 + b^2)/12$$	(3.156) (3.157) (3.158)	
Solid circular cone, base radius r, height h[b]	$$I_1 = I_2 = \frac{3}{20} m \left(r^2 + \frac{h^2}{4}\right)$$ $$I_3 = \frac{3}{10} mr^2$$	(3.159) (3.160)	
Solid ellipsoid, semi-axes a,b,c	$$I_1 = m(b^2 + c^2)/5$$ $$I_2 = m(c^2 + a^2)/5$$ $$I_3 = m(a^2 + b^2)/5$$	(3.161) (3.162) (3.163)	
Elliptical lamina, semi-axes a,b	$$I_1 = mb^2/4$$ $$I_2 = ma^2/4$$ $$I_3 = m(a^2 + b^2)/4$$	(3.164) (3.165) (3.166)	
Disk, radius r	$$I_1 = I_2 = mr^2/4$$ $$I_3 = mr^2/2$$	(3.167) (3.168)	
Triangular plate[c]	$$I_3 = \frac{m}{36}(a^2 + b^2 + c^2)$$	(3.169)	

[a]With respect to principal axes for bodies of mass m and uniform density. The radius of gyration is defined as $k = (I/m)^{1/2}$.
[b]Origin of axes is at the centre of mass ($h/4$ above the base).
[c]Around an axis through the centre of mass and perpendicular to the plane of the plate.

Centres of mass

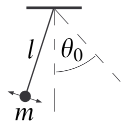
Solid hemisphere, radius r	$d = 3r/8$ from sphere centre	(3.170)
Hemispherical shell, radius r	$d = r/2$ from sphere centre	(3.171)
Sector of disk, radius r, angle 2θ	$d = \frac{2}{3} r \frac{\sin\theta}{\theta}$ from disk centre	(3.172)
Arc of circle, radius r, angle 2θ	$d = r \frac{\sin\theta}{\theta}$ from circle centre	(3.173)
Arbitrary triangular lamina, height h[a]	$d = h/3$ perpendicular from base	(3.174)
Solid cone or pyramid, height h	$d = h/4$ perpendicular from base	(3.175)
Spherical cap, height h, sphere radius r	solid: $d = \frac{3}{4} \frac{(2r-h)^2}{3r-h}$ from sphere centre	(3.176)
	shell: $d = r - h/2$ from sphere centre	(3.177)
Semi-elliptical lamina, height h	$d = \frac{4h}{3\pi}$ from base	(3.178)

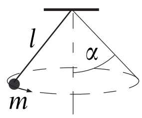
[a] h is the perpendicular distance between the base and apex of the triangle.

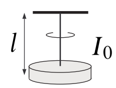
Pendulums

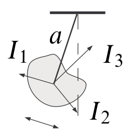
Simple pendulum	$P = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16} + \cdots\right)$	(3.179)	P period; g gravitational acceleration; l length; θ_0 maximum angular displacement
Conical pendulum	$P = 2\pi\left(\frac{l\cos\alpha}{g}\right)^{1/2}$	(3.180)	α cone half-angle
Torsional pendulum[a]	$P = 2\pi\left(\frac{lI_0}{C}\right)^{1/2}$	(3.181)	I_0 moment of inertia of bob; C torsional rigidity of wire (see page 81)
Compound pendulum[b]	$P \simeq 2\pi\left[\frac{1}{mga}(ma^2 + I_1\cos^2\gamma_1 + I_2\cos^2\gamma_2 + I_3\cos^2\gamma_3)\right]^{1/2}$	(3.182)	a distance of rotation axis from centre of mass; m mass of body; I_i principal moments of inertia; γ_i angles between rotation axis and principal axes
Equal double pendulum[c]	$P \simeq 2\pi\left[\frac{l}{(2\pm\sqrt{2})g}\right]^{1/2}$	(3.183)	

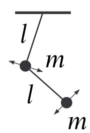
[a] Assuming the bob is supported parallel to a principal rotation axis.
[b] I.e., an arbitrary triaxial rigid body.
[c] For very small oscillations (two eigenmodes).



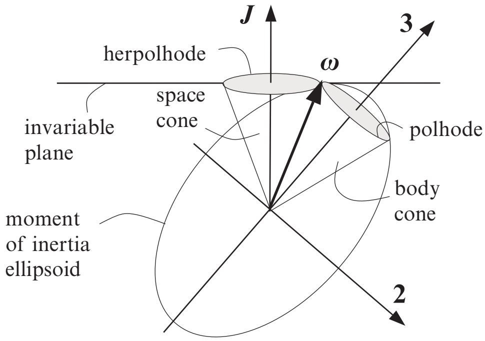




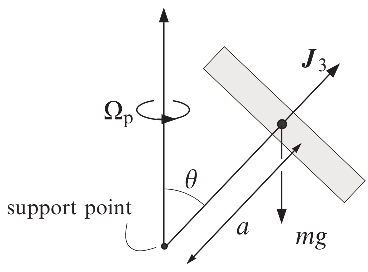




Tops and gyroscopes



prolate symmetric top



gyroscope

Euler's equations[a]	$G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$	(3.184)	G_i external couple (=0 for free rotation)
	$G_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$	(3.185)	I_i principal moments of inertia
	$G_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$	(3.186)	ω_i angular velocity of rotation
Free symmetric top[b] ($I_3 < I_2 = I_1$)	$\Omega_b = \frac{I_1 - I_3}{I_1}\omega_3$	(3.187)	Ω_b body frequency; Ω_s space frequency
	$\Omega_s = \frac{J}{I_1}$	(3.188)	J total angular momentum
Free asymmetric top[c]	$\Omega_b^2 = \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2}\omega_3^2$	(3.189)	
Steady gyroscopic precession	$\Omega_p^2 I_1' \cos\theta - \Omega_p J_3 + mga = 0$	(3.190)	Ω_p precession angular velocity; θ angle from vertical; J_3 angular momentum around symmetry axis
	$\Omega_p \simeq \begin{cases} Mga/J_3 & \text{(slow)} \\ J_3/(I_1'\cos\theta) & \text{(fast)} \end{cases}$	(3.191)	m mass; g gravitational acceleration
Gyroscopic stability	$J_3^2 \geq 4 I_1' mga \cos\theta$	(3.192)	a distance of centre of mass from support point; I_1' moment of inertia about support point
Gyroscopic limit ("sleeping top")	$J_3^2 \gg I_1' mga$	(3.193)	
Nutation rate	$\Omega_n = J_3/I_1'$	(3.194)	Ω_n nutation angular velocity
Gyroscope released from rest	$\Omega_p = \frac{mga}{J_3}(1 - \cos\Omega_n t)$	(3.195)	t time

[a]Components are with respect to the principal axes, rotating with the body.
[b]The body frequency is the angular velocity (with respect to principal axes) of $\boldsymbol{\omega}$ around the **3**-axis. The space frequency is the angular velocity of the **3**-axis around $\boldsymbol{J}$, i.e., the angular velocity at which the body cone moves around the space cone.
[c]$\boldsymbol{J}$ close to **3**-axis. If $\Omega_b^2 < 0$, the body tumbles.