TRIGONOMETRY

1. INTRODUCTION TO TRIGONOMETRY :

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

(a) **Measurement of angles :** Commonly two systems of measurement of angles are used.

(i) **Sexagesimal or English System :** Here 1 right angle = 90° (degrees)

1° = 60' (minutes)

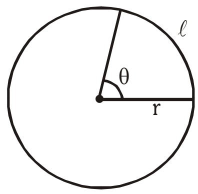
1' = 60" (seconds)

(ii) **Circular system :** Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

(b) Relation between the these systems : $\frac{\pi}{2}$ radian = 90°

(c) If θ is the angle subtended at the centre of a circle of radius 'r',

by an arc of length 'ℓ' then $\frac{\ell}{r} = \theta$.



Note that here ℓ, r are in the same units and θ is always in radians.

SOLVED EXAMPLE

Example 1 : If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.

Solution : Let r_1 and r_2 be the radii of the given circles and let their arcs of same length 's' subtend angles of 60° and 75° at their centres.

Now, $60° = \left(60 \times \frac{\pi}{180}\right) = \left(\frac{\pi}{3}\right)$ and $75° = \left(75 \times \frac{\pi}{180}\right) = \left(\frac{5\pi}{12}\right)$

$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1}$ and $\frac{5\pi}{12} = \frac{s}{r_2}$

$\Rightarrow \quad \frac{\pi}{3} r_1 = s$ and $\frac{5\pi}{12} r_2 = s \Rightarrow \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow$ $r_1 : r_2 = 5 : 4$ **Ans.**

Problems for Self Practise - 01 :

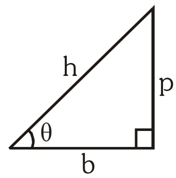
(1) The radius of a circle is 30 cm. Find the length of an arc of this circle if the length of the chord of the arc is 30 cm.

Answers : (1) 10π cm

2. T-RATIOS (OR TRIGONOMETRIC RATIOS) :

In a right angle triangle

$\sin\theta = \frac{p}{h}$; $\cos\theta = \frac{b}{h}$; $\tan\theta = \frac{p}{b}$; $\text{cosec}\,\theta = \frac{h}{p}$; $\sec\theta = \frac{h}{b}$ and $\cot\theta = \frac{b}{p}$



'p' is perpendicular ; 'b' is base and 'h' is hypotenuse.

Note : The quantity by which the cosine falls short of unity i.e. $1 - \cos\theta$, is called the versed sine θ of θ and also by which the sine falls short of unity i.e. $1 - \sin\theta$ is called the coversed sine of θ.

3. BASIC TRIGONOMETRIC IDENTITIES :

(a) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(b) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$ or $\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

or $\text{cosec}\theta + \cot\theta = \frac{1}{\text{cosec}\theta - \cot\theta}$

SOLVED EXAMPLE

Example 2 : If $\sin\theta + \sin^2\theta = 1$, then prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$

Solution : Given that $\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

L.H.S. $= \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1 = (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$

Example 3 : $4(\sin^6\theta + \cos^6\theta) - 6(\sin^4\theta + \cos^4\theta)$ is equal to

(A) 0 (B) 1 (C) –2 (D) none of these

Solution : $4[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 6[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$

$= 4[1 - 3\sin^2\theta\cos^2\theta] - 6[1 - 2\sin^2\theta\cos^2\theta]$

$= 4 - 12\sin^2\theta\cos^2\theta - 6 + 12\sin^2\theta\cos^2\theta = -2$ **Ans.(C)**

Problems for Self Practise - 02 :

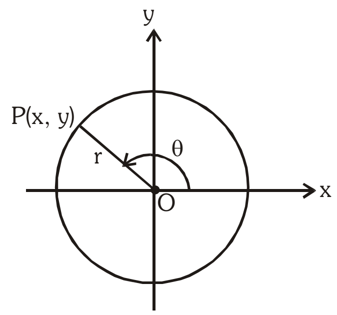
(1) If $\cot\theta = \frac{4}{3}$, then find the value of $\sin\theta$, $\cos\theta$ and $\text{cosec}\theta$ in first quadrant.

(2) If $\sin\theta + \text{cosec}\theta = 2$, then find the value of $\sin^8\theta + \text{cosec}^8\theta$

Answers : (1) $\frac{3}{5}, \frac{4}{5}, \frac{5}{3}$ (2) 2

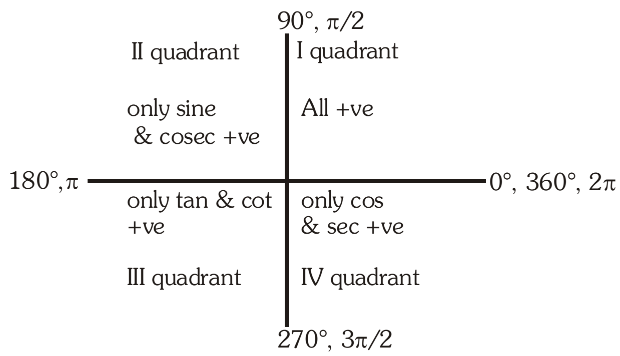
4. NEW DEFINITION OF T-RATIOS :

By using rectangular coordinates the definitions of trigonometric functions can be extended to angles of any size in the following way (see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle θ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms of r and the coordinates x and y.



$\sin\theta = y/r$,
$\cos\theta = x/r$
$\tan\theta = y/x$,
(The other function are reciprocals of these)
This can give negative values of the trigonometric functions.

5. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :



6. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

If θ is any angle, then $-\theta$, $\frac{\pi}{2} \pm \theta$, $\pi \pm \theta$, $\frac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$ etc. are called allied angles.

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

- **Trigonometric Ratios of** $(-\theta)$:
  $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$, $\cot(-\theta) = -\cot\theta$,
  $\sec(-\theta) = \sec\theta$, $\text{cosec}(-\theta) = -\text{cosec}\,\theta$.
- **Trigonometric Ratios of** $(\pi - \theta)$ **:**
  $\sin(\pi-\theta) = \sin\theta$, $\cos(\pi-\theta) = -\cos\theta$, $\tan(\pi-\theta) = -\tan\theta$, $\cot(\pi-\theta) = -\cot\theta$,
  $\sec(\pi-\theta) = -\sec\theta$, $\text{cosec}(\pi-\theta) = \text{cosec}\,\theta$.
- **Trigonometric Ratios of** $\left(\frac{\pi}{2}-\theta\right)$ **:**

  $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$, $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$, $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$,

  $\sec\left(\frac{\pi}{2}-\theta\right) = \text{cosec}\,\theta$, $\text{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta$,

