CHAPTER TWO

Complex Numbers

DEFINITIONS

A number of the form $a + ib$ where $a, b \in \mathbf{R}$, the set of real numbers, and $i = \sqrt{-1}$, is called a *complex number*. A complex number can also be defined as an *ordered pair* of real numbers a and b, and may be written as (a, b), where the first number denotes the *real part* and the second number denotes the *imaginary part*. If $z = a + ib$, then the real part of z is denoted by $\text{Re}(z)$ and the imaginary part of z is denoted by $\text{Im}(z)$.

A complex number z is said to be *purely real* if $\text{Im}(z) = 0$ and is said to be *purely imaginary* if $\text{Re}(z) = 0$. Note that the complex number $0 = 0 + i0$ is both purely real and purely imaginary. It is the only complex number with this property.

We denote the set of all complex numbers by $\mathbf{C}$. That is, $\mathbf{C} = \{a + ib \mid a, b \in \mathbf{R}\}$. Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are said to be *equal* if $a_1 = a_2$ and $b_1 = b_2$.

Very Important

- The main advantage of complex number is to use **one** symbol z for an ordered pair (x, y) of real numbers. We lose this advantage whenever we replace z by $x + iy$.
- As far as possible do not write $z = x + iy$ while solving the problems. Putting $z = x + iy$ should be the last resort rather than the first option.

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBERS

1. *Addition:* $(a + ib) + (c + id) = (a + c) + i(b + d)$
2. *Subtraction:* $(a + ib) - (c + id) = (a - c) + i(b - d)$
3. *Multiplication:* $(a + ib)(c + id)$
$$= (ac - bd) + i(ad + bc)$$
4. *Reciprocal:* If at least one of a, b is non-zero then the reciprocal of $a + ib$ is given by
$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$
5. *Quotient:* If at least one of c, d is non-zero, then quotient of $a + ib$ and $c + id$ is given by
$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$
$$= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

CONJUGATE OF COMPLEX NUMBER

Let $z = a + ib$ be a complex number. We define conjugate of z, denoted by $\bar{z}$ to be the complex number $a - ib$. That is, if $z = a + ib$, then $\bar{z} = a - ib$.

Properties of Conjugate of a Complex Number

(i) $z_1 = z_2 \Leftrightarrow \bar{z}_1 = \bar{z}_2$

(ii) $\overline{(\bar{z})} = z$

(iii) $z + \bar{z} = 2\,\text{Re}(z)$

(iv) $z - \bar{z} = 2i\,\text{Im}(z)$

(v) $z = \bar{z} \Leftrightarrow z$ is purely real

(vi) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(vii) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(viii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ix) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(x) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(xi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$

(xii) If $P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ where $a_0, a_1, \dots a_n$ and z are complex number, then
$$\overline{P(z)} = \bar{a}_0 + \bar{a}_1(\bar{z}) + \bar{a}_2(\bar{z})^2 + \cdots + \bar{a}_n(\bar{z})^n$$
$$= \bar{P}(\bar{z})$$
where $\bar{P}(z) = \bar{a}_o + \bar{a}_1 z + \bar{a}_2 z^2 + \cdots + \bar{a}_n z^n$

(xiii) If $R(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials in z, and $Q(z) \neq 0$, then

$$\overline{R(z)} = \frac{\bar{P}(\bar{z})}{\bar{Q}(\bar{z})}$$

Illustration 1

$$\overline{\left(\frac{z+3z^2}{z-1}\right)} = \frac{\bar{z}+3\bar{z}^2}{\bar{z}-1}$$

(xiv) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where a_i, b_i, c_i $(i = 1, 2, 3)$ are complex numbers.

MODULUS OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number. We define the *modulus* or the *absolute value* of z to be the real number $\sqrt{a^2+b^2}$ and denote it by $|z|$.
Note that $|z| \geq 0 \; \forall \; z \in \mathbf{C}$

Properties of Modulus

If z is a complex number, then

(i) $|z| = 0 \Leftrightarrow z = 0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$

(iv) $-|z| \leq \text{Im}(z) \leq |z|$

(v) $z\bar{z} = |z|^2$

In particular, note that if $z \neq 0$, then $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$ so that $\text{Re}\left(\dfrac{1}{z}\right) = \dfrac{\text{Re}(z)}{|z|^2}$ and $\text{Im}\left(\dfrac{1}{z}\right) = -\dfrac{\text{Im}(\bar{z})}{|z|^2}$.

If z_1, z_2 are two complex numbers, then

(vi) $|z_1 z_2| = |z_1|\,|z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

(viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1\bar{z}_2$
$= |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

(ix) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1\bar{z}_2$
$= |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(xi) If a and b are real numbers and z_1, z_2 are complex numbers, then

$$|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

(xii) If $z_1, z_2 \neq 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow z_1/z_2$ is purely imaginary.

(xiii) *Triangle Inequality*. If z_1 and z_2 are two complex numbers, then $|z_1 + z_2| \leq |z_1| + |z_2|$.

The equality holds if and only if $z_1\bar{z}_2 \geq 0$.

In general, $|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$ and the equality sign holds if and only if the ratio of any two non-zero terms is positive.

(xiv) $|z_1 - z_2| \leq |z_1| + |z_2|$

(xv) $\big||z_1| - |z_2|\big| \leq |z_1| + |z_2|$

(xvi) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(xvii) If a_1, a_2, a_3 and a_4 are four complex numbers, then $|z - a_1| + |z - a_2| + |z - a_3| + |z - a_4| \geq \max\{|a_1 - a_l| + |a_m - a_n|: l, m, n$ are distinct integers lying in $\{2, 3, 4\}$ and $m < n\}$.

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

A complex number $z = x + iy$ can be represented by a point P whose Cartesian co-ordinates are (x, y) referred to rectangular axes Ox and Oy, usually called the *real* and *imaginary* axes respectively. The plane is called the *Argand plane, complex plane* or *Gaussian plane*. The point $P(x, y)$ is called the *image* of the complex number z and z is said to be the *affix* or *complex co-ordinate* of point P.

Note

All purely real numbers lie on the real axis and all purely imaginary numbers lie on the imaginary axis. The complex number $0 = 0 + i0$ lies at the origin O.

We have $OP = \sqrt{x^2 + y^2} = |z|$. Thus, $|z|$ is the length of OP.

ARGUMENT OF A COMPLEX NUMBER

If z is a non-zero complex numbers represented by point P in the complex plane, then argument of z is the angle which OP makes with the positive direction of the real axis. See Fig. 2.1.

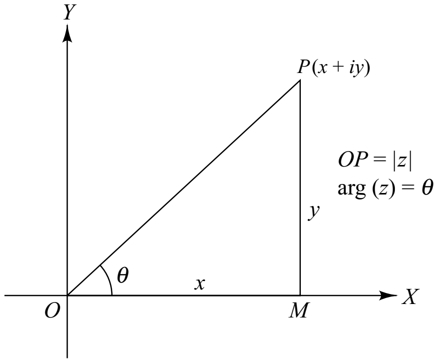
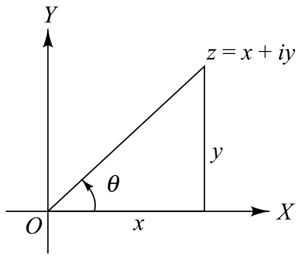


Fig. 2.1

> **Note**
>
> Argument of a non-zero complex number is not unique, since, if θ is a value of the argument, then $2n\pi + \theta$ where $n \in$ **I**, the set of integers, are also values of the argument of z. The value θ of the argument which satisfies the inequality $-\pi < \theta \le \pi$ is called the ***principal value*** of the argument or ***principal argument***.

Principal Value of the Argument for Different Positions of *z* in the Complex Plane

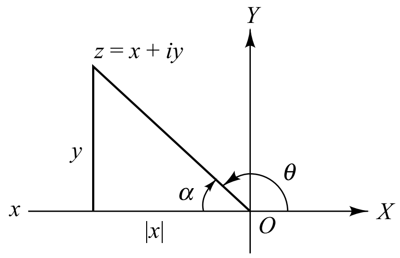
1. When z lies in the first quadrant



$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

Fig. 2.2 (i)

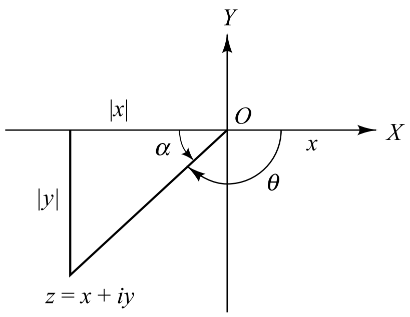
2. When z lies in the second quadrant



$$\arg(z) = \pi - \tan^{-1}\left(\frac{y}{|x|}\right)$$

Fig. 2.2 (ii)

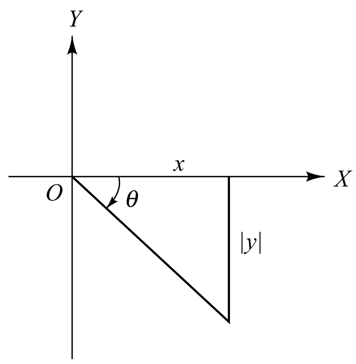
3. When z lies in the third quadrant



$$\arg(z) = -\pi + \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

Fig. 2.2 (iii)

4. When z lies in the fourth quadrant



$$\arg(z) = -\tan^{-1}\left(\frac{|y|}{x}\right)$$

Fig. 2.2 (iv)

Thus, if $z = x + iy$, then

$$\arg(z) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0, y \ge 0 \\ \tan^{-1}(y/x) - \pi & \text{if } x < 0, y < 0 \\ \pi/2 & \text{if } x = 0, y > 0 \\ -\pi/2 & \text{if } x = 0, y < 0 \end{cases}$$

> **Common Mistake**
>
> A usual mistake committed by the students is to take the argument of $z = x + iy$ as $\tan^{-1}(y/x)$ irrespective of the values of x and y. Kindly remember that $\tan^{-1}(y/x)$ lies in the interval $(-\pi/2, \pi/2)$ whereas the principal value of argument of z lies in the interval $(-\pi, \pi]$.

POLAR FORM OF A COMPLEX NUMBER

Let z be a non-zero complex number, then we can write

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg(z)$.

Illustration 2

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

and $$-1 - i = \sqrt{2}\left\{\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right\}$$

In fact, if $z = r(\cos\theta + i\sin\theta)$, then z is also given by

$$z = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$$

where k is any integer.

Euler's Formula

The complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$ or cis θ. That is

$$e^{i\theta} = \text{cis}\,\theta = \cos\theta + i\sin\theta$$

Some Important Results Involving Argument

If z, z_1 and z_2 are non-zero complex numbers, then

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) *$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

In fact,

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$

where

$$k = \begin{cases} 0 & \text{if } -\pi < \arg(z_1) + \arg(z_2) \le \pi \\ 1 & \text{if } -2\pi < \arg(z_1) + \arg(z_2) \le -\pi \\ -1 & \text{if } \pi < \arg(z_1) + \arg(z_2) \le 2\pi \end{cases}$$

(iii) *$\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(iv) *$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

In fact,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$$

where k is defined as in (ii) with + sign between $\arg(z_1)$ and $\arg(z_2)$ replaced by –

(v) $|z_1 + z_2| = |z_1 - z_2|$

$\Leftrightarrow \arg(z_1) - \arg(z_2) = \pi/2$

(vi) $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow \arg(z_1) = \arg(z_2)$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

where $r_1 = |z_1|$, $r_2 = |z_2|$

and $\theta_1 = \arg(z_1)$, $\theta_2 = \arg(z_2)$, then

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$

$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$

$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \cos(\theta_1 - \theta_2) = 0$

$\Leftrightarrow Oz_1$ is at right angles to Oz_2.

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$

$\Leftrightarrow Oz_1$ is perpendicular to Oz_2.

* L.H.S. and R.H.S. may differ by a multiple of 2π.

VECTOR REPRESENTATION OF COMPLEX NUMBERS

We can also represent the complex number $z = x + iy$ by using the vector $\overrightarrow{OP}$ joining the origin O of the complex plane to the point $P(x, y)$, instead of using point P itself. The length of the vector $\overrightarrow{OP}$, that is, $|\overrightarrow{OP}|$ is the modulus of z. The angle between the positive real axis and the vector $\overrightarrow{OP}$, more exactly, the angle through which the positive real axis must be rotated to cause it to have the same direction as $\overrightarrow{OP}$ (considered positive if the rotation is counter-clockwise and negative otherwise) is the argument of the complex number z.

GEOMETRICAL REPRESENTATION OF ALGEBRAIC OPERATIONS ON COMPLEX NUMBERS

Let z_1 and z_2 be two complex numbers represented by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively.

Sum

By definition, $z_1 + z_2$ should be represented by the point $(x_1 + x_2, y_1 + y_2)$. This point is nothing but the vertex P which completes the parallelogram with the line segments joining the origin with z_1 and z_2 as the adjacent sides. See Fig. 2.3

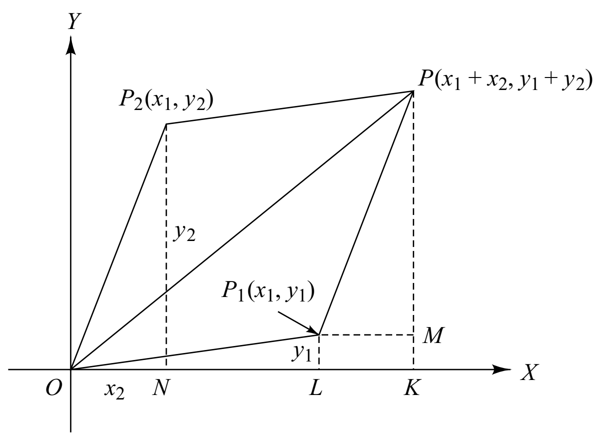


Fig. 2.3

Note that the addition of two complex numbers z_1 and z_2 follows the same law of addition as that of vectors, represented both in magnitude and direction by the line segments joining the origin and the points representing z_1 and z_2, for

$$\mathbf{OP_1} + \mathbf{OP_2} = \mathbf{OP_1} + \mathbf{P_1P} = \mathbf{OP}$$

Difference

We first represent $-z_2$ by P'_2 so that $P_2P'_2$ is bisected at O. Complete the parallelogram $OP_1\,PP'_2$. Then, it can be easily seen that P represents the difference $z_1 - z_2$. See Fig. 2.4.

As $OP_1\,PP'_2$ is a parallelogram so $P_1P = OP'_2$. Using vector notation, we have

$$z_1 - z_2 = \mathbf{OP_1} - \mathbf{OP_2} = \mathbf{OP_1} + \mathbf{P_2O} = \mathbf{OP_1} + \mathbf{OP'_2}$$
$$= \mathbf{OP_1} + \mathbf{P_1P} = \mathbf{OP} = \mathbf{P_2P_1}$$

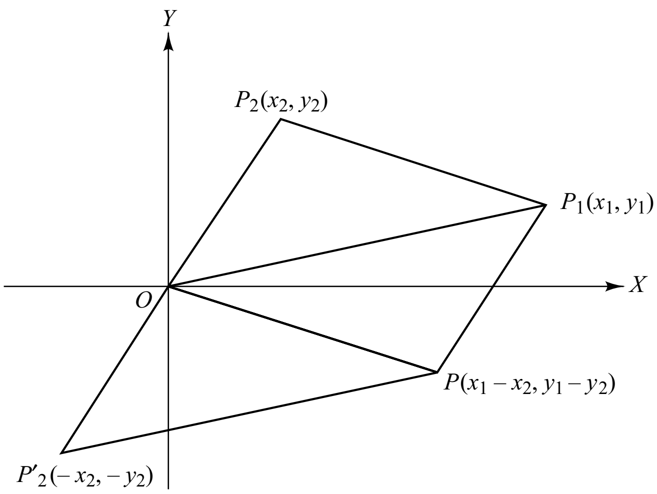


Fig. 2.4

Note the complex number $z_1 - z_2$ is represented by the vector $\mathbf{P_2P_1}$, where the points P_2 and P_1 represent the complex number z_2 and z_1 respectively.

Note that $\arg(z_1 - z_2)$ is the angle through which OX must be rotated in the anticlockwise direction so that it becomes parallel to P_2P_1.

Product (Multiplication)

Let $z_1 = r_1(\cos\,\theta_1 + i\sin\,\theta_1)$ and $z_2 = r_2\,(\cos\,\theta_2 + i\,\sin\,\theta_2)$. Then

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\,\theta_1)\,(\cos\,\theta_2 + i\,\sin\,\theta_2)$$
$$= r_1\,r_2\,\{\cos(\theta_1 + \theta_2) + i\,\sin\,(\theta_1 + \theta_2)\}$$

Thus $|z_1\,z_2| = r_1\,r_2$ and $\arg(z_1\,z_2) = \theta_1 + \theta_2$

This shows that modulus of the product of two complex numbers is the product of their moduli, and the argument of the product of any two complex numbers is the sum of their arguments. Using this, we shall derive a geometrical interpretation of the product of two complex numbers.

Let E be a point on the x-axis such that $OE = 1$ unit. (See Fig 2.5.) Complete the triangle OP_1E. Now, taking OP_2 as the base, construct a triangle OPP_2 similar to OP_1E so that

$$OP : OP_1 = OP_2 : OE$$

i.e., $\quad OP = OP_1 \cdot OP_2 \qquad [\because\ OE = 1]$

Also $\quad \angle P_2OP = \angle EOP_1 = \angle XOP_1 = \theta_1$

Thus $\quad \angle XOP = \theta_1 + \theta_2$

Hence P represents the complex number for which the modulus is r_1r_2 and the argument is $\theta_1 + \theta_2$. That is, it represents the complex number $z_1\,z_2$.

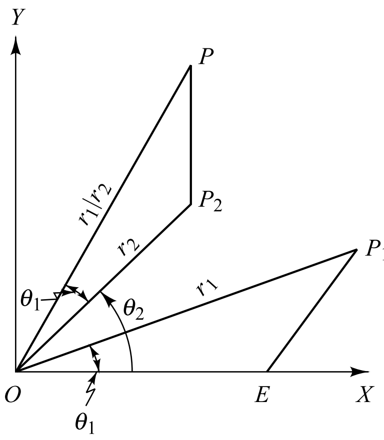


Fig. 2.5

Quotient

Let $z_1 = r_1\,(\cos\,\theta_1 + i\,\sin\,\theta_1)$ and $z_2 = r_2\,(\cos\,\theta_2 + i\,\sin\,\theta_2)$. We take $z_2 \neq 0$, so that $r_2 \neq 0$. Now

$$\frac{z_1}{z_2} = \frac{r_1\,(\cos\theta_1 + i\sin\theta_1)}{r_2\,(\cos\theta_2 + i\sin\theta_2)}$$
$$= \frac{r_1}{r_2}\,\{\cos\,(\theta_1 - \theta_2) + i\,\sin\,(\theta_1 - \theta_2)\}$$

We shall use this to get a geometrical interpretation of the quotient of a complex number by a non-zero complex number.

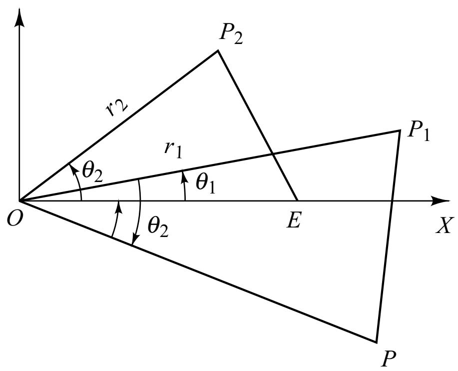


Fig. 2.6

Let P_1 and P_2 represent z_1 and z_2 respectively. On OP_1 construct the triangle OPP_1 similar to OEP_2, where E lies on the x-axis and $OE = 1$ unit. (See Fig. 2.6.)

Now, $\quad OP : OE = r_1 : r_2 \quad \Rightarrow \quad OP = \dfrac{r_1}{r_2}$

Also $\quad \angle XOP = \theta_1 - \theta_2$

The point P thus represents the quotient z_1/z_2, since its modulus is r_1/r_2 and its arguments is $\theta_1 - \theta_2$.

Remark

Note that if θ_1 and θ_2 are the principal values of arg z_1 and arg z_2 then $\theta_1 + \theta_2$ is not necessarily the principal value of $\arg(z_1 z_2)$, nor is $\theta_1 - \theta_2$ necessarily the principal value of $\arg(z_1/z_2)$.

Interpretation of $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$

If z_1, z_2, z_3 are the vertices of a triangle ABC described in the counter-clockwise sense, then

(i) $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \angle BAC = \alpha$ (say), and

(ii) $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{CA}{BA}(\cos\alpha + i\sin\alpha)$

See Fig. 2.7.

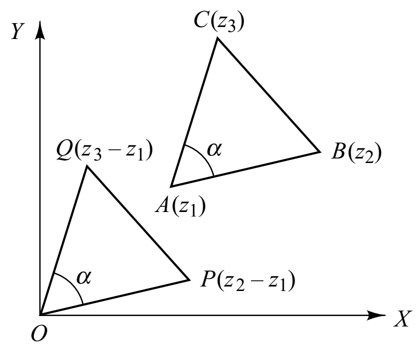


Fig. 2.7

Corollary The points z_1, z_2, z_3 will be collinear if and only if angle $\alpha = 0$ or π, i.e., if and only if $\dfrac{z_3 - z_1}{z_2 - z_1}$ is purely real.

Interpretation of $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right)$

Let z_1, z_2, z_3 and z_4 be four complex numbers. Then the line joining z_4 and z_3 is inclined to the line joining z_2 and z_1 at the following angle:

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$$

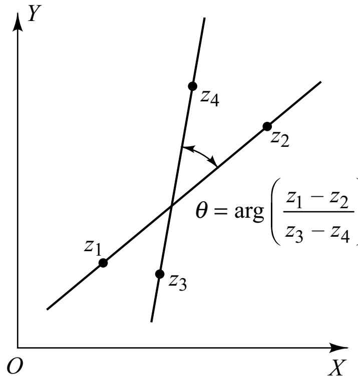


Fig. 2.8

Corollary The line joining z_4 and z_3 is inclined at 90° to the line joining z_2 and z_1 if

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\frac{\pi}{2}$$

i.e., if $z_1 - z_2 = \pm ik(z_3 - z_4)$, where k is a non-zero real number. (Fig. 2.9).

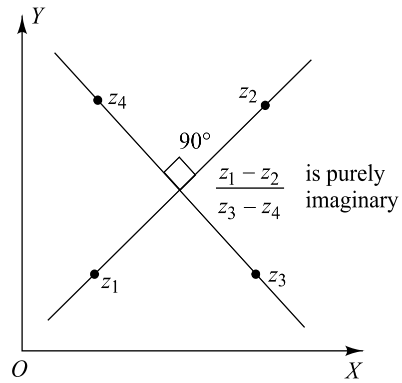


Fig. 2.9

SOME IMPORTANT GEOMETRICAL RESULTS AND EQUATIONS

1. **Distance Formula**

   Distance between $A(z_1)$ and $B(z_2)$ is given by

   $AB = |z_2 - z_1|$

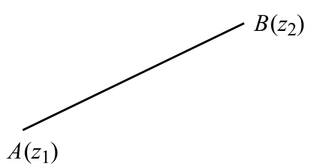


Fig. 2.10

2. **Section Formula**

   The point $P(z)$ which divides the join of the segment AB in the ratio $m : n$ is given by

$$z = \frac{mz_2 + nz_1}{m+n}$$

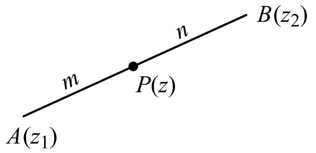


Fig. 2.11

3. Mid-point Formula

Mid-point $M(z)$ of the segment AB is given by

$$z = \frac{1}{2}(z_1 + z_2)$$

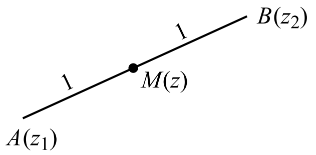


Fig. 2.12

4. Condition(s) *for four non-collinear $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ to represent vertices of a*

(i) Parallelogram

The diagonals AC and BD must bisect each other

$$\Leftrightarrow \quad \frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$$

$$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$$

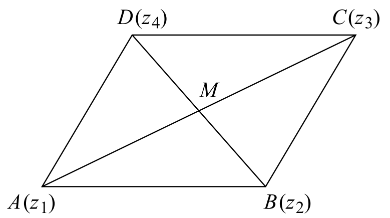


Fig. 2.13

(ii) Rhombus

(a) the diagonals AC and BD bisect each other

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$, and

(b) a pair of two adjacent sides are equal, for instance, $AD = AB$

$$\Leftrightarrow \quad |z_4 - z_1| = |z_2 - z_1|$$

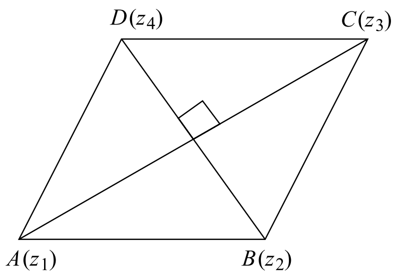


Fig. 2.14

(iii) Square

(a) the diagonals AC and BD bisect each other

$$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$$

(b) a pair of adjacent sides are equal; for instance, $AD = AB$

$$\Leftrightarrow \quad |z_4 - z_1| = |z_2 - z_1|$$

(c) the two diagonals are equal, that is,

$AC = BD \Leftrightarrow |z_3 - z_1| = |z_4 - z_2|$

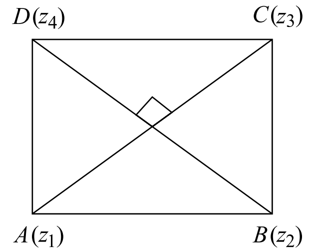


Fig. 2.15

(iv) Rectangle

(a) the diagonals AC and BD bisect each other

$$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$$

(b) the diagonals AC and BD are equal

$$\Leftrightarrow \quad |z_3 - z_1| = |z_4 - z_2|.$$

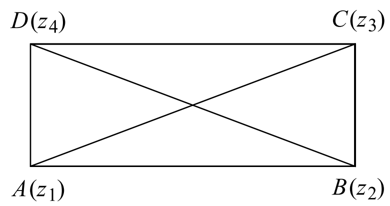
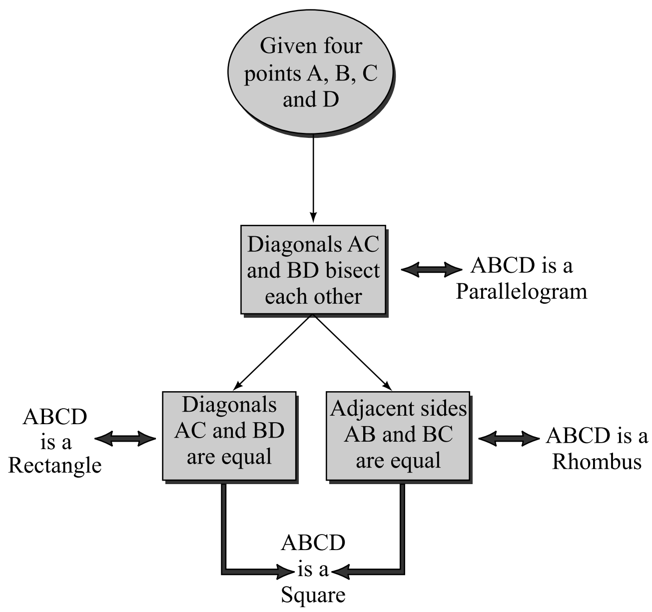


Fig. 2.16

Showing that four points, no three of which are collinear, form a Parallelogram/Rhombus/Square/Rectangle



5. Centroid, Incentre, Orthocentre and Circumcentre of a Triangle

Let ABC be a triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$,

(i) **Centroid** $G(z)$ of the ΔABC is the point of concurrence of medians of ΔABC and is given by

$$z = \frac{1}{3}\,(z_1 + z_2 + z_3)$$

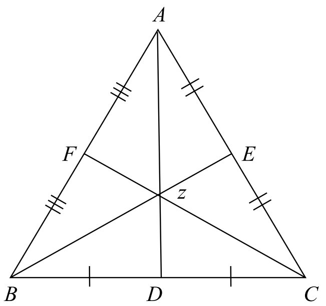


Fig. 2.17

(ii) **Incentre** $I(z)$ of the ΔABC is the point of concurrence of internal bisectors of angles of ΔABC and is given by

$$z = \frac{az_1 + bz_2 + cz_3}{a + b + c}$$

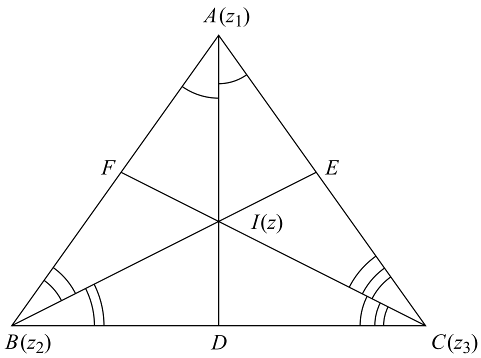


Fig. 2.18

(iii) **Circumcentre** $S(z)$ of the ΔABC is the point of concurrence of perpendicular bisectors of sides of ΔABC and is given by

$$z = \frac{(z_2 - z_3)|z_1|^2 + (z_3 - z_1)|z_2|^2 + (z_1 - z_2)|z_3|^2}{\overline{z}_1(z_2 - z_3) + \overline{z}_2(z_3 - z_1) + \overline{z}_3(z_1 - z_2)}$$

$$z = \frac{\begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \overline{z}_1 & z_1 & 1 \\ \overline{z}_2 & z_2 & 1 \\ \overline{z}_3 & z_3 & 1 \end{vmatrix}}$$

Also $$z = \frac{z_1(\sin 2A) + z_2(\sin 2B) + z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$$

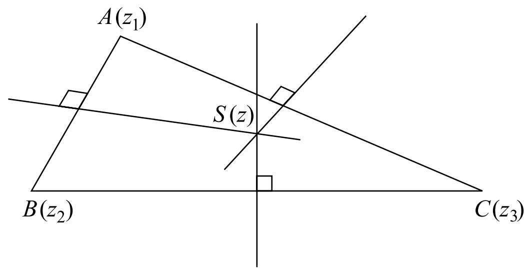


Fig. 2.19

(iv) **Orthocentre** $H(z)$ of the ΔABC is the point of concurrence of altitudes of ΔABC and is given by

$$z = \frac{\begin{vmatrix} z_1^2 & \overline{z}_1 & 1 \\ z_2^2 & \overline{z}_2 & 1 \\ z_3^2 & \overline{z}_3 & 1 \end{vmatrix} + \begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \overline{z}_1 & z_1 & 1 \\ \overline{z}_2 & z_2 & 1 \\ \overline{z}_3 & z_3 & 1 \end{vmatrix}}$$

or $$z = \frac{(\tan A)\,z_1 + (\tan B)\,z_2 + (\tan C)z_3}{\tan A + \tan B + \tan C}$$

or $$z = \frac{(a \sec A)\,z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C}$$

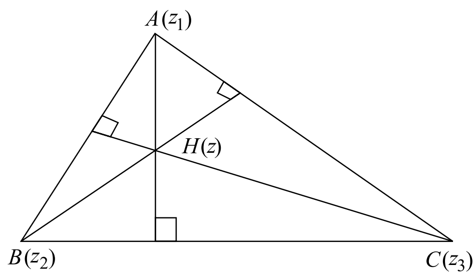


Fig. 2.20

Remark

In case circumcentre of ΔABC is at the origin, then orthocentre of triangle is given by $z_1 + z_2 + z_3$.

Important Note

It is not necessary to remember formulae for circumcentre and orthocentre of a triangle.

Euler's Line

The centroid G of a triangle lies on the segment joining the orthocentre H and the circumcentre S of the triangle. G divides the join of H and S in the ratio 2 : 1.

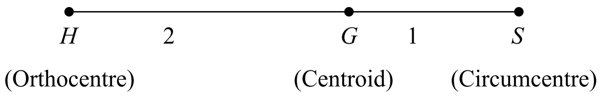


Fig. 2.21

Thus,

$$z_G = \frac{1}{3}(z_H + 2z_S)$$

6. **Area of a Triangle**

   Area of ΔABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \left| \frac{1}{4i} \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \right|$$

$$= \left| \frac{1}{2} \operatorname{Im}(\bar{z}_1 z_2 + \bar{z}_2 z_3 + \bar{z}_3 z_1) \right|$$

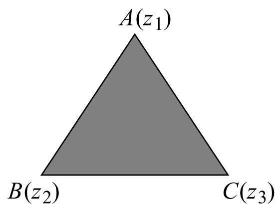


Fig. 2.22

7. **Condition for Triangle to be Equilateral**

   Triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is equilateral if and only if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

$$\Leftrightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$$

$$\Leftrightarrow \quad z_1 \bar{z}_2 = z_2 \bar{z}_3 = z_3 \bar{z}_1$$

$$\Leftrightarrow \quad z_1^2 = z_2 z_3 \text{ and } z_2^2 = z_1 z_3$$

$$\Leftrightarrow \quad \begin{vmatrix} 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \\ 1 & z_1 & z_2 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad \frac{z_2 - z_1}{z_3 - z_2} = \frac{z_3 - z_2}{z_1 - z_3}$$

$$\Leftrightarrow \frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3} = 0$$

where $\quad z = \frac{1}{3}(z_1 + z_2 + z_3)$

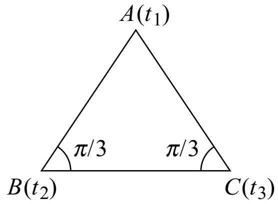


Fig. 2.23

8. **Equation of a Straight Line**

   (i) Non-parametric form

   An equation of a straight line joining the two points $A(z_1)$ and $B(z_2)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $\quad \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

or $\quad z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$

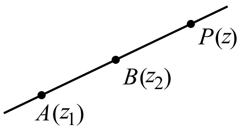


Fig. 2.24

   (ii) Parametric form

   An equation of the straight line joining the points $A(z_1)$ and $B(z_2)$ is

$$z = tz_1 + (1 - t)z_2$$

   where t is a real parameter.