- **Trigonometric Ratios of $\left(\frac{\pi}{2}+\theta\right)$:**

$\sin\left(\frac{\pi}{2}+\theta\right)$ = cos θ, $\cos\left(\frac{\pi}{2}+\theta\right)$ = – sin θ, $\tan\left(\frac{\pi}{2}+\theta\right)$ = – cot θ, $\cot\left(\frac{\pi}{2}+\theta\right)$ = – tan θ,

$\sec\left(\frac{\pi}{2}+\theta\right)$ = – cosec θ, $\text{cosec}\left(\frac{\pi}{2}+\theta\right)$ = sec θ,

- **Trigonometric Ratios of (π + θ) :**
sin (π + θ) = – sin θ, cos (π + θ) = – cos θ, tan (π + θ) = tan θ, cot (π + θ) = cot θ,
sec (π + θ) = – sec θ, cosec (π + θ) = – cosec θ,

- **Trigonometric Ratios of $\left(\frac{3\pi}{2}-\theta\right)$:**

$\sin\left(\frac{3\pi}{2}-\theta\right)$ = – cos θ, $\cos\left(\frac{3\pi}{2}-\theta\right)$ = – sin θ, $\tan\left(\frac{3\pi}{2}-\theta\right)$ = cot θ, $\cot\left(\frac{3\pi}{2}-\theta\right)$ = tan θ,

$\sec\left(\frac{3\pi}{2}-\theta\right)$ = – cosec θ, $\text{cosec}\left(\frac{3\pi}{2}-\theta\right)$ = – sec θ

- **Trigonometric Ratios of $\left(\frac{3\pi}{2}+\theta\right)$**

$\sin\left(\frac{3\pi}{2}+\theta\right)$ = – cos θ, $\cos\left(\frac{3\pi}{2}+\theta\right)$ = sin θ, $\tan\left(\frac{3\pi}{2}+\theta\right)$ = – cot θ, $\cot\left(\frac{3\pi}{2}+\theta\right)$ = – tan θ,

$\sec\left(\frac{3\pi}{2}+\theta\right)$ = cosec θ, $\text{cosec}\left(\frac{3\pi}{2}+\theta\right)$ = – sec θ

7. VALUES OF T-RATIOS OF SOME STANDARD ANGLES :

Angles / T-ratio	0°	30°	45°	60°	90°	180°	270°
	0	π/6	π/4	π/3	π/2	π	3π/2
sin θ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1
cos θ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0
tan θ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.	0	N.D.
cot θ	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0	N.D.	0
sec θ	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.	–1	N.D.
cosecθ	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1	N.D.	–1

N.D. $\rightarrow$ Not Defined

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$; $\cos(2n+1)\frac{\pi}{2} = 0$ where $n \in I$

SOLVED EXAMPLE

Example 4 : $\cos(540° - \theta) - \sin(630° - \theta)$ is equal to

(A) 0 (B) $2\cos\theta$ (C) $2\sin\theta$ (D) $\sin\theta - \cos\theta$

Solution : $\cos(540° - \theta) - \sin(630° - \theta) = -\cos\theta + \cos\theta = 0$ **Ans. (A)**

Problems for Self Practise - 03 :

(1) If $\cos\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, then find the value of $4\tan^2\theta - 3\text{cosec}^2\theta$.

(2) Find value of $\cos 570° \sin 510° + \sin(-330°)\cos(-390°)$

Answers : (1) 8 (2) 0

8. TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES :

(a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

(b) $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

(c) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(d) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(e) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(f) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(g) $\cot(A + B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$

(h) $\cot(A - B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$

8.1 Some more results :

(a) $\sin^2 A - \sin^2 B = \sin(A + B).\sin(A - B) = \cos^2 B - \cos^2 A$.

(b) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A - B)$.

8.2 Trigonometric ratios of sum of more than two angles :

(a) $\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$
$= \Sigma \sin A \cos B \cos C - \Pi \sin A$
$= \cos A \cos B \cos C [\tan A + \tan B + \tan C - \tan A \tan B \tan C]$

(b) $\cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$
$= \Pi \cos A - \Sigma \sin A \sin B \cos C$
$= \cos A \cos B \cos C [1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$

(c) $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$

(d) $\tan(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) = \frac{S_1 - S_3 + S_5 - ......}{1 - S_2 + S_4 - ....}$

where S_i denotes sum of product of tangent of angles taken i at a time

SOLVED EXAMPLE

Example 5 : Prove that $\sqrt{3}$ cosec20° – sec20° = 4.

Solution : L.H.S. = $\frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°.\cos 20°}$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20°\cos 20°} = \frac{4(\sin 60.\cos 20° - \cos 60°.\sin 20°)}{\sin 40°}$$

$$= 4.\frac{\sin(60° - 20°)}{\sin 40°} = 4.\frac{\sin 40°}{\sin 40°} = 4 = \text{R.H.S.}$$

Example 6 : Prove that tan70° = cot70° + 2cot40°.

Solution : L.H.S. = $\tan 70° = \tan(20° + 50°) = \frac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$

or tan70° – tan20° tan50° tan70° = tan20° + tan50°
or tan70° = tan70° tan50° tan20° + tan20° + tan50° = 2 tan 50° + tan20°
= cot70° + 2cot40° = R.H.S.

Example 7 : Prove that sin (45º + A) cos (45º – B) + cos (45º + A) sin (45º – B) = cos (A – B)

Solution : Clearly sin (45º + A) cos (45º – B) + cos (45º + A) sin (45º – B)
= sin (45º + A + 45º – B) = sin (90º + A – B) = cos (A – B)

Example 8 : Show that : $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \left(\frac{1}{\sqrt{2}}\right)\sin A$

Solution : LHS = $\sin\left\{\frac{\pi}{8} + \frac{A}{2} + \frac{\pi}{8} - \frac{A}{2}\right\}\sin\left\{\frac{\pi}{8} + \frac{A}{2} - \frac{\pi}{8} + \frac{A}{2}\right\} = \sin\frac{\pi}{4}\,.\sin A = \frac{1}{\sqrt{2}}\sin A$ = RHS

<u>Problems for Self Practise</u> - 04 :

(1) If $\sin A = \frac{3}{5}$ and $\cos B = \frac{9}{41}$, $0 < A, B < \frac{\pi}{2}$, then find the value of the following :

(a) sin(A + B) (b) sin(A – B) (c) cos(A + B) (d) cos(A – B)

(2) If x + y = 45°, then prove that :

(a) (1 + tanx)(1 + tany) = 2 (b) (cotx – 1)(coty – 1) = 2

(Remember these results)

(3) Prove that $1 + \tan A \tan\frac{A}{2} = \tan A \cot\frac{A}{2} - 1 = \sec A$

Answers : (1) (a) $\frac{187}{205}$ (b) $\frac{-133}{205}$ (c) $\frac{-84}{205}$ (d) $\frac{156}{205}$

9. FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE :

(a) $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$.

(b) $2 \cos A \sin B = \sin (A+B) - \sin (A-B)$.

(c) $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

(d) $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$

10. FORMULAE TO TRANSFORM SUM OR DIFFERENCE INTO PRODUCT :

(a) $\sin C + \sin D = 2 \sin \left(\frac{C+D}{2}\right) \cos \left(\frac{C-D}{2}\right)$

(b) $\sin C - \sin D = 2 \cos \left(\frac{C+D}{2}\right) \sin \left(\frac{C-D}{2}\right)$

(c) $\cos C + \cos D = 2 \cos \left(\frac{C+D}{2}\right) \cos \left(\frac{C-D}{2}\right)$

(d) $\cos C - \cos D = 2 \sin \left(\frac{C+D}{2}\right) \sin \left(\frac{D-C}{2}\right)$

SOLVED EXAMPLE

Example 9 : Prove that $\sin 5A + \sin 3A = 2\sin 4A \cos A$

Solution : L.H.S. $\sin 5A + \sin 3A = 2\sin 4A \cos A$ = R.H.S.

$[\because \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}]$

Example 10 : Find the value of $2 \sin 3\theta \cos \theta - \sin 4\theta - \sin 2\theta$

Solution : $2 \sin 3\theta \cos \theta - \sin 4\theta - \sin 2\theta = 2 \sin 3\theta \cos \theta - [2 \sin 3\theta \cos \theta] = 0$

Example 11 : Prove that

(a) $\frac{\sin 8\theta \cos \theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos \theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

(b) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4 \cos 2\theta \cos 4\theta$

Solution : (a) $\frac{2\sin 8\theta \cos \theta - 2\sin 6\theta \cos 3\theta}{2\cos 2\theta \cos \theta - 2\sin 3\theta \sin 4\theta}$

$= \frac{\sin 9\theta + \sin 7\theta - \sin 9\theta - \sin 3\theta}{\cos 3\theta + \cos \theta - \cos \theta + \cos 7\theta} = \frac{2 \sin 2\theta \cos 5\theta}{2\cos 5\theta \cos 2\theta} = \tan 2\theta$

(b) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = \frac{\sin 5\theta \cos 3\theta + \sin 3\theta \cos 5\theta}{\sin 5\theta \cos 3\theta - \sin 3\theta \cos 5\theta} = \frac{\sin 8\theta}{\sin 2\theta} = 4 \cos 2\theta \cos 4\theta$

Problems for Self Practise - 05 :

(1) Prove that :

(a) $\frac{\sin A+\sin 2A}{\cos A-\cos 2A} = \cot\frac{A}{2}$ (b) $\frac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A} = \frac{\sin 3A}{\sin 5A}$

(2) Prove that :

(a) $\cos A \sin (B - C) + \cos B \sin (C - A) + \cos C \sin (A - B) = 0$

(b) $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$

11. TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES :

11.1 Trigonometric ratios of an angle 2θ in terms of the angle θ :

(a) $\sin 2\theta = 2 \sin \theta \cos \theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(b) $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(c) $1 + \cos 2\theta = 2 \cos^2 \theta$

(d) $1 - \cos 2\theta = 2 \sin^2 \theta$

(e) $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta}$

(f) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

11.2 Trigonometric ratios of an angle 3θ in terms of the angle θ :

(a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(c) $\tan 3\theta = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

11.3 Trigonometric ratios of sub multiple angles :

Since the trigonometric relations are true for all values of angle θ, they will be true if instead of θ be substitute $\frac{\theta}{2}$

(a) $\sin \theta = 2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(b) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(c) $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

(d) $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

(e) $\tan\frac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$

(f) $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(g) $\sin\frac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$

(h) $\cos\frac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$

(i) $\tan\frac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

11.4 Important results :

(a) $\sin\theta \sin(60° - \theta)\sin(60° + \theta) = \frac{1}{4}\sin 3\theta$

(b) $\cos\theta . \cos(60° - \theta)\cos(60° + \theta) = \frac{1}{4}\cos 3\theta$

(c) $\tan\theta \tan(60° - \theta)\tan(60° + \theta) = \tan 3\theta$

(d) (i) $\cot A - \tan A = 2\cot 2A$ (ii) $\cot A + \tan A = 2\,\mathrm{cosec}\,2A$

11.5 Trigonometric ratios of some standard angles :

(a) $\sin 18° = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \cos 72° = \cos\frac{2\pi}{5}$

(b) $\cos 36° = \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54° = \sin\frac{3\pi}{10}$

(c) $\sin 72° = \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \cos 18° = \cos\frac{\pi}{10}$

(d) $\sin 36° = \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54° = \cos\frac{3\pi}{10}$

(e) $\sin 15° = \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \cos\frac{5\pi}{12}$

(f) $\cos 15° = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \sin\frac{5\pi}{12}$

(g) $\tan 15° = \tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75° = \cot\frac{5\pi}{12}$

(h) $\tan 75° = \tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15° = \cot\frac{\pi}{12}$

(i) $\tan(22.5°) = \tan\frac{\pi}{8} = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8}$

(j) $\tan(67.5°) = \tan\frac{3\pi}{8} = \sqrt{2}+1 = \cot(22.5°) = \cot\frac{\pi}{8}$

SOLVED EXAMPLE

Example 12 : Prove that : $\frac{2\cos 2A+1}{2\cos 2A-1} = \tan(60°+A)\tan(60°-A)$.

Solution *:* R.H.S. = tan(60° + A) tan(60° – A)

$$=\left(\frac{\tan 60°+\tan A}{1-\tan 60°\tan A}\right)\left(\frac{\tan 60°-\tan A}{1+\tan 60°\tan A}\right)=\left(\frac{\sqrt{3}+\tan A}{1-\sqrt{3}\tan A}\right)\left(\frac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}\right)$$

$$=\frac{3-\tan^2 A}{1-3\tan^2 A}=\frac{3-\frac{\sin^2 A}{\cos^2 A}}{1-3\frac{\sin^2 A}{\cos^2 A}}=\frac{3\cos^2 A-\sin^2 A}{\cos^2 A-3\sin^2 A}==\frac{2\cos^2 A+\cos^2 A-2\sin^2 A+\sin^2 A}{2\cos^2 A-2\sin^2 A-\sin^2 A-\cos^2 A}$$

$$=\frac{2(\cos^2 A-\sin^2 A)+\cos^2 A+\sin^2 A}{2(\cos^2 A-\sin^2 A)-(\sin^2 A+\cos^2 A)}=\frac{2\cos 2A+1}{2\cos 2A-1}=\text{L.H.S.}$$

Example 13 : Prove that : tanA + tan(60° + A) + tan(120° + A) = 3tan3A

Solution : L.H.S. = tanA + tan(60° + A) + tan(120° + A)