   (iii) General Equation of a Straight Line

   The general equation of a straight line is

$$\bar{a}z + a\bar{z} + b = 0$$

   where a is a non-zero complex number and b is a real number.

9. **Complex Slope of a Line**

   If $A(z_1)$ and $B(z_2)$ are two points in the complex plane, then complex slope of AB is defined to be

$$\mu = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

   Two lines with complex slopes μ_1 and μ_2 are

   (i) parallel, if $\mu_1 = \mu_2$

   (ii) perpendicular, if $\mu_1 + \mu_2 = 0$

   The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is given by $-(a/\bar{a})$

10. Length of Perpendicular from a Point to a Line

Length of perpendicular of point $A(\omega)$ from the line $\bar{a}z + a\bar{z} + b = 0$

($a \in \mathbf{C} - \{0\}, b \in \mathbf{R}$) is given by

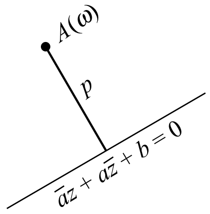


Fig. 2.25

$$p = \frac{|\bar{a}\omega + a\bar{\omega} + b|}{2|a|}$$

11. Some Results on Circle

(i) Equation of a circle

An equation of a circle with centre at z_0 and radius r is

$$|z - z_0| = r$$

or $z = z_0 + re^{i\theta}, 0 \le \theta < 2\pi$ (parametric form)

or $z\bar{z} - z_0\,\bar{z} - \bar{z}_0\,z + z_0\,\bar{z}_0 - r^2 = 0$

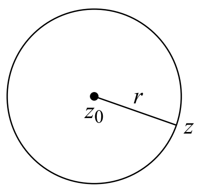


Fig. 2.26

(ii) General equation of a circle

General equation of a circle is

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0 \qquad (1)$$

where a is a complex number and b is a real number such that $a\bar{a} - b \ge 0$

Centre of (1) is $-a$ and its radius is $\sqrt{a\bar{a} - b}$.

(iii) Diameter form of a circle

An equation of the circle one of whose diameter is the line segment joining $A(z_1)$ and $B(z_2)$ is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) = 0$$

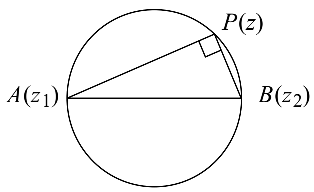


Fig. 2.27

(iv) An equation of circle through two points

An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$ is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2)$$

$$+ ik\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

where k is real a parameter.

(v) Equation of a circle passing through three non-collinear points.

Let three non-collinear points be $A(z_1)$, $B(z_2)$ and $C(z_3)$. Let $P(z)$ be any point on the circle. Then either

$\angle ACB = \angle APB$

[when angles are in the same segment]

$\angle ACB + \angle APB = \pi$

[when angles are in the opposite segment]

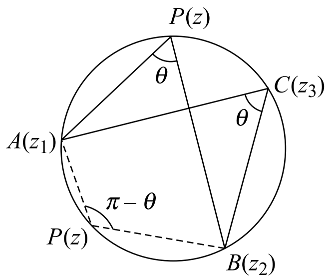


Fig. 2.28

$$\Rightarrow \quad \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) - \arg\left(\frac{z - z_2}{z - z_1}\right) = 0$$

or

$$\arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) + \arg\left(\frac{z - z_1}{z - z_2}\right) = \pi$$

$$\Rightarrow \quad \arg\left[\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\left(\frac{z - z_1}{z - z_2}\right)\right] = 0$$

$$\text{or} \quad \arg\left[\left(\frac{z - z_1}{z - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\right] = \pi$$

[using $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$]

In any case, we get $\frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \quad \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}$$

(vi) Condition for four points to be concyclic.

Four points z_1, z_2, z_3 and z_4 will lie on the same circle if and only if $\frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \quad \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \text{ is purely real.}$$

RECOGNIZING SOME LOCI BY INSPECTION

(i) arg $(z) = \alpha$ $(-\pi < \alpha \leq \pi)$
arg $(z) = \alpha$ represents a ray starting at the origin (excluding the origin) and making an angle α with the positive direction of the real axis. See Fig. 2.29.

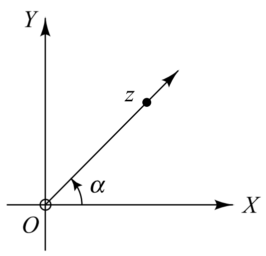


Fig. 2.29

(ii) arg $(z - z_0) = \alpha$ $(-\pi < \alpha \leq \pi)$
arg $(z - z_0) = \alpha$ represents a ray starting at the fixed point z_0 (excluding the point z_0) and making an angle α with the positive direction of the real axis.

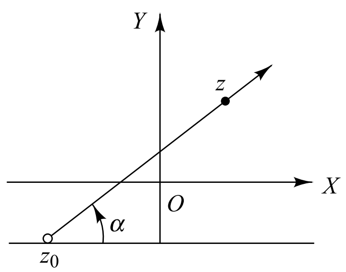


Fig. 2.30

(iii) If z_1 and z_2 are two fixed points, then

$$|z - z_1| = |z - z_2|$$

represents the perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

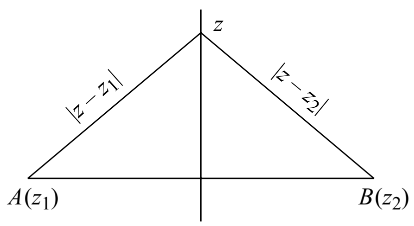


Fig. 2.31

(iv) If z_1 and z_2 are two fixed points, and $k > 0$, $k \neq 1$ is a real number, then

$$\frac{|z - z_1|}{|z - z_2|} = k$$

represents a circle.

For $k = 1$, it represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

(v) $|z - z_1| + |z - z_2| = k$
Let z_1 and z_2 be two fixed points and k be a positive real number.

(a) If $k > |z_1 - z_2|$, then
$|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of major axis $= k$. See Fig. 2.32

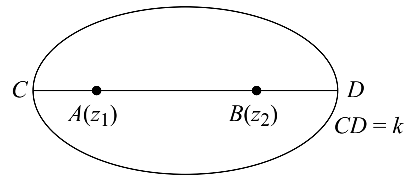


Fig. 2.32

(b) If $k = |z_1 - z_2|$, then
$|z - z_1| + |z - z_2| = k$
represents the segment joining z_1 and z_2.

(c) If $k < |z_1 - z_2|$, then
$|z - z_1| + |z - z_2| = k$
does not represent any curve in the *Argand* plane.

(vi) $|z - z_1| - |z - z_2| = k$
Let z_1 and z_2 be two fixed points, k be a positive real number.

(a) If $k < |z_1 - z_2|$, then
$||z - z_1| - |z - z_2|| = k$
represents a hyperbola with foci at $A(z_1)$ and $B(z_2)$. See Fig 2.33

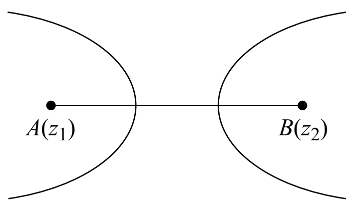


Fig. 2.33

(b) If $k = |z_1 - z_2|$, then $||z - z_1| - |z - z_2|| = k$ represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB. See Fig. 2.34.

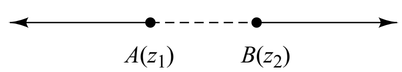


Fig. 2.34

Remark

If $k > |z_1 - z_2|$, then
$||z - z_1| - |z - z_2|| = k$
does not represent any curve in the *Argand* plane.

(vii) If z_1 and z_2 are two fixed points, then

$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

represent a circle with z_1 and z_2 as the extremities of a diameter. See Fig. 2.35.

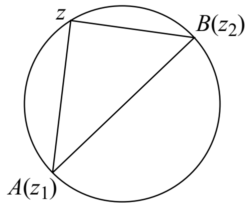


Fig. 2.35

(viii) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$

Let z_1 and z_2 be two fixed points, and α be a real number such $0 \le \alpha \le \pi$.

(a) If $0 < \alpha < \pi$ and $\alpha \ne \pi/2$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$$

represents a segment of the circle passing through $A(z_1)$ and $B(z_2)$. See Fig. 2.36.

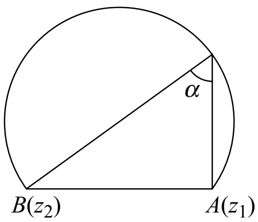


Fig. 2.36

(b) If $\alpha = \pi/2$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\left(= \frac{\pi}{2}\right)$$

represents a circle with diameter as the segment joining $A(z_1)$ and $B(z_2)$. See Fig. 2.37.

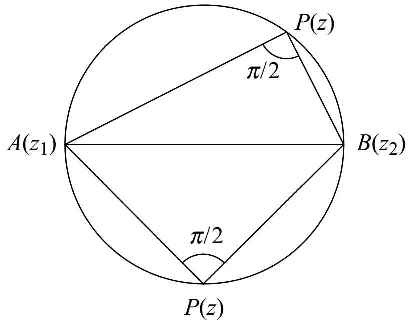


Fig. 2.37

(c) If $\alpha = \pi$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\ (= \pi)$$

represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB. See Fig. 2.38

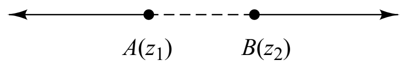


Fig. 2.38

(d) If $\alpha = 0$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\ (= 0)$$

represents the segment joining $A(z_1)$ and $B(z_2)$ see Fig. 2.39.

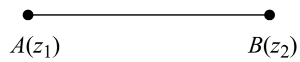


Fig. 2.39

GEOMETRIC INTERPRETATION OF MULTIPLYING A COMPLEX NUMBER BY $e^{i\alpha}$.

Let z be a non-zero complex number. We can write z in the polar form as follows:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where $r = |z|$ and $\arg(z) = \theta$

We have

$$ze^{i\alpha} = re^{i\theta}\, e^{i\alpha} = re^{i(\theta + \alpha)}$$

Thus, $ze^{i\alpha}$ represents the complex number whose modulus is r and argument is $\theta + \alpha$. Geometrically, $ze^{i\alpha}$ can be obtained by rotating the segment joining O and $P(z)$ through an angle α in the anticlockwise direction. See Fig. 2.40

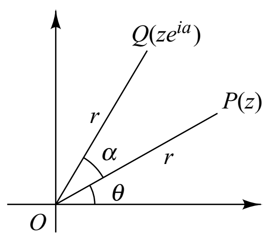


Fig. 2.40

In particular iz is obtained by rotating OP through an angle of $\pi/2$ in the anticlockwise direction.

Corollary If $A(z_1)$ and $B(z_2)$ are two complex number such that angle $\angle AOB = \theta$, then we can write

$$z_2 = \frac{|z_2|}{|z_1|}\, z_1 e^{i\theta}$$

Suppose $z_1 = r_1 e^{i\alpha}$ and $z_2 = r_2 e^{i\beta}$,

where $|z_1| = r_1$, $|z_2| = r_2$.

then $\dfrac{z_2}{z_1} = \dfrac{r_2 e^{i\beta}}{r_1 e^{i\alpha}} = \dfrac{r_2}{r_1} e^{i(\beta-\alpha)}$

Note that

$$\beta - \alpha = \theta$$

Thus,

$$\frac{z_2}{z_1} = \frac{r_2}{r_1} e^{i\theta}$$

$$\Rightarrow \quad z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta}$$

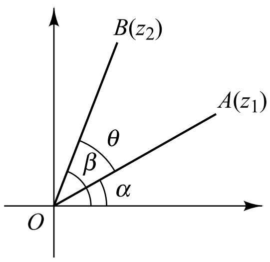


Fig. 2.41

Square Roots of a Complex Number

Let $z = x + iy$. If $a + ib$ is a square root of z, then

$$(a + ib)^2 = x + iy$$

$$\Rightarrow \quad a^2 - b^2 = x,\ 2ab = y$$

Also, $a^2 + b^2 = \sqrt{x^2 + y^2}$

Thus, $a = \pm\dfrac{1}{\sqrt{2}}\sqrt{\sqrt{x^2+y^2}+x}$

$$b = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{x^2+y^2}-x}$$

If $y > 0$, then a and b both have the same signs.
If $y < 0$, then a and b have opposite signs.

Illustration 3

$$\sqrt{3+4i} = \pm\frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{3^2+4^2}+3} + \left(\sqrt{\sqrt{3^2+4^2}-3}\right)i\right]$$

$$= \pm\frac{1}{\sqrt{2}}\left[2\sqrt{2} + \sqrt{2}i\right] = \pm(2+i)$$

Alternatively we can use De Moivre's Theorem.

DE MOIVRE'S THEOREM AND ITS APPLICATIONS

(a) *De Moivre's Theorem for integral index.*
If n is an integer, then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

(b) *De Moivre's Theorem for rational index.*
If n is a rational number, then value of or one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos(n\theta) + i\sin(n\theta)$. In fact, if $n = p/q$ where $p, q \in \mathbf{I}, q > 0$ and p,q have no factors in common, then $(\cos\theta + i\sin\theta)^n$ has q distinct values, one of which is $\cos(n\theta) + i\sin(n\theta)$.

Note

The values of $(\cos\theta + i\sin\theta)^{p/q}$ where $p, q \in \mathbf{I}$, $q > 0$, hcf $(p,q) = 1$ are given by

$$\cos\left[\frac{p}{q}(2k\pi+\theta)\right] + i\sin\left[\frac{p}{q}(2k\pi+\theta)\right]$$

where $k = 0, 1, 2, \ldots, q-1$.

*n*th Roots of Unity

By an nth root of unity we mean any complex number z which satisfies the equation

$$z^n = 1 \qquad (1)$$

Since, an equation of degree n has n roots, there are n values of z which satisfy the equation (1). To obtain these n values of z, we write

$$1 = \cos(2k\pi) + i\sin(2k\pi)$$

where $k \in \mathbf{I}$ and

$$\Rightarrow \quad z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

[using the De Moivre's Theorem]

where $k = 0, 1, 2, \ldots, n-1$.

Note

We may use any n consecutive integral values to k. For instance, in case of 3, we may take – 1, 0 and 1 and in case of 4, we may take – 1, 0, 1 and 2 or – 2, – 1, 0 and 1.

Notation Let $\omega = \cos\left(\dfrac{2\pi}{n}\right) + i\sin\left(\dfrac{2\pi}{n}\right)$

By using the De Moivre's theorem, we can write the nth roots of unity as

$$1, \omega, \omega^2, \ldots, \omega^{n-1}.$$

Sum of the Roots of Unity is Zero

We have

$$1 + \omega + \ldots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega}$$

But $\omega^n = 1$ as ω is a nth root of unity.

$$\therefore \quad 1 + \omega + \ldots + \omega^{n-1} = 0$$

Also, note that

$$\frac{1}{x-1} + \frac{1}{x-\omega} \ldots + \frac{1}{x-\omega^{n-1}} = \frac{nx^{n-1}}{x^n - 1}$$

Writing *n*th Roots of Unity When *n* is Odd

If $n = 2m + 1$, then nth roots of unity are also given by

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

where $k = -m, -(m-1), \ldots, -1, 0, 1, 2, \ldots m$.

Since $$\cos\left(-\frac{2k\pi}{n}\right) = \cos\left(\frac{2k\pi}{n}\right)$$

and $$\sin\left(-\frac{2k\pi}{n}\right) = -\sin\left(\frac{2k\pi}{n}\right)$$

we may take the roots as

$$1, \cos\left(\frac{2k\pi}{n}\right) \pm i\sin\left(\frac{2k\pi}{n}\right)$$

where $k = 1, 2, \ldots, m$.

In terms, ω we may take nth roots of unity to be $1, \omega^{\pm 1}, \omega^{\pm 2}, \ldots \omega^{\pm m}$.

Writing *n*th Roots of Unity When *n* is Even

If $n = 2m$, then nth roots of unity are given

$$z = \pm 1, \pm \omega, \pm \omega^2, \ldots \pm \omega^{m-1}$$

where $\omega = \cos\left(\frac{2\pi}{2m}\right) + i\sin\left(\frac{2\pi}{2m}\right) = \cos\left(\frac{\pi}{m}\right) + i\sin\left(\frac{\pi}{m}\right)$

Cube Roots of Unity

Cube roots of unity are given by $1, \omega, \omega^2$, where

$$\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2} \text{ and } \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

Some Results Involving Complex Cube Root of Unity (ω)

(i) $\omega^3 = 1$, $\bar{\omega} = \omega^2$ and $\frac{1}{\omega} = \omega^2$

(ii) $1 + \omega + \omega^2 = 0$

(iii) $x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$

(iv) ω and ω^2 are roots of $x^2 + x + 1 = 0$

(v) $a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$
$= (a-b)(a\omega - b\omega^2)(a\omega^2 - b\omega)$

(vi) $a^2 + b^2 + c^2 - bc - ca - ab$
$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(vii) $a^3 + b^3 + c^3 - 3abc$
$= (a+b+c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(viii) $x^3 + 1 = (x+1)(x+\omega)(x+\omega^2)$

(ix) $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

(x) Cube roots of real number a are $a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2$.

To obtain cube roots of a real number a, we write $x^3 = a$ as $y^3 = 1$ where $y = x/a^{1/3}$.

Solution of $y^3 = 1$ are $1, \omega, \omega^2$.

$\therefore \quad x = a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2$.

Illustration 4

To obtain cube roots of -27, we write $\left(\frac{x}{-3}\right)^3 = 1$

$\Rightarrow \quad x = -3, -3\omega, -3\omega^2$

*n*th Roots of a Complex Number

Let $z \neq 0$ be a complex number. We can write z in the polar form as follows:

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg(z)$. Recall $-\pi < \theta \leq \pi$.

The nth root of z has n values one of which is equal to $z_0 = \sqrt[n]{|z|}\left[\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right]$ and is called as the principal value of $\sqrt[n]{|z|}$. To obtain other values of $\sqrt[n]{|z|}$, we write z as

$$z = r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$$

$$\Rightarrow \quad z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$$= z_0\,\omega^k \text{ where } k = 0, 1, 2, \ldots, n-1.$$

and $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is a complex nth root of unity.

Thus, all the nth roots of z can be obtained by multiplying the principal value of $\sqrt[n]{|z|}$ by different roots of unity.

Rational Power of a Complex Number

If z is a complex number and m/n is a rational number such that m and n are relatively prime integers and $n > 0$. We define

$$z^{m/n} = \left(\sqrt[n]{z}\right)^m$$

Thus, $z^{m/n}$ has n distinct values which are given by

$$z^{m/n} = \left(\sqrt[n]{|z|}\right)^m\left[\cos\left(\frac{m}{n}(\theta + 2k\pi)\right) + i\sin\left(\frac{m}{n}(\theta + 2k\pi)\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: If $a + ib = \sum_{k=1}^{101} i^k$, then (a, b) equals

(a) (0, 1) (b) (0, 0)
(c) (0, –1) (d) (1, 1)

Ans. (a)

Solution: Write

$$a + ib = i + (i^2 + i^3 + i^4 + i^5) + (i^6 + i^7 + i^8 + i^9) + \dots (i^{98} + i^{99} + i^{100} + i^{101})$$

TIP Use

$$i^m + i^{m+1} + i^{m+2} + i^{m+3} = 0 \ \forall \ m \in \mathbf{I}$$

to obtain

$$a + ib = 0 + i \quad \Rightarrow \quad a = 0, b = 1$$

Example 2: If $\left(\frac{1+i}{1-i}\right)^n = -1$, $n \in \mathbf{N}$, then least value of n is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (b)

Solution: Write $1 = -i^2$ in the numerator, to obtain

$$\frac{1+i}{1-i} = \frac{-i^2+i}{1-i} = \frac{i(1-i)}{1-i} = i$$

$$\therefore \quad -1 = \left(\frac{1+i}{1-i}\right) = i^n$$

The least value of n is 2.

Example 3: The conjugate of a complex number z is $\frac{2}{1-i}$. Then Re (z) equals

(a) –1 (b) 0
(c) 1 (d) 2

Ans. (c)

Solution: $\bar{z} = \frac{2}{1-i} = \frac{1-i^2}{1-i} = 1 + i$

Now, Re (z)= Re $(\bar{z})$ = 1.

Example 4: The number of complex numbers z such that $|z - i| = |z + i| = |z + 1|$ is

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (b)

Solution:

TIP
Note z is circum-centre of the circle passing through three non-collinear points A $(0 + i)$, B $(0 - i)$ and $C(-1 + 0i)$

Thus, there is only one value of z.

Example 5: If $z + 2|z| = \pi + 4i$, then Im (z) equals

(a) π (b) 4
(c) $\sqrt{\pi^2 + 16}$ (d) none of these

Ans. (b)

Solution: The given equation can be written as

$$z = (\pi - 2|z|) + 4i$$

As $\pi - 2|z|$ is real, we get

$$\text{Im}(z) = 4$$

Example 6: If $|z| = z + 3 - 2i$, then z equals

(a) $7/6 + i$ (b) $-7/6 + 2i$
(c) $-5/6 + 2i$ (d) $5/6 + i$

Ans. (c)

Solution:

$$z = |z| - 3 + 2i$$

$$\Rightarrow \quad |z|^2 = (|z| - 3)^2 + 4$$

$$= |z|^2 - 6|z| + 9 + 4$$

$$\Rightarrow \quad |z| = 13/6$$

Thus, $z = -5/6 + 2i$

Example 7: If ω $(\neq 1)$ is a cube root of unity and $(1 + \omega^2)^{11} = a + b\omega + c\omega^2$, then (a, b, c) equals

(a) (1, 1, 0) (b) (0, 1, 1)
(c) (1, 0, 1) (d) (1, 1, 1)

Ans. (a)

Solution:

$$a + b\omega + c\omega^2 = (1 + \omega^2)^{11} = (-\omega)^{11}$$

$$= -(\omega^3)^3\omega^2 = 1 + \omega \qquad [\because 1 + \omega + \omega^2 = 0]$$

$$\Rightarrow \quad a = 1, b = 1, c = 0$$

Example 8: If $x^2 + y^2 = 1$ and $x \neq -1$ then $\frac{1+y+ix}{1+y-ix}$ equals

(a) 1 (b) $x + iy$
(c) 2 (d) $y + ix$

Ans. (d)

◎ **Solution:**

> **TIP**
> Sometimes, it is helpful to write $x^2 + y^2 = z\bar{z}$
> where $z = x + iy$ or $y + ix$

Put $z = y + ix$, then $1 = x^2 + y^2 = z\bar{z}$

Now,

$$\frac{1+y+ix}{1+y-ix} = \frac{z\bar{z}+z}{1+\bar{z}} = \frac{z(\bar{z}+1)}{1+\bar{z}} = z = y + ix$$

◉ **Example 9:** If z is a non-zero complex number, then $\arg(z) + \arg(\bar{z})$ equals

(a) 0 (b) π
(c) 2π (d) None of these

Ans. (d)

◎ **Solution:** If $z \in \mathbf{R}$ and $z < 0$, then

$$\arg(z) = \arg(\bar{z}) = \pi$$

$$\Rightarrow \quad \arg(z) + \arg(\bar{z}) = 2\pi$$

Suppose $z \in \mathbf{C}$, $z \neq 0$ and z is not a negative real number.

Let $\arg(z) = \alpha$, where $-\pi < \alpha < \pi$

In this case $\arg(\bar{z}) = -\alpha$, so that

$\arg(z) + \arg(\bar{z}) = 0$

◉ **Example 10:** If $z \in \mathbf{C}$ and $2z = |z| + i$, then z equals

(a) $\frac{\sqrt{3}}{6} + \frac{1}{2}i$ (b) $\frac{\sqrt{3}}{6} + \frac{1}{3}i$
(c) $\frac{\sqrt{3}}{6} + \frac{1}{4}i$ (d) $\frac{\sqrt{3}}{6} + \frac{1}{6}i$

Ans. (a)

◎ **Solution:**

$$|2z|^2 = |z|^2 + 1$$

$$\Rightarrow \quad 3|z|^2 = 1 \Rightarrow |z| = 1/\sqrt{3}$$

Thus, $z = \frac{\sqrt{3}}{6} + \frac{1}{2}i$

◉ **Example 11:** If $z = \left(\frac{1}{\sqrt{3}} + \frac{1}{2}i\right)^7 + \left(\frac{1}{\sqrt{3}} - \frac{1}{2}i\right)^7$, then

(a) Re $(z) = 0$
(b) Im $(z) = 0$
(c) Re $(z) > 0$, Im $(z) < 0$
(d) Re $(z) < 0$, Im $(z) > 0$

Ans. (b)

◎ **Solution:**

$$\bar{z} = \left(\frac{1}{\sqrt{3}} - \frac{1}{2}i\right)^7 + \left(\frac{1}{\sqrt{3}} + \frac{1}{2}i\right)^7 = z$$

$\Rightarrow$ z is purely real.

$\therefore$ Im $(z) = 0$

◉ **Example 12:** If $\omega\,(\neq 1)$ is a complex cube root of unity and $(1 + \omega^4)^n = (1 + \omega^8)^n$, then the least positive integral value of n is

(a) 2 (b) 3
(c) 6 (d) 12

Ans. (b)

◎ **Solution:**

As $\omega^4 = \omega$, $\omega^8 = \omega^2$, we get

$$(1 + \omega)^n = (1 + \omega^2)^n$$

$$\Rightarrow \quad (-\omega^2)^n = (-\omega)^n$$

$$\Rightarrow \quad \omega^n = 1$$

$$\therefore \quad n = 3$$

◉ **Example 13:** If $z = \frac{1+\cos\theta + i\sin\theta}{\sin\theta + i(1+\cos\theta)}$ $(0 < \theta < \pi/2)$

then $|z|$ equals

(a) $2|\sin\theta|$ (b) $2|\cos\theta|$
(c) 1 (d) $|\cot(\theta/2)|$

Ans. (c)

◎ **Solution:** Using $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$, we get

$$|z|^2 = \frac{(1+\cos\theta)^2 + \sin^2\theta}{\sin^2\theta + (1+\cos\theta)^2} = 1$$

$$\Rightarrow \quad |z| = 1$$

◉ **Example 14:** All the roots of $(z + 1)^4 = z^4$ lie on

(a) a straight line parallel to x-axis
(b) a straight line parallel to y-axis
(c) a circle with centre at $-1 + 0i$
(d) a circle with centre at $1 + i$

Ans. (b)

◎ **Solution:**

> **TIP**
> It is unnecessary to find roots of $(z + 1)^4 = z^4$.

If z is a root of $(z + 1)^4 = z^4$, then $|(z + 1)^4| = |z^4|$

$$\Rightarrow \quad |z + 1| = |z|$$

$$\Rightarrow \quad |z - (-1)| = |z - 0|$$

$\Rightarrow$ z lies on the perpendicular bisector of the segment joining $-1 + 0i$ and 0 i.e. z lies on the line Re $(z) = -1/2$.

◉ **Example 15:** If $\alpha\,(\neq 1)$ is a fifth root of unity and $\beta\,(\neq 1)$ is a fourth root of unity then

$$z = (1 + \alpha)(1 + \beta)(1 + \alpha^2)(1 + \beta^2)(1 + \alpha^3)(1 + \beta^3)$$

equals

(a) α (b) β
(c) $\alpha\beta$ (d) 0

Ans. (d)

◎ **Solution:** As $\beta \neq 1$ is a fourth root of unity,

$$\beta^4 = 1 \Rightarrow (1 - \beta)(1 + \beta + \beta^2 + \beta^3) = 0$$

As $\beta \neq 1$, $1 + \beta + \beta^2 (1 + \beta) = 0$

$\Rightarrow$ $(1 + \beta)(1 + \beta^2) = 0$

$\therefore$ $z = 0$

Example 16: Suppose z_1, z_2, z_3 are vertices of an equilateral triangle whose circumcentre $-3 + 4i$, then $|z_1 + z_2 + z_3|$ is equal to

(a) 5 (b) $10\sqrt{3}$

(c) 15 (d) $15\sqrt{3}$

Ans. (c)

Solution: As triangle is equilateral, circumcentre and centroid of the triangle coincides, therefore,

$$\frac{1}{3}(z_1 + z_2 + z_3) = -3 + 4i$$

$$\Rightarrow \quad |z_1 + z_2 + z_3| = 3\sqrt{(-3)^3 + 4^2} = 15$$

Example 17: If $z \neq 0$ lies on the circle $|z - 1| = 1$ and $\omega = 5/z$, then ω lies on

(a) a circle (b) an ellipse

(c) a straight line (d) a parabola

Ans. (c)

Solution: $z = 5/\omega$ and $|z - 1| = 1$

$$\Rightarrow \quad \left|\frac{5}{\omega} - 1\right| = 1 \Rightarrow |\omega - 5| = |\omega|$$

that is, ω lies on the perpendicular bisector of the segment joining 0 and $5 + 0i$.

Therefore ω lies on the straight line Re $(\omega) = 5/2$

Example 18: If $\bar{z} = 3i + \dfrac{25}{z + 3i}$, then $|z|$ cannot exceed

(a) 3 (b) 8

(c) 16 (d) 18

Ans. (b)

Solution: $(\bar{z} - 3i)(z + 3i) = 25$

$\Rightarrow$ $|z - 3i|^2 = 25$ or $|z - 3i| = 5$

Now, $|z| = |(z - 3i) + 3i|$

$\leq |z - 3i| + |3i| = 5 + 3 = 8$

Example 19: If $|z - 1| = |z + 1| = |z - 2i|$, then value of $|z|$ is

(a) 1 (b) 2

(c) 5/4 (d) 3/4

Ans. (d)

Solution: z is centre of the circle passing through $1 + 0i$, $-1 + 0i$ and $0 + 2i$. Clearly centre lies on the y-axis.

If $z = 0 + ai$ the centre, then

$\sqrt{1 + a^2} = |a + 2|$

$\Rightarrow$ $1 + a^2 = a^2 + 4a + 4$

$\Rightarrow$ $a = 3/4$

$\therefore$ $|z| = 3/4$

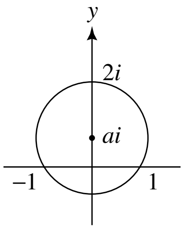


Fig. 2.42

Example 20: The number of complex numbers satisfying $\bar{z} = iz^2$ is

(a) 1 (b) 2

(c) 3 (d) 4

Ans. (d)

Solution: $|\bar{z}| = |iz^2|$ $\Rightarrow$ $|z| = |z|^2$

$\Rightarrow$ $|z| = 0$ or $|z| = 1$.

If $|z| = 0$, then $z = 0$.

If $|z| = 1$, we get $\bar{z} = 1/z$, so that the equation becomes $1/z = iz^2$

or $z^3 = -i = i^3 \Rightarrow z = i, i\omega, i\omega^2$

where ω $(\neq 1)$ is a cube root of unity.

LEVEL 1

Straight Objective Type Questions

Example 21: If $z \in \mathbf{C}$, $z \notin \mathbf{R}$, and $a = z^2 + 3z + 5$, then a cannot take value

(a) $-2/5$ (b) $5/2$

(c) $11/4$ (d) $-11/5$

Ans. (c)

Solution: $a = \left(z + \dfrac{3}{2}\right)^2 + \dfrac{11}{4}$

As $z \notin \mathbf{R}$, $z \neq -\dfrac{3}{2}$, thus, $a \neq \dfrac{11}{4}$

Example 22: Suppose $a, b, c \in \mathbf{C}$, and $|a| = |b| = |c| = 1$ and $abc = a + b + c$, then $bc + ca + ab$ is equal to

(a) 0 (b) -1

(c) 1 (d) none of these

Ans. (c)

Solution: $|a|^2 = |b|^2 = |c|^2 = 1$

$\Rightarrow$ $a\bar{a} = b\bar{b} = c\bar{c} = 1$

Now, $abc = a + b + c$

$\Rightarrow$ $\bar{a}\,\bar{b}\,\bar{c} = \bar{a} + \bar{b} + \bar{c}$

$\Rightarrow \quad \dfrac{1}{abc} = \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

$\Rightarrow \quad bc + ca + ab = 1$

Example 23: The number of complex numbers z which satisfy $z^2 + 2|z|^2 = 2$ is

(a) 0 (b) 2
(c) 3 (d) 4

Ans. (d)

Solution: As $z^2 = 2(1 - |z|^2)$ is real, z is either purely real or purely imaginary.
If z is purely real, then

$$z^2 = 2(1 - z^2) \quad \Rightarrow \quad z = \pm\sqrt{\frac{2}{3}}$$

TIP
If z is purely imaginary, then $|z|^2 = -z^2$.