= tanA + tan(60° + A) + tan{180° –(60° – A)}

= tanA + tan(60° + A) – tan(60° – A) [∵ tan(180° – θ) = –tanθ]

$$= \tan A + \frac{\tan 60° + \tan A}{1 - \tan 60° \tan A} - \frac{\tan 60° - \tan A}{1 + \tan 60° \tan A} = \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A + 3\tan A + \sqrt{3}\tan^2 A - \sqrt{3} + \tan A + 3\tan A - \sqrt{3}\tan^2 A}{(1 - \sqrt{3}\tan A)(1 + \sqrt{3}\tan A)}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{\tan A - 3\tan^3 A + 8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A} = 3\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = 3\tan 3A = \text{R.H.S.}$$

Example 14 : Show that sin12°.sin48°.sin54° = 1/8

Solution : L.H.S. = $\frac{1}{2}[\cos 36° - \cos 60°]\sin 54° = \frac{1}{2}\left[\cos 36° \sin 54° - \frac{1}{2}\sin 54°\right]$

$$= \frac{1}{4}[2\cos 36° \sin 54° - \sin 54°] = \frac{1}{4}[\sin 90° + \sin 18° - \sin 54°]$$

$$= \frac{1}{4}[1 - (\sin 54° - \sin 18°)] = \frac{1}{4}[1 - 2\sin 18° \cos 36°]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 18°}{\cos 18°}\cos 18° \cos 36°\right] = \frac{1}{4}\left[1 - \frac{\sin 36° \cos 36°}{\cos 18°}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 36° \cos 36°}{2\cos 18°}\right] = \frac{1}{4}\left[1 - \frac{\sin 72°}{2\sin 72°}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8} = \text{R.H.S.}$$

Problems for Self Practise - 06 :

(1) Prove that :

(a) $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$ (b) $\frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \cot\theta$

(2) Prove that :

(a) $\cot\theta \cot(60° - \theta) \cot(60° + \theta) = \cot 3\theta$

(b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(c) $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

(3) Prove that : cos20°cos40°cos60°cos80°= $\frac{1}{16}$

(4) If 3 cos x + 2 cos 3x = cos y, 3 sin x + 2 sin 3x = sin y, then the value of cos 2x is

(A) – 1 (B) $\frac{1}{8}$ (C) $-\frac{1}{8}$ (D) $\frac{7}{8}$

Answers : (4) (A)

12. CONDITIONAL TRIGONOMETRIC IDENTITIES :

If A + B + C = 180°, then

(a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(b) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(c) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

(d) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(e) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(f) $\cos 2A + \cos 2B + \cos 2C = -1-4 \cos A \cos B \cos C$

(g) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(h) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(i) (i) If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, then $A + B + C = n\pi, n \in I$

(ii) If $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

SOLVED EXAMPLE

Example 15 : For all values of α, β, γ prove that,

$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}.\cos\frac{\beta+\gamma}{2}.\cos\frac{\gamma+\alpha}{2}.$$

Solution : LHS $= \cos\alpha + \cos\beta + 2\cos\left(\frac{2\gamma+\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}+\gamma\right)\right]$

$= 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left\{\frac{\alpha-\beta+\alpha+\beta+2\gamma}{4}\right\}\cos\left\{\frac{\alpha-\beta-\alpha-\beta-2\gamma}{4}\right\}$

$= 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)$ = RHS

Example 16 : If $x + y + z = \frac{\pi}{2}$ show that, $\sin 2x + \sin 2y + \sin 2z = 4\cos x \cos y \cos z$.

Solution : LHS $= 2\sin\left(\frac{\pi}{2}-z\right)\cos(x-y) + 2\sin z\cos z$; $x + y = \left(\frac{\pi}{2}-z\right)$

$\Rightarrow$ $2\cos z\{\cos(x-y)+\cos(x+y)\}$ ($\because z = \frac{\pi}{2}-(x+y)$)

$= 2\cos z \times 2\cos x\cos y = 4\cos x\cos y\cos z$

Example 17 : If $x + y = \pi + z$, then prove that $\sin^2x + \sin^2y - \sin^2z = 2 \sin x \sin y \cos z$.

Solution : LHS = $\sin^2x + \sin (y + z) \sin(y - z) = \sin^2x + \sin(y+z) \sin(\pi - x) = \sin x [\sin(\pi - (y - z)) + \sin(y + z)]$
$= \sin x \cdot 2 \sin y \cos z = 2 \sin x \sin y \cos z$

Problems for Self Practise - 07 :

(1) If ABCD is a cyclic quadrilateral, then find the value of sinA + sinB – sinC – sinD

(2) If $A + B + C = \dfrac{\pi}{2}$, then find the value of tanA tanB + tanBtanC + tanC tanA

(3) If A + B + C = 180°, prove that

(a) $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B) = 4\sin\dfrac{B-C}{2}\sin\dfrac{C-A}{2}\sin\dfrac{A-B}{2}$

(b) $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

Answers : (1) 0 (2) 1

13. TRIGONOMETRIC SERIES:

13.1 Addition series

(a) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots\ldots + \sin\{\alpha + (n-1)\beta\} = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \dfrac{n-1}{2}\beta\right)$

(b) $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots. + \cos\{\alpha + (n-1)\beta\} = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \dfrac{n-1}{2}\beta\right)$

where : $\beta \neq 2m\pi,\ m \in I$

13.2 Product series of cosine angles

$\cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \ldots\ldots \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n \sin\theta}$

SOLVED EXAMPLE

Example 18 : Prove that :

(a) $\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7}\cos\dfrac{6\pi}{7} = \dfrac{1}{8}$

(b) If ϕ is the exterior angle of a regular polygon of n sides and θ is any constant, then prove that $\sin\theta + \sin(\theta + \phi) + \sin(\theta + 2\phi) + \ldots\ldots\ldots\ldots$ up to n terms = 0

Solution : (a) By using Series formulae $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots\ldots\ldots. \cos 2^{n-1}A = \dfrac{\sin 2^n A}{2^n \sin A}$

LHS $= \cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7}\cos\dfrac{6\pi}{7} = -\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7} = -\dfrac{\sin\frac{8\pi}{7}}{2^3 \cdot \sin\frac{\pi}{7}} = \dfrac{1}{8}$ = RHS

(b) LHS = $\sin\theta + \sin(\theta+\phi) + \sin(\theta+2\phi) + ... + \sin(\theta+(n-1)\phi)$

$$= \frac{\sin\frac{n\phi}{2}}{\sin\frac{\phi}{2}} \cdot \sin\left(\frac{2\theta+(n-1)\phi}{2}\right)$$

$\because$ $\phi = \frac{2\pi}{n}$ (External angle of regular polygon)

So LHS = $\frac{\sin\frac{n}{2}(2\pi/n)}{\sin(\pi/n)} \sin\left(\frac{2\theta+\frac{(n-1)2\pi}{n}}{2}\right) = 0$ = RHS

Problems for Self Practise - 08 :

Find the summation of the following series

(1) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

(2) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$

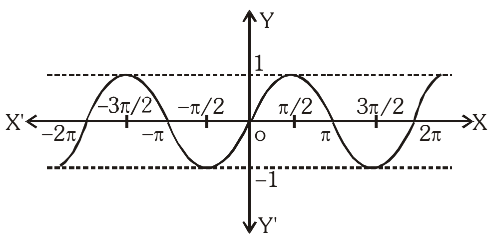
(3) $\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + ...$ up to n terms.

(4) $\sin2\alpha + \sin3\alpha + \sin4\alpha + ... + \sin n\alpha$, where $(n+2)\alpha = 2\pi$

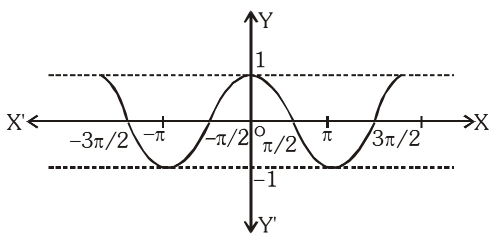
Answers : (1) $-\frac{1}{2}$ (2) $\frac{1}{2}$ (3) $\frac{1}{2}$ (4) 0

14. GRAPH OF TRIGONOMETRIC FUNCTIONS :

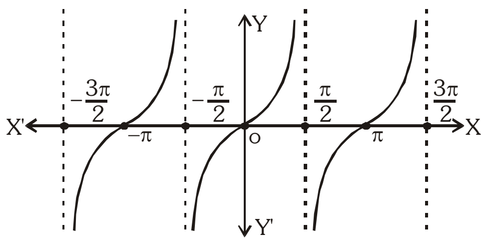
(i) y = sinx



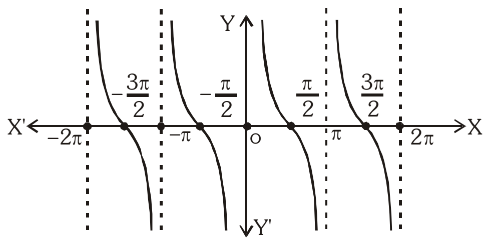
(ii) y = cosx



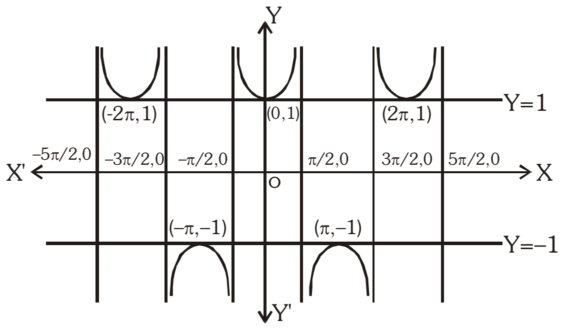
(iii) y = tanx



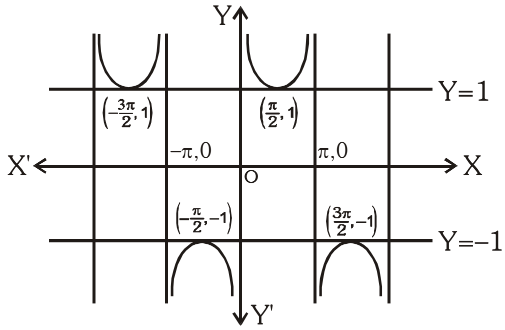
(iv) y = cotx



(v) **y = secx**



(vi) **y = cosecx**



15. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

T-Ratio	**Domain**	**Range**	**Period**
sin x	R	[–1,1]	2π
cos x	R	[–1,1]	2π
tan x	$R-\{(2n+1)\pi/2\ ;\ n\in I\}$	R	π
cot x	$R-\{n\pi : n \in I\}$	R	π
sec x	$R-\{(2n+1)\ \pi/2 : n \in I\}$	$(-\infty,-1] \cup [1,\infty)$	2π
cosec x	$R-\{n\pi : n \in I\}$	$(-\infty,-1] \cup [1,\infty)$	2π

16. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(a) $E = a \sin\theta + b\cos\theta$

$\Rightarrow$ $$E = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta\right\}$$

Let $\dfrac{b}{\sqrt{a^2+b^2}} = \sin\alpha$ & $\dfrac{a}{\sqrt{a^2+b^2}} = \cos\alpha$

$\Rightarrow$ $E = \sqrt{a^2+b^2}\ \sin(\theta+\alpha)$, where $\tan\alpha = \dfrac{b}{a}$

Hence for any real value of θ,

$$-\sqrt{a^2+b^2} \le E \le \sqrt{a^2+b^2}$$

Hence $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\,]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(b) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $a, b > 0$

(c) In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

SOLVED EXAMPLE

Example 19 : Find the maximum and minimum values of trigonometric functions cos 2x + $\cos^2$x

Solution : y = cos 2x + $\cos^2$x $\Rightarrow$ y = 3 $\cos^2$x – 1 $\Rightarrow$ $0 \le \cos^2 x \le 1$

y_{max} = 3 – 1 = 2 $\Rightarrow$ y_{min} = 0 – 1 = – 1

Example 20 : Prove that : $-4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$, for all values of θ.

Solution : We have, $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3} = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

Since, $-\sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow$ $-7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow$ $-7 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) \le 7$ for all θ.

$\Rightarrow$ $-7 + 3 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 7 + 3$ for all θ.

$\Rightarrow$ $-4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$ for all θ.

Problems for Self Practise - 09 :

(1) Find maximum and minimum value of $5\cos\theta + 3\sin\left(\theta + \frac{\pi}{6}\right)$ for all real values of θ.

(2) Find the minimum value of $\cos\theta + \cos2\theta$ for all real values of θ.

(3) Find maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

Answers : (1) 7 & –7 (2) $-\frac{9}{8}$ (3) $4 + \sqrt{10}$ & $4 - \sqrt{10}$

17. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation.

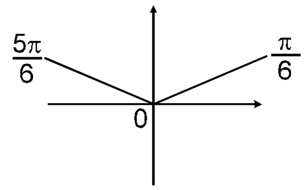
17.1 Solution of Trigonometric Equation :

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) Principal solution :- The solutions of a trigonometric equation which lie in the interval [0, 2π) are called Principal solutions.

e.g. Find the Principal solutions of the equation $\sin x = \frac{1}{2}$.

Solution :



$\because$ $\sin x = \frac{1}{2}$

$\because$ there exists two values

i.e. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$

$\therefore$ Principal solutions of the equation $\sin x = \frac{1}{2}$ are $\frac{\pi}{6}, \frac{5\pi}{6}$

(b) **General solution :-** Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

(c) **Particular solution :-** The solution of the trigonometric equation lying in the given interval.