In this case $z^2 = 2(1 + z^2) \quad \Rightarrow \quad z = \pm\sqrt{2}i$

Thus, there are four complex numbers satisfying

$$z^2 + 2|z|^2 = 2$$

Example 24: Suppose $a \in \mathbf{R}$ and the equation $z + a|z| + 2i = 0$ has no solution in $\mathbf{C}$, then a satisfies the relation.

(a) $|a| > 1$ (b) $|a| \geq 1$
(c) $|a| > \sqrt{2}$ (d) $|a| \geq \sqrt{2}$

Ans. (b)

Solution: $z = -a\,|z| - 2i$

$\Rightarrow \quad |z|^2 = a^2\,|z|^2 + 4$

$\Rightarrow \quad |z|^2(1 - a^2) = 4$

This equation has no solution if $1 - a^2 \leq 0$ or $|a| \geq 1$

For $|a| < 1$, $|z| = \dfrac{2}{\sqrt{1-a^2}}$ and

$$z = \frac{2a}{\sqrt{1-a^2}} - 2i$$

Example 25: Suppose z is a complex number and $n \in \mathbf{N}$ be such that $z^n = (1 + z)^n = 1$, then the least value of n is

(a) 3 (b) 6
(c) 9 (d) 18

Ans. (b)

Solution: $|z|^n = |1 + z|^n = 1 \quad \Rightarrow \quad |z| = |z + 1| = 1$

$\Rightarrow$ $|z| = 1$ and z lies on the perpendicular bisector of the segment joining $0 + 0i$ and $-1 + 0i$, that z lies on the line $\text{Re}(z) = -1/2$.

Let $\quad z = -\dfrac{1}{2} + iy$, then $|z| = 1$

$$\frac{1}{4} + y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{\sqrt{3}}{2}$$

$\therefore \quad z = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2} = \omega, \omega^2$

where ω is complex cube root of unity.
If $z = \omega$, $\omega^n = (1 + \omega)^n = (-1)^n\,\omega^{2n}$
$\Rightarrow n$ is even and multiple of 3. Thus, least value of n is 6.
Similarly, for $z = \omega^2$.

Example 26: Let $z \neq \pm i$ be a complex number such that $\dfrac{z-i}{z+i}$ is purely imaginary number, then $z + \dfrac{1}{z}$ is

(a) a non-zero real number other than 1
(b) a purely imaginary number
(c) a non-zero real number
(d) 0

Ans. (c)

Solution: Let $\dfrac{z-i}{z+i} = ik$, where $k \in \mathbf{R}$.

$\Rightarrow \quad z - i = ikz - k$

$\Rightarrow \quad z(1 - ik) = -k + i$

$\Rightarrow \quad z = \dfrac{-k+i}{1-ik}$

Note that $\quad |z|^2 = \dfrac{k^2+1}{1+k^2} = 1$

$\Rightarrow \quad z\bar{z} = 1 \quad \Rightarrow \quad \bar{z} = 1/z$

Thus, $\quad z + \dfrac{1}{z} = z + \bar{z}$, which is a real number.

Also, $\quad z + \bar{z} = 0$

$\Rightarrow \quad 2\text{Re}(z) = 0 \quad \Rightarrow \quad \text{Re}(z) = 0$

$\Rightarrow \quad z = ai$ for some $a \in \mathbf{R}$.

But in this case

$\dfrac{z-i}{z+i}$ is a real number

Therefore, $z + \bar{z} \neq 0$.

Example 27: The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if

(a) $z_1 + z_4 = z_2 + z_3$
(b) $z_1 + z_3 = z_2 + z_4$
(c) $z_1 + z_2 = z_3 + z_4$
(d) None of these

Ans. (b)

Solution: See theory.

Example 28: If the complex numbers z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then

(a) $z_1 + z_2 + z_3 = 0$ (b) $z_1 + z_2 - z_3 = 0$
(c) $z_1 - z_2 + z_3 = 0$ (d) $z_1 + z_2 + z_3 \neq 0$

Ans. (a)

◎ **Solution:** Let $|z_1| = |z_2| = |z_3| = k$ (say),

$\Rightarrow z_1, z_2, z_3$ lie on a circle with centre at the origin and radius k. As z_1, z_2, z_3 are vertices of an equilateral triangle, the circumcentre and the centroid of the triangle coincide. Therefore,

$$\frac{1}{3}(z_1 + z_2 + z_3) = 0 \quad \Rightarrow \quad z_1 + z_2 + z_3 = 0$$

◉ **Example 29:** The value of $S = \sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is

(a) -1 (b) 0
(c) $-i$ (d) i

Ans. (d)

◎ **Solution:** $\sin\left(\frac{2\pi k}{7}\right) - i\cos\left(\frac{2\pi k}{7}\right)$

$$= -i\left[\cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right)\right]$$

$$= -i\,\omega^k \text{ [De Moivre's Theorem]}$$

where $\omega = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$.

Note that $\omega^7 = 1$.

$$\therefore \quad S = -i\sum_{k=1}^{6}\omega^k = \frac{-i\,\omega(1-\omega^6)}{1-\omega}$$

$$= \frac{-i(\omega - \omega^7)}{1-\omega} = \frac{-i(\omega - 1)}{1-\omega} = i$$

◉ **Example 30:** The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for

(a) $x = n\pi, n \in \mathbf{I}$ (b) $x = \left(n + \frac{1}{2}\right)\pi, n \in \mathbf{I}$
(c) $x = 0$ (d) no value of x.

Ans. (d)

◎ **Solution:** $\overline{\sin x + i\cos 2x} = \cos x - i\sin 2x$

$\Rightarrow \sin x - i\cos 2x = \cos x - i\sin 2x$

$\Rightarrow \sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow \tan x = 1$ and $\tan 2x = 1$

These two equations cannot hold simultaneously.

◉ **Example 31:** If z_1 and z_2 are two complex numbers and a, b are two real numbers, then $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$ equals

(a) $(a^2 + b^2)|z_1 z_2|$
(b) $(a^2 + b^2)(z_1^2 + z_2^2)$
(c) $(a^2 + b^2)(|z_1|^2 + |z_2|^2$
(d) $2ab|z_1 z_2|$

Ans. (c)

◎ **Solution:** We have

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2$$
$$= a^2|z_1|^2 + b^2|z_2|^2 - abz_1\bar{z}_2 - ab\bar{z}_1 z_2$$
$$+ b^2|z_1|^2 + a^2|z_2|^2 + ba\,z\bar{z}_2 + ba\bar{z}_1 z_2$$
$$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

◉ **Example 32:** If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then

(a) $a = b = 2 - \sqrt{3}$ should look like this
(b) $a = 2 - \sqrt{3}$, $b = \sqrt{3} - 1$
(c) $a = \sqrt{3} - 1$, $b = 2 - \sqrt{3}$
(d) none of these

Ans. (a)

◎ **Solution:** By the hypothesis $0 < a, b < 1$

and $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$

$\Rightarrow |(a-1) + i(1-b)| = |1 + ib| = |a + i|$

$\Rightarrow (a-1)^2 + (1-b)^2 = 1 + b^2 = a^2 + 1$

$\Rightarrow a^2 - 2a + 1 - 2b = 0$ and $b^2 = a^2$

As $0 < a, b < 1$ and $a^2 = b^2$, we get $a = b$.

$\therefore \quad a^2 - 2a + 1 - 2a = 0 \Rightarrow a^2 - 4a + 1 = 0$

$\Rightarrow \quad a = 2 \pm \sqrt{3}$ As $0 < a < 1$, $a = 2 - \sqrt{3}$.

◉ **Example 33:** If $z \neq 0$ is a complex number such that $\arg(z) = \pi/4$, then

(a) $\text{Re}(z^2) = 0$ (b) $\text{Im}(z^2) = 0$
(c) $\text{Re}(z^2) = \text{Im}(z^2)$ (d) none of these

Ans. (a)

◎ **Solution:** As $\arg(z) = \pi/4$, we can write

$$z = r\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \text{where } r = |z|$$

$$\Rightarrow \quad z^2 = r^2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

[Using De Moivre's Theorem]

$$= r^2(0 + i) = ir^2$$

$\Rightarrow \quad \text{Re}(z^2) = 0$

◉ **Example 34:** Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$. Then z equal

(a) w (b) $-w$
(c) $\bar{w}$ (d) $-\bar{w}$

Ans. (d)

◎ **Solution:** Let $|z| = |w| = r$ and $\arg(w) = \theta$, so that $\arg(z) = \pi - \theta$.

We have

$$z = r[\cos(\pi - \theta) + i\sin(\pi - \theta)]$$

$$= r(-\cos\theta + i\sin\theta)$$

$$= \overline{-r(\cos\theta + i\sin\theta)} = -\overline{w}$$

Example 35: If $|z| = 1$ and $w = \dfrac{z-1}{z+1}$ (where $z \neq -1$), then Re(w) equals

(a) 0 (b) $-\dfrac{1}{|z+1|^2}$

(c) $\left|\dfrac{z}{z+1}\right| \dfrac{1}{|z+1|^2}$ (d) $\dfrac{\sqrt{2}}{|z+|^2}$

Ans. (a)

Solution: $|z| = 1 \Rightarrow z\overline{z} = 1$.

$$2\text{Re}(w) = w + \overline{w} = \frac{z-1}{z+1} + \frac{\overline{z}-1}{\overline{z}+1}$$

$$= \frac{z-1}{z+1} + \frac{(1/z)-1}{(1/z)+1}$$

$$= \frac{z-1}{z+1} + \frac{1-z}{1+z} = 0$$

$\Rightarrow \quad \text{Re}(w) = 0$

Example 36: Let z and w be two complex numbers such that $|z| = |w| = 1$ and $|z + iw| = |z - i\overline{w}| = 2$. Then z equals

(a) 1 or i (b) i or $-i$
(c) 1 or -1 (d) i or -1

Ans. (c)

Solution: We have $|-iw| = |-i|\,|w| = 1$
and $|i\overline{w}| = |i|\,|\overline{w}| = 1$

$\Rightarrow -iw$ and $i\overline{w}$ lie on the circle $|z| = 1$.

As $|z - (-iw)| = |z - i\overline{w}| = 2$ we get z and $-iw$, as well as z and $i\overline{w}$ are the end points of the same diameter, with one end point at z.

$\therefore \quad -iw = i\overline{w} \Rightarrow w + \overline{w} = 0$

$\Rightarrow \quad w$ is purely imaginary.

Let $\quad w = ik$ where $k \in \mathbf{R}$.

As $\quad |w| = 1$, we get $|ik| = 1$

$\Rightarrow \quad |k| = 1 \Rightarrow k = \pm 1$.

$\therefore \quad w = \pm i \Rightarrow -iw = i\overline{w} = \pm 1$

When $i\overline{w} = 1$, then $z = -1$ and

when $i\overline{w} = -1$ then $z = 1$

Example 37: The complex numbers $z = x + iy$ which satisfy the equation $\left|\dfrac{z-5i}{z+5i}\right| = 1$, lie on

(a) the x – axis
(b) the straight line $y = 5$
(c) a circle passing through origin
(d) none of these

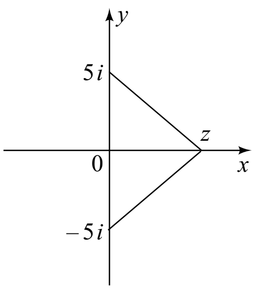


Fig. 2.43

Ans. (a)

Solution: $\left|\dfrac{z-5i}{z+5i}\right| = 1 \Rightarrow |z - 5i| = |z - (-5i)|$

$\Rightarrow$ z lies on the perpendicular bisector of the segment joining $5i$ and $-5i$, i.e., z lies on the x-axis.

Example 38: The inequality $|z - 4| < |z - 2|$ represents the region given by

(a) $\text{Re}(z) \geq 0$ (b) $\text{Re}(z) < 3$
(c) $\text{Re}(z) \leq 0$ (d) $\text{Re}\,z > 3$

Ans. (d)

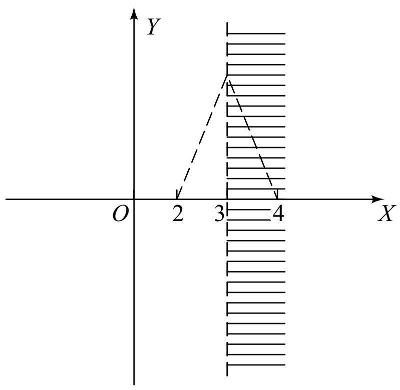


Fig. 2.44

Solution: If z satisfies $|z - 4| = |z - 2|$, then z lies on the perpendicular bisector of the segment joining $z = 2$ and $z = 4$.

i.e., $|z - 4| = |z - 2| \Rightarrow \text{Re}(z) = 3$.

As $z = 0$ does not satisfy $|z - 4| < |z - 2|$, we get $|z - 4| < |z - 2|$ represents the region $\text{Re}(z) > 3$.

Example 39: If z_1 and z_2 are two complex numbers such that $\left|\dfrac{z_1 - z_2}{z_1 + z_2}\right| = 1$, then

(a) $z_2 = kz_1, k \in \mathbf{R}$ (b) $z_2 = ikz_1, k \in \mathbf{R}$
(c) $z_1 = z_2$ (d) none of these

Ans. (b)

Solution: Note that at least one of z_1, z_2 is different from 0. Suppose $z_2 \neq 0$. We can write

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1, \text{ as } \left|\frac{z_1/z_2 - 1}{z_1/z_2 + 1}\right| = 1 \text{ or as } \left|\frac{z_1}{z_2} - 1\right| = \left|\frac{z_1}{z_2} + 1\right|.$$

This shows that z_1/z_2 lies on the perpendicular bisector of the segment joining $A(-1 + i0)$ and $B(1 + i0)$ [See Theory]. Thus, z_1/z_2 lies on the imaginary axis.

$\therefore \quad \dfrac{z_1}{z_2} = ia$ for some $a \in \mathbf{R}$. $\Rightarrow \dfrac{z_2}{z_1} = \dfrac{1}{ia} = \dfrac{-i}{a}$

$\Rightarrow \quad z_2 = ikz_1$ for some $k \in \mathbf{R}$

◉ **Example 40:** For any complex number z, the minimum value of $|z| + |z - 2i|$ is

(a) 0 (b) 1
(c) 2 (d) none of these

Ans. (c)

◎ **Solution:** We have, for $z \in \mathbf{C}$

$$|2i| = |z + (2i - z)| \le |z| + |2i - z|$$

$$\Rightarrow \quad 2 \le |z| + |z - 2i|$$

Thus, minimum value of $|z| + |z - 2i|$ is 2 and it is attained when $z = i$.

◉ **Example 41:** If $x = 2 + 5i$, then the value of $x^3 - 5x^2 + 33x - 19$ is equal to

(a) – 5 (b) – 7
(c) 7 (d) 10

Ans. (d)

◎ **Solution:** $x = 2 + 5i \Rightarrow x - 2 = 5i$

$\Rightarrow (x - 2)^2 = (5i)^2 \Rightarrow x^2 - 4x + 4 = -25$

$\Rightarrow x^2 - 4x + 29 = 0$. We now divide $x^3 - 5x^2 + 33x - 19$ by $x^2 - 4x + 29$.

$$\begin{array}{r|l} & x - 1 \\ x^2 - 4x + 29 & x^3 - 5x^2 + 33x - 19 \\ & x^3 - 4x^2 + 29x \\ & \ominus \quad \oplus \quad \ominus \\ \hline & -x^2 + 4x - 19 \\ & -x^2 + 4x - 29 \\ & \oplus \quad \ominus \quad \oplus \\ \hline & 10 \end{array}$$

Thus, $x^3 - 5x^2 + 33x - 19 = (x - 1)(x^2 - 4x + 29) + 10$

$= (x - 1)(0) + 10 = 10$

◉ **Example 42:** If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$, then $|w| = 1$ implies, that, in the complex plane

(a) z lies on the imaginary axis
(b) z lies on the real axis
(c) z lies on the unit circle
(d) none of these

Ans. (b)

◎ **Solution:** $|w| = 1 \quad \Rightarrow \quad \left|\frac{1 - iz}{z - i}\right| = 1$

$\Rightarrow \quad |-i^2 - iz| = |z - i| \quad \Rightarrow \quad |(-i)(z + i)| = |z - i|$

$\Rightarrow \quad |z + i| = |z - i|$

$\Rightarrow$ z lies on the perpendicular bisector of the segment joining i and $-i$.

$\Rightarrow$ z lies on the real axis.

◉ **Example 43:** The real part of $z = \frac{1}{1 - \cos\theta + i\sin\theta}$ is

(a) $\frac{1}{1 - \cos\theta}$ (b) $\frac{1}{2}$
(c) $\frac{1}{2}\tan\theta$ (d) 2

Ans. (b)

◎ **Solution:**

> **TIP**
>
> If $z = \frac{1}{\omega}$, then
>
> $\text{Re}(z) = \frac{\text{Re}(\omega)}{|\omega|^2}$

$$\therefore \quad \text{Re}(z) = \frac{1 - \cos\theta}{(1 - \cos\theta)^2 + \sin^2\theta}$$

$$= \frac{1 - \cos\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{1 - \cos\theta}{2(1 - \cos\theta)} = \frac{1}{2}$$

◉ **Example 44:** If the imaginary part of $\frac{2z + 1}{iz + 1}$ is – 4, then the locus of the point representing z in the complex plane is

(a) a straight line (b) a parabola
(c) a circle (d) an ellipse

Ans. (c)

◎ **Solution:** Let $z = x + iy$, then

$$\frac{2z + 1}{iz + 1} = \frac{2(x + iy) + 1}{i(x + iy) + 1} = \frac{(2x + 1) + 2iy}{(1 - y) + ix}$$

$$= \frac{[(2x + 1) + 2iy][(1 - y) - ix]}{(1 - y)^2 + x^2}$$

As $\text{Im}\left(\frac{2z + 1}{iz + 1}\right) = -4$, we get

$$\frac{2y(1 - y) - x(2x + 1)}{x^2 + (1 - y)^2} = -4$$

$$\Rightarrow \quad 2x^2 + 2y^2 + x - 2y = 4x^2 + 4(y^2 - 2y + 1)$$

$$\Rightarrow \quad 2x^2 + 2y^2 - x - 6y + 4 = 0$$

which represents a circle.

◉ **Example 45:** The area of the triangle whose vertices are the points represented by the complex number z, iz and $z + iz$ is

(a) $\frac{1}{4}|z|^2$ (b) $\frac{1}{8}|z|^2$
(c) $\frac{1}{2}|z|^2$ (d) $\frac{1}{2}|z|$

Ans. (c)

◎ **Solution:** Area of the triangle is given by

$$\Delta = \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ z+iz & \bar{z}-i\bar{z} & 1 \end{vmatrix}\right|$$

Applying $R_3 \to R_3 - R_1 - R_2$, we get

$$\Delta = \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ 0 & 0 & -1 \end{vmatrix}\right| = \frac{1}{4}\left|(-1)\begin{vmatrix} z & \bar{z} \\ iz & -i\bar{z} \end{vmatrix}\right|$$

$$= \frac{1}{4}\left|(-1)\,(i)z\,\bar{z}\right|\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \frac{1}{4}|z|^2\,(2) = \frac{1}{2}|z|^2$$

◉ **Example 46:** If ω is a complex cube root of unity, then a root of the equation $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$ is

(a) $x = 1$ (b) $x = \omega$
(c) $x = \omega^2$ (d) $x = 0$

Ans. (d)

◎ **Solution:** Let us denote the given determinant by Δ. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+1+\omega+\omega^2 & \omega & \omega^2 \\ x+1+\omega+\omega^2 & x+\omega^2 & 1 \\ x+1+\omega+\omega^2 & 1 & x+\omega \end{vmatrix} = \begin{vmatrix} x & \omega & \omega^2 \\ x & x+\omega^2 & 1 \\ x & 1 & x+\omega \end{vmatrix}$$

Clearly $\Delta = 0$ for $x = 0$.

◉ **Example 47:** Let z_1 and z_2 be two non-zero complex numbers such that $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$, then the origin and points represented by z_1 and z_2

(a) lie on a straight line
(b) form a right triangle
(c) form an equilateral triangle
(d) none of these

Ans. (c)

◎ **Solution:** Let $z = \frac{z_1}{z_2}$, then $z + \frac{1}{z} = 1 \Rightarrow z^2 - z + 1 = 0$

$$\Rightarrow \quad z = \frac{1 \pm \sqrt{3}\,i}{2} \quad \Rightarrow \quad \frac{z_1}{z_2} = \frac{1 \pm \sqrt{3}i}{2}$$

If z_1 and z_2 are represented by A and B respectively and O be the origin, then

$$\frac{OA}{OB} = \frac{|z_1|}{|z_2|} = \left|\frac{1 \pm \sqrt{3}i}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$\Rightarrow \quad OA = OB$

Also, $$\frac{AB}{OB} = \frac{|z_2 - z_1|}{|z_2|} = \left|1 - \frac{z_1}{z_2}\right|$$

$$= \left|1 - \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\right| = \left|\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$\Rightarrow \quad AB = OB$

Thus, $\quad OA = OB = AB$.

$\therefore$ ΔOAB is an equilateral triangle.

◉ **Example 48:** If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}\,x^{2n}$, then value of $a_0 + a_3 + a_6 + \ldots$ is

(a) 1 (b) 2^n
(c) 2^{n-1} (d) 3^{n-1}

Ans. (d)

◎ **Solution:** Putting $x = 1, \omega, \omega^2$ in

$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}\,x^{2n}$, we get

$(1 + 1 + 1)^n = a_0 + a_1 + a_2 + a_3 + \ldots + a_{2n}$,

$(1 + \omega + \omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \ldots + a_{2n}\,\omega^{2n}$,

and $(1 + \omega^2 + \omega^4)^n = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \ldots + a_{2n}\,\omega^{4n}$

Adding the above three equations and using $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$ we get

$$3^n = 3(a_0 + a_3 + a_6 + \ldots)$$

$\Rightarrow a_0 + a_3 + a_6 + \ldots = 3^{n-1}$.

◉ **Example 49:** Let

$$z = \begin{vmatrix} 1 & 1-2i & 3+5i \\ 1+2i & -5 & 10i \\ 3-5i & -10i & 11 \end{vmatrix}, \text{ then}$$

(a) z is purely imaginary
(b) z is purely real
(c) $z = 0$
(d) none of these

Ans. (b)

◎ **Solution:** Conjugate of z equals the determinant obtained by taking conjugate of each of its element. Therefore,

$$\bar{z} = \begin{vmatrix} 1 & 1+2i & 3-5i \\ 1-2i & -5 & -10i \\ 3+5i & 10i & 11 \end{vmatrix} = \begin{vmatrix} 1 & 1-2i & 3+5i \\ 1+2i & -5 & 10i \\ 3-5i & -10i & 11 \end{vmatrix} = z$$

Thus, z is purely real.

◉ **Example 50:** If $(x + iy)^{1/3} = a + ib$, then $\frac{x}{a} + \frac{y}{b}$ equals

(a) $4(a^2 - b^2)$ (b) $2(a^2 - b^2)$
(c) $2(a^2 + b^2)$ (d) none of these

Ans. (a)

◎ **Solution:** $(x + iy)^{1/3} = a + ib$

$\Rightarrow \quad x + iy = (a + ib)^3 = a^3 + 3a^2(ib) + 3a(ib)^2 + (ib)^3$

$\Rightarrow \quad x + iy = (a^3 - 3ab^2) + (3a^2b - b^3)i$

Equating real and imaginary parts, we get

$$x = a^3 - 3ab^2 \text{ and } y = 3a^2b - b^3$$

$$\Rightarrow \quad \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2.$$

Thus, $\frac{x}{a} + \frac{y}{b} = 4a^2 - 4b^2 = 4(a^2 - b^2)$

Example 51: If $z \in \mathbf{C}$, the minimum value of $|z| + |z - i|$ is attained at

(a) exactly one point
(b) exactly two points
(c) infinite number of points
(d) none of these

Ans. (c)

Solution: We have

$$1 = |i| = |z + (i - z)| \le |z| + |i - z|$$

$$\Rightarrow \quad |z| + |z - i| \ge 1$$

The minimum value 1 is attained at all points $z = i\,t$ where $t \in [0, 1]$.

Example 52: For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

(a) 0 (b) 2
(c) 7 (d) 17

Ans. (b)

Solution: $|z_1| = 12$ implies that z_1 lies on the circle with centre C_1 at the origin and radius 12 whereas $|z_2 - 3 - 4i| = 5$ implies z_2 lies on the circle with centre at C_2 $(3 + 4i)$ and radius 5. See Fig. 2.45. The quantity $|z_1 - z_2|$ will be least if z_1 and z_2 lie on the line joining C_1 and C_2 i.e. on the line $z = (3 + 4i)t$

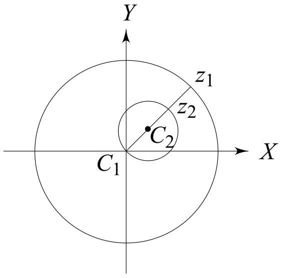


Fig. 2.45

In fact, when we take $z_2 = 6 + 8i$ and $z_1 = \frac{12}{5}(3 + 4i)$, we obtain $|z_1 - z_2| = \left|\frac{12}{5}(3+4i) - (6+8i)\right| = \frac{2}{5}|3+4i| = 2.$

Example 53: If z lies on the circle $|z - 1| = 1$, then $\frac{z-2}{z}$ equals

(a) 0
(b) 2
(c) – 1
(d) none of these

Ans. (d)

Solution: Note that $|z - 1| = 1$ represents a circle with the segment joining $z = 0$ and $z = 2 + 0i$ as a diameter. See Fig. 2.46. If z lies on the circle, then $\frac{z-2}{z-0}$ is purely imaginary.

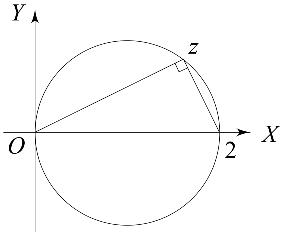


Fig. 2.46

Example 54: If $1, \omega, \ldots, \omega^{n-1}$ are the nth roots of unity, then value of $\frac{1}{2-\omega} + \frac{1}{2-\omega^2} + \cdots + \frac{1}{2-\omega^{n-1}}$ equals

(a) $\frac{1}{2^n - 1}$ (b) $\frac{n(2^n - 1)}{2^n + 1}$

(c) $\frac{(n-2)\,2^{n-1}}{2^n - 1}$ (d) none of these

Ans. (d)

Solution: We know that

$$\frac{1}{x-1} + \frac{1}{x-\omega} + \frac{1}{x-\omega^2} + \cdots + \frac{1}{x-\omega^{n-1}} = \frac{n(x^{n-1})}{x^n - 1}$$

Putting $x = 2$, we get

$$\frac{1}{2-\omega} + \frac{1}{2-\omega^2} + \cdots + \frac{1}{2-\omega^{n-1}} = \frac{n(2^{n-1})}{2^n - 1}$$

Example 55: If $\omega = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$, then value of $1 + \omega + \omega^2 + \ldots + \omega^{n-1}$ is

(a) $1 + i\cot\left(\frac{\pi}{2\pi}\right)$ (b) $1 + i\tan\left(\frac{\pi}{n}\right)$

(c) $1 + i$ (d) none of these

Ans. (a)

Solution: We have

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega} = \frac{2}{1-\omega}.$$

as $\omega^n = \cos\left(\frac{n\pi}{n}\right) + i\sin\left(\frac{n\pi}{n}\right) = \cos\pi + i\sin\pi = -1$

$$\text{Now,} \quad \frac{2}{1-\omega} = \frac{2(1-\bar{\omega})}{(1-\omega)(1-\bar{\omega})} = \frac{2(1-\bar{\omega})}{1-(\omega+\bar{\omega})+\omega\bar{\omega}}$$

$$= \frac{2(1-\bar{\omega})}{2-2\,\text{Re}(\omega)} \qquad [\because \omega\bar{\omega} = |\omega|^2 = 1]$$

$$= \frac{1-\text{Re}(\omega)+i\,\text{Im}(\omega)}{1-\text{Re}(\omega)} = 1 + i\frac{\text{Im}(\omega)}{1-\text{Re}(\omega)}$$

$$= 1 + i\frac{\sin\left(\frac{\pi}{n}\right)}{1-\cos\left(\frac{\pi}{n}\right)} = 1 + i\frac{2\sin\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)}{2\sin^2\left(\frac{\pi}{2n}\right)}$$

$$= 1 + i\cot(\pi/2n)$$

Example 56: If $|z| = 1$ and $z \ne \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on:

(a) a line not passing through the origin.
(b) $|z| = 2$
(c) the x – axis
(d) the y – axis

Ans. (d)

Solution: As $|z| = 1$, we get $z\bar{z} = 1$.

Let $w = \dfrac{z}{1-z^2}$, then

$$w + \bar{w} = \frac{z}{1-z^2} + \frac{\bar{z}}{1-\bar{z}^2} = \frac{z}{1-z^2} + \frac{1/z}{1-(1/z)^2}$$

$$= \frac{z}{1-z^2} + \frac{z}{z^2-1} = 0$$

$\Rightarrow \quad 2\text{Re}(w) = 0 \quad \Rightarrow \quad \text{Re}(w) = 0$

Thus, w lies on the y-axis.

Alternate Solution

$$w = \frac{z}{1-z^2} = \frac{z}{z\bar{z}-z^2} = \frac{1}{\bar{z}-z}$$

$$= \frac{1}{-2i\,\text{Im}(z)}$$

$\Rightarrow w$ is purely imaginary, that is, w lies on the y-axis.

Example 57: The locus of the centre of a circle which touches the circle $|z - z_1| = a$ and $|z - z_2| = b \neq a$ externally is

(a) an ellipse (b) a hyperbola

(c) a circle (d) a pair of straight lines.

Ans. (b)

Solution: Suppose $|z - w| = r$ touches $|z - z_1| = a$ and $|z - z_2| = b$ externally.

Then $|w - z_1| = a + r$, $|w - z_2| = b + r$

$\Rightarrow \quad |w - z_1| - |w - z_2| = a - b$

$\Rightarrow \quad w$ lies on a hyperbola with focii at z_1 and z_2

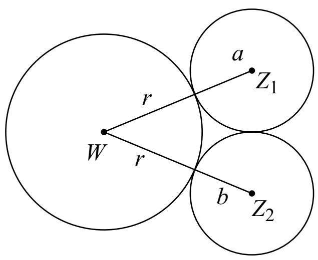


Fig. 2.47

Example 58: If $|z^2 - 1| = |z|^2 + 1$, then z lies on

(a) a circle (b) the imaginary axis

(c) the real axis (d) an ellipse

Ans. (b)

Solution: $|z^2 - 1| = |z|^2 + 1$ can be written as $|z^2 + (-1)| = |z^2| + |-1|$

$\Leftrightarrow \dfrac{z^2}{-1}$ is a non-negative real number.

$\Leftrightarrow z^2$ is a non-positive real number.

$\Leftrightarrow z$ lies on the imaginary axis.

Example 59: If $z^2 + z + 1 = 0$, where z is a complex number, then values of

$$S = \left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \text{---} + \left(z^6 + \frac{1}{z^6}\right)^2 \text{ is}$$

(a) 12 (b) 18

(c) 54 (d) 6

Ans. (a)

Solution: $z^2 + z + 1 = 0 \quad \Rightarrow \quad z = \omega, \omega^2$.

Let $z = \omega$ then $(\omega + \omega^2)^2$

$= (\omega^2 + \omega^4)^2 = (\omega^4 + \omega^8)^2$

$= (\omega^5 + \omega^{10})^2 = 1$

and $(\omega^3 + \omega^6)^2 = (\omega^6 + \omega^{12})^2 = 4$.

Thus, $S = 12$.

Example 60: If $|z + 4| \leq 3$, then maximum value of $|z + 1|$ is

(a) 4 (b) 10

(c) 6 (d) 0

Ans. (c)

Solution: $|z+4| \leq 3$ represents the interior and boundary of the circle with center at -4 and radius equal to 3. As -1 is an end point of a diameter of the circle, maximum possible value of $|z + 1|$ is 6 which is attained when $z = -7$. See Fig. 2.48

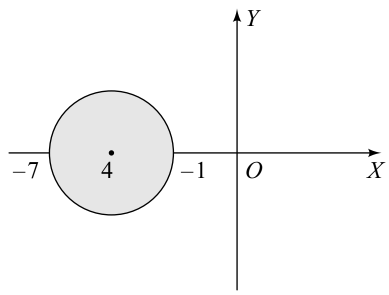


Fig. 2.48

Example 61: Let z, w be two complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\arg(zw) = \pi$, then $\arg z$ equals

(a) $\dfrac{3\pi}{4}$ (b) $\dfrac{\pi}{2}$

(c) $\dfrac{\pi}{4}$ (d) $\dfrac{5\pi}{4}$

Ans. (a)

Solution: $\bar{z} + i\bar{w} = 0 \quad \Rightarrow \quad z - iw = 0$

Now, $\arg(zw) = \pi \quad \Rightarrow \quad \arg\left(\dfrac{z^2}{i}\right) = \pi$

$\Rightarrow \quad 2\arg(z) - \arg(i) = \pi$

$\Rightarrow \quad 2\arg(z) = \pi + \dfrac{\pi}{2} = \dfrac{3\pi}{2} \quad \Rightarrow \quad \arg(z) = \dfrac{3\pi}{4}$

Example 62: If $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then value of $z_1^2 + z_2^2 + z_3^2$ equals

(a) – 1 (b) 0
(c) 1 (d) 3

Ans. (b)

Solution: $z_1^2 + z_2^2 + z_3^2$

$$= (z_1 + z_2 + z_3)^2 - 2(z_2 z_3 + z_3 z_1 + z_1 z_2)$$
$$= 0 - 2z_1 z_2 z_3 \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$$
$$= -2z_1 z_2 z_3 (\bar{z}_1 + \bar{z}_2 + \bar{z}_3) \quad [\because |z_1| = |z_2| = |z_3| = 1]$$
$$= -2z_1 z_2 z_3(0)$$
$$= 0$$

Example 63: If z satisfies the relation

$$|z - i|z|| = |z + i|z||, \quad (1)$$

then

(a) Im $(z) = 0$ (b) $|z| = 1$
(c) Re$(z) = 0$ (d) none of these

Ans. (a)

Solution: $z = 0$ clearly satisfies (1).
For $z \neq 0$, (1) can be written as

$$\left|\frac{z}{|z|} - 1\right| = \left|\frac{z}{|z|} + i\right|$$

$\Rightarrow \frac{z}{|z|}$ is equidistant from i and $-i$

$\Rightarrow \frac{z}{|z|}$ lies on the real axis.