17.2 General Solutions of some Trigonometric Equations (to be remembered) :

(a) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

(b) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(c) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(d) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha$, $n \in I$

Note :

(a) $\sin\theta = 0 \Rightarrow \theta = n\pi, \quad n \in I$

(b) $\sin\theta = 1 \Rightarrow \theta = (4n + 1)\frac{\pi}{2}, \; n \in I$

(c) $\sin\theta = -1 \Rightarrow \theta = (4n - 1)\frac{\pi}{2}, \; n \in I$

(d) $\cos\theta = 0 \Rightarrow \theta = (2n + 1)\frac{\pi}{2}, \; n \in I$

(e) $\cos\theta = 1 \Rightarrow \theta = 2n\pi, n \in I$

(f) $\cos\theta = -1 \Rightarrow \theta = (2n + 1)\pi, \quad n \in I$

(g) $\tan\theta = 0 \Rightarrow \theta = n\pi, \quad n \in I$

SOLVED EXAMPLE

Example 21 : Solve $\sin\theta = \frac{\sqrt{3}}{2}$.

Solution : $\because$ $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow$ $\sin\theta = \sin\frac{\pi}{3}$

$\therefore$ $\theta = n\pi + (-1)^n \frac{\pi}{3}$, $n \in I$

Example 22 : Solve : $\sec 2\theta = -\frac{2}{\sqrt{3}}$

Solution : $\because$ $\sec 2\theta = -\frac{2}{\sqrt{3}}$ $\Rightarrow$ $\cos 2\theta = -\frac{\sqrt{3}}{2}$

$\Rightarrow$ $\cos 2\theta = \cos\frac{5\pi}{6}$ $\Rightarrow$ $2\theta = 2n\pi \pm \frac{5\pi}{6}$, $n \in I$

$\Rightarrow$ $\theta = n\pi \pm \frac{5\pi}{12}$, $n \in I$

Example 23 : Solve $\tan\theta = 2$

Solution : $\because$ $\tan\theta = 2$ ....(i)

Let $2 = \tan\alpha$ $\Rightarrow$ $\tan\theta = \tan\alpha$

$\Rightarrow$ $\theta = n\pi + \alpha$, where $\alpha = \tan^{-1}(2)$, $n \in I$

Example 24 : Solve $4\tan^2\theta = 3\sec^2\theta$

Solution : $\because$ $4\tan^2\theta = 3\sec^2\theta$ ....(i)

For equation (i) to be defined $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in I$

$\because$ equation (i) can be written as:

$$\frac{4\sin^2\theta}{\cos^2\theta} = \frac{3}{\cos^2\theta}$$

$\because$ $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in I$

$\therefore$ $\cos^2\theta \neq 0$

$\Rightarrow$ $4\sin^2\theta = 3$ $\Rightarrow$ $\sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow$ $\sin^2\theta = \sin^2\frac{\pi}{3}$ $\Rightarrow$ $\theta = n\pi \pm \frac{\pi}{3}$, $n \in I$

Problems for Self Practise - 10 :

(1) Find general solutions of the following equations :

(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\left(\frac{3\theta}{2}\right) = 0$ (c) $\tan\left(\frac{3\theta}{4}\right) = 0$ (d) $\cos^2 2\theta = 1$

(e) $\sqrt{3}\sec 2\theta = 2$ (f) $\text{cosec}\left(\frac{\theta}{2}\right) = -1$

Answers : (1) (a) $\theta = n\pi + (-1)^n \frac{\pi}{6},\ n \in I$ (b) $\theta = (2n+1)\frac{\pi}{3},\ n \in I$

(c) $\theta = \frac{4n\pi}{3},\ n \in I$ (d) $\theta = \frac{n\pi}{2},\ n \in I$

(e) $\theta = n\pi \pm \frac{\pi}{12},\ n \in I$ (f) $\theta = 2n\pi + (-1)^{n+1}\pi$, $n \in I$

Note : Important points to be remembered while solving trigonometric equations :

(a) Many trigonometrical equations can be solved by different methods. The form of solution obtained in different methods may be different. From these different forms of solutions, the students should not think that the answer obtained by one method are wrong and those obtained by another method are correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

To test the equivalence of two solutions obtained from two methods, the simplest way is to put values of n = ...–2, –1, 0, 1, 2, 3....... etc. and then to find the angles in $[0, 2\pi]$. If all the angles in both solutions are same, the solutions are equivalent.

(b) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(c) Avoid squaring the equations, if possible, because it may lead to extraneous solutions. Reject extra solutions if they do not satisfy the given equation.

(d) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(e) Some necessary restrictions :

If the equation involves tanx, secx, take $\cos x \ne 0$. If cot x or cosec x appear, take $\sin x \ne 0$.

If log appear in the equation, i.e. log $[f(\theta)]$ appear in the equation, use $f(\theta) > 0$ and base of log $> 0, \ne 1$.

Also note that $\sqrt{[f(\theta)]}$ is always positive, for example $\sqrt{\sin^2\theta} = |\sin\theta|$, not $\pm \sin\theta$.

(f) Verification : Student are advice to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.

17.3 Different strategies for solving trigonometric equations :

17.3.1 Solving trigonometric equations by factorisation or by reducing them in quadratic equations.

SOLVED EXAMPLE

Example 25 : Solve : $\tan^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 0$

Solution : $\tan^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 0$

After factorization we get

$\Rightarrow \tan\theta = 1, \sqrt{3} \Rightarrow \theta = n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{3}$.

Example 26 : Solve : $4\cos\theta - 3\sec\theta = 2\tan\theta$

Solution : $4\cos\theta - 3\sec\theta = 2\tan\theta \quad \Rightarrow 4\cos\theta - \frac{3}{\cos\theta} = \frac{2\sin\theta}{\cos\theta} \quad \Rightarrow 4\cos^2\theta - 3 = 2\sin\theta$

$\Rightarrow 4 - 4\sin^2\theta - 3 = 2\sin\theta \quad \Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4} \Rightarrow \sin\theta = \frac{-(\sqrt{5}+1)}{4}, \frac{\sqrt{5}-1}{4} = -\cos 36^\circ, \sin 18^\circ$

$= -\sin 54^\circ, \sin 18^\circ = \sin\left(\frac{-3\pi}{10}\right), \sin\frac{\pi}{10} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{10}$ or $n\pi - (-1)^n\frac{3\pi}{10}$.