$\Rightarrow z$ is real.

Thus, Im$(z) = 0$.

Example 64: If α, β are distinct complex numbers with $|\beta| = 1$, then value of $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$ equals

(a) 1 (b) $|\alpha|$
(c) 2 (d) none of these

Ans. (a)

Solution: $|\beta| = 1 \Rightarrow \beta\bar{\beta} = 1$

$$\therefore \left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{1 - \bar{\alpha}/\bar{\beta}}\right| = \left|\frac{\bar{\beta}(\beta - \alpha)}{\bar{\beta} - \bar{\alpha}}\right| = |\beta| \frac{|\beta - \alpha|}{|\bar{\beta} - \bar{\alpha}|}$$
$$= |\beta|(1) = 1. \quad [\because |z| = |\bar{z}|]$$

Example 65: Suppose $z \in \mathbf{C}$, and $z \notin \mathbf{R}$. If $w = \frac{1 - z + z^2}{1 + z + z^2}$ is a real number, then $|z|$ equals

(a) 1 (b) 2
(c) $\sqrt{3}$ (d) $2\sqrt{3}$

Ans. (a)

Solution: Note that $z \neq 0$ and $1 - w = \frac{2z}{1 + z + z^2} \in \mathbf{R}$.

$$\Rightarrow \frac{z}{1 + z + z^2} \in \mathbf{R} \Rightarrow \frac{1}{1 + z + 1/z} \in \mathbf{R} \Rightarrow z + \frac{1}{z} \in \mathbf{R}$$

$$\Rightarrow z + \frac{1}{z} = \bar{z} + \frac{1}{\bar{z}} \quad \Rightarrow \quad z - \bar{z} = \frac{1}{\bar{z}} - \frac{1}{z} = \frac{z - \bar{z}}{|z|^2}$$

As $z \notin \mathbf{R}$, $z \neq \bar{z}$, therefore $|z|^2 = 1$ or $|z| = 1$

Example 66: If $\left|z - \frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to

(a) 1 (b) $2 + \sqrt{2}$
(c) $\sqrt{3} + 1$ (d) $\sqrt{5} + 1$

Ans. (d)

Solution: $$|z| - \frac{4}{|z|} \leq \left||z| - \frac{4}{|z|}\right| \leq \left|z - \frac{4}{z}\right| = 2$$

$$\Rightarrow \quad |z|^2 - 2|z| - 4 \leq 0$$

$$\Rightarrow \quad (|z| - 1)^2 \leq 5 \Rightarrow |z| \leq 1 + \sqrt{5}.$$

The maximum value $\sqrt{5} + 1$ is attained when $z = \sqrt{5} + 1$.

Example 67: If $|\omega| = 2$, then the set of points $x + iy = \omega - \frac{1}{\omega}$ lie on

(a) circle (b) ellipse
(c) parabola (d) hyperbola

Ans. (b)

Solution: $|\omega| = 2 \Rightarrow \omega = 2e^{i\theta} = 2(\cos\theta + i\sin\theta)$

$$\Rightarrow \quad x + iy = \omega - \frac{1}{\omega} = 2e^{i\theta} - \frac{1}{2}e^{-i\theta}$$

$$\Rightarrow \quad x = \frac{3}{2}\cos\theta,\ y = \frac{5}{2}\sin\theta \Rightarrow \frac{x^2}{9/4} + \frac{y^2}{25/4} = 1$$

which represents an ellipse.

Example 68: If $|z| = 1$, $z \neq 1$, then value of arg $\left(\frac{1}{1-z}\right)$ cannot exceed

(a) $\pi/2$ (b) π
(c) $3\pi/2$ (d) 2π

Ans (a)

Solution: As $|z| = 1$, $z \neq 1$, $z = \cos\theta + i\sin\theta$, $-\pi < \theta \leq \pi$, $\theta \neq 0$. Now

$$\omega = \frac{1}{1-z} = \frac{1}{1 - \cos\theta - i\sin\theta}$$

$$= \frac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} = \frac{(1-\cos\theta)+i\sin\theta}{2(1-\cos\theta)}$$

$$= \frac{1}{2}+\frac{i}{2}\cot\left(\frac{\theta}{2}\right) = \frac{1}{2}+\frac{i}{2}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

This shows that ω lies on the line $x = 1/2$ and $-\pi/2 < \arg(\omega) < \pi/2$, Arg $(\omega) \neq 0$, The maximum value of Arg(ω) is never attained.

Example 69: If $z \neq 1$, $\frac{z^2}{z-1}$ is real, then point represented by the complex number z lies

(a) on circle with centre at the origin.
(b) either on the real axis or on a circle not passing through the origin.
(c) on the imaginary axis.
(d) either on the real axis or on a circle passing through the origin

Ans (b)

Solution: As $\frac{z^2}{z-1}$ is real, we get

$$\frac{z^2}{z-1} = \frac{\bar{z}^2}{\bar{z}-1}$$

$\Leftrightarrow \quad z^2(\bar{z} - 1) = \bar{z}^2 (z - 1)$

$\Leftrightarrow \quad z\bar{z}(z - \bar{z}) - (z - \bar{z})(z + \bar{z}) = 0$

$\Leftrightarrow \quad (z - \bar{z})(z\bar{z} - z - \bar{z}) = 0$

$\Rightarrow \quad z - \bar{z} = 0$ or $z\bar{z} - z - \bar{z} = 0$

$\Rightarrow \quad z$ lies on the real axis

or z lies on a circle through the origin.

Example 70: If $3^{49}(x + iy) = \left(\frac{3}{2}+\frac{i}{2}\sqrt{3}\right)^{100}$, $y \in \mathbf{N}$, and $x = ky$, then value of k is

(a) $\pm 1/3$ (b) $\pm 2\sqrt{2}$
(c) $\pm 1/\sqrt{3}$ (d) $\pm 1/\sqrt{3}$

Ans (b)

Solution: We have

$3^{49}|x + iy| = |3/2 + \sqrt{3}i/2|^{100}$

$\Rightarrow \quad 3^{49}\sqrt{x^2+y^2} = (9/4 + 3/4)^{50}$

$\Rightarrow \quad \sqrt{x^2+y^2} = 3 \Rightarrow (y)\sqrt{1+k^2} = 3$

$\Rightarrow \quad \sqrt{1+k^2} = 3, 3/2, 1,$ as $y \in \mathbf{N}$

$\Rightarrow \quad k = \pm 2\sqrt{2}, \pm \sqrt{5}/2, 0$

Out of the given values, we have $k = \pm 2\sqrt{2}$.

Example 71: If $(4 + i)(z + \bar{z}) - (3 + i)(z - \bar{z}) + 26i = 0$, then the value of $|z|^2$ is

(a) 13 (b) 17
(c) 19 (d) 11

Ans (b)

Solution: Let $z = x + iy$, then

$2(4 + i)x - (3 + i)2iy + 26i = 0$

$\Rightarrow \quad 4x + y = 0, x - 3y + 13 = 0$

$\Rightarrow \quad x = -1, y = 4$

$\therefore \quad |z|^2 = 17$

Example 72: Let $z = a\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$, $a \in \mathbf{R}$, $|a| < 1$, then $S = z^{2015} + z^{2016} + z^{2017} + \ldots$ equals

(a) $\frac{a^{2015}}{z-1}$ (b) $\frac{a^{2015}}{1-z}$
(c) $\frac{z^{2015}}{1-a}$ (d) $\frac{z^{2015}}{a-1}$

Ans (a)

Solution: We have $|z| = |a| < 1$, thus

$$S = \frac{z^{2015}}{1-z}$$

But $\quad z^{2015} = a^{2015}[\cos(403\pi) + i\sin(403\pi)] = -a^{2015}$

$\therefore \quad S = \frac{a^{2015}}{z-1}$

Example 73: If $z = \sqrt{20i-21}+\sqrt{20i+21}$, than one of the possible value of $|\arg(z)|$ equals

(a) $\pi/4$ (b) $\pi/2$
(c) $3\pi/8$ (d) π

Ans (a)

Solution: $20i + 21 = (5 + 2i)^2$

and $\quad 20i - 21 = (2 - 5i)^2$

$\therefore \quad z = \pm(5 + 2i) \pm (2 + 5i)$

$\Rightarrow \quad z = 7(1 + i), 3(1 - i), 3(-1 + i), 7(-1 - i)$

$\Rightarrow \quad \arg(z) = \pi/4, -\pi/4, 3\pi/4, -3\pi/4$

Thus, one of the possible value of $|\arg(z)|$ is $\pi/4$.

Example 74: If $(a + bi)^{11} = x + iy$, where $a, b, x, y \in \mathbf{R}$, then $(b + ai)^{11}$ equals

(a) $y + ix$ (b) $-y - ix$
(c) $-x - iy$ (d) $x + iy$

Ans (b)

Solution: $\overline{(a+bi)^{11}} = \overline{x+iy}$

$\Rightarrow \quad (a - bi)^{11} = x - iy$

$\Rightarrow \quad [(-i)(b + ia)]^{11} = -i(y + ix)$

$\Rightarrow \quad (-i)^{11}(b + ia)^{11} = -i(y + ix)$

As $\quad (-i)^{11} = (-i)^8(-1)^3 i^3 = i,$

we get $\quad (b + ia)^{11} = -(y + ix) = -y - ix$

Example 75: If $a, b, x, y \in \mathbf{R}$, $\omega \neq 1$, is a cube root of unity and $(a + b\omega)^7 = x + y\omega$, then $(b + a\omega)^7$ equals

(a) $y + x\omega$ (b) $-y - x\omega$

(c) $x + y\omega$ (d) $-x - y\omega$

Ans (a)

Solution: Taking conjugate, we get

$$(a + b\omega^2)^7 = x + y\,\omega^2 \Rightarrow (a\omega^3 + b\omega^2)^7 = x\omega^3 + y\omega^2$$

$$\omega^{14}\,(a\omega + b)^7 = \omega^2\,(x\omega + y)$$

$$\Rightarrow \quad (a\omega + b)^7 = x\omega + y$$

Assertion-Reason Type Questions

Example 76: Suppose z_1, z_2 are two distinct complex numbers and a, b are real numbers.

Statement-1: If $z_1 + z_2 = a$, $z_1 z_2 = b$, then $\arg(z_1 z_2) = 0$

Statement-2: If $z_1 + z_2 = a$, $z_1 z_2 = b$, then $\bar{z}_1 = z_2$

Ans. (c)

Solution: Statement-2 is false. This can be seen by taking z_1 and z_2 to be distinct real numbers.

For statement-1, take $z_1 = \alpha_1 + i\beta_1$, $z_2 = \alpha_2 + i\beta_2$. We have

$$\beta_1 + \beta_2 = 0,\ \alpha_1\beta_2 + \alpha_2\beta_1 = 0$$

If $\beta_2 = 0$, then $\beta_1 = 0$, and statement-1 is true.

If $\beta_2 \neq 0$, then $\beta_1 = -\beta_2$ and $(\alpha_1 - \alpha_2)\beta_2 = 0$

$\Rightarrow \quad \alpha_1 - \alpha_2 = 0$ or $\alpha_1 = \alpha_2$

Thus, $\alpha_2 + i\beta_2 = \alpha_1 - i\beta_1 \Rightarrow z_2 = \bar{z}_1$

In this case also $\arg(z_1 z_2) = \arg(z_1 \bar{z}_1)$

$= \arg(|z_1|^2) = 0$

Example 77: Suppose z_1 and z_2 are two distinct non-zero complex numbers.

Statement-1: $|z_1 - z_2| = \left|\dfrac{1}{z_1} - \dfrac{1}{z_2}\right|$

$\Rightarrow \quad |z_1|\,|z_2| = 1$

Statement-2: $|z_1 - z_2| = \left|\dfrac{1}{z_1} - \dfrac{1}{z_2}\right|$

then both of z_1, z_2 have modulus 1

Ans (c)

Solution: $|z_1 - z_2| = \left|\dfrac{1}{z_1} - \dfrac{1}{z_2}\right| = \dfrac{|z_2 - z_1|}{|z_2\,z_1|}$

As $z_1 \neq z_2$, we get $|z_1\,z_2| = 1$

$\Rightarrow \quad |z_1|\,|z_2| = 1$

$\therefore$ Statement-1 is true.

Statement-2 is false. For example take $z_1 = 3$, $z_2 = 1/3$.

Example 78: Suppose $a, b, c \in \mathbf{R}$.

Statement-1: If $z = a + (b + ic)^{2017} + (b - ic)^{2017}$ then z is real.

Statement-2: If $z = \bar{z}$, then z is real

Ans (a)

Solution: Statement-2 is true. See theory

We have

$$\bar{z} = a + (\overline{b + ic})^{2017} + (\overline{b - ic})^{2017}$$

$$= a + (b - ic)^{2017} + (b + ic)^{2017}$$

$$= z$$

$\Rightarrow \quad z$ is real

$\therefore$ Statement-1 is true and statement-2 is correct explanation for it.

Example 79: Let $\omega \neq 1$, be a cube root of unity, and $a, b \in \mathbf{R}$.

Statement-1: $a^3 + b^3 = (a + b)\,(a\omega + b\omega^2)\,(a\omega^2 + b\omega)$

Statement-2: $x^3 - 1 = (x - 1)\,(x\omega^2 - \omega)\,(x\omega - \omega^2)$

for each $x \in \mathbf{R}$.

Ans (a)

Solution: We have

$$(x\omega^2 - \omega)\,(x\omega - \omega^2)$$

$$= x^2\omega^3 - x\omega^2 - x\omega^4 + \omega^3$$

$$= x^2 - (\omega^2 + \omega)x + 1$$

$$= x^2 + x + 1 \qquad [\because\ \omega^2 + \omega = -1]$$

$\therefore \quad (x - 1)\,(x\omega^2 - \omega)\,(x\omega - \omega^2)$

$= (x - 1)\,(x^2 + x + 1) = x^3 - 1$

Thus, Statement-2 is true.

Replacing x by $-x$, we get

$$(-x)^3 - 1 = (-x - 1)\,(-x\,\omega^2 - \omega)\,(-x\omega - \omega^2)$$

$$\Rightarrow \quad x^3 + 1 = (x + 1)\,(x\,\omega^2 + \omega)\,(x\omega + \omega^2)$$

Taking conjugate of both the sides, we get

$$x^3 + 1 = (x + 1)\,(x\,\omega + \omega^2)\,(x\omega^2 + \omega)$$

$$[\because\ \omega = \omega^2]$$

If $b = 0$, statement-1 is clearly true. Suppose $b \neq 0$.

Replacing x by a/b we get

$$\left(\frac{a}{b}\right)^3 + 1 = \left(\frac{a}{b} + 1\right)\left(\frac{a}{b}w + w^2\right)\left(\frac{a}{b}w^2 + w\right)$$

$$\Rightarrow \quad a^3 + b^3 = (a + b)\,(aw + bw^2)\,(aw^2 + bw)$$

Thus, statement-1 is also true and statement-2 is a correct explanation for it.

◉ **Example 80:** Let A, B, C be three set of complex numbers as defined below:

$$A = \{z : \text{Im } z \geq 1\}$$

$$B = \{z : |z - 2 - i| = 3\}$$

$$C = \{z : \text{Re } ((1 - i)\, z)\} = \sqrt{2}\ \}$$

Statement-1: $|z + 1 - i|^2 + |z - 5 - i|^2 = 37$

$\forall\, z \in A \cap B \cap C$

Statement-2: $A \cap B \cap C$ consists of exactly one point.

Ans (d)

◎ **Solution:** The set A consists of all points in the half plane Im $(z) \geq 1$, that is, all the points above and including the line through S and parallel to the real axis. The set B is the set of all points on the circle with centre at $2 + i$ and radius 3, and the set C consists of all the points on the line $x + y = \sqrt{2}$. The regions in A, B and C intersect in exactly one point *viz*. R. see Fig. 2.49.

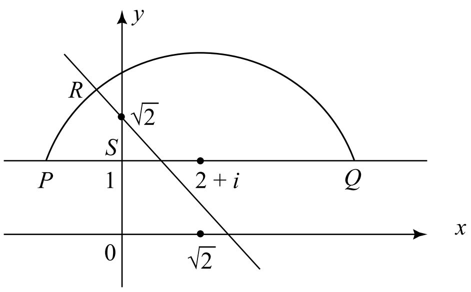


Fig. 2.49

$\therefore$ Statement-2 is true. Points $P\,(-1 + i)$ and $Q(5 + i)$ are the end points of diameter of the circle $|z - (2 + i)| = 3$

Now, $|z + 1 - i|^2 + |z - 5 - i|^2$

$$= PR^2 + QR^2 = PQ^2 = 36$$

$\therefore$ Statement-1 is false.

◉ **Example 81:** Let z_1, z_2, z_3 be three distinct non-zero complex numbers such that $a = |z_1|$, $b = |z_2|$, $c = |z_3|$. and

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Statement-1: If $z_1 + z_2 + z_3 = 0$, then triangle with vertices z_1, z_2 and z_3 is an equilateral triangle.

Statement-2: Area of triangle with vertices z_1, z_2, and z_3 is $\dfrac{\sqrt{3}}{4}|z_1 - z_2|^2$

Ans (c)

◎ **Solution:** Applying $C_1 \to C_1 + C_2 + C_3$,

we get

$$\Delta = (a + b + c)\,\Delta_1 \text{ where}$$

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= -(b - c)^2 - (a - b)\,(a - c)$$

$$= -[a^2 + b^2 + c^2 - bc - ca - ab]$$

$$= -\frac{1}{2}\,[(b - c)^2 + (c - a)^2 + (a - b)^2]$$

Now, $\Delta = 0$

$\Rightarrow \quad (a + b + c)\,[(b - c)^2 + (c - a)^2 + (a - b)^2]$

As $a + b + c > 0$, we get

$$a = b = c$$

$\Rightarrow \quad |z_1| = |z_2| = |z_3|$

$\Rightarrow$ z_1, z_2, z_3 lie on a circle with centre at the origin and radius equal to a.

If we take z_1, z_2 as opposite vertices of a diameter, then triangle is a right triangle with right angle at z_3, and its area is $\dfrac{1}{2}\,|z_3 - z_1|\,|z_3 - z_2|$

$\therefore$ Statement-2 is false.

If $z_1 + z_2 + z_3 = 0$, then $|z_1 - z_2|^2 + |z_3|^2$

$$= |z_1 - z_2|^2 + |-z_1 - z_2|^2$$

$$= 2|z_1|^2 + 2|z_2|^2 = 4a^2$$

$\Rightarrow \quad |z_1 - z_2| = \sqrt{3}a$

Similarly, $|z_2 - z_3| = |z_3 - z_1| = \sqrt{3}a$

Thus, triangle is an equilateral triangle.

◉ **Example 82: Statement-1:** If z_1, z_2, z_3 are such that $|z_1| = |z_2| = |z_3| = 1$, then maximum value of $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$ is 9.

Statement-2: If z_1, z_2, z_3 are such that $|z_1| = |z_2| = |z_3| = 1$, then

$$\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2) \geq -3/2$$

Ans. (a)

◎ **Solution:** $0 \leq |z_1 + z_2 + z_3|^2$

$\Rightarrow \quad 0 \leq |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2)$

$\Rightarrow \quad \text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2) \geq -3/2$

$[\because\ |z_1| = |z_2| = |z_3| = 1]$

Therefore, statement-2 is true.

Next, $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$

$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2) - 2\,\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2)$

$\le 2(1 + 1 + 1) + 2(3/2) = 9$

Maximum value is obtained when $z_1 = 1$, $z_2 = \omega$, $z_3 = \omega^2$, where ω is a cube root of unity.

Statement-1 is also true and statement-2 is correct explanation for it.

Example 83: Statement-1: If $\omega \ne 1$ is a cube root of unity and z is a complex number such that $|z| = 1$, then

$$\left|\frac{2 + 3\omega + 4z\,\omega^2}{4\omega + 3\omega^2\,z + 2z}\right| = 1.$$

Statement-2: If z_1, z_2 are two complex numbers, then $|z_1| = |z_2| \Leftrightarrow z_1 = \bar{z}_2$

Ans. (c)

Solution: $\left|\dfrac{2 + 3\omega + 4z\,\omega^2}{4\omega + 3\omega^2\,z + 2z}\right| = \left|\dfrac{1}{z}\,\dfrac{2 + 3\omega + 4\omega^2 z}{2 + 3\omega^2 + 4\omega z^{-1}}\right|$

$$= \left|\frac{1}{z}\right|\left|\frac{2 + 3\omega + 4\omega^2\,z}{2 + 3\bar{\omega} + 4\bar{\omega}^2\bar{z}}\right| = \frac{1}{|z|} = 1$$

$[\because\ \bar{\omega} = \omega^2,\ \bar{z} = 1/z]$

Thus, Statement-1 is true.

Statement-2 is false as $|z_1| = |z_2|$ does not imply $z_1 = \bar{z}_2$. However, $z_1 = \bar{z}_2 \Rightarrow |z_1| = |z_2|$

Example 84: Statement-1: If z is a root of the equation $x^7 + 2x + 3 = 0$, then $1 \le |z| < 3/2$.

Statement-2: If z lies in the annular region $1 < |z| \le 3/2$, then z satisfies the

$$\frac{1}{z-1} + \frac{1}{z-\omega} + \frac{1}{z-\omega^2} = 1$$

where $\omega \ne 1$ is a cube root of unity.

Ans. (c)

Solution: Suppose $|z| < 1$ and $z^7 + 2z + 3 = 0$, then

$$3 = |-3| = |z^7 + 2z| \le |z^7| + 2|z|$$

$$\Rightarrow \quad 3 \le |z|^7 + 2|z| < 1 + 2(1) = 3.$$

A contradiction.

Next, suppose that $|z| \ge 3/2$ and $z^7 + 2z + 3 = 0$, then $\omega = 1/z$ satisfies the equation $1 + 2\omega^6 + 3\omega^7 = 0$. Now, $1 = |-1| = |2\omega^6 + 3\omega^7| \le 2|\omega|^6 + 3|\omega^7|$

$$\Rightarrow \quad 1 \le 2\left(\frac{2}{3}\right)^6 + 3\left(\frac{2}{3}\right)^7 = \left(\frac{2}{3}\right)^6[2+2] = \frac{2^8}{3^6} < 1$$

A contradiction.

Thus, if z satisfies the equation $z^7 + 2z + 3 = 0$, then $1 \le |z| < 3/2$.

Thus, Statement-1 is true.

Statement-2 is false, as the given relation implies $z^3 - 3z^2 + 1 = 0$ which is satisfied by just 3 values of z where as $1 < |z| \le 3/2$ contains infinite number of points.

Example 85: Statement-1 If $\omega \ne 1$, is a cube root of unity, then $A^2 = O$, where

$$A = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}$$

Statement-2 If $\omega \ne 1$, is a cube root of unity, then

$$\Delta = \begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = x^3$$

Ans. (b)

Solution: It is easy to show

$$A^2 = \begin{pmatrix} y & y & y \\ y & y & y \\ y & y & y \end{pmatrix} = O$$

where $\quad y = 1 + \omega + \omega^2 = 0.$

$\therefore$ Statement-1 is true.

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = x\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & x+\omega^2 & 1 \\ 1 & 1 & x+\omega \end{vmatrix} = x\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & x+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & x+\omega-\omega^2 \end{vmatrix}$$

$= x[(x + \omega^2 - \omega)(x + \omega - \omega^2) - (1 - \omega)(1 - \omega^2)]$

$= x[x^2 - (\omega - \omega^2)^2 - \{1 - \omega - \omega^2 + 1\}] = x^3$

$\therefore$ Statement-2 is also true but is not the correct reason for the statement-1.

LEVEL 2

Straight Objective Type Questions

Example 86: Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}\left(z^{2m-1}\right)$ at $\theta = 2°$ is

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$

(c) $\frac{1}{\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

Ans. (d)

Solution: $\sum_{m=1}^{15} \text{Im}\left(z^{2m-1}\right) = \text{Im}\sum_{m=1}^{15}\left(z^{2m-1}\right)$

$$= \text{Im}\left[\frac{z\left(1-\left(z^2\right)^{15}\right)}{1-z^2}\right] = \text{Im}\left(\frac{1-z^{30}}{\bar{z}-z}\right)$$

$$= \text{Im}\left[\frac{1}{-2i\sin\theta}\{1-\cos(30\theta)-i\sin(30\theta)\}\right]$$

$$= \frac{1}{2\sin\theta}[1-\cos(30\theta)] = \frac{1}{2\sin 2°}[1-\cos(60°)]$$

$$= \frac{1}{4\sin 2°}$$

Example 87: Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is

(a) 48 (b) 32

(c) 40 (d) 80

Ans. (a)

Solution: $z\bar{z}^3 + \bar{z}z^3 = 350 \Rightarrow z\bar{z}\left(\bar{z}^2 + z^2\right) = 350$

$$\Rightarrow \left(x^2+y^2\right)\left[(x-iy)^2+(x+iy)^2\right] = 350$$

$$\Rightarrow \left(x^2+y^2\right)\left(2x^2-2y^2\right) = 350$$

$$\Rightarrow x^4 - y^4 = 175$$

$$\Rightarrow x^4 \ge 175 \Rightarrow x^4 \ge 256$$

Let us try $x = \pm 4$

Thus, $y^4 = 256 - 175 = 81 \Rightarrow y = \pm 3$

$\therefore$ roots of $z\bar{z}^3 + \bar{z}z^3 = 350$ are $z = \pm 4 \pm 3i$

Area of rectangle whose vertices are $\pm 4 \pm 3i$ is (8) (6) = 48.

Example 88: Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)\,z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a nonzero complex number w, then which of the following is not true?

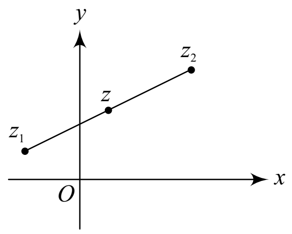


Fig. 2.50

(a) $|z-z_1|+|z-z_2| = |z_1-z_2|$

(b) Arg $(z-z_1)$ = Arg $(z-z_2)$

(c) $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix} = 0$

(d) Arg $(z-z_1)$ = Arg (z_2-z_1)

Ans. (b)

Solution: $z - z_1 = t\,(z_2 - z_1)$

$\Rightarrow \quad |z-z_1| = t|z_2-z_1| = t|z_1-z_2| \quad (1)$

$[\because\ t > 0]$

and $\quad z - z_2 = (1-t)\,(z_1-z_2)$

$\Rightarrow \quad |z-z_2| = (1-t)\,|z_1-z_2| = (1-t)\,|z_2-z_1| \quad (2)$

$[\because\ 1-t > 0]$

From (1) and (2), we get

$$|z-z_1| + |z-z_2| = |z_2-z_1|$$

Next,

$$\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix} = \begin{vmatrix} t(z_2-z_1) & t(\bar{z}_2-\bar{z}_1) \\ z_2-z_1 & (\bar{z}_2-\bar{z}_1) \end{vmatrix} = 0$$

$[\because\ R_1$ and R_2 are identical$]$

Also, Arg $(z-z_1)$ = Arg (z_2-z_1) since z_1, z and z_2 lie on the same straight line and on the same side of z_1.

Example 89: For complex numbers $z_1 = x_1 + iy$, and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$ if $x_1 \le x_2$ and $y_1 \le y_2$. Let z be a complex number such that $1 \cap z$, then

(a) $\frac{1-z}{1+z} \cap -i$ (b) $1 \cap \frac{1-z}{1+z}$

(c) $\frac{1-z}{1+z} \cap 0$ (d) $\frac{1+z}{1-z} \cap 0$

Ans. (c)

Solution: Let $z = x + iy$. As $1 \cap z$, we get $1 \le x$ and $0 \le y$

Now, $$\frac{1-z}{1+z} = \frac{(1-x)-iy}{(1+x)+iy} = \frac{[(1-x)-iy][(1+x)-iy]}{(1+x)^2+y^2}$$

$$= \frac{(1-x^2)-y^2-iy(1+x+1-x)}{(1+x)^2+y^2}$$

$$= \frac{1-(x^2+y^2)-2iy}{(1+x)^2+y^2}$$

As $x \ge 1$, and $y \ge 0$, we get

$$\frac{1-(x^2+y^2)}{(1+x)^2+y^2} \le 0 \text{ and } \frac{-2y}{(1+x)^2+y^2} \le 0$$

Thus, $\frac{1-z}{1+z} \cap 0$

Example 90: The complex number z_1, z_2, z_3 are the vertices of an equilateral triangle. If z_0 is the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2$ is equal to

(a) z_0^2 (b) $3z_0^2$

(c) z_0^3 (d) $3z_0^3$

Ans. (b)

Solution: As the triangle with vertices z_1, z_2 and z_3 is an equilateral triangle, the circumcentre and the centroid of the triangle coincides. Thus,

$$z_0 = \frac{1}{3}(z_1 + z_2 + z_3)$$

$$\Rightarrow \quad (3z_0)^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_2 z_3 + z_3 z_1 + z_1 z_2) \quad (1)$$

As the triangle is an equilateral triangle,

$$z_2 z_3 + z_2 z_1 + z_1 z_3 = z_1^2 + z_2^2 + z_3^2 \quad (2)$$

From (1) and (2) we get

$$9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^3)$$

$$\Leftrightarrow \quad 3z_0^2 = z_1^2 + z_2^2 + z_3^2.$$

Example 91: If ω is an imaginary cube root of unity, then value of the expression

$1(2 - \omega)(2 - \omega^2) + 2(3 - \omega)(3 - \omega^2) + \ldots + (n - 1)(n - \omega)(n - \omega^2)$ is

(a) $\frac{1}{4}n^2(n+1)^2 - n$ (b) $\frac{1}{4}n^2(n+1)^2 + n$

(c) $\frac{1}{4}n^2(n+1) - n$ (d) $\frac{1}{4}n(n+1)^2 - n$

Ans. (a)

Solution: rth term of the given expression is $r(r + 1 - \omega)(r + 1 - \omega^2)$

$$= (r + 1 - 1)(r + 1 - \omega)(r + 1 - \omega^2)$$
$$= (r + 1)^3 - 1$$
$$[\because (x - 1)(x - \omega)(x - \omega^2) = x^3 - 1]$$

$$\therefore \quad S = \sum_{r=1}^{n-1}\left[(r+1)^3 - 1\right] = \sum_{r=0}^{n-1}\left[(r+1)^3 - 1\right]$$

$$= \sum_{r=0}^{n} r^3 - n = \frac{1}{4}n^2(n+1)^2 - n$$

Example 92: The number of complex numbers z such that $|z| < 1/3$, and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$, is

(a) 0 (b) 1

(c) 4 (d) infinite

Ans. (a)

Solution: We have

$$1 = \left|\sum_{r=1}^{n} a_r z^r\right| \le \sum_{r=1}^{n} |a_r||z|^r$$

$$< \sum_{r=1}^{n} 2\left(\frac{1}{3}\right)^r < \sum_{r=1}^{\infty} 2\left(\frac{1}{3}\right)^r = \frac{2/3}{1-2/3} = 1$$

A contradiction.

Example 93: If a, b, c are integers, not all equal and $\omega(\ne 1)$ is a cube root of unity, then minimum value of $|a + b\omega + c\omega^2|$ is

(a) $\sqrt{3}$ (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: $\left|a + b\omega + c\omega^2\right|^2$

$$= (a + b\omega + c\omega^2)(a + b\bar{\omega} + c\bar{\omega}^2)$$
$$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$
$$= a^2 + b^2 + c^2 + (bc + ca + ab)(\omega + \omega^2)$$
$$= a^2 + b^2 + c^2 - bc - ca - ab$$
$$= \frac{1}{2}[(b - c)^2 + (c - a)^2 + (a - b)^2]$$

As a, b, c are integers not all equal, at least two of $b - c$, $c - a$ and $a - b$ are of different from zero. Therefore minimum value of $(b - c)^2 + (c - a)^2 + (a - b^2)$ is 2.

Thus, $\left|a + b\omega + c\omega^2\right|^2 \ge 1 \quad \Rightarrow \quad |a + b\omega + c\omega^2| \ge 1$.

Example 94: The region of the argand plane defined by $|z - i| + |z + i| \le 4$ is

(a) interior of an ellipse
(b) exterior of a circle
(c) interior and boundary of an ellipse
(d) none of these

Ans. (c)

Solution: Let A and B be the point $0 + i$ and $0 - i$ and $P(z)$ be any point satisfying $|z - i| + |z + i| \le 4$.

$\Rightarrow \quad PA + PB \le 4$

Thus, P lies in the interior or on the boundary on the ellipse with foci A and B and length of major axis = 4. See Fig. 2.51

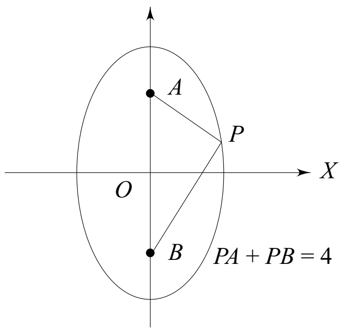


Fig. 2.51

Example 95: If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to

(a) $-\pi$ (b) $-\pi/2$
(c) $\pi/2$ (d) 0

Ans. (d)

Solution: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ where $r_1 = |z_1|$, $r_2 = |z_2|$, $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

We have

$$|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$$
$$= (r_1 + r_2)^2 + 2r_1r_2\{\cos(\theta_1 - \theta_2) - 1\}$$

Now, $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow \cos(\theta_1 - \theta_2) = 1 \Leftrightarrow \theta_1 - \theta_2 = 0 \Leftrightarrow \theta_1 = \theta_2$.

Example 96: If $|z - 25i| \le 15$, then

$|$maximum $\arg(z)$ – minimum $\arg(z)|$ equals

(a) $2\cos^{-1}(3/5)$ (b) $2\cos^{-1}(4/5)$
(c) $\pi/2 + \cos^{-1}(3/5)$ (d) $\sin^{-1}(3/5) - \cos^{-1}(3/5)$

Ans. (b)

Solution: If $|z - 25i| \le 15$, then z lies either in the interior and or on the boundary of the circle with centre at $C(25i)$ and radius equal to 15.

From Fig. 2.52 it is clear that least argument is for point A and the greatest argument is for point B.

From right ΔOAC, $\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{OA}{OC} = \dfrac{20}{25}$

$\Rightarrow \quad \pi/2 - \theta = \cos^{-1}(4/5)$

Now, for $|z - 25i| \le 15$

$|$maximum $(\arg z)$ – minimum $(\arg z)|$

$= |\arg(B) - \arg(A)|$

$= \angle BOA = \angle BOX - \angle AOX$

$= \pi/2 + (\pi/2 - \theta) - \theta = \pi - 2\theta$

$= 2\cos^{-1}(4/5)$

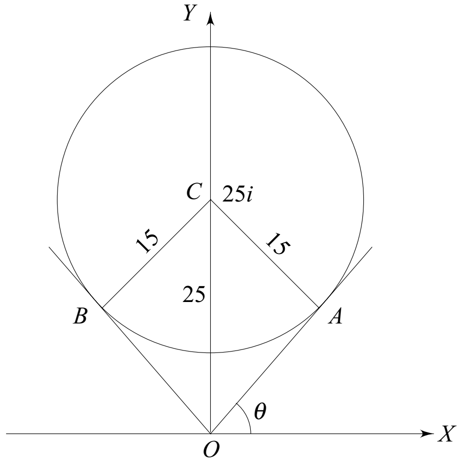


Fig. 2.52

Example 97: If $|z| = 3$, the area of the triangle whose sides are z, ωz and $z + \omega z$ (where ω is a complex cube root of unity) is

(a) $9\sqrt{3}/4$ (b) $3\sqrt{3}/2$
(c) $5/2$ (d) $8\sqrt{3}/3$

Ans. (a)

Solution: We have $|z| = 3$, $|\omega z| = |\omega|\,|z| = (1)(3) = 3$ and

$$|z + \omega z| = |(1 + \omega)z| = |(-\omega^2)z|$$
$$= |-\omega^2|\,|z| = (1)(3) = 3.$$

$\therefore$ The given triangle is equilateral and its area is

$$\frac{\sqrt{3}}{4}|z|^2 = \frac{9\sqrt{3}}{4}.$$

Example 98: The greatest and the least value of $|z_1 + z_2|$ if $z_1 = 24 + 7i$ and $|z_2| = 6$ are respectively

(a) 31, 19 (b) 25, 19
(c) 31, 25 (d) none of these

Ans. (a)

Solution: Note that $|z| = 6$ represents a circle. As $|z_2| = 6$, $|z_1 + z_2| = |z_2 - (-24 - 7i)|$ represent distance between a point on the circle $|z| = 6$ and the point $(-24 - 7i)$.

$|z_1 + z_2|$ will be greatest and least at points B and A which are the end points of the diameter of the circle through C. As $OC = 25$, $CA = OC - OA = 25 - 6 = 19$ and $CB = OC + OB = 25 + 6 = 31$. See Fig. 2.53

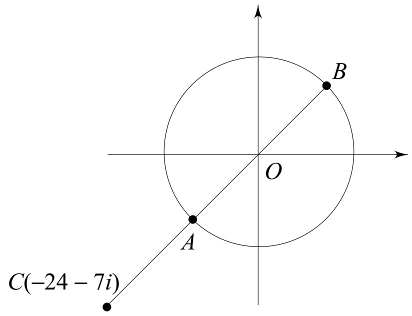


Fig. 2.53

Alternate Solution

$$|z_2| = 6 \Rightarrow z_2 = 6e^{i\theta} \text{ where } \theta \in \mathbf{R}.$$

$$\therefore \quad |z_1 + z_2|^2 = |24 + 7i + 6(\cos\theta + i\sin\theta)|^2$$
$$= (24 + 6\cos\theta)^2 + (7 + 6\sin\theta)^2$$
$$= 576 + 36\cos^2\theta + 288\cos\theta + 49 + 36\sin^2\theta + 84\sin\theta$$
$$= 625 + 36 + 12(24\cos\theta + 7\sin\theta)$$
$$= 661 + 12(25)\sin(\theta + \alpha)$$
[put $7 = r\cos\alpha$ and $24 = r\sin\alpha$]
$$= 661 + 300\sin(\theta + \alpha)$$

Thus, greatest possible value of $|z_1 + z_2|^2$ is 661 + 300 = 961 and the least possible value of $|z_1 + z_2|^2$ is 361.

$\therefore$ greatest and least possible values of $|z_1 + z_2|$ are 31 and 19 respectively.

Example 99: If α, β are the roots of $x^2 + px + q = 0$, and ω is a cube root of unity, then value of $(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta)$ is

(a) p^2 (b) $3q$
(c) $p^2 - 2q$ (d) $p^2 - 3q$

Ans. (d)

Solution: We have $\alpha + \beta = -p$, $\alpha\beta = q$

Now $(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta)$
$$= \omega^3\alpha^2 + \omega^4\alpha\beta + \omega^2\alpha\beta + \omega^3\beta^2$$
$$= \alpha^2 + \beta^2 + (\omega + \omega^2)\alpha\beta = \alpha^2 + \beta^2 - \alpha\beta$$
$$= (\alpha + \beta)^2 - 3\alpha\beta = p^2 - 3q$$

Example 100: Maximum distance from the origin of the points z satisfying the relation $|z + 1/z| = 1$ is

(a) $(\sqrt{5} + 1)/2$ (b) $(\sqrt{5} - 1)/2$
(c) $3 - \sqrt{5}$ (d) $(3 + \sqrt{5})/2$

Ans. (a)

Solution: We may assume $|z| \ge \frac{1}{|z|}$ for otherwise, we may interchange z and $1/z$ in the given equation.

We have

$$|z| - \frac{1}{|z|} = \left||z| - \frac{1}{|z|}\right| = \left||z| - \left|-\frac{1}{z}\right|\right| \le \left|z - \left(-\frac{1}{z}\right)\right| = \left|z + \frac{1}{z}\right| = 1$$

Thus, $|z| - \frac{1}{|z|} \le 1 \Rightarrow |z|^2 - |z| - 1 \le 0$

$\Rightarrow$ $|z|$ lies between the roots of $|z|^2 - |z| - 1 = 0$

$$\Rightarrow \frac{1}{2}(1 - \sqrt{5}) \le |z| \le \frac{1}{2}(1 + \sqrt{5})$$

As $z \ne 0$, $|z| > 0$, therefore, $0 < |z| \le \frac{1}{2}(\sqrt{5} + 1)$.

Taking $z = \frac{i}{2}(\sqrt{5} + 1)$, we get $\left|z + \frac{1}{z}\right| = 1$. Thus, maximum possible value of $|z|$ is $(\sqrt{5} + 1)/2$.

Example 101: If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then area of the triangle whose vertices are z_1, z_2, z_3 is

(a) $3\sqrt{3}/4$ (b) $\sqrt{3}/4$
(c) 1 (d) 2

Ans. (a)

Solution: $|z_2 - z_3|^2 + |z_2 + z_3|^2 = 2|z_2|^2 + 2|z_3|^2$

$\Rightarrow$ $|z_2 - z_3|^2 + |-z_1|^2 = 2(1) + 2(1)$

[$\because$ $z_1 + z_2 + z_3 = 0$]

$\Rightarrow$ $|z_2 - z_3| = \sqrt{3}$

Similarly, $|z_3 - z_1| = |z_1 - z_2| = \sqrt{3}$

Thus, area of triangle with vertices z_1, z_2, z_3 is

$$\frac{\sqrt{3}}{4}\left(\sqrt{3}\right)^2 = \frac{3\sqrt{3}}{4}$$

Example 102: An equation of straight line joining the complex numbers a and ib (where $a, b \in \boldsymbol{R}$ and $a, b \ne 0$) is

(a) $z\left(\frac{1}{a} - \frac{i}{b}\right) + \bar{z}\left(\frac{1}{a} + \frac{i}{b}\right) = 2$

(b) $z(a - ib) + \bar{z}(a + ib) = 2(a^2 + b^2)$

(c) $z(a + ib) + \bar{z}(a - ib) = 2ab$

(d) none of these

Ans. (a)

Solution: An equation of straight line joining a and ib is

$$\begin{vmatrix} z & \bar{z} & 1 \\ a & a & 1 \\ ib & -ib & 1 \end{vmatrix} = 0$$

$$\Rightarrow z(a + ib) - \bar{z}(a - ib) - 2iab = 0$$

$$\Rightarrow z\left(\frac{1}{a} - \frac{i}{b}\right) + \bar{z}\left(\frac{1}{a} + \frac{i}{b}\right) = 2$$

Example 103: Two non-parallel lines meet the circle $|z| = r$ in the points a, b and c, d respectively. The point of intersection of these lines is

(a) $\dfrac{a^{-1}+b^{-1}+c^{-1}+d^{-1}}{a^{-1}b^{-1}+c^{-1}d^{-1}}$

(b) $\dfrac{ab+cd}{a+b+c+d}$

(c) $\dfrac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$

(d) none of these

Ans. (c)

◎ **Solution:** An equation of straight line passing through $A(a)$ and $B(b)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad z(\bar{a} - \bar{b}) - \bar{z}(a-b) + a\bar{b} - \bar{a}b = 0 \qquad (1)$$

As a, b lie on $|z| = r$, we get

$$|a| = |b| = r \Rightarrow a\bar{a} = b\bar{b} = r^2.$$

Equation (1), now can be written as

$$z\left(\frac{r^2}{a} - \frac{r^2}{b}\right) - \bar{z}(a-b) + \frac{ar^2}{b} - \frac{br^2}{a} = 0$$

$$\Rightarrow \quad \frac{zr^2}{ba}(b-a) + \bar{z}\,(b-a) = (b^2 - a^2)\,\frac{r^2}{ab}$$

$$\Rightarrow \quad za^{-1}\,b^{-1} + \frac{\bar{z}}{r^2} = a^{-1} + b^{-1} \qquad (2)$$

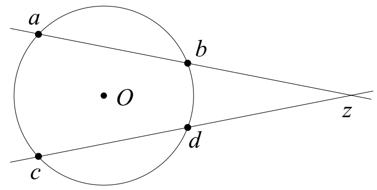


Fig. 2.54

Similarly, equation of straight line joining c and d is

$$z\,c^{-1}\,d^{-1} + \frac{\bar{z}}{r^2} = c^{-1} + d^{-1} \qquad (3)$$

Subtracting (3) from (2), we get

$$z(a^{-1}\,b^{-1} - c^{-1}\,d^{-1}) = a^{-1} + b^{-1} - c^{-1} - d^{-1}$$

$$\Rightarrow \quad z = \frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$$

EXERCISE

Concept-based Straight Objective Type Questions

1. The number of complex numbers z such that $(1+i)z = i\,|z|$
   (a) 0 (b) 1
   (c) 2 (d) infinite
2. Suppose $a, b, c \in \mathbf{R}$ and $C < 0$. Let $z = a + (b + ic)^{2015} + (b - ic)^{2015}$, then
   (a) Re $(z) = 0$
   (b) Im $(z) = 0$
   (c) Re $(z) > 0$, Im $(z) < 0$
   (d) Re $(z) < 0$, Im $(z) > 0$
3. The number of solutions of $z^2 + |z| = 0$ is
   (a) 1 (b) 2
   (c) 3 (d) infinite
4. The equation $\left|\dfrac{(1+i)z-2}{(1+i)z+4}\right| = k$
   does not represent a circle when k is
   (a) 2 (b) π
   (c) e (d) 1
5. If $|z| \geq 5$, then least value of $\left|z - \dfrac{1}{z}\right|$ is
   (a) 5 (b) 24/5
   (c) 8 (d) 8/3
6. Principal argument of $z = \dfrac{i-1}{i\left(1-\cos\dfrac{2\pi}{7}\right)+\sin\dfrac{2\pi}{7}}$ is
   (a) $\dfrac{\pi}{28}$ (b) $\dfrac{3\pi}{28}$
   (c) $\dfrac{17\pi}{28}$ (d) $\dfrac{19\pi}{28}$
7. If $x + iy = \sqrt{\dfrac{a+ib}{c+id}}$, then $(x^2 + y^2)^2\,(c^2 + d^2)$ equals
   (a) $a^2 + b^2$ (b) $\sqrt{a^2+b^2}$
   (c) $\dfrac{a^2+b^2}{c^2+d^2}$ (d) $\sqrt{\dfrac{a^2+b^2}{c^2+d^2}}$

8. Suppose z_1, z_2, z_3 are three complex numbers, and

$$\Delta = \frac{1}{4i}\begin{vmatrix} 1 & z_1 & \bar{z}_1 \\ 1 & z_2 & \bar{z}_2 \\ 1 & z_2 & \bar{z}_3 \end{vmatrix},$$

then
(a) Re $(\Delta) = 0$ (b) Im $(\Delta) = 0$
(c) Re $(\Delta) \geq 0$ (d) Im $(\Delta) \leq 0$

9. If $x, y, a, b \in \mathbf{R}$, $a \neq 0$ and
$(a + ib)(x + iy) = (a^2 + b^2)\, i$,
then (x, y) equals
(a) (a, b) (b) $(a, 0)$
(c) $(0, b)$ (d) (b, a)

10. If ω ($\neq 1$) is a cube root of unity, then the value of tan $[(\omega^{2017} + \omega^{2225})\,\pi - \pi/3]$
(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $-\sqrt{3}$ (d) $\sqrt{3}$

11. If z is purely imaginary and Im $(z) < 0$, then arg $(i\bar{z})$ + arg (z) is equal to
(a) π (b) 0
(c) $\pi/2$ (d) $-\pi/2$

12. The inequality $a + ib > c + id$ is true when
(a) $a > c, b > d > 0$ (b) $a > c, b = d = 0$
(c) $a > c, b = d > 0$ (d) none of these

13. Let $z \in \mathbf{C}$ be such that Re$(z^2) = 0$, then
(a) $|\text{Re}(z)| + \text{Im}(z) = 0$
(b) $|\text{Re}(z)| = |\text{Im}(z)|$
(c) $\text{Re}(z) + |\text{Im}(z)| = 0$
(d) $\text{Re}(z) = 0$ or $\text{Im}(z) = 0$

14. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$ equals
(a) $0, \pi$ (b) $\pi, -\pi$
(c) $\frac{\pi}{2}, \frac{3\pi}{2}$ (d) $0, 2\pi$

15. If $z = x + iy$ and $0 \leq \sin^{-1}\left(\frac{z-4}{2i}\right) \leq \frac{\pi}{2}$ then
(a) $x = 4, 0 \leq y \leq 2$ (b) $0 \leq x \leq 4, 0 \leq y \leq 2$
(c) $x = 0, 0 \leq y \leq 2$ (d) none of these

16. If $a > 0$ and $z|z| + az + 3i = 0$, then z is
(a) 0
(b) purely imaginary
(c) a positive real number
(d) a negative real number

17. If $z \neq 0$ is a complex number such that Re$(z) = 0$, then
(a) Re $(z^2) = 0$ (b) Im $(z^2) = 0$
(c) Re (z^2) = Im (z^2) (d) Im $(z^2) < 0$

18. If $z_k = \cos\left(\frac{k\pi}{10}\right) + i\sin\left(\frac{k\pi}{10}\right)$, then $z_1 z_2 z_3 z_4$ is equal to
(a) -1 (b) 2
(c) -2 (d) 1

19. If $|z_1| = |z_2| = 1$, $z_1 z_2 \neq -1$ and $z = \frac{z_1 + z_2}{1 + z_1 z_2}$ then
(a) z is a purely real number
(b) z is a purely imaginary number
(c) $|z| = 1$
(d) none of these

20. If $z \in \mathbf{C}$, then Re$(\bar{z}^2) = k^2$, $k > 0$, represents
(a) an ellipse (b) a parabola
(c) a circle (d) a hyperbola

LEVEL 1
Straight Objective Type Questions

21. If $\omega \neq 1$ is a cube root of unity, then $1, \omega, \omega^2$
(a) are vertices of an equilateral triangle
(b) lie on a straight line
(c) lie on a circle of radius $\sqrt{3/2}$
(d) none of these

22. If α, β, γ are the cube roots of p, $p < 0$, then for any x, y and z which does not make denominator zero, the expression $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$ equals
(a) $\omega, 1$ (b) ω, ω^2
(c) $\omega^2, 1$ (d) $1, \omega, \omega^2$

23. $ABCD$ is a rhombus, its diagonal AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M are the complex numbers $1 + i$ and $2 - i$ respectively, then A represent the complex number.

(a) $3-\frac{1}{2}i, 1+\frac{3}{2}i$ (b) $3+\frac{1}{2}i, 1+\frac{3}{2}i$

(c) $3-\frac{1}{2}i, 1-\frac{3}{2}i$ (d) $3+\frac{1}{2}i, 1-\frac{3}{2}i$

24. Let α and β be the roots of the equation $x^2 + x + 1 = 0$. The equation whose roots are α^{19}, β^7 is

(a) $x^2 - x - 1 = 0$ (b) $x^2 - x + 1 = 0$
(c) $x^2 + x - 1 = 0$ (d) $x^2 + x + 1 = 0$

25. If ω is an imaginary cube root of unity, then the value of

$$\sin\left\{(\omega^{10} + \omega^{23})\pi - \frac{\pi}{4}\right\}$$

is

(a) $-\frac{\sqrt{3}}{2}$ (b) $-\frac{1}{\sqrt{2}}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

26. If $\omega(\neq 1)$ is a cube root of unity and $(1 + \omega)^{2017} = A + B\omega$. Then A and B are respectively the numbers

(a) 0,1 (b) 1, 1
(c) 1, 0 (d) – 1, 1

27. If $\omega(\neq 1)$ is a cube root of unity, then

$$\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$$

equals

(a) 0 (b) 1
(c) i (d) ω

28. If ω ($\neq 1$), is a cube root of unity, then value of $(1 + \omega - \omega^2)^7$ equals

(a) 128ω (b) -128ω
(c) $128\omega^2$ (d) $-128\omega^2$

29. If $\Delta = \begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$
(c) $x = 0, y = 3$ (d) $x = 0, y = 0$

30. If $\arg(z) < 0$, then $\arg(-z) - \arg(z)$ equals

(a) π (b) $-\pi$

(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

31. If z_1, z_2, z_3 are complex numbers such that

$$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1,$$

then $|z_1 + z_2 + z_3|$ is

(a) equal to 1 (b) less than 1
(c) greater than 3 (d) equal to 3

32. Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin. Then n must be of the form

(a) $4k + 1$ (b) $4k + 2$
(c) $4k + 3$ (d) $4k$

33. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1}{2}(1 - \sqrt{3}i)$ are vertices of a triangle which is

(a) of area zero (b) right-angled isosceles
(c) equilateral (d) obtuse-angle isosceles

34. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then the value of the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} \text{ is}$$

(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $3\omega(1 - \omega)$

35. The inequality $|z - i| < |z + i|$ represents the region

(a) Re$(z) > 0$ (b) Re$(z) < 0$
(c) Im$(z) > 0$ (d) Im$(z) < 0$

36. If $iz^3 + z^2 - z + i = 0$, then

(a) Re $z = 0$ (b) Im $z = 0$
(c) $|z| = 1$ (d) none of these.

37. If $x + iy = \frac{1}{1 - \cos\theta + 2i\sin\theta}$, $\theta \neq 2n\pi$, $n \in \boldsymbol{I}$, then maximum value of x is

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) $\frac{1}{3}$

38. The equation $z^3 = \bar{z}$ has

(a) no solution
(b) two solutions
(c) five solutions
(d) infinite number of solutions

39. If $z = 5 + t + i\sqrt{25 - t^2}$, $(-5 \le t \le 5)$, then locus of z is a curve which passes through

(a) $5 + 0i$ (b) $-2 + 3i$
(c) $2 + 4i$ (d) $-2 - 3i$

40. If $\omega \neq 1$ is a cube root of unity and satisfies

$$\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = 2\omega^2$$

and $\dfrac{1}{a+\omega^2}+\dfrac{1}{b+\omega^2}+\dfrac{1}{c+\omega^2} = 2\omega$,

then the value of $\dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1}$ is

(a) 2 (b) −2
(c) $-1+\omega^2$ (d) none of these

41. If $|z - i\,\mathrm{Re}(z)| = |z - \mathrm{Im}(z)|$, then z lies on
(a) $\mathrm{Re}(z) = 2$ (b) $\mathrm{Im}(z) = 2$,
(c) $\mathrm{Re}(z) + \mathrm{Im}(z) = 2$ (d) none of these

42. If ω is a complex cube root of unity, then value of expression
$$\cos\left[\{(1-\omega)(1-\omega^2) + \dots + (12-\omega)(12-\omega^2)\}\frac{\pi}{370}\right]$$
(a) −1 (b) 0
(c) 1 (d) $\sqrt{3}/2$

43. If roots of the equation $z^2 + az + b = 0$ are purely imaginary then
(a) $(b-\bar{b})^2 + (a+\bar{a})(\bar{a}b + a\bar{b}) = 0$
(b) $(b-\bar{b})^2 + (a-\bar{a})^2 = 0$
(c) $(b+\bar{b})^2 - (a-\bar{a})^2 = 0$
(d) none of these

44. The system of equations $|z + 1 - i| = \sqrt{2}$ and $|z| = 3$ has
(a) no solution
(b) one solution
(c) two solutions
(d) infinite number of solutions

45. If $8i\,z^3 + 12z^2 - 18z + 27i = 0$, then
(a) $|z| = \dfrac{3}{2}$ (b) $|z| = \dfrac{2}{3}$
(c) $|z| = 1$ (d) $|z| = \dfrac{3}{4}$

46. If the complex number z lies on the boundary of the circle of radius 5 and centre at 4, then the greatest value of $|z + 1|$
(a) 4 (b) 5
(c) 10 (d) 9

47. If $x + iy = \dfrac{3}{\cos\theta + i\sin\theta + 2}$, then $4x - x^2 - y^2$ reduces to
(a) 2 (b) 3
(c) 4 (d) 5

48. Suppose z_1, z_2, z_3 represent the vertices A, B, and C respectively of a ΔABC with centroid at G. If the mid point of AG is the origin, then
(a) $z_1 + z_2 + z_3 = 0$ (b) $2z_1 + z_2 + z_3 = 0$
(c) $z_1 + z_2 + 4z_3 = 0$ (d) $4z_1 + z_2 + z_3 = 0$

49. Suppose that three points z_1, z_2, z_3 are connected by the relation $az_1 + bz_2 + cz_3 = 0$, where $a + b + c = 0$, then the points are
(a) vertices of a right triangle
(b) vertices of an isosceles triangle
(c) vertices of an equilateral triangle
(d) collinear

50. If the complex number $\dfrac{z-1}{z+1}$ is purely imaginary, then
(a) $|z| = 1$ (b) $|z| < 1$
(c) $|z| > 1$ (d) $|z| \geq 2$

51. If z is a complex number such that $-\pi/2 \leq \arg z \leq \pi/2$, then which of the following inequality is true.
(a) $|z - \bar{z}| \leq |z|\,|\arg(z) - \arg(\bar{z})|$
(b) $|z - \bar{z}| \leq |\arg(z) - \arg(\bar{z})|$
(c) $|z - \bar{z}| > |z|\,|\arg(z) - \arg(\bar{z})|$
(d) none of these

52. If $|\omega| = 1$, then the set of points
$$z = \omega + \frac{1}{\omega} \text{ is satisfies}$$
(a) $|\mathrm{Re}(z)| \leq 2$ (b) $|z| \leq 1$
(c) $|z| = 1$ (d) $|\mathrm{Im}(z)| \geq 2$

53. Number of complex numbers satisfying $|z| = 1$ and $\left|\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}\right| = 1$, is
(a) 0 (b) 2
(c) 4 (d) 8

54. If z_1, z_2 are two complex numbers such that $|z_1| = |z_2| = \sqrt{2}$ and $|z_1 + z_2| = \sqrt{3}$, then $|z_1 - z_2|$ equals:
(a) $2\sqrt{2}$ (b) $\sqrt{5}$
(c) 3 (d) $2-\sqrt{2}$

55. Let z_1, z_2, z_3 be three non-zero complex numbers such that $z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1$, then z_1, z_2, z_3
(a) are vertices of an equilateral triangle
(b) are vertices of an isosceles triangle
(c) lie on a straight line
(d) none of these

56. If $|z_1| = |z_2| = |z_3| = 1$, then value of $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$ cannot exceed
(a) 3 (b) 6
(c) 9 (d) 12

57. Let z_1, z_2, z_3 be three complex number such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then $\left|z_1^2 + 2z_2^2 + z_3^2\right|$ equals

(a) 1 (b) 2
(c) 3 (d) 4

58. Let z_1, z_2, z_3 be three complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and
$$z = (z_1 + z_2 + z_3)\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right),$$ then $|z|$ cannot exceed
(a) 1 (b) 3
(c) 6 (d) 9

59. Suppose z is a complex number such that $z \neq -1$, $|z| = 1$, and $\arg(z) = \theta$. Let $\omega = \frac{z(1-\bar{z})}{\bar{z}(1+z)}$, then Re (ω) is equal to
(a) $1 + \cos(\theta/2)$ (b) $1 - \sin(\theta/2)$
(c) $-2\sin^2(\theta/2)$ (d) $2\cos^2(\theta/2)$

60. Let $a = \text{Im}\left(\frac{1+z^2}{2iz}\right)$, where z is any non-zero complex number. Then the set
$$A = \{a : |z| = 1 \quad \text{and} \quad z \neq \pm 1\}$$
is equal to
(a) $(-1, 1)$ (b) $[-1, 1]$
(c) $[0, 1)$ (d) $(-1, 0]$

61. Number of complex numbers such that
$|z| = 1$ and $z = 1 - 2\bar{z}$ is
(a) 0 (b) 1
(c) 2 (d) infinite

62. Let z_1, z_2 be two complex numbers such that $z_1 \neq 0$ and z_2/z_1 is purely real, then $\left|\frac{2iz_1 + 5z_2}{2iz_1 - 5z_2}\right|$ is equal to
(a) 3 (b) 2
(c) 1 (d) 0

63. If $z = i(1+\sqrt{3})$, then $z^4 + 2z^3 + 4z^2 + 5$ is equal to
(a) 5 (b) -5
(c) $2\sqrt{3}\,i$ (d) $-2\sqrt{3}\,i$

64. Suppose $a < 0$ and z_1, z_2, z_3, z_4 be the fourth roots of a. Then $z_1^2 + z_2^2 + z_3^2 + z_4^2$ is equal
(a) $-a^2$ (b) $|a| - a$
(c) $a + |a|$ (d) a^2

65. Suppose $\arg(z) = -5\pi/13$, then $\arg\left(\frac{z+\bar{z}}{1+z\bar{z}}\right)$ is
(a) $-5\pi/13$ (b) $5\pi/13$
(c) π (d) 0

66. The number of values of $\theta \in (0, \pi]$, such that $(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)(\cos 5\theta + i\sin 5\theta)(\cos 7\theta + i\sin 7\theta)(\cos 9\theta + i\sin 9\theta) = -1$ is
(a) 11 (b) 13
(c) 14 (d) 16

67. If $z \in \mathbf{C} - \{0, -2\}$ is such that
$\log_{(1/7)} |z - 2| > \log_{(1/7)} |z|$
then
(a) Re $(z) > 1$ (b) Re $(z) < 1$
(c) Im $(z) > 1$ (d) Im $(z) < 1$

68. $\text{Im}\left(\frac{2z+1}{iz+1}\right) = 5$ represents
(a) a circle (b) a straight line
(c) a parabola (d) an ellipse

69. Let z_1, z_2 be two complex numbers such that Im $(z_1 + z_2) = 0$ and Im $(z_1 z_2) = 0$ then
(a) $z_1 = -\bar{z}_2$ (b) $z_1 = z_2$
(c) $z_1 = \bar{z}_2$ (d) none of these

70. If $n \in \mathbf{N}$, then $\frac{(1+i)^n}{(1-i)^{n-2}}$ is equal to
(a) i^{n+1} (b) $-2i^{n+1}$
(c) i^{n+2} (d) $-2i^{n+2}$

71. Let $\omega \neq 1$, be a cube root of unity, and $f : \mathbf{I} \to \mathbf{C}$ be defined by
$f(n) = 1 + \omega^n + \omega^{2n}$, then range of f is
(a) $\{0\}$ (b) $\{0, 3\}$
(c) $\{0, 1, 3\}$ (d) $\{0, 1\}$

72. If $z + \frac{1}{z} = 2\cos\theta$, $z \in \mathbf{C}$ then $z^{2n} - 2z^n \cos(n\theta)$ is equal to
(a) 1 (b) 0
(c) -1 (d) $-n$

73. If $\omega \neq 1$ is a cube root of unity, then
$$z = \sum_{k=1}^{60} \omega^k - \prod_{k=1}^{30} \omega^k \text{ is equal to}$$
(a) 0 (b) ω
(c) ω^2 (d) -1

74. Let $g(x)$ and $h(x)$ be two polynomials with real coefficients. If $P(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then
(a) $g(1) = 0, h(1) = 1$
(b) $g(1) = 1, h(1) = 0$
(c) $g(1) = 0, h(1) = 0$
(d) $g(1) = 1, h(1) = 0$

75. If $x^2 - x + 1$ divides the polynomial $x^{n+1} - x^n + 1$, then n must be of the form
(a) $3k + 1$ (b) $6k + 1$
(c) $6k - 1$ (d) $3k - 1$

Assertion-Reason Type Questions

76. **Statement-1:** $z^2 + z\,|z-1| + |z-1|^2 = 0$, $z \in \mathbf{C}$ has no solution in C.

Statement-2: $z^2 + az + a^2 = 0$, $z \in \mathbf{C}$, $a > 0$ has no solution in C.

77. Let $z \in C$ satisfy the relations

$|z| = 1$ and $z = 2i + \bar{z}$

Statement-1: z is purely imaginary.

Statement-2: $|\arg(z)| \le \pi/6$.

78. Let $\alpha = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\frac{3\pi}{11}\right)$

Statement-1: $\text{Re}\,(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5) = -\frac{1}{2}$

Statement-2: $\sum_{k=0}^{10} \alpha^k = 0$

79. Let $z = \cos\theta + i\sin\theta$. $\theta \in (0, \pi)$, $\alpha = \pi/60$

and $f(\theta) = \sum_{k=1}^{15} \sin[(2k-1)\theta]$.

Statement-1: $f(\alpha) = \frac{1}{2\sin\alpha}$

Statement-2: $z + z^3 + \dots + z^{29} = \frac{z(1-z^{30})}{1-z^2}$

80. Let $\omega \ne 1$ be a cube root of unity and S be the set of all non-singular matrices of the form

$$\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$$

where each of a, b and c is either ω or ω^2.

Statement-1: S contains exactly two distinct matrices.

Statement-2: A square matrix A with complex entries is non-singular if and only if $|A| \ne 0$

81. **Statement-1:** If $|z_1|\,|z_2| = 1$, then

$$|z_1 - z_2| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$$

Statement-2: $|z| = |\bar{z}|\ \forall\ z \in C$.

82. Let $f(z)$ be a polynomial in z with complex coefficients. Suppose, when $f(z)$ is divided by $z - i$, the remainder is i and when $f(z)$ is divided by $z + i$, the remainder is $1 + i$.