Example 27 : Solve $2\cos^2x + 4\cos x = 3\sin^2x$

Solution :

$\because \quad 2\cos^2x + 4\cos x - 3\sin^2x = 0$

$\Rightarrow \quad 2\cos^2x + 4\cos x - 3(1 - \cos^2x) = 0$

$\Rightarrow \quad 5\cos^2x + 4\cos x - 3 = 0$

$\Rightarrow \quad \left\{\cos x - \left(\frac{-2+\sqrt{19}}{5}\right)\right\}\left\{\cos x - \left(\frac{-2-\sqrt{19}}{5}\right)\right\} = 0 \quad$ ....(ii)

$\because \quad \cos x \in [-1, 1] \quad \forall x \in R$

$\therefore \quad \cos x \neq \frac{-2-\sqrt{19}}{5}$

$\therefore$ equation (ii) will be true if $\cos x = \frac{-2+\sqrt{19}}{5}$

$\Rightarrow \quad \cos x = \cos\alpha$, where $\cos\alpha = \frac{-2+\sqrt{19}}{5}$

$\Rightarrow \quad x = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(\frac{-2+\sqrt{19}}{5}\right), n \in I$

Example 28 : Solve the equation $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cdot \cos x$.

Solution : $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cdot \cos x \Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = \frac{7}{2}\sin x \cdot \cos x$

$\Rightarrow \quad 1-\frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x) \Rightarrow 2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow \quad (2\sin 2x - 1)(\sin 2x + 4) = 0 \Rightarrow \sin 2x = \frac{1}{2}$ or $\sin 2x = -4$ (which is not possible)

$\Rightarrow \quad 2x = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

i.e., $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}, n \in I$ **Ans.**

Example 29 : The general solution of the equation $2\cos 2x = 3.2\cos^2 x - 4$ is

(A) $x = 2n\pi, n \in I$ (B) $x = n\pi, n \in I$ (C) $x = n\pi/4, n \in I$ (D) $x = n\pi/2, n \in I$

Solution : $2\cos 2x = 6\cos^2 x - 4$

$\Rightarrow \quad 2(2\cos^2 x - 1) = 6\cos^2 x - 4$

$\Rightarrow \quad 2\cos^2 x = 2 \Rightarrow \cos^2 x = 1 \Rightarrow x = n\pi$. **Ans. (B)**

Example 30 : If $2\cos^2(\pi + x) + 3\sin(\pi + x)$ vanishes then the values of x lying in the interval from 0 to 2π are

(A*) $x = \pi/6$ or $5\pi/6$ (B) $x = \pi/3$ or $5\pi/3$ (C) $x = \pi/4$ or $5\pi/4$ (D) $x = \pi/2$ or $5\pi/2$

Solution : $2\cos^2(\pi + x) + 3\sin(\pi + x) = 0 \Rightarrow 2\cos^2 x - 3\sin x = 0 \Rightarrow 2 - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow 2\sin^2 x + 3\sin x - 2 = 0 \Rightarrow \sin x = -2, \frac{1}{2} \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$ **Ans. (A)**

Example 31 : The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is-

(A) 0 (B) 5 (C) 6 (D) 10

Solution : $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow \quad (3\sin x - 1)(\sin x - 2) = 0$

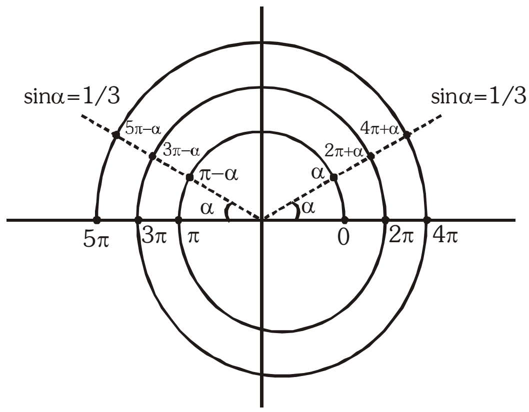
$\because \quad \sin x \neq 2$

$\Rightarrow \quad \sin x = \frac{1}{3} = \sin\alpha$ (say)

where α is the least positive value of x

such that $\sin\alpha = \frac{1}{3}$.

Clearly $0 < \alpha < \frac{\pi}{2}$. We get the solution,



$x = \alpha, \pi - \alpha, 2\pi + \alpha, 3\pi - \alpha, 4\pi + \alpha$ and $5\pi - \alpha$.

Hence total six values in $[0, 5\pi]$ **Ans. (C)**

Problems for Self Practise - 11 :

(1) Solve $\cos 2\theta - (\sqrt{2} + 1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right) = 0$

(2) Solve the following equations :
(a) $3\sin x + 2\cos^2 x = 0$ (b) $\sec^2 2\alpha = 1 - \tan 2\alpha$ (c) $7\cos^2\theta + 3\sin^2\theta = 4$

Answers : (1) $2n\pi \pm \frac{\pi}{3}, n \in I$ or $2n\pi \pm \frac{\pi}{4}, n \in I$

(2) (a) $x = n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in I$

(b) $\alpha = \frac{n\pi}{2}$ or $\alpha = \frac{k\pi}{2} + \frac{3\pi}{8},\ n, k \in I$

(c) $\theta = n\pi \pm \frac{\pi}{3},\ n \in I$

17.3.2 Trigonometric equations which can be solved by transforming a sum or difference of trigonometric ratios into their product or vice-versa

SOLVED EXAMPLE

Example 32 : Solve $\cos 3x + \sin 2x - \sin 4x = 0$

Solution : : $\cos 3x + \sin 2x - \sin 4x = 0$ $\Rightarrow$ $\cos 3x + 2\cos 3x.\sin(-x) = 0$

$\Rightarrow$ $\cos 3x - 2\cos 3x.\sin x = 0$ $\Rightarrow$ $\cos 3x(1 - 2\sin x) = 0$

$\Rightarrow$ $\cos 3x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow$ $3x = (2n+1)\frac{\pi}{2}, n \in I$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ $x = (2n+1)\frac{\pi}{6}, n \in I$ or $x = n\pi + (-1)^n\frac{\pi}{6}, n \in I$

$\therefore$ solution of given equation is

$(2n+1)\frac{\pi}{6}, n \in I$ or $n\pi + (-1)^n\frac{\pi}{6}, n \in I$

Example 33 : Solve $\sin 5x.\cos 3x = \sin 6x.\cos 2x$

Solution : $\because$ $\sin 5x.\cos 3x = \sin 6x.\cos 2x$ $\Rightarrow$ $2\sin 5x.\cos 3x = 2\sin 6x.\cos 2x$

$\Rightarrow$ $\sin 8x + \sin 2x = \sin 8x + \sin 4x$ $\Rightarrow$ $\sin 4x - \sin 2x = 0$

$\Rightarrow$ $2\sin 2x.\cos 2x - \sin 2x = 0$ $\Rightarrow$ $\sin 2x(2\cos 2x - 1) = 0$

$\Rightarrow$ $\sin 2x = 0$ or $2\cos 2x - 1 = 0$

$\Rightarrow$ $2x = n\pi, n \in I$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow$ $x = \frac{n\pi}{2}, n \in I$ or $2x = 2n\pi \pm \frac{\pi}{3}, n \in I$

$\Rightarrow$ $x = n\pi \pm \frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is

$\frac{n\pi}{2}, n \in I$ or $n\pi \pm \frac{\pi}{6}, n \in I$

Example 34 : Solve : cosθ cos2θ cos3θ = $\frac{1}{4}$; where $0 \le \theta \le \pi$.