Statement-1: The remainder when $f(z)$ is divided by $z^2 + 1$ is $\frac{1}{2}(iz+1)+1$

Statement-2: If $r(z)$ is remainder when $f(z)$ is divided by $p(z)$, then
$r(z) = 0$ or $\deg(r(z)) < \deg(p(z))$.

83. Let $a, b \in \mathbf{C}$, $a \ne 0$. Suppose z_1, z_2, z_3 are the roots of the equation $(z + ab)^3 = a^3$.

Statement-1: Length of a side of the triangle with vertices z_1, z_2 and z_3 is $\sqrt{3}\,|a|$

Statement-2: Roots of $z^3 = a^3$ lie on a circle of radius $|a|$.

84. **Statement-1:** The equation $z^2 - z + \pi = 0$.

has no solution in the unit disc $|z| < 1$.

Statement-2: $|z_1 + z_2| \le |z_1| + |z_2|\ \forall\ z_1, z_2 \in \mathbf{C}$.

85. Let $z_1 = r_1 e^{i\theta}$, $z_2 = r_2 e^{i\varphi}$ where $\theta, \varphi \in (-\pi, \pi]$.

Statement-1: $|z_1 - z_2|^2 \le (r_1 - r_2)^2 + (\theta - \phi)^2$

Statement-2: $|\sin\theta| \le |\theta|\ \forall\ \theta \in \mathbf{R}$.

LEVEL 2

Straight Objective Type Questions

86. If the complex numbers z_1, z_2, z_3, are the vertices of a parallelogram $ABCD$, then the fourth vertex is

(a) $\frac{1}{2}(z_1 + z_2)$ (b) $\frac{1}{4}(z_1 + z_2 - z_3 - z_4)$

(c) $\frac{1}{3}(z_1 + z_2 + z_3)$ (d) $z_1 + z_3 - z_2$

87. If a, b and c are three integers such that at least two of them are unequal and $\omega\ (\ne 1)$ is a cube root of unity, then the least value of the expression $|a + b\omega + c\omega^2|$ is

(a) 0 (b) 1

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

88. The shaded region in Fig. 2.55 is given by

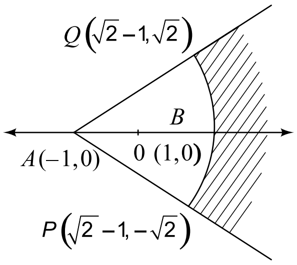


Fig. 2.55

(a) $\left\{z : |z-1| < 2, |\arg(z+1)| < \dfrac{\pi}{2}\right\}$

(b) $\left\{z : |z+1| < 2, |\arg(z+1)| < \dfrac{\pi}{2}\right\}$

(c) $\left\{z : |z-1| > 2, |\arg(z-1)| < \dfrac{\pi}{4}\right\}$

(d) $\left\{z : |z+1| > 2, |\arg(z+1)| < \dfrac{\pi}{4}\right\}$

89. Let $w = \alpha + i\beta$, $\beta \neq 0$ and $z \neq 1$. If $\dfrac{w - \bar{w}z}{1-z}$ is purely real, then the set of value of z is
(a) $\{z : |z| = 1\}$ (b) $\{z : \bar{z} = z\}$
(c) $\{z : |z| \neq 1\}$ (d) $\{z : |z| = 1, z \neq 1\}$

90. If z and w are non-zero complex numbers such that $zw = |z|^2$, then $|z - \bar{z}| + |w + \bar{w}| = 4$ represents a
(a) rectangle (b) square
(c) rhombus (d) trapezium

91. Let z be a complex number such that $z = 1 - t + i\sqrt{t^2 + t + 2}$, where $t \in \boldsymbol{R}$, then locus of z on the Argand plane is
(a) a parabola (b) an ellipse
(c) a hyperbola (d) a pair of straight line.

92. Let $z_1, z_2, \in \boldsymbol{C}$ and $a, b > 0$ be such that $a|z_1| = b|z_2|$, then $w = \dfrac{az_1 - bz_2}{az_1 + bz_2}$ lies
(a) in the 1st quadrant (b) in the 3rd quadrant
(c) on the real axis (d) on the imaginary axis

93. If $z \in \boldsymbol{C}$, then least value of the expression $|z| + |1 - z| + |z - 2|$ is
(a) 1 (b) 3/2
(c) 2 (d) cannot be determined

94. If $k > 0$, $k \neq 1$, and $z_1, z_2 \in \boldsymbol{C}$, then $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ represents
(a) a circle (b) an ellipse
(c) a parabola (d) a hyperbola

95. If $z_1, z_2, z_3 \in \boldsymbol{C}$ are distinct and are such that $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$, then z_1, z_2, z_3
(a) are vertices of a right triangle
(b) an equilateral triangle
(c) an obtuse angled triangle
(d) none of these

96. If $w = \cos(\pi/n) + i\sin(\pi/n)$, then value of $1 + w + w^2 + \cdots + w^{n-1}$ is
(a) $1 + i$
(b) $1 + i\tan(\pi/2n)$
(c) $1 + i\cot(\pi/2n)$
(d) none of these

97. Let z_1, z_2 be two non-zero complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, then $\dfrac{z_1}{\bar{z}_1} + \dfrac{z_2}{\bar{z}_2}$ equals
(a) 0 (b) 1
(c) − 1 (d) none of these

98. If $|z_1| = 2$ and $(1 - i)z_2 + (1 + i)\bar{z}_2 = 8\sqrt{2}$, then
(a) minimum value of $|z_1 - z_2|$ is 1
(b) minimum value of $|z_1 - z_2|$ is 2
(c) maximum value of $|z_1 - z_2|$ is 8
(d) maximum value of $|z_1 - z_2|$ is 4

99. If z_1 lies on $|z| = r$, then equation of tangent at z_1 is
(a) $\dfrac{z}{z_1} + \dfrac{\bar{z}}{\bar{z}_1} = 2$ (b) $\dfrac{z}{\bar{z}_1} + \dfrac{\bar{z}}{z_1} = r$
(c) $\dfrac{z}{\bar{z}_1} + \dfrac{\bar{z}}{z_1} = 2$ (d) $\dfrac{z}{z_1} + \dfrac{\bar{z}}{\bar{z}_1} = r$

100. If $z \in \boldsymbol{C}$, then minimum value of $|z - 2 + 3i| + |z - 1 + i|$ is
(a) $\sqrt{5}$ (b) $2\sqrt{5}$
(c) $\sqrt{13} - \sqrt{2}$ (d) 0

101. If $a > 0$ and $|z - a^2| + |z - 2a| = 3$ represents an ellipse then a lies in
(a) (0, 3) (b) $(0, \infty)$
(c) (1, 3) (d) $(3, \infty)$

102. If the points $A(z)$, $B(-z)$ and $C(1 - z)$ are the vertices of an equilateral triangle, then value of z is
(a) $1 \pm \dfrac{i\sqrt{3}}{2}$ (b) $\dfrac{1}{2}(1 \pm i)$
(c) $\dfrac{1}{4}(1 \pm \sqrt{3}i)$ (d) $\dfrac{1}{3}(1 \pm \sqrt{3}i)$

103. If $|z + 1| + |z - 3| \leq 10$, then the range of values of $|z - 7|$ is
(a) [0, 10] (b) [3, 13]
(c) [2, 12] (d) [7, 9]

104. If $\omega \neq 1$ is a cube root of unity and $|z-1|^2 + 2|z-\omega|^2 = 3|z-\omega^2|^2$ then z lies on
(a) a straight line
(b) a parabola
(c) an ellipse
(d) a rectangular hyperbola

105. If $x > 0$, the least value of $n \in N$ such that $\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is
(a) 2 (b) 4
(c) 8 (d) 32

Previous Years' AIEEE/JEE Main Questions

1. z and w are two non-zero complex numbers such that $|z| = |w|$ and Arg z + Arg $w = \pi$, then z equals
(a) $\overline{w}$ (b) $-\overline{w}$
(c) w (d) $-w$ **[2002]**

2. If $|z-4| < |z-2|$, then
(a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$
(c) $\text{Re}(z) > 3$ (d) $\text{Re}(z) > 2$ **[2002]**

3. The locus of the centre of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$, $a \neq b$, externally is
(a) an ellipse (b) a hyperbola
(c) a circle (d) a pair of straight lines **[2002]**

4. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then
(a) $x = 2n$, where n is any positive integer
(b) $x = 4n + 1$, where n is any positive integer
(c) $x = 2n + 1$, where n is any positive integer
(d) $x = 4n$, where n is any positive integer **[2003]**

5. If z and w are two non-zero complex numbers such that $|zw| = 1$, and $\text{Arg}(z) - \text{Arg}(w) = \frac{\pi}{2}$, then $\overline{z}w$ is equal to
(a) -1 (b) i
(c) $-$i (d) 1 **[2003]**

6. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. If the origin, z_1 and z_2 form an equilateral triangle, then
(a) $a^2 = 2b$ (b) $a^2 = 3b$
(c) $a^2 = 4b$ (d) $a^2 = b$ **[2003]**

7. Let z, w be two complex numbers such that $\overline{z} + i\overline{w} = 0$ and $\arg(zw) = \pi$, then $\arg(z)$ equals
(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{4}$ **[2004]**

8. If $z = x - iy$ and $z^{1/3} = p + iq$, then
$$\frac{1}{p^2+q^2}\left(\frac{x}{p}+\frac{y}{q}\right)$$
is equal to
(a) 2 (b) -1
(c) 1 (d) -2 **[2004]**

9. If $|z^2 - 1| = |z|^2 + 1$, then z lies on
(a) a circle (b) the imaginary axis
(c) the real axis (d) an ellipse **[2004]**

10. If the cube roots of unity $1, \omega, \omega^2$, then the roots of the equation $(x-1)^3 + 8 = 0$, are
(a) $-1, 1-2\omega, 1-2\omega^2$
(b) $-1, 1+2\omega, 1+2\omega^2$
(c) $-1, -1+2\omega, -1-2\omega^2$
(d) $-1, -1, -1$ **[2005]**

11. If $w = \frac{z}{z-\frac{1}{3}i}$ and $|w| = 1$, then z lies on
(a) straight line (b) a parabola
(c) an ellipse (d) a circle **[2005]**

12. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg z_1 − arg z_2 is equal to
(a) 0 (b) $-\frac{\pi}{2}$
(c) $\frac{\pi}{2}$ (d) π **[2005]**

13. The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$ is
(a) $-i$ (b) i
(c) 1 (d) -1 **[2006]**

14. If $z^2 + z + 1 = 0$, where z is a complex number, then value of

$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\cdots+\left(z^6+\frac{1}{z^6}\right)^2 \text{ is}$$

(a) 12 (b) 18
(c) 54 (d) 6 **[2006]**

15. If $|z + 4| \le 3$, then the maximum value of $|z + 1|$ is

(a) 4 (b) 10
(c) 6 (d) 0 **[2007]**

16. The conjugate of a complex number is $\frac{1}{i-1}$. Then that number is

(a) $\frac{-1}{i-1}$ (b) $\frac{1}{i+1}$
(c) $\frac{-1}{i+1}$ (d) $\frac{1}{i-1}$ **[2008]**

17. If $\left|z-\frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to

(a) 1 (b) $2+\sqrt{2}$
(c) $\sqrt{3}+1$ (d) $\sqrt{5}+1$ **[2009]**

18. The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals

(a) 2 (b) ∞
(c) 0 (d) 1 **[2010]**

19. If $\omega(\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$, there (A, B)

(a) (–1, 1) (b) (0,1)
(c) (1, 1) (d) (1, 0) **[2011]**

20. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re $(z) = 1$, then it is necessary that

(a) $\beta \in (1, \infty)$ (b) $\beta \in (0, 1)$
(c) $\beta\,(-1, 0)$ (d) $|\beta| = 1$ **[2011]**

21. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies:

(a) on a circle with centre at the origin.
(b) either on the real axis or on a circle not passing through the origin.
(c) on the imaginary-axis.
(d) either on the real axis or on a circle passing through the origin. **[2012]**

22. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals

(a) $\frac{\pi}{2}-\theta$ (b) θ
(c) $\pi-\theta$ (d) $-\theta$ **[2013]**

23. If $z_1 \neq 0$ and z_2 be two complex numbers such that z_2/z_1 is purely imaginary number, then $\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|$ is equal to:

(a) 2 (b) 5
(c) 3 (d) 1 **[2013, online]**

24. Let $a = \text{Im}\left(\frac{1+z^2}{2iz}\right)$, where z is any non-zero complex number. The set $A = \{a : |z| = 1 \text{ and } z \neq \pm 1\}$ is equal to:

(a) (–1, 1) (b) [–1, 1]
(c) [0, 1) (d) (–1, 0] **[2013, online]**

25†. Let $z \in \mathbf{C}$ satisfy $|z| = 1$ and $z = 1 - \bar{z}$.

Statement-1: z is a real number

Statement-2: Principal argument of z is $\pm \pi/3$.

[2013, online]

26. If a complex number z satisfies $z + \sqrt{2}\,|z + 1| + i = 0$, then $|z|$ is equal to:

(a) 2 (b) $\sqrt{3}$
(c) $\sqrt{5}$ (d) 1 **[2013, online]**

27. If z is a complex number such that $|z| \ge 2$, then minimum value of $\left|z+\frac{1}{2}\right|$:

(a) is strictly greater than 5/2
(b) is strictly greater that 3/2 but less than 5/2
(c) is equal to 5/2
(d) lies in the interval (1, 2) **[2014]**

28. Let w (Im $w \neq 0$) be a complex number. Then the set of all complex numbers z satisfying the equation $w - \bar{w}z = k\,(1 - z)$, for some real number k, is:

(a) $\{z : |z| = 1\}$ (b) $\{z : z = \bar{z}\}$
(c) $\{z : z \neq 1\}$ (d) $\{z : |z| = 1, z \neq 1\}$
[2014, online]

29†. If z_1, z_2 and z_3, z_4 are 2 pairs of complex conjugate numbers and $z_1, z_3 \notin \mathbf{R}$, then

$\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)$ equals:

† Slightly modified version.

(a) 0 (b) $\dfrac{\pi}{2}$

(c) $\dfrac{3\pi}{2}$ (d) π **[2014, online]**

30. Let $z \neq -i$ be any complex number such that $\dfrac{z-i}{z+i}$ is a purely imaginary number. Then $z+\dfrac{1}{z}$ is:
(a) 0
(b) any non-zero real number other than 1
(c) any non-zero real number
(d) a purely imaginary number **[2014, online]**

31. For all complex numbers z of the form $1 + i\alpha$, $\alpha \in \mathbf{R}$ if $z^2 = x + iy$, then
(a) $y^2 - 4x + 2 = 0$ (b) $y^2 + 4x - 4 = 0$
(c) $y^2 - 4x + 4 = 0$ (d) $y^2 + 4x + 2 = 0$
[2014, online]

32. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\dfrac{z_1 - 2z_2}{2 - z_1 z_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a:
(a) straight line parallel to the x-axis.
(b) straight line parallel to the y-axis.
(c) circle of radius 2.
(d) circle of radius $\sqrt{2}$. **[2015]**

33. The largest value of r for which the region represented by the set $\{w \in \mathbf{C}: |w - 4 - i| \leq r\}$ is contained in the region represented by the set $\{z \in \mathbf{C}: |z - 1| \leq |z + i|\}$, is equal to:
(a) $\sqrt{17}$ (b) $\sqrt{2}$
(c) $\dfrac{3}{2}\sqrt{2}$ (d) $\dfrac{5}{2}\sqrt{2}$ **[2015, online]**

34. If z is a non-real complex number, then the minimum value of $\dfrac{\text{Im}\, z^5}{(\text{Im}\, z)^5}$ is:
(a) – 1 (b) – 2
(c) – 4 (d) – 5 **[2015, online]**

35. A value of θ for which $z = \dfrac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is
(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{6}$
(c) $\sin^{-1}\left(\dfrac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ **[2016]**

36. The point represented by $2 + i$ in the Argand plane moves 1 unit eastwards, then 2 units northwards and finally from there $2\sqrt{2}$ units in the south-westwards direction. Then its new position in the Argand plane is at the point represented by:
(a) $1 + i$ (b) $2 + 2i$
(c) $-2 - 2i$ (d) $-1 - i$ **[2016, online]**

37. Let $z = 1 + ai$ be a complex number, $a > 0$, such that z^3 is a real number. Then the sum $1 + z + z^2 + \ldots + z^{11}$ is equal to:
(a) $1365\sqrt{3}\, i$ (b) $-1365\sqrt{3}\, i$
(c) $-1250\sqrt{3}\, i$ (d) $1250\sqrt{3}\, i$
[2016, online]

Previous Years' B-Architecture Entrance Examination Questions

1. $\dfrac{5+i\sin\theta}{5-3i\sin\theta}$ is a real number when
(a) $\theta = \pi/4$ (b) $\theta = -\pi$
(c) $\theta = -\pi/2$ (d) $\theta = \pi/2$ **[2006]**

2. Two points P and Q in the Argand diagram represent complex numbers z and $3z + 2 + i$. If P moves arround the circle with centre at the origin and radius 2, then Q moves on the circle, whose centre and radius are
(a) $-2 - i$, 6 (b) $2 - i$, 3
(c) $2 + i$, 6 (d) $2 + i$, 3 **[2007]**

3. Let z be a complex number such that $|z| = 2$, then maximum possible value of $\left|z+\dfrac{2}{z}\right|$ is
(a) 1 (b) 2
(c) 3 (d) 4 **[2008]**

4. If $i = \sqrt{-1}$, then $4+3\left(-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)^{127}+5\left(-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)^{124}$ is equal to
(a) $4\sqrt{3}\, i$ (b) $2\sqrt{3}\, i$
(c) $1-\sqrt{3}\, i$ (d) $1+\sqrt{3}\, i$ **[2009]**

5. The real part of a complex number z having minimum principal argument and satisfying $|z - 5i| \leq 1$ is
(a) $\dfrac{2}{5}\sqrt{6}$ (b) 0
(c) $\dfrac{2}{\sqrt{5}}$ (d) $-\dfrac{1}{5}\sqrt{6}$ **[2010]**

6. Area of a triangle with vertices given by z, iz, $z + iz$, where z is a complex number, is
   (a) 0 (b) $\frac{1}{2}|z|^2$
   (c) $|z|^2$ (d) $2|z|^2$ **[2011]**
7. Two circles in the complex plane are
   $C_1 : |z - i| = 2$
   $C_2 : |z - 1 - 2i| = 4$
   then
   (a) C_1 and C_2 touch each other
   (b) C_1 and C_2 intersect at two distinct points
   (c) C_1 lies within C_2
   (d) C_2 lies within C_1 **[2012]**
8. If $z = i(i + \sqrt{2})$, then value of $z^4 + 4z^3 + 6z^2 + 4z$ is
   (a) –5 (b) 3
   (c) 6 (d) – 9 **[2013]**
9. Suppose z is a complex number such that $z \neq -1$, $|z| = 1$ and $\arg(z) = \theta$. Let $w = \frac{z(1-\bar{z})}{\bar{z}(1+z)}$, then Re($w$) is equal to
   (a) $1+\cos\left(\frac{\theta}{2}\right)$ (b) $1-\sin\left(\frac{\theta}{2}\right)$
   (c) $-2\sin^2\left(\frac{\theta}{2}\right)$ (d) $2\cos^2\left(\frac{\theta}{2}\right)$ **[2014]**
10. If $|z_1| = |z_2| = |z_3| = 1$
    and $z_1 + z_2 + z_3 = \sqrt{2} + i$, then the number
    $z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1$ is:
    (a) a positive real number
    (b) a negative real number
    (c) always zero
    (d) a purely imaginary number **[2015]**
11. Let $S = \{z \in \mathbf{C}: z(iz_1 - 1) = z_1 + 1, [z_1] < 1\}$. Then, for all $z \in S$, which one of the following is always true?
    (a) Re z + Im z < 0 (b) Re z < 0
    (c) Re z – Im z > 1 (d) Re z – Im z < 0 **[2016]**

Answers

Concept-based

1. (b)	**2.** (b)	**3.** (c)	**4.** (d)
5. (b)	**6.** (c)	**7.** (a)	**8.** (b)
9. (d)	**10.** (c)	**11.** (c)	**12.** (b)
13. (b)	**14.** (d)	**15.** (a)	**16.** (b)
17. (b)	**18.** (a)	**19.** (a)	**20.** (d)

Level 1

21. (a)	**22.** (b)	**23.** (c)	**24.** (d)
25. (c)	**26.** (b)	**27.** (a)	**28.** (d)
29. (d)	**30.** (a)	**31.** (a)	**32.** (d)
33. (c)	**34.** (b)	**35.** (c)	**36.** (c)
37. (c)	**38.** (c)	**39.** (c)	**40.** (a)
41. (d)	**42.** (c)	**43.** (a)	**44.** (a)
45. (a)	**46.** (c)	**47.** (b)	**48.** (d)
49. (d)	**50.** (a)	**51.** (a)	**52.** (a)
53. (d)	**54.** (b)	**55.** (a)	**56.** (c)
57. (a)	**58.** (d)	**59.** (c)	**60.** (a)
61. (a)	**62.** (c)	**63.** (a)	**64.** (c)
65. (c)	**66.** (b)	**67.** (a)	**68.** (a)
69. (d)	**70.** (b)	**71.** (b)	**72.** (c)
73. (d)	**74.** (c)	**75.** (b)	**76.** (c)
77. (c)	**78.** (a)	**79.** (a)	**80.** (a)
81. (b)	**82.** (a)	**83.** (b)	**84.** (a)
85. (a)			

Level 2

86. (d)	**87.** (b)	**88.** (d)	**89.** (d)
90. (b)	**91.** (c)	**92.** (d)	**93.** (c)
94. (a)	**95.** (b)	**96.** (c)	**97.** (a)
98. (b)	**99.** (a)	**100.** (a)	**101.** (a)
102. (c)	**103.** (b)	**104.** (a)	**105.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (c)	**3.** (b)	**4.** (d)
5. (c)	**6.** (b)	**7.** (a)	**8.** (d)
9. (b)	**10.** (a)	**11.** (a)	**12.** (a)
13. (a)	**14.** (a)	**15.** (c)	**16.** (c)
17. (d)	**18.** (d)	**19.** (c)	**20.** (a)
21. (d)	**22.** (b)	**23.** (d)	**24.** (a)
25. (d)	**26.** (c)	**27.** (d)	**28.** (d)
29. (a)	**30.** (no answer)		**31.** (b)
32. (c)	**33.** (d)	**34.** (c)	**35.** (d)
36. (a)	**37.** (a)		

Previous Years' B-Architecture Entrance Examination Questions

1. (b)	**2.** (c)	**3.** (c)	**4.** (a)
5. (a)	**6.** (b)	**7.** (c)	**8.** (b)
9. (c)	**10.** (c)	**11.** (*)	

Hints and Solutions

Concept-based

1. $|(1+i)z| = |i|z||$

$\Rightarrow \sqrt{2}\,|z| = |z| \Rightarrow (\sqrt{2}-1)\,|z| = 0$

$\Rightarrow |z| = 0 \Rightarrow z = 0$

2. $\bar{z} = \bar{a} + (\overline{b+ic})^{2015} + (\overline{b-ic})^{2015}$

$= a + (b-ic)^{2015} + (b+ic)^{2015} = z$

$\therefore$ z is real and thus, $\text{Im}(z) = 0$

3. $|z^2| = |-|z|| \Rightarrow |z|^2 = |z|$

$\Rightarrow |z| = 0$ or $|z| = 1$

If $|z| = 0$, we get $z = 0$. If $|z| = 1$, we get $z^2 = -1 \Rightarrow z = \pm i$

4. We can write the equation as

$$\left|\frac{z-\dfrac{2}{(1+i)}}{z+\dfrac{4}{(1+i)}}\right| = k \qquad (1)$$

But $\dfrac{2}{1+i} = \dfrac{1^2-i^2}{1+i} = 1-i$

and $\dfrac{4}{1+i} = 2(1-i)$

Therefore, (1) can be written as

$$\left|\frac{z-(1-i)}{z+2(1-i)}\right| = k \qquad (2)$$

This will not represent a circle if $k = 1$. When $k=1$, (2) represents perpendicular bisector of the segment joining $-2(1-i)$ and $1-i$.

5. As $|z| \geq 5$, $\left|\dfrac{1}{z}\right| \leq \dfrac{1}{5}$. Now

$$\left|z-\frac{1}{z}\right| \geq \left||z|-\frac{1}{|z|}\right| = |z|-\frac{1}{|z|} \geq 5-\frac{1}{5} = \frac{24}{5}$$

The least value is attained when $z = 5$.

6. $i\left(1-\cos\dfrac{2\pi}{7}\right)+\sin\dfrac{2\pi}{7}$

$= 2i\sin^2\dfrac{\pi}{7}+2\sin\dfrac{\pi}{7}\cos\dfrac{\pi}{7}$

$= 2\sin\left(\dfrac{\pi}{7}\right)\left[\cos\dfrac{\pi}{7}+i\sin\dfrac{\pi}{7}\right]$

Also, $i-1 = \sqrt{2}\left(-\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

$\therefore \quad z = \dfrac{\sqrt{2}}{2\sin\left(\dfrac{\pi}{7}\right)}\left[\cos\left(\dfrac{3\pi}{4}-\dfrac{\pi}{7}\right)+i\sin\left(\dfrac{3\pi}{4}-\dfrac{\pi}{7}\right)\right]$

$= \dfrac{\sqrt{2}}{2\sin\left(\dfrac{\pi}{7}\right)}\left[\cos\left(\dfrac{17\pi}{28}\right)+i\sin\left(\dfrac{17\pi}{28}\right)\right]$

Thus, $\arg(z) = \dfrac{17\pi}{28}$

7. $(x+iy)^2\,(c+id) = a+ib$

$\Rightarrow |(x+iy)^2|\,|c+id| = |a+ib|$

$\Rightarrow |x+iy|^2\,|c+id| = |a+ib|$

$\Rightarrow (x^2+y^2)\,\sqrt{c^2+d^2} = \sqrt{a^2+b^2}$

8. $\bar{\Delta} = \dfrac{1}{-4i}\begin{vmatrix} 1 & \bar{z}_1 & z_1 \\ 1 & \bar{z}_2 & z_2 \\ 1 & \bar{z}_3 & z_3 \end{vmatrix} = \Delta$

$\Rightarrow \Delta$ is purely real $\Rightarrow \text{Im}(\Delta) = 0$

9. $(a+ib)\,(x+iy) = (a+ib)\,(a-ib)\,i$

As $a \neq 0$

$x+iy = (a-ib)\,i = b+ia \Rightarrow (x, y) = (b, a)$

10. $\omega^{2017} = (\omega^3)^{672}\,\omega = \omega$

and $\omega^{2225} = (\omega^3)^{741}\,\omega^2 = \omega^2$

$\therefore \tan[(\omega^{2017}+\omega^{2225})\pi - \pi/3]$

$= \tan[(\omega+\omega^2)\pi - \pi/3]$

$= \tan(-\pi-\pi/3) = -\tan(\pi+\pi/3)$

$= -\tan(\pi/3) = -\sqrt{3}$

11. Let $z = -i\,t$ where $t > 0$, then

$i\bar{z} = i(i\,t) = -t$

$\therefore \arg(i\bar{z}) + \arg(z) = \pi - \pi/2 = \pi/2$

12. The inequality $a+ib < c+id$ is true if and only if $b = d = 0$ and $a > c$.

13. Let $z = x+iy$, then $z^2 = x^2-y^2+2ixy$

$\therefore \text{Re}(z^2) = 0 \Rightarrow x^2-y^2 = 0$

$\Rightarrow |\text{Re}(z)| = |\text{Im}(z)|$

14. $\arg\left(\dfrac{z_1}{z_4}\right)+\arg\left(\dfrac{z_2}{z_3}\right)$

$= \arg\left(\dfrac{z_1z_2}{z_4z_3}\right)+2k\pi = \arg\left(\dfrac{|z_1|^2}{|z_3|^2}\right)+2k\pi = 2k\pi$

where $k = 0$ or 1

[$\because$ $z_1 = z_3 = i$, gives answer 2π and $z_1 = z_3 = 1$, gives answer 0]

15. $\dfrac{z-4}{2i} = \dfrac{x-4}{2i} + \dfrac{y}{2}$

For $0 \le \sin^{-1}\left(\dfrac{z-4}{2i}\right) \le \dfrac{\pi}{2}$, we must

$x - 4 = 0, \quad 0 \le \dfrac{y}{2} \le 1$

$\Rightarrow \quad x = 4, \quad 0 \le y \le 2.$

16. $z = \dfrac{-3i}{|z| + a} \quad \Rightarrow \quad z$ is purely imaginary.

17. Let $z = bi$, $b \in \mathbf{R}$, $\quad b \ne 0$,

Then $z^2 = -b^2 \in \mathbf{R}$,

Therefore $\text{Im}(z^2) = 0$

18. We have $z_k = \omega^k$ where

$\omega = \cos\left(\dfrac{\pi}{10}\right) + i\sin\left(\dfrac{\pi}{10}\right)$

Thus, $z_1 z_2 z_3 z_4 = \omega . \omega^2 . \omega^3 . \omega^4 = \omega^{10}$

$= \cos\left(\dfrac{10\pi}{10}\right) + i\sin\left(\dfrac{10\pi}{10}\right)$

[by the De Movire's Theorem]

$= \cos \pi + i \sin \pi = -1.$

19. $\bar{z} = \dfrac{\bar{z}_1 + \bar{z}_2}{1 + \bar{z}_1 \bar{z}_2} = \dfrac{1/z_1 + 1/z_2}{1 + 1/z_1 z_2} = \dfrac{z_1 + z_2}{1 + z_1 z_2} = z$

Thus, z is purely real.

20. Let $z = x + iy$, then $\bar{z}^2 = x^2 - y^2 - 2ixy$

$\therefore \quad \text{Re}(\bar{z}^2) = k^2 \Rightarrow \quad x^2 - y^2 = k^2$

which represents a hyperbola.

Level 1

21. $|1 - \omega| = |\omega - \omega^2| = |\omega^2 - 1|$

Alternatively plot the points on an argand diagram.

22. $x^3 = p = \left(p^{\frac{1}{3}}\right)^3 \Rightarrow x = p^{\frac{1}{3}},\ p^{\frac{1}{3}}\omega,\ p^{\frac{1}{3}}\omega^2.$

Let $\alpha = p^{\frac{1}{3}}$, $\beta = p^{\frac{1}{3}}\omega$, $\gamma = p^{\frac{1}{3}}\omega^2$.

$$\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = \frac{x + y\omega + \omega^2 z}{x\omega + y\omega^2 + z} = \frac{1}{\omega} = \omega^2$$

If $\alpha = p^{\frac{1}{3}}\omega$, $\beta = p^{\frac{1}{3}}$, $\gamma = p^{\frac{1}{3}}\omega^2$, then

$$\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = \frac{1}{\omega^2} = \omega.$$

23. $\dfrac{z-(2-i)}{1+i-(2-i)} = \dfrac{AM}{MD} e^{\pm\pi i/2} \Rightarrow \dfrac{z-(2-i)}{-1+2i} = \dfrac{1}{2}(\pm i)$

$\Rightarrow z = (2 - i) + \dfrac{i}{2}(-1 + 2i)$

or $z = (2 - i) - \dfrac{i}{2}(-1 + 2i)$

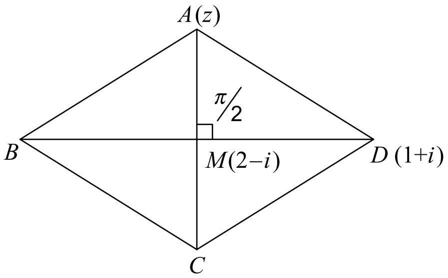


Fig. 2.56

$\Rightarrow z = 1 - \dfrac{3}{2}i$ or $z = 3 - \dfrac{1}{2}i$

24. Let $\alpha = \omega$, $\beta = \omega^2$ so that $\alpha^{19} = \omega$ and $\beta^7 = \omega^2$.

$\therefore$ equation whose roots are α^{19}, β^7 is $x^2 + x + 1 = 0$.

25. Using $\omega^{10} = \omega$, $\omega^{23} = \omega^2$, we get

$$\sin\left\{(\omega^{10} + \omega^{23})\pi - \frac{\pi}{4}\right\} = \sin\left(-\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

26. $(1 + \omega)^{2017} = (-\omega^2)^{2017} = -\omega^{4034} = -\omega^2$

$\therefore \quad A + B\omega = 1 + \omega \quad \Rightarrow \quad A = 1, B = 1.$

27. Use $\quad R_2 \to R_2 - R_1 - R_3$

28. $(1 + \omega - \omega^2)^7 = (-\omega^2 - \omega^2)^7 = -2^7\omega^{14} = -128\omega^2$

29. Use $\quad C_2 \to C_2 + 3i\, C_3$.

30. Let $\alpha = \arg z < 0$, then $\arg(-z) = \pi + \alpha$,

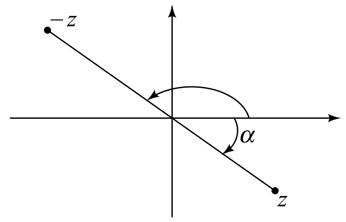


Fig. 2.57

31. $\qquad |z_1| = |z_2| = |z_3| = 1$

$\Rightarrow \qquad z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = 1$

$\therefore |z_1 + z_2 + z_3| = \left|\dfrac{1}{\bar{z}_1} + \dfrac{1}{\bar{z}_2} + \dfrac{1}{\bar{z}_3}\right| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$

32. nth roots of unity are given by

$$\cos\left(\frac{2m\pi}{n}\right) + i\sin\left(\frac{2m\pi}{n}\right) = e^{2m\pi i/n} \text{ for } m = 0, 1, 2, \cdots, n-1.$$

Let $z_1 = e^{2m_1\pi i/n}$ and $z_2 = e^{2m_2\pi i/n}$

where $0 \le m_1, m_2 < n$, $m_1 \ne m_2$.

As the join of z_1 and z_2 subtend a right angle at the origin z_1/z_2 is purely imaginary we get

$\frac{e^{2m_1\pi i/n}}{e^{2m_2\pi i/n}} = il$ for some real $l \Rightarrow \quad e^{2(m_1-m_2)\pi i/n} = il$

$\Rightarrow \cos\left[\frac{2(m_1-m_2)\pi}{n}\right] + i\sin\left[\frac{2(m_1-m_2)\pi}{n}\right] = il$

$\Rightarrow \cos\left[\frac{2(m_1-m_2)\pi}{n}\right] = 0 \Rightarrow \frac{2(m_1-m_2)\pi}{n} = \frac{\pi}{2}$

$\Rightarrow n = 4(m_1 - m_2)$

Thus, n must be of the form $4k$.

33. $$\left|\frac{z_1-z_3}{z_2-z_3}\right| = \left|\frac{z_1-z_3}{z_2-z_3}\right| = \left|\frac{1-i\sqrt{3}}{2}\right|$$

$$= \sqrt{\frac{1}{4}(1+3)} = 1$$

$\Rightarrow \quad |z_1 - z_3| = |z_2 - z_3|$

Also, $\frac{z_1-z_3}{z_2-z_3} - 1 = \frac{1-i\sqrt{3}}{2} - 1$

$\Rightarrow \quad \frac{z_1-z_2}{z_2-z_3} = \frac{-1-i\sqrt{3}}{2}$

$\Rightarrow \quad \frac{|z_1-z_2|}{|z_2-z_3|} = \sqrt{\frac{1}{4}(1+3)} = 1$

$\Rightarrow \quad |z_1 - z_2| = |z_2 - z_3|$

Thus, $|z_1 - z_3| = |z_2 - z_3| = |z_2 - z_1|$

Hence, z_1, z_2 and z_3 are the vertices of an equilateral triangle.

34. Using $1 + \omega^2 = -\omega$, $\omega^4 = \omega$ and applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \omega-1 & \omega^2-1 \\ 1 & \omega^2-1 & \omega-1 \end{vmatrix}$$

$= (\omega - 1)^2 - (\omega^2 - 1)^2$

$= (\omega + \omega^2 - 2)(\omega - 1 - \omega^2 + 1)$

$= (-3)(\omega - \omega^2) = 3\omega(\omega - 1)$

35. $|z - i| = |z + i|$ represents the real axis. As $z = i$ satisfies $|z - i| < |z + i|$, we get $|z - i| < |z + i|$, represents Im $(z) > 0$

36. $iz^3 + z^2 - z + i = 0 \Rightarrow iz^3 - i^2z^2 - (z - i) = 0$

$\Rightarrow \quad iz^2(z - i) - (z - i) = 0$

$\Rightarrow \quad (iz^2 - 1)(z - i) = 0$

$\Rightarrow \quad z^2 = 1/i$ or $z = i$

In any case $|z| = 1$.

37. $2x = (x + iy) + (x - iy) = \frac{2(1-\cos\theta)}{(1-\cos\theta)^2 + 4\sin^2\theta}$

$\Rightarrow \quad x = \frac{1}{5+3\cos\theta}$

Thus, maximum value of x is 1/2. It is attained at $\theta = \pi$.

38. $z^3 = \bar{z} \Rightarrow |z^3| = |\bar{z}| \Rightarrow |z^3| = |\bar{z}|$

$\Rightarrow \quad |z| = 0$ or $|z| = 1 \Rightarrow z = 0$ or $\bar{z} = 1/z$.

$\therefore \quad z^3 = \bar{z} \Rightarrow z^3 = \frac{1}{z} \Rightarrow z^4 = 1$

$\Rightarrow \quad z = 1, -1, i, -i$.

Thus, solutions of $z^3 = \bar{z}$ are 0, 1, –1, i, $-i$

39. Let $z = x + iy$ so that

$x = 5 + t, \quad y = \sqrt{25 - t^2}$

$\Rightarrow \quad (x - 5)^2 + y^2 = 25$,

This clearly passes through $2 + 4i$

40. Note that ω, ω^2 are roots of

$$\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} = \frac{2}{x}$$

$\Leftrightarrow \quad x[bc + ca + ab + 2(a + b + c)x + 3x^2]$

$= 2[abc + (bc + ca + ab)x + (a + b + c)x^2 + x^3]$

$\Leftrightarrow \quad x^3 - (bc + ca + ab)x - 2abc = 0$.

If α is the third root of this equation, then

$\alpha + \omega + \omega^2 = 0 \quad \Rightarrow \quad \alpha = 1$.

41. $z = x + iy$, $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$

$\Rightarrow |x + iy - ix|^2 = |x + iy - y|^2$

$\Rightarrow x^2 + (y - x)^2 = (x - y)^2 + y^2$

$\Rightarrow x = \pm y$.

42. We have

$(r - \omega)(r - \omega^2) = r^2 - (\omega + \omega^2)r + 1$

$= r^2 + r + 1 = (r + 1)^2 - r$

$\therefore \quad \sum_{r=1}^{12}(r-\omega)(r-\omega^2) = \sum_{r=1}^{12}\left[(r+1)^2 - r\right]$

$= \frac{1}{6}(13)(13+1)(26+1) - 1 - \frac{1}{2}(12)(13) = 740$

43. Let ix ($x \in \boldsymbol{R}$) be root of $z^2 + az + b = 0$, then

$-x^2 + aix + b = 0$ (1)

$\Rightarrow \quad -x^2 - \bar{a}ix + \bar{b} = 0$

Subtracting we get

$(a + \bar{a})ix + b - \bar{b} = 0$

$\Rightarrow \quad x = -\frac{b-\bar{b}}{i(a+\bar{a})} = \frac{i(b-\bar{b})}{a+\bar{a}}$

Putting this in (1), we get

$$\frac{(b-\bar{b})^2}{(a+\bar{a})^2} - \frac{a(b-\bar{b})}{a+\bar{a}} + b = 0$$

$\Rightarrow (b - \bar{b})^2 - a(b - \bar{b})(a + \bar{a}) + b(a + \bar{a})^2 = 0$

$\Rightarrow (b - \bar{b})^2 - (a + \bar{a})\{ab - a\bar{b} - ab - \bar{a}b\} = 0$

$\Rightarrow (b - \bar{b})^2 + (a + \bar{a})(a\bar{b} + \bar{a}b) = 0$

44. The circle $|z - (-1 + i)| = \sqrt{2}$ completely lies inside the circle $|z| = 3$.

45. $8iz^3 + 12z^2 - 18z + 27i = 0$

$\Rightarrow \quad 8iz^3 - 12i^2z^2 - 18z + 27i = 0$

$\Rightarrow \quad 4iz^2(2z - 3i) - 9(2z - 3i) = 0$

$\Rightarrow \quad (4iz^2 - 9)\ (2z - 3i) = 0$

$\Rightarrow \quad z^2 = \dfrac{9}{4i}$ or $z = \dfrac{3i}{2}$

In any case $|z| = 3/2$.

46. As -1 lies on the circle $|z - 4| = 5$, the real number $|z + 1|$ is maximum when z is the other end point of the diameter.

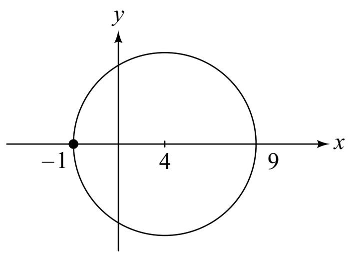


Fig. 2.58

47. $\dfrac{1}{x+iy} = \dfrac{\cos\theta + 2}{3} + i\dfrac{\sin\theta}{3}$

$\Rightarrow \dfrac{x - iy}{x^2 + y^2} = \dfrac{\cos\theta + 2}{3} + i\dfrac{\sin\theta}{3}$

$\Rightarrow \dfrac{x}{x^2 + y^2} = \dfrac{\cos\theta + 2}{3}$

Also, $x^2 + y^2 = \dfrac{9}{(2+\cos\theta)^2 + \sin^2\theta} = \dfrac{9}{5 + 4\cos\theta}$

$\therefore\ 4x - (x^2 + y^2)$

$= \left(\dfrac{4(\cos\theta + 2)}{3} - 1\right)(x^2 + y^2)$

$= \dfrac{4\cos\theta + 5}{3} \cdot \dfrac{9}{5 + 4\cos\theta} = 3$

48. Affix of G is $\dfrac{1}{3}(z_1 + z_2 + z_3)$.

As origin is the mid-point of AG,

$0 = \dfrac{1}{2}\left[\dfrac{1}{3}(z_1 + z_2 + z_3) + z_1\right]$

$\Rightarrow 4z_1 + z_2 + z_3 = 0.$

49. $z_3 = -\dfrac{1}{c}(az_1 + bz_2) = \dfrac{az_1 + bz_2}{a+b}$

50. $\dfrac{z-1}{z+1}$ = it where $t \in \mathbf{R}$. $\Rightarrow \dfrac{\bar{z}-1}{\bar{z}+1} = -$ it

$\Rightarrow \dfrac{z-1}{z+1} + \dfrac{\bar{z}-1}{\bar{z}+1} = 0$

$\Rightarrow (z - 1)(\bar{z} + 1) + (\bar{z} - 1)(z + 1) = 0$

$\Rightarrow 2(z\bar{z} - 1) = 0 \quad \Rightarrow \quad |z| = 1.$

51. $\left|\dfrac{z}{|z|} - \dfrac{\bar{z}}{|z|}\right| \le |\arg(z) - \arg(\bar{z})|$

$\Rightarrow |z - \bar{z}| \le |z|\ |\arg(z) - \arg(\bar{z})|$

52. As $|\omega| = 1 \Rightarrow |\omega|^2 = 1 \Rightarrow \omega\bar{\omega} = 1 \Rightarrow \dfrac{1}{\omega} = \bar{\omega}$.

Thus, $\quad z = \omega + 1/\omega = \omega + \bar{\omega} = 2\,\mathrm{Re}(\omega)$

Now, $\quad |\mathrm{Re}(z)| = |2\,\mathrm{Re}(\omega)| = |2\,\mathrm{Re}(\omega)| \le 2|\omega| = 2$

53. $|z| = 1 \Rightarrow z = \cos\theta + i\sin\theta$ for some $\theta \in [0, 2\pi)$

Now, $\quad |z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar{z} = 1.$

Thus, $\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z} = z^2 + \dfrac{1}{z^2}$

$= (\cos\theta + i\sin\theta)^2 + (\cos\theta - i\sin\theta)^2$

$= 2(\cos^2\theta + i^2\sin^2\theta) = 2(\cos 2\theta)$

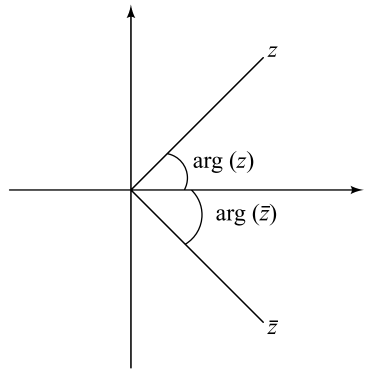


Fig. 2.59

$\therefore \quad \left|\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}\right| = 1 \Rightarrow |\cos 2\theta| = \dfrac{1}{2}$

$\Rightarrow \quad \cos 2\theta = \pm 1/2$

$\Rightarrow \quad 2\theta = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3, 7\pi/3, 8\pi/3, 10\pi/3, 11\pi/3$

$\Rightarrow \quad \theta = \pi/6, \pi/3, 2\pi/3. 5\pi/6, 7\pi/6, 4\pi/3, 5\pi/3, 11\pi/6$

Hence, there are 8 values of z

54. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

$\Rightarrow \quad 3 + |z_1 - z_2|^2 = 2(2) + 2(2)$

$\Rightarrow \quad |z_1 - z_2|^2 = 5 \Rightarrow |z_1 - z_2| = \sqrt{5}$.

55. $|z_1\bar{z}_2| = |z_2\bar{z}_3| = |z_3\bar{z}_1|$

$\Rightarrow \quad |z_1|\,|z_2| = |z_2|\,|z_3| = |z_3|\,|z_1| \Rightarrow |z_1| = |z_2| = |z_3| = r$(say)

$[\because z_1, z_2, z_3 \ne 0]$

Thus, $\quad z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3$

Now, $\quad z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1$

$\Rightarrow \quad \dfrac{z_1}{z_2} = \dfrac{z_2}{z_3} = \dfrac{z_3}{z_1}$

$\Rightarrow \quad z_1^2 = z_2z_3, \quad z_2^2 = z_3z_1,\ z_3^2 = z_1z_2$

Hence, $z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$

$\Rightarrow z_1, z_2, z_3$ are vertices of an equilateral triangle.

56. $|z_1 + z_2 + z_3| \ge 0$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\mathrm{Re}\left(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1\right) \ge 0$

$\Rightarrow \quad \mathrm{Re}\left(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1\right) \ge -\dfrac{3}{2}$

Now, $\quad |z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$

$= 6 - 2\mathrm{Re}\left(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1\right)$

$\leq \; 6-(-3)=9$

57. $z_1+z_2+z_3=0 \Rightarrow \bar{z}_1+\bar{z}_2+\bar{z}_3=0$

$\Rightarrow \dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}=0 \qquad [\because \; z_1\bar{z}_1=1 \text{ etc.}]$

Now, $0=(z_1+z_2+z_3)^2=z_1^2+z_2^2+z_3^2+2(z_2z_3+z_3z_1+z_1z_2)$

$\Rightarrow \quad 0=z_1^2+z_2^2+z_3^2+2z_1z_2z_3\left(\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}\right)$

$= z_1^2+z_2^2+z_3^2$

Thus, $\quad \left|z_1^2+2z_2^2+z_3^2\right| = \left|z_1^2+z_2^2+z_3^2+z_2^2\right|$

$= \left|0+z_2^2\right|=|z_2|^2=1$

58. Note that $\dfrac{1}{z_1}=\bar{z}_1$ etc. Thus,

$z=(z_1+z_2+z_3)\left(\bar{z}_1+\bar{z}_2+\bar{z}_3\right)$

$=|z_1+z_2+z_3|^2 \leq (|z_1|+|z_2|+|z_3|)^2=9$

The maximum value is obtained when $z_1=z_2=z_3=1$.

59. $\omega = \dfrac{z-z\bar{z}}{\bar{z}+\bar{z}z}=\dfrac{z-1}{1/z+1}=\dfrac{(z-1)z}{z+1} \qquad [\because \; |z|=1]$

$=\dfrac{z^2-1}{z+1}-\dfrac{z-1}{z+1}=z-1+\dfrac{1-z}{1+z}$

$\text{Re}(\omega) = \text{Re}(z)-1+\text{Re}\left(\dfrac{1-z}{1+z}\right)$

But $\quad \text{Re}\left(\dfrac{1-z}{1+z}\right)=\dfrac{1}{2}\left(\dfrac{1-z}{1+z}+\dfrac{1-\bar{z}}{1+\bar{z}}\right)$

$= \dfrac{1}{2}\left(\dfrac{1-z}{1+z}+\dfrac{z-1}{z+1}\right)=0$

$\therefore \quad \text{Re}(\omega) = \cos\theta-1=-2\sin^2(\theta/2)$

60. Let $\omega = \dfrac{1+z^2}{2iz}$. $\text{Im}(\omega) = \dfrac{1}{2}(\omega-\bar{\omega})$

For $|z|=1$, $\quad \bar{\omega}=\dfrac{1+\bar{z}^2}{-2i\bar{z}}=\dfrac{1+(1/z)^2}{-2i(1/z)}$

$= \dfrac{z^2+1}{-2iz}=-\omega$

Thus, $\text{Im}(\omega) = \dfrac{1}{2i}(\omega+\omega)=-i\omega$

$= -\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)=-\dfrac{1}{2}(z+\bar{z})$

$= -\dfrac{1}{2}(2\cos\theta) = -\cos\,\theta$

$[\because \; |z|=1 \Rightarrow z=\cos\,\theta+i\sin\,\theta]$

As $z\neq 1$, $\;\theta\neq 0\;$ and $\;$ as $z\neq -1$, $\theta\neq\pi$,

$\therefore \quad A=(-1,\,1)$

61. $|z|=1 \Rightarrow \; z\bar{z}=1$. Therefore $z=1-2\bar{z}$

$\Rightarrow \quad z=1-2/z \quad \Rightarrow \quad z^2-z+2=0$

$\Rightarrow \quad z=\dfrac{1\pm\sqrt{7}\,i}{2}$, $\quad$ but then $|z|\neq 1$.

62. Suppose $z_2/z_1=a$, where $a\in\mathbf{R}$.

Now, $\left|\dfrac{2iz_1+5z_2}{2iz_1-5z_2}\right|=\left|\dfrac{2i+5(z_2/z_1)}{2i-5(z_2/z_1)}\right|=\left|\dfrac{2i+5a}{2i-5a}\right|=1$

63. $z=i\left(1+\sqrt{3}\right)=-1+\sqrt{3}\,i=2\omega$

where $\omega\neq 1$ is a cube root of unity.

$\therefore \quad z^4+2z^3+4z^2+5$

$=(2\omega)^4+2(2\omega)^3+4(2\omega)^2+5$

$=16\omega^4+16\omega^3+16\omega^2+5$

$=16(\omega+1+\omega^2)+5=16(0)+5=5$

64. Let $a=-b^4$ where $b>0$.

Then $z^4=a=b^4(-1)$

$\Rightarrow z=b\left(\pm\cos\dfrac{\pi}{4}\pm i\sin\dfrac{\pi}{4}\right)$

$\therefore \quad z_1^2+z_2^2+z_3^2+z_4^2$

$= 2b^2\left[\cos\left(\dfrac{\pi}{2}\right)-i\sin\left(\dfrac{\pi}{2}\right)+\cos\left(\dfrac{\pi}{2}\right)+i\sin\left(\dfrac{\pi}{2}\right)\right]$

$=0=a+|a| \qquad [\because \; a<0]$

65. As $\dfrac{z+\bar{z}}{1+z\bar{z}}$ is a positive real number,

$\arg\left(\dfrac{z+\bar{z}}{1+z\bar{z}}\right)=0.$

66. Using $(\cos\,\alpha+i\,\sin\,\alpha)(\cos\,\beta+i\,\sin\,\beta)$

$=\cos(\alpha+\beta)+i\,\sin(\alpha+\beta)$,

we get

$\cos(25\theta)=-1, \quad \sin(25\,\theta)=0$

$\Rightarrow \quad 25\theta=(2k+1)\pi,\; k\in\mathbf{I}.$

Now, $0<\dfrac{(2k+1)\pi}{25}\leq\pi$

$\Rightarrow \quad 1\leq(2k+1)\leq 25$

$\Rightarrow \quad k=0,1,2,\ldots,12$

67. $\log_{(1/7)}|z-2|>\log_{(1/7)}|z|$

$\Rightarrow \quad |z-2|<|z| \qquad (1)$

But $|z-2|=|z|$ represents perpendicular bisector of the segment joining 0 and 2, that is, $|z-2|=|z|$ represents the line $\text{Re}(z)=1$. As 0 does not satisfy (1), we get (1) represents $\text{Re}(z)>1$.

68. $\dfrac{1}{2i}\left[\dfrac{2z+1}{iz+1}-\dfrac{2\bar{z}+1}{-i\bar{z}+1}\right]=5$

$\Rightarrow\quad (2z + 1)(i\bar{z} - 1) + (2\bar{z} + 1)(iz + 1)$

$= 10\,i\,(iz + 1)(i\bar{z} - 1)$

$\Rightarrow\quad 4iz\bar{z} + (-2 + i)z + (2 + i)\bar{z}$

$= 10\,i\,(-z\bar{z} - iz + i\bar{z} - 1)$

$\Rightarrow\quad 14iz\bar{z} + (8 + i)z + (-8 + i)\bar{z} - 10\,i = 0$

$$\Rightarrow\quad z\bar{z} + \frac{1}{14}(1 - 8i)z + \frac{1}{14}(1 + 8i)\bar{z} - \frac{5}{7} = 0$$

This represents a circle.

69. Take z_1 and z_2 as two real numbers.

70. $$\frac{(1+i)^n}{(1-i)^{n-2}} = \left(\frac{1+i}{1-i}\right)^{n-2}(1+i)^2 = \left(\frac{-i^2+i}{1-i}\right)^{n-2}(2i)$$

$= i^{n-2}(2i) = 2i^{n-1} = -2i^{n+1}$

71. If $n = 3k$, $k \in \mathbf{I}$, then $f(n) = 1 + 1 + 1 = 3$

If $n = 3k + 1$, $k \in \mathbf{I}$, then $f(n) = 1 + \omega + \omega^2 = 0$

If $n = 3k + 2$ $k \in \mathbf{I}$, then $f(n) = 1 + \omega^2 + \omega = 0$

Thus, range of f is $\{0, 3\}$.

72. $z + \dfrac{1}{z} = 2\cos\theta \quad \Rightarrow z^2 - 2z\cos\theta + 1 = 0$

$\Rightarrow\quad z = \cos\theta \pm i\sin\theta$

Now, $z^{2n} - 2z^n\cos(n\theta)$

$= z^n[z^n - 2\cos(n\theta)]$

$= z^n[\cos(n\theta) \pm i\sin(n\theta) - 2\cos(n\theta)]$

$= -z^n\bar{z}^n = -1$

73. Using $\omega^k + \omega^{k+1} + \omega^{k+2} = 0\ \forall\ k \in \mathbf{I}$,

we get $\displaystyle\sum_{k=1}^{60}\omega^k = 0$

Next, $\displaystyle\prod_{k=1}^{30}\omega^k = \omega^m =$ where

$m = 1 + 2 + \ldots + 30 = \dfrac{1}{2}(30)(31) = 465$

$\therefore\quad \displaystyle\prod_{k=1}^{30}\omega^k = \omega^{465} = 1$

74. $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

Therefore, ω, ω^2 are zeros of $P(x)$

$0 = g(1) + \omega h(1)$ and $0 = g(1) + \omega^2 h(1)$

$\Rightarrow g(1) = 0,\ h(1) = 0.$

75. $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

Therefore, $-\omega$, $-\omega^2$ must be zeros of $x^{n+1} - x^n + 1$. Now,

$(-\omega)^{n+1} - (-\omega)^n + 1 = (-1)^{n+1}\omega^n(\omega + 1) + 1$

$= (-1)^{n+1}\omega^n(-\omega^2) + 1$

$= (-1)^{n+2}\omega^{n+2} + 1$

$= 0$ if $n = 6k + 1$

76. Clearly $z \neq 0, 1$. We can write the given equation as $t^2 + t + 1 = 0$ where

$t = z/|z - 1|$

$\Rightarrow\quad t = \omega, \omega^2$ where $\omega \neq 1$ is a cube root of unity.

Thus, $\dfrac{z}{|z-1|} = \omega$

$\Rightarrow\quad |z| = |z - 1|\,|\omega| = |z - 1|$

$\Rightarrow\quad z$ lies on the perpendicular bisector of the segment joining $z = 0$ and $z = 1$, that is, z lies on $\mathrm{Re}(z) = 1/2$. Let $z = 1/2 + ai$ where $a \in \mathbf{R}$.

Note that $a \neq 0$ since $z = 1/2$ does not satisfy $z^2 + z|z - 1| + |z - 1|^2 = 0$

Putting $z = 1/2 + ai$ in $z^2 + z|z - 1| + |z - 1|^2 = 0$ and equating imaginary parts of both sides, we get

$a + a|z - 1| = 0 \Rightarrow |z - 1| = -1.$

A contradiction.

Thus $z^2 + z\,|z - 1| - |z + 1|^2 = 0$ has no solution in C,

that is, statement-1 is true.

Statement-2 is false as $a\omega$, $a\omega^2$ are solutions of $z^2 + az + a^2 = 0$.

77. $z = 2i + \bar{z} \quad \Rightarrow \quad z - \bar{z} = 2i$

$\Rightarrow\quad 2i\,\mathrm{Im}(z) = 2i \quad \Rightarrow \quad \mathrm{Im}(z) = 1$

As $|z| = 1$, we get $z = \pm i$

Thus, statement-1 is true.

As $\arg(i) = \pi/2$ and $\arg(-i) = -\pi/2$,

$\therefore$ Statement-2 is false.

78. Note that

$$\alpha = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\pi - \frac{8\pi}{11}\right)$$

$$= \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\frac{8\pi}{11}\right)$$

$\therefore\ \alpha^{11} = \cos(8\pi) + i\sin(8\pi) = 1$

[De Moivre's Theorem]

Now, $\displaystyle\sum_{k=0}^{10}\alpha^k = \frac{1 - \alpha^{11}}{1 - \alpha} = 0$

$\therefore$ Statement-2 is true

We have $\bar{\alpha} = \dfrac{\alpha\bar{\alpha}}{\alpha} = \dfrac{1}{\alpha} = \alpha^{10}$ etc.

Thus,

$\mathrm{Re}\,(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$

$$= \frac{1}{2}(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5)$$

$$= \frac{1}{2}\sum_{k=1}^{10}\alpha^k = \frac{1}{2}(-1) = -\frac{1}{2} \qquad \text{[Use statement-2]}$$

$\therefore$ Statement-1 is also correct and Statement-2 is a correct explanation for it.

Statement-2 is true as it is formula for sun of a G.P.

79. $z^k = \cos(k\theta) + i \sin(k\theta)$

$$f(\theta) = \sum_{k=1}^{15} \text{Im}(z^{2k-1})$$

$$= \text{Im}\left(\sum_{k=1}^{15} z^{2k-1}\right)$$

$$= \text{Im}\left(\frac{z(1-z^{30})}{1-z^2}\right)$$

But $z^{30} = \cos(30\ \theta) + i \sin(30\ \theta)$

When $\theta = \alpha = \pi/60$, $z^{30} = 0 + i = i$

$$\therefore f(\alpha) = \text{Im}\left(\frac{z(1-z^{30})}{1-z^2}\right)$$

$$= \text{Im}\left(\frac{1-i}{1/z - z}\right)$$

$$= \text{Im}\left(\frac{1-i}{-2i \sin\alpha}\right)$$

$$= \frac{1}{2\sin\alpha}\text{Im}(1+i) = \frac{1}{2\sin\alpha}$$

80. Statement-2 is true. See theory of chapter 5 on matrices.

Let $A = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$,

$$|A| = 1\begin{vmatrix} 1 & c \\ \omega & 1 \end{vmatrix} - a\begin{vmatrix} \omega & c \\ \omega^2 & 1 \end{vmatrix} + b\begin{vmatrix} \omega & 1 \\ \omega^2 & \omega \end{vmatrix}$$

$= 1 - c\omega - a(\omega - c\omega^2)$

$= (1 - a\omega)(1 - c\omega)$

Note that $|A| = 0$ if $a = \omega^2$ or $c = \omega^2$

Thus, $|A| \neq 0$ if $a = c = \omega$ and $b = \omega$ or ω^2

$\therefore$ *S* contains exactly two distinct elements.

81. $|z_1 - z_2| - \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$

$$= |z_1 - z_2| - \frac{|z_2 - z_1|}{|z_1||z_2|}$$

$$= |z_1 - z_2| - |z_2 - z_1| = 0$$

$\therefore$ Statement-1 is true.

Statement-2 is also true but a correct explanation for truth of statement-1.

82. Statement-2 is a true statement.
Suppose $f(z) = (z^2 + 1)\, q(z) + r(z)$.
If $r(z) \neq 0$, then $r(z) = az + b$
where $a, b \in \mathbf{C}$.
We have

$$i = f(i) = (i^2 + i)\, q(i) + ai + b$$

$\Rightarrow \quad ai + b = i$

Also, $1 + i = f(-i) = ((-i)^2 + 1)\, q(-i) - ai + b$

$\Rightarrow \quad -ai + b = 1 + i$

$\therefore \quad b = \frac{1}{2} + i, \ a = \frac{1}{2}i$

Thus, $az + b = \frac{1}{2}(iz + 1) + i$

83. Statement-2 is true, as $z^3 = a^3 \Rightarrow |z|^3 = |a|^3$
$\Rightarrow |z| = |a|$
Next,
$(z + ab)^3 = a^3$
$\Rightarrow \quad z + ab = a, aw, aw^2$
Let $z_1 = a - ab$, $z_2 = aw - ab$
and $z_3 = aw^2 - ab$

We gave $|z_2 - z_1| = |a||w - 1| = \sqrt{3}\,a$,

$|z_3 - z_2| = \sqrt{3}\,a$, $|z_1 - z_3| = \sqrt{3}\,a$

Thus, statement-1 is also true, but statement-2 is not a correct explanation of statement-1.

84. Statement-2 is true. See Theory. Let α be a root of $z^2 - z + \pi = 0$ and suppose $|\alpha| < 1$. We have

$$\pi = |\pi| = |\alpha - \alpha^2| \leq |\alpha| + |\alpha|^2 < 2$$

A contradiction.

Thus, statement-1 is also true and statement-2 is a correct explanation for it.

85. We know that $|\sin\theta| \leq \theta \quad \forall\ \theta \geq 0$.

If $\theta < 0$, then $|\sin(-\theta)| \leq -\theta \Rightarrow |\sin\theta| \leq |\theta|$

Thus, $|\sin\theta| \leq |\theta|\ \forall\ \theta \in \mathbf{R}$ and statement-2 is true.

Now, $|z_1 - z_2|^2 = (r_1 \cos\theta - r_2 \cos\varphi)^2 + (r_1 \sin\theta - r_2 \sin\varphi)^2$

$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta - \varphi)$$

$$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta - \varphi}{2}\right)$$

$$\leq (r_1 - r_2)^2 + 4(1)(1)\left(\frac{\theta - \phi}{2}\right)^2$$

Thus, Statement-1 is also true and Statement-2 is a correct explanation for it.

Level 2

86. As $ABCD$ is a parallelogram,

mid point of AC = mid point of BD

$\Rightarrow \frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$

$\Rightarrow z_4 = z_1 + z_3 - z_2$

87. $|a + b\omega + c\omega^2|^2 = (a + b\omega + c\bar{\omega})(a + b\bar{\omega} + c\omega)$

$= a^2 + b^2 + c^2 - bc - ca - ab$

$= \frac{1}{2}\left[(b-c)^2 + (c-a)^2 + (a-b)^2\right]$

As a, b, c are integers and at least two of them are unequal, we get,

$(b - c)^2 + (c - a)^2 + (a - b) \geq 2.$

Thus, $|a + b\omega + c\omega^2|^2 \geq 1 \Rightarrow |a + b\omega + c\omega^2| \geq 1$

Least value 1 is attained when $a = 2$, $b = 1$, $c = 1$.

88. We have $AP = AB = AQ = 2$

Thus, for the shaded region $|z + 1| > 2$

Also, $\angle BAQ = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}-1+1}\right) = \frac{\pi}{4}$

and $\angle BAP = -\pi/4$

Hence, for the shaded region

$|z + 1| > 2$ and $|\arg(z + 1)| < \pi/4$

89. As $\frac{w - \bar{w}z}{1-z}$ is purely real,

$\frac{w - \bar{w}z}{1-z} = \frac{\bar{w} - w\bar{z}}{1-\bar{z}}$

$\Rightarrow (1 - \bar{z})(w - \bar{w}z) = (1 - z)(\bar{w} - w\bar{z})$

$\Rightarrow (w - \bar{w})(1 - z\bar{z}) = 0$

As $w \neq \bar{w}$, we get $z\bar{z} = 1$

Thus, set of values of z is

$\{z: |z| = 1, z \neq 1\}$.

90. $zw = |z|^2 \Rightarrow zw = z\bar{z} \Rightarrow w = \bar{z}$

Thus, $|z - \bar{z}| + |w + \bar{w}| = 4$

$\Rightarrow |z - \bar{z}| + |\bar{z} + z| = 4$

$\Rightarrow |2iy| + |2x| = 4$

$\Rightarrow |x| + |y| = 2$

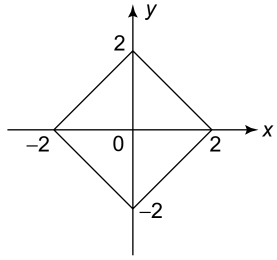


Fig. 2.60

This represents a square. See Fig. 2.60.

91. Let $z = x + iy$, then

$x = 1 - t$, $y = \sqrt{t^2 + t + 2}$

$\Rightarrow t = 1 - x$ and $y^2 = t^2 + t + 2 = (t + 1/2)^2 + 7/4$

$\Rightarrow y^2 = (x - 3/2)^2 + 7/4.$

This represents a hyperbola.