Solution : $\frac{1}{2}$ (2cosθ cos3θ) cos2θ = $\frac{1}{4}$

$\Rightarrow$ (cos2θ + cos4θ) cos2θ = $\frac{1}{2}$

$\Rightarrow$ $\frac{1}{2}$ [2cos^{2}2θ + 2cos4θ cos2θ] = $\frac{1}{2}$

$\Rightarrow$ 1 + cos4θ + 2cos4θ cos2θ = 1

$\therefore$ cos4θ (1+ 2cos2θ) = 0

cos4θ = 0 or (1 + 2cos2θ) = 0

Now from the first equation : 2cos4θ = 0 = cos(π/2)

$\therefore$ 4θ = $\left(n+\frac{1}{2}\right)\pi \Rightarrow \theta = (2n+1)\frac{\pi}{8}$, n ∈ I

for n = 0, $\theta = \frac{\pi}{8}$; n = 1, $\theta = \frac{3\pi}{8}$; n = 2, $\theta = \frac{5\pi}{8}$; n = 3, $\theta = \frac{7\pi}{8}$ ($\because$ $0 \le \theta \le \pi$)

and from the second equation :

cos2θ = $-\frac{1}{2}$ = –cos(π/3) = cos(π–π/3) = cos (2π/3)

$\therefore$ 2θ = 2kπ ± 2π/3 $\therefore$ θ = kπ ± π/3, k ∈ I

again for k = 0, $\theta = \frac{\pi}{3}$; k = 1, $\theta = \frac{2\pi}{3}$ ($\because$ $0 \le \theta \le \pi$)

$\therefore$ $\theta = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ **Ans.**

Problems for Self Practise - 12 :

(1) Solve sin7θ = sin3θ + sinθ

(2) Solve 5sinx + 6sin2x +5sin3x + sin4x = 0

(3) Solve cosθ – sin3θ = cos2θ

Answers :

(1) $\frac{n\pi}{3}$, n ∈ I or $\frac{n\pi}{2} \pm \frac{\pi}{12}$, n ∈ I

(2) $\frac{n\pi}{2}$, n ∈ I or $2n\pi \pm \frac{2\pi}{3}$, n ∈ I

(3) $\frac{2n\pi}{3}$, n ∈ I or $2n\pi - \frac{\pi}{2}$, n ∈ I or $n\pi + \frac{\pi}{4}$, n ∈ I

17.3.3 Trigonometric Equations of the form a sinx + b cosx = c, where a, b, c $\in$ R, can be solved by dividing both sides of the equation by $\sqrt{a^2+b^2}$.

SOLVED EXAMPLE

Example 35 : Solve sinx + cosx = $\sqrt{2}$

Solution : $\because$ sinx + cosx = $\sqrt{2}$ ....(i)

Here a = 1, b = 1.

$\therefore$ divide both sides of equation (i) by $\sqrt{2}$, we get

$$\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} = 1 \Rightarrow \sin x.\sin\frac{\pi}{4} + \cos x.\cos\frac{\pi}{4} = 1 \Rightarrow \cos\left(x-\frac{\pi}{4}\right) = 1$$

$\Rightarrow$ $x - \frac{\pi}{4} = 2n\pi,\ n \in I \Rightarrow x = 2n\pi + \frac{\pi}{4},\ n \in I$

$\therefore$ Solution of given equation is $2n\pi + \frac{\pi}{4},\ n \in I$

Note : Trigonometric equation of the form a sinx + b cosx = c can also be solved by changing sinx and cosx into their corresponding tangent of half the angle.

Example 36 : Solve 3cosx + 4sinx = 5

Solution : $\because$ 3cosx + 4sinx = 5 ....(i)

$\because$ $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\therefore$ equation (i) becomes

$$\Rightarrow \quad 3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = 5 \quad \text{........(ii)}$$

Let $\tan\frac{x}{2} = t$

$\therefore$ equation (ii) becomes $3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) = 5$

$\Rightarrow$ $4t^2 - 4t + 1 = 0 \Rightarrow (2t-1)^2 = 0$

$\Rightarrow$ $t = \frac{1}{2}$ $\because$ $t = \tan\frac{x}{2}$

$\Rightarrow \quad \tan\frac{x}{2} = \frac{1}{2} \qquad \Rightarrow \quad \tan\frac{x}{2} = \tan\alpha,$ where $\tan\alpha = \frac{1}{2}$

$\Rightarrow \quad \frac{x}{2} = n\pi + \alpha \qquad \Rightarrow \quad x = 2n\pi + 2\alpha$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right), n \in I$

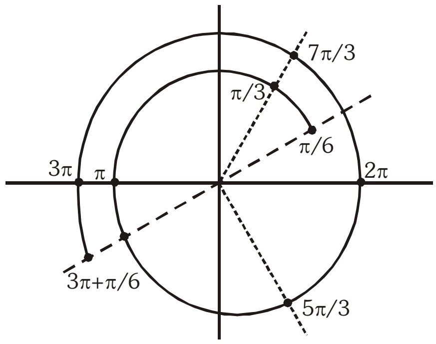
Example 37 : Find the number of distinct solutions of secx + tanx = $\sqrt{3}$, where $0 \le x \le 3\pi$.

Solution *:* Here, sec x + tanx = $\sqrt{3}$

$\Rightarrow \quad 1 + \sin x = \sqrt{3}\cos x$

or $\quad \sqrt{3}\cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2+b^2}$ i.e. $\sqrt{4} = 2$, we get



$\Rightarrow \quad \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$

$\Rightarrow \quad \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x = \frac{1}{2} \Rightarrow \cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$

As $\quad 0 \le x \le 3\pi$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le 3\pi + \frac{\pi}{6}$$

$\Rightarrow \quad x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$

But at $x = \frac{3\pi}{2}$, tanx and secx is not defined.

$\therefore \quad$ Total number of solutions are 2. **Ans.**

<u>Problems for Self Practise</u> - 13 :

Solve the following equations :

(1) $\sin x + \sqrt{2} = \cos x$.

(2) $\text{cosec}\theta = 1 + \cot\theta$.

Answers : (1) $x = 2n\pi - \frac{\pi}{4}, \ n \in I$

(2) $2m\pi + \frac{\pi}{2}, \ m \in I$

17.3.4 Trigonometric equations of the form P(sinx ± cosx, sinx cosx) = 0, where p(y, z) is a polynomial, can be solved by using the substitution sinx ± cosx = t.

SOLVED EXAMPLE

Example 38 : Solve sinx + cosx = 1 + sinx.cosx

Solution : $\because$ sinx + cosx = 1 + sinx.cosx ....(i)

Let sinx + cosx = t

$\Rightarrow$ $\sin^2x + \cos