92. $w + \bar{w} = \frac{az_1 - bz_2}{az_1 + bz_2} + \frac{a\bar{z}_1 - b\bar{z}_2}{a\bar{z}_1 + b\bar{z}_2}$

$= \frac{(az_1 - bz_2)(a\bar{z}_1 + b\bar{z}_2) + (a\bar{z}_1 - b\bar{z}_2)(az_1 + bz_2)}{|az_1 + bz_2|^2}$

$= \frac{a^2 z_1 \bar{z}_1 - b^2 z_2 \bar{z}_2}{|az_1 + bz_2|^2} = 0 \; [\because a|z_1| = b|z_2|]$

$\Rightarrow w$ lies on the imaginary axis.

93. $|z| + |1 - z| + |z - 2|$

$\geq \max\{|z|, |(1 - z) + (z - 2)|, |z + (1 - z)| + |z - 2|, |1 - z| + |z - (z - 2)|\} = 2$

The value 2 is attained when $z = 1$

94. See Theory.

95. $|z_2 - z_3|^2 + |z_1|^2 = |z_2 - z_3|^2 + |-z_2 - z_3|^2$

$\Rightarrow |z_2 - z_3|^2 + 1 = 2|(|z_2|^2 + |z_3|^2) = 4$

$\Rightarrow |z_2 - z_3| = \sqrt{3}$

Similarly, $|z_3 - z_1| = |z_1 - z_2| = \sqrt{3}$

Thus, z_1, z_2, z_3 are vertices of an equilateral triangle.

96. $1 + w + w^2 + \ldots + w^{n-1}$

$= \frac{1 - w^n}{1 - w} = \frac{1 - \cos\pi + i\sin\pi}{1 - \cos(\pi/n) - i\sin(\pi/n)}$

$= \frac{2}{2\sin^2(\pi/2n) - 2i\sin(\pi/2n)\cos(\pi/2n)}$

$= \frac{2}{-2i\sin(\pi/2n)[\cos(\pi/2n) + i\sin(\pi/2n)]}$

$$= \frac{\cos(\pi/2n) - i\sin(\pi/2n)}{-i\sin(\pi/2n)} = 1 + i\cot\left(\frac{\pi}{2n}\right)$$

97. $|z_1 + z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2$

$\Rightarrow 2(z_1\bar{z}_2 + \bar{z}_1 z_2) = 0 \Rightarrow \frac{z_1}{\bar{z}_1} + \frac{z_2}{\bar{z}_2} = 0$

98. z_1 lies on the circle $|z| = 2$ and z_2 lies on the line $x + y = 4\sqrt{2}$

Distance of $x + y = 4\sqrt{2}$ from (0, 0) is 4

Thus, minimum distance between z_1 and z_2 is 2.

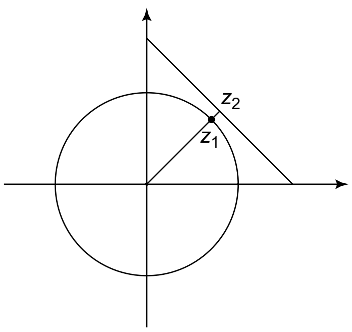


Fig. 2.61

Alternative Solution

Distance of $z_1 = 2(\cos\theta + i\sin\theta)$ from $x + y = 4\sqrt{2}$, is

$$\frac{|2\cos\theta + 2\sin\theta - 4\sqrt{2}|}{\sqrt{2}}$$

$= \sqrt{2}\ (2\sqrt{2} - (\cos\theta + \sin\theta))$

Maximum possible value of $\cos\theta + \sin\theta$ is $\sqrt{2}$.

99. Let $z_1 = r(\cos\theta + i\sin\theta)$.

Equation of tanget to $x^2 + y^2 = r^2$ at $(r\cos\theta, r\sin\theta)$ is $x\cos\theta + y\sin\theta = r$

$$\Rightarrow \left(\frac{z + \bar{z}}{2}\right)\left(\frac{z_1 + \bar{z}_1}{2}\right) + \left(\frac{z - \bar{z}}{2i}\right)\left(\frac{z_1 - \bar{z}_1}{2i}\right) = r^2$$

$\Rightarrow z\bar{z}_1 + \bar{z} z_1 = 2r^2$

$\Rightarrow \frac{z}{z_1} + \frac{\bar{z}}{\bar{z}_1} = 2$.

100. Using $|z_1| + |z_2| \geq |z_1 - z_2|$, we get

$|z - 2 + 3i| + |z - 1 + i| \geq |-1 + 2i| = \sqrt{5}$

101. $|z - a^2| + |z - 2a| = 3$ will represent an ellipse if

$|a^2 - 2a| < 3$

$\Leftrightarrow -3 < a^2 - 2a < 3$

$\Leftrightarrow -2 < (a - 1)^2 < 4$

$\Leftrightarrow (a - 1)^2 < 4 \Leftrightarrow -1 < a < 3 \Leftrightarrow a \in (0, 3)$

102. As $AB = BC = CA$, we get

$2|z| = |1| = |1 - 2z|$

$\Rightarrow |z| = \frac{1}{2}$ and $|z - \frac{1}{2}| = \frac{1}{2}$

$\Rightarrow z$ is the point of intersection of circles

$|z| = 1/2$ and $|z - 1/2| = 1/2$

$\Rightarrow z = \frac{1}{4}\left(1 \pm \sqrt{3}i\right)$

103. $|z + 1| + |z - 3| \leq 10$ represents the ellipse with focii at (–1, 0), and (3, 0) and length of major axis 10. Its centre is (1, 0), and its equation is

$$\frac{(x-1)^2}{25} + \frac{y^2}{21} = 1$$

Any point on the ellipse is $P(1 + 5\cos\theta, \sqrt{21}\sin\theta)$.

Its distance from A (7, 0) is given by

$AP^2 = (5\cos\theta + 8)^2 + 21\sin^2\theta$

$= (2\cos\theta + 20)^2 - 20^2 + 85$

$\Rightarrow 18^2 - 20^2 + 85 \leq AP^2 \leq 22^2 - 20^2 + 85$

$\Rightarrow 9 \leq AP^2 \leq 169$

$\Rightarrow 3 \leq AP \leq 13$

104. $|z - 1|^2 + 2|z - w|^2 = 3|z - w^2|^2$

$\Rightarrow |z|^2 + 1 - z - \bar{z} + 2[|z|^2 + 1 - z\bar{w} - \bar{z}w]$

$= 3[|z|^2 + 1 - z\bar{w}^2 - \bar{z}w^2]$

$\Rightarrow (3w - 2w^2 - 1)z + (3w^2 - 2w - 1)\bar{z} = 0$

which represents a straight line.

105. As R.H.S is real, L.H.S must be real.

Also, $\left(\frac{1+i}{1-i}\right) = \left(\frac{i - i^2}{1-i}\right)^n = i^n$ is real when n is even.

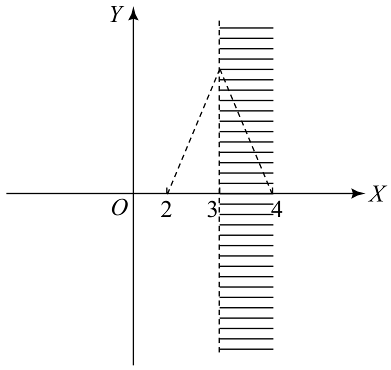
As $x > 0$ and $\frac{1+x^2}{2x} = \frac{1}{2}\left(x + \frac{1}{x}\right) > 1$ for $x \neq 1$.

Thus, we get only possible value of x is 1.

$\therefore$ RHS = 1, thus, least value of n is 4.

Previous Years' AIEEE/JEE Main Questions

1. Let $|z| = |w| = r$ and Arg $(w) = \theta$,
   so that Arg $(z) = \pi - \theta$. We have
   $$z = r[\cos(\pi - \theta) + i\sin(\pi - \theta)]$$
   $$= r[-\cos\theta + i\sin\theta]$$
   $$= -r(\cos\theta - i\sin\theta) = -\bar{w}$$

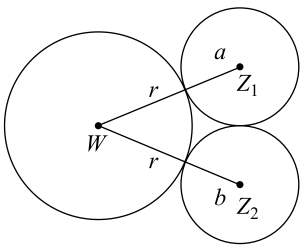
2. 

   If z satisfies $|z - 4| = |z - 2|$, then z lies on the perpendicular bisector of the segment joining $z = 2$ and $z = 4$.
   i.e., $|z - 4| = |z - 2| \Rightarrow \text{Re}(z) = 3$.

   As $z = 0$ does not satisfy $|z - 4| < |z - 2|$, we get $|z - 4| < |z - 2|$ represents the region $\text{Re}(z) > 3$.

3. Suppose $|z - w| = r$ touches $|z - z_1| = a$ and $|z - z_2| = b$ externally.

   Then $|w - z_1| = a + r$, $|w - z_2| = b + r$
   $\Rightarrow |w - z_1| - |w - z_2| = a - b$
   $\Rightarrow$ w lies on a hyperbola with foci at z_1 and z_2



4. As $1 = -i^2$,
   $$1 = \left(\frac{1+i}{1-i}\right)^x = \left(\frac{-i^2+i}{1-i}\right)^x = \left(\frac{i(1-i)}{1-i}\right)^x = i^x$$
   $\Rightarrow x = 4n$ for some $n \in \mathbf{N}$.

5. $|\bar{z}w| = |\bar{z}||w| = |z||w| = |zw| = 1$
   $$\text{Arg}(\bar{z}w) = \arg(w) + \arg(\bar{z})$$
   $$= \arg(w) - \arg(z) = -\pi/2$$
   $$\therefore\ \bar{z}w = |\bar{z}w|[\cos(-\pi/2) + i\sin(-\pi/2)] = -i$$

6. $z_1 + z_2 = -a$, $z_1 z_2 = b$
   As 0, z_1, z_2 for an equilateral triangle,
   $$0^2 + z_1^2 + z_2^2 = 0(z_1) + 0(z_2) + z_1 z_2$$
   $$\Rightarrow\ (z_1 + z_2)^2 = 3z_1 z_2 \Rightarrow a^2 = 3b.$$

7. $\bar{z} + i\bar{w} = 0 \Rightarrow \overline{\bar{z} + i\bar{w}} = \bar{0} \Rightarrow z - iw = 0$
   Now, $\arg(zw) = \pi$
   $$\Rightarrow\ \arg\left(\frac{z^2}{i}\right) = \pi$$
   $$\Rightarrow \arg(z^2) - \arg(i) = \pi$$
   $$\Rightarrow\ 2\arg(z) - \pi/2 = \pi$$
   $$\Rightarrow\ \arg(z) = 3\pi/4$$

8. $z^{1/3} = p + iq$
   $$\Rightarrow\ x - iy = (p + iq)^3$$
   $$\Rightarrow\ = p^3 + 3p^2(iq) + 3p(iq)^2 + (iq)^3$$
   $$\Rightarrow\ x = p^3 - 3pq^2 \text{ and } -y = 3p^2q - q^3$$
   $$\Rightarrow\ \frac{x}{p} = p^2 - 3q^2 \text{ and } \frac{-y}{q} = 3p^2 - q^2$$
   $$\Rightarrow\ \frac{x}{p} + \frac{y}{q} = -2(p^2 + q^2)$$
   $$\therefore\ \left(\frac{x}{p} + \frac{y}{q}\right)\Big/\left(p^2 + q^2\right) = -2$$

9. $|z^2 - 1| = |z|^2 + 1$ can be written as $|z^2 + (-1)| = |z^2| + |-1|$

   $\Leftrightarrow \dfrac{z^2}{-1}$ is a non-negative real number.
   $\Leftrightarrow z^2$ is a non-positive real number.
   $\Leftrightarrow z$ lies on the imaginary axis.

   Alternative solution

   Let $z = x + iy$ then $|z^2 - 1| = |z|^2 + 1$, we get
   $|(x^2 - y^2 - 1) + 2ixy| = x^2 + y^2 + 1$
   $$\Rightarrow\ \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2} = x^2 + y^2 + 1$$
   $$\Rightarrow\ (x^2 - y^2)^2 + 1 - 2(x^2 - y^2) + 4x^2y^2 = (x^2 + y^2 + 1)^2$$
   $$\Rightarrow\ (x^2 + y^2)^2 + 1 - 2(x^2 - y^2) = (x^2 + y^2)^2 + 1 + 2(x^2 + y^2)$$
   $$\Rightarrow\ -4x^2 = 0 \Rightarrow x = 0$$
   $\therefore$ z lies on the imaginary axis.

10. $(x - 1)^3 = -8$
    $$\Rightarrow\ \left(-\frac{x-1}{2}\right)^3 = 1$$
    $$\Rightarrow\ -\frac{x-1}{2} = 1, \omega, \omega^2$$

$\Rightarrow \quad x = -1,\ 1 - 2\omega,\ 1 - 2\omega^2$

11. $|w| = 1 \Rightarrow |z| = \left|z - \frac{1}{3}i\right|$

$\Rightarrow z$ is equidistant from $z = 0$ and $z = \frac{1}{3}i$. Thus, z lies on the perpendicular bisector of the segment joining $z = 0$ and $z = i/3$. Therefore, z lies on a straight line.

12. $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow$ z_1, z_2 lie on a ray through the origin O and same side of the origin

$\Leftrightarrow$ $\arg(z_1) = \arg(z_2)$

13. $\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}$

$= i\left(\cos\frac{2k\pi}{11} - i\sin\frac{2k\pi}{11}\right)$

$= i\omega^k$

where $\omega = \cos\frac{2\pi}{11} - i\sin\frac{2\pi}{11}$

Thus, $S = \sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$

$= i\sum_{k=1}^{10}\omega^k = i\left(\frac{\omega - \omega^{11}}{1-\omega}\right)$

But $\omega^{11} = \cos 2\pi - i\sin 2\pi = 1$

$\therefore \quad S = -i$

14. $z^2 + z + 1 = 0 \quad \Rightarrow z = \omega$ or ω^2

where ω is complex cube root of unity.

Let $z = \omega$, so that $\frac{1}{z} = \omega^2$

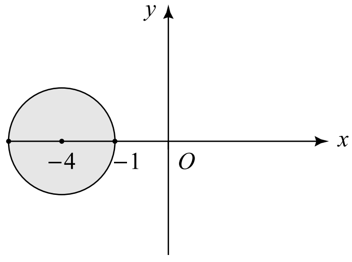
Thus,

$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \cdots + \left(z^6 + \frac{1}{z^6}\right)^2$

$= (-1)^2 + (-1)^2 + (2)^2 + (-1)^2 + (-1)^2 + (2)^2$

$= 12.$

15. $|z + 4| \le 3$ represents the interior and boundary of the circle with centre at $(-4, 0)$ and radius $= 3$. See Figure.



As -1 is an end point of a diameter of the circle, maximum possible value of $|z + 1|$ is 6 which is attained when $z = -7$

16. $\bar{z} = \frac{1}{i-1} \quad \Rightarrow z = \frac{1}{\bar{i} - 1} = \frac{1}{-i-1} = \frac{-1}{i+1}$

17. $|z| - \frac{4}{|z|} \le \left||z| - \frac{4}{|z|}\right| \le \left|z - \frac{4}{z}\right| = 2$

$\Rightarrow \quad |z|^2 - 2|z| - 4 \le 0$

$\Rightarrow \quad (|z| - 1)^2 \le 5 \quad \Rightarrow \quad |z| \le \sqrt{5} + 1$

18. z is equidistant from $A(1 + 0i)$, $B(-1 + 0i)$ and $C(0 + i)$. Thus, z is circumcentre of ΔABC, that is, there is exaclty one such z.

19. $(1 + \omega)^7 = (-\omega^2)^7 = (-\omega^2)^6(-\omega^2) = -\omega^2$

$= 1 + \omega$

$\therefore A = 1, B = 1$

20. As $\alpha, \beta \in \mathbf{R}$, the roots of $z^2 + \alpha z + \beta = 0$ are of the form

$1 + ia,\ 1 - ia$, where $a \in \mathbf{R}$, $a \ne 0$.

Now, $\beta = (1 + ia)(1 - ia) = 1 + a^2 > 1$

$\Rightarrow \beta \in (1, \infty)$

21. As $\frac{z^2}{z-1}$ is real, we get

$\frac{z^2}{z-1} = \frac{\bar{z}^2}{\bar{z}-1}$

$\Leftrightarrow \quad z^2(\bar{z} - 1) = \bar{z}^2(z - 1)$

$\Leftrightarrow \quad z\bar{z}(z - \bar{z}) - (z - \bar{z})(z + \bar{z}) = 0$

$\Leftrightarrow \quad (z - \bar{z})(z\bar{z} - z - \bar{z}) = 0$

$\Leftrightarrow \quad z - \bar{z} = 0$ or $z\bar{z} - z - \bar{z} = 0$

$\Rightarrow$ z lies on the real axis

or z lies on a circle through the origin.

22. $|z| = 1 \quad \Rightarrow \quad z\bar{z} = 1$

$\arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+1/z}\right)$

$= \arg(z) = \theta$

23. Let $z_2/z_1 = ik$ where k is a real number.

$\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right| = \left|\frac{2/3 + z_2/z_1}{2/3 - z_2/z_1}\right| = \left|\frac{2/3 + ik}{2/3 - ik}\right| = 1$

24. Let $\omega = \frac{1+z^2}{2iz}$. $\operatorname{Im}(\omega) = \frac{1}{2}(\omega - \bar{\omega})$

For $|z| = 1$, $\bar{\omega} = \frac{1+\bar{z}^2}{-2i\bar{z}} = \frac{1+(1/z)^2}{-2i(1/z)}$

$= \frac{z^2+1}{-2iz} = -\omega$

Thus, Im(ω) = $\frac{1}{2i}(\omega+\omega)=-i\omega$

$= -\frac{1}{2}\left(z+\frac{1}{z}\right)=-\frac{1}{2}(z+\bar{z})$

$=-\frac{1}{2}(2\cos\theta)=-\cos\theta$

[$\because$ $|z| = 1 \Rightarrow z = \cos\theta + i\sin\theta$]

As $z \neq 1$, $\theta \neq 0$ and as $z \neq -1$, $\theta \neq \pi$

$\therefore$ $a \in (-1, 1)$

25. $z=1-\bar{z} \Rightarrow z+\bar{z}=1$

$\Rightarrow 2\,\text{Re}(z) = 1 \Rightarrow \text{Re}(z) = \frac{1}{2}=\cos\frac{\pi}{3}$

As $|z| = 1$, Re(z) = cos(π/3),

Im (z) = ± sin (π/3)

Thus, arg (z) = ± π/3

$\therefore$ statement-1 is false and statement-2 is true .

26. Im(z) + 1 = 0 Im(z) = –1

Let $z = a - i$

Now, $z+\sqrt{2}\,|z+1|+i=0$

$\Rightarrow a-i+\sqrt{2}\sqrt{(a+1)^2+1}+i=0$

$\Rightarrow a^2 = 2(a^2 + 2a + 2)$

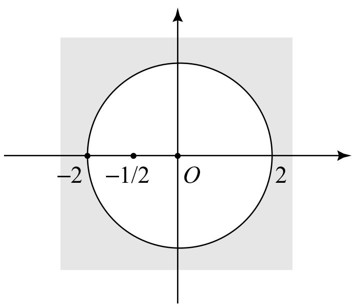
$\Rightarrow a^2 + 4a + 4 = 0 \Rightarrow (a + 2)^2 = 0$

$\therefore$ $a = -2$

Thus, $z = -2 - i \Rightarrow |z| = \sqrt{5}$

27. $\left|z+\frac{1}{2}\right| \geq |z|-\frac{1}{2} \geq \frac{3}{2}$

Minimum value 3/2 of $|z + 1/2|$ is attained when $z = -2$, and $3/2 \in (1, 2)$.



28. $w-\bar{w}z = k(1 - z)$

$\Rightarrow$ $w - k = z(\bar{w}-k)$

$\Rightarrow$ $|w - k| = |z||\bar{w}-k|$

$= |z||w - k|$

As Im (w) ≠ 0, $|w - k| \neq 0$

$\therefore$ $|z| = 1$

Also, $z \neq 1$ for otherwise $w=\bar{w}$

$\Rightarrow$ Im(w) = 0

29. $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$

$= \arg\left(\frac{z_1z_2}{z_4z_3}\right) = \arg\left(\frac{|z_1|^2}{|z_3|^2}\right) = 0$

30. Let $\frac{z-i}{z+i}=ik$, where $k \in$ **R.**

$\Rightarrow$ $z - i = ikz - k$

$\Rightarrow$ $z(1- ik) = -k + i$

$\Rightarrow$ $z = \frac{-k+i}{1-ik}$

Note that $|z|^2=\frac{k^2+1}{1+k^2}=1$

$\Rightarrow$ $z\bar{z}=1 \Rightarrow \bar{z}=1/z$

Thus, $z+\frac{1}{z}=z+\bar{z}$, which is a real number.

Also, $z+\bar{z}=0$

$\Rightarrow$ 2Re(z) = 0 $\Rightarrow$ Re(z) = 0

$\Rightarrow$ $z = ai$ for some $a \in$ **R.**

But in this case

$\frac{z-i}{z+i}$ is a real number

Therefore, $z+\bar{z}\neq 0$.

31. $(1+\alpha i)^2 = x + iy$

$\Rightarrow$ $1 - \alpha^2 + 2\alpha i = x + iy$

$\Rightarrow$ $1 - \alpha^2 = x$, $2\alpha = y$

$\Rightarrow$ $1-(y/2)^2 = x$

$\Rightarrow$ $4 - y^2 = 4x$

$\Rightarrow$ $y^2 + 4x - 4 = 0$

32. $\left|\frac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$

$\Leftrightarrow |z_1-2z_2|^2=|2-z_1\bar{z}_2|^2$

$\Leftrightarrow |z_1|^2+4|z_2|^2-2\bar{z}_1z_2-2z_1\bar{z}_2$

$= 4 + |z_1|^2|\bar{z}_2|^2-2z_1\bar{z}_2-2\bar{z}_1z_2$

$\Leftrightarrow |z_1|^2|z_2|^2 - |z_1|^2 - 4|z_2|^2 + 4 = 0$

$\Leftrightarrow (|z_1|^2 - 4)(|z_1|^2 - 1) = 0$

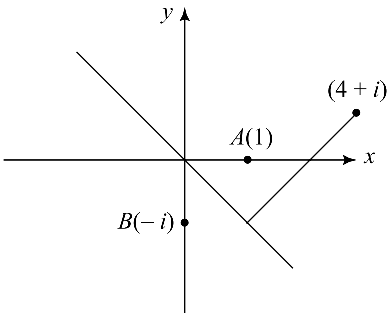
As $|z_2| \neq 1$, $|z_1|^2 - 4 = 0 \Rightarrow |z_1| = 2$

$\Rightarrow z_1$ lies on a circle of radius 2.

33. $|z - 1| = |z + i|$

$\Rightarrow$ z is equidistant from $A(1)$ and $B(-i)$

$\Rightarrow$ z lies on the line $y = -x$



As $z = 1$ satisfies the inequality

$|z - 1| \le |z + i|$,

we get $|z - 1| \le |z + i|$ represents the region lying above the line $x + y = 0$

Largest value of r is length of perpendicular from (4, 1) to the line $x + y = 0$, that is,

largest $r = \frac{|4+1|}{\sqrt{1+1}} = \frac{5}{\sqrt{2}} = \frac{5}{2}\sqrt{2}$.

34. Let $z = r(\cos\theta + i\sin\theta)$, so that

$\Rightarrow$ As z is non-real complex number, $\text{Im}(z) \ne 0$

$z^5 = r^5 [\cos(5\theta) + i\sin(5\theta)]$

$\Rightarrow \quad (\text{Im}(z))^5 = r^5 \cos^5\theta$

and $\quad \text{Im}(z^5) = r^5 \cos(5\theta)$

Thus, $\frac{\text{Im}(z^5)}{(\text{Im}(z))^5} = \frac{\cos(5\theta)}{\cos^5\theta}$

But $\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$$\Rightarrow \frac{\cos(5\theta)}{\cos^5\theta} = 16 - 20\sec^2\theta + 5\sec^4\theta$$

$$= 5(\sec^4\theta - 4\sec^2\theta + 4) - 4$$

$$= 5(\sec^2\theta - 2)^2 - 4 \ge -4$$

Thus, least value of

$\frac{\cos(5\theta)}{\cos^5\theta}$ is -4 which is attained when $\theta = \pi/4$.

35. As $z = \frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, $\text{Re}(z) = 0$

that is, $z + \bar{z} = 0$

$$\Rightarrow \frac{2+3i\sin\theta}{1-2i\sin\theta} + \frac{2-3i\sin\theta}{1+2i\sin\theta} = 0$$

$\Rightarrow (2 + 3i\sin\theta)(1 + 2i\sin\theta) + (2 - 3i\sin\theta)(1 - 2i\sin\theta)$

$= 0$

$\Rightarrow 2(2 - 6\sin^2\theta) = 0 \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$

Thus, a value of θ for which z is purely imaginary is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

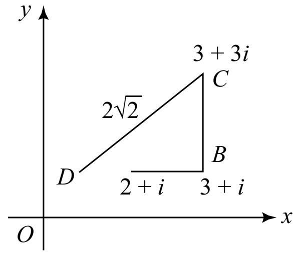
36. Refer figure

Coordinates of A are (2, 1) that of B are (3, 1) and C are (3, 3).

Coordinates of D are $(3 + 2\sqrt{2}\cos(5\pi/4), 3 + 2\sqrt{2}\sin(5\pi/4))$

$= (3 - 2, 3 - 2) = (1, 1)$

Thus, D is represented by $1 + i$



37. $z^3 = (1 + ai)^3$

$= 1 + 3ai - 3a^2 - ia^3$

As z^3 is real $3a - a^3 = 0$

$\Rightarrow a(\sqrt{3} - a)(a + \sqrt{3}) = 0$

$\Rightarrow a = \sqrt{3}$

Thus, $z = 1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

We have $S = 1 + z + z^2 + \ldots + z^{11}$

$$= \frac{1-z^{12}}{1-z}$$

But $z^{12} = 2^{12}(\cos(4\pi) + i\sin(4\pi))$

$= 2^{12}(1 + i0) = 2^{12}$

and $1 - z = -\sqrt{3}i$

Thus, $S = \frac{1-2^{12}}{-\sqrt{3}i}$

$$= \frac{4095}{\sqrt{3}}i = 1365\sqrt{3}i$$

Previous Years' B-Architecture Entrance Examination Questions

1. $\frac{5+i\sin\theta}{5-3i\sin\theta}$ is a real number

$$\Leftrightarrow \frac{5+i\sin\theta}{5-3i\sin\theta} = \frac{5-i\sin\theta}{5+3i\sin\theta}$$

$\Leftrightarrow \quad (5 + i\sin\theta)(5 + 3i\sin\theta)$

$= (5 - 3\, i\sin\theta)(5 - i\, \sin\theta)$

$\Leftrightarrow \quad 25 + 20\, i\sin\theta - 3\sin^2\theta$

$= 25 - 20\, i\sin\theta - 3\sin^2\theta$

$\Leftrightarrow 40\, i\sin\, \theta = 0 \Leftrightarrow \sin\, \theta = 0$

This is possible when $\theta = -\pi$.

TIP Amongst the choices given, $-\pi$ is the only value which makes imaginary part of the numerator and denominator 0.

2. $|z| = 2$

Let $w = 3z + 2 + i$, then $z = (w - (2 + i))/3$

$|z| = 2 \quad \Rightarrow \quad |w - (2 + i)| = 6$

$\Rightarrow$ w lies on a circle with centre at $(2 + i)$ and radius 6.

3. Let $z = 2(\cos\, \theta + i\, \sin\, \theta)$. Now

$z + \frac{2}{z} = 2(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$

$= 3\, \cos\, \theta + i\sin\, \theta$

$\Rightarrow \quad \left|z + \frac{2}{z}\right|^2 = 9\, \cos^2\theta + \sin^2\theta$

$= 1 + 4(1 + \cos\, 2\theta) \le 9$

$\therefore$ maximum possible value of $\left|z + \frac{2}{z}\right|$ is 3 which is attained when $z = 1$

4. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ then $\omega^3 = 1$

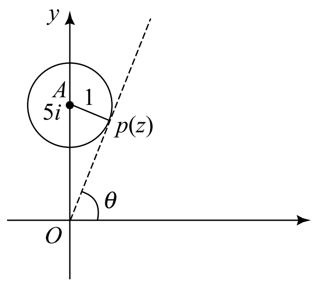
$z = 4 + 3\omega^{127} + 5\omega^{124}$

$= 4 + 3(\omega^3)^{42}\omega + 5(\omega^3)^{41}\omega$

$= 4 + 3\omega + 5\omega = 4 + 8\omega$

$= 4 - 4 + 4\sqrt{3}i = 4\sqrt{3}i$

5. Principal argument is least at point P.



$\sin(\pi/2 - \theta) = \frac{AP}{OA}$

$\Rightarrow \quad \cos\, \theta = 1/5$

$\Rightarrow \quad \sin\theta = \sqrt{24}/5$

Now, $z = \sqrt{24}(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \text{Re}(z) = \frac{\sqrt{24}}{5} = \frac{2}{5}\sqrt{6}$

6. Area of triangle is

$$\Delta = \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ z + iz & \bar{z} - i\bar{z} & 1 \end{vmatrix}\right|$$

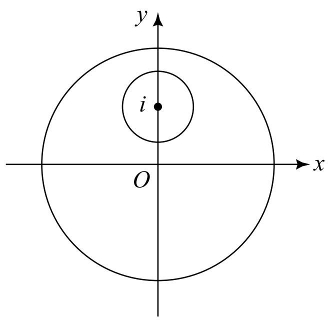
Using $R_3 \to R_3 - R_1 - R_2$, we get

$$\Delta = \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ 0 & 0 & -1 \end{vmatrix}\right|$$

$$= \frac{1}{4}|-iz\bar{z} - iz\bar{z}| = \frac{1}{2}|z|^2$$

7. As $|i - (1 + 2i)| = \sqrt{2} < |\, 4 - 2\, | = 2$,

C_1 lies inside C_2



8. $z = i\left(i + \sqrt{2}\right) = -1 + \sqrt{2}i$

$\Rightarrow \quad (z + 1)^2 = -2 \quad \Rightarrow z^2 + 2z + 3 = 0$

We now divide

$z^4 + 4\, z^3 + 6\, z^{\,2} + 4\, z$ by $z^2 + 2z + 3$.

$z^2 + 2z + 3 \;\overline{)\; z^4 + 4z^3 + 6z^2 + 4z \;}\; z^2 + 2z - 1$

$\quad z^4 + 2z^3 + 3z^2$

$\ominus \quad \ominus \quad \ominus$

$\quad 2z^3 + 3z^2 + 4z$

$\quad 2z^3 + 4z^2 + 6z$

$\ominus \quad \ominus \quad \ominus$

$\quad -z^2 - 2z$

$\quad -z^2 - 2z - 3$

$\oplus \quad \oplus \quad \oplus$

$\quad 3$

$\therefore \quad z^4 + 4z^3 + 6z^2 + 4z$

$= (z^2 + 2z + 3)\,(z^2 + 2z - 1) + 3$

$= 0 + 3 = 3$

9. $|z| = 1$, $\arg(z) = \theta$

$\Rightarrow z = \cos\, \theta + i\sin\, \theta$

Now, $w = \frac{z(1 - \bar{z})}{\bar{z}(1 + z)} = \frac{z - z\bar{z}}{z + z\bar{z}}$

$$= \frac{z-1}{\bar{z}+1}$$

$$\bar{w} = \frac{\bar{z}-1}{z+1}$$

Now, $\quad 2\text{Re}(w) = w + \bar{w} = \frac{z-1}{\bar{z}+1} + \frac{\bar{z}-1}{z+1}$

$$= \frac{z^2 - 1 + \bar{z}^2 - 1}{(z+1)(\bar{z}+1)}$$

$$= \frac{2\cos 2\theta - 2}{z\bar{z} + z + \bar{z} + 1}$$

$$= \frac{2(\cos 2\theta - 1)}{2(\cos\theta + 1)}$$

$$= \frac{-2\sin^2(\theta)}{2\cos^2(\theta/2)} = -4\sin^2\left(\frac{\theta}{2}\right)$$

$\Rightarrow \quad \text{Re}(w) = -2\sin^2\left(\frac{\theta}{2}\right)$

10. $\left|\sqrt{2}+i\right|^2 = |z_1 + z_2 + z_3|^2$

$\Rightarrow 3 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\text{Re}\,(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1)$

$\Rightarrow \quad 3 = 1 + 1 + 1 + 2\ \text{Re}\,(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1)$

$\Rightarrow \text{Re}\,(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1) = 0$

11. $z\ (iz_1 - 1) = z_1 + 1$

$\Rightarrow z_1(iz - 1) = 1 + z$

$\Rightarrow z_1 = \frac{1+z}{iz-1}$

$\Rightarrow \left|\frac{1+z}{iz-1}\right| = |z_1| < 1$

$\Rightarrow |1 + z|^2 < |1 - iz|^2$

$\Rightarrow 1 + z + \bar{z} + z\bar{z} < 1 - iz + i\bar{z} + z\bar{z}$

$\Rightarrow 2\ \text{Re}(z) < -\ i(2i\ \text{Im}(z))$

$\Rightarrow \text{Re}(z) - \text{Im}(z) < 0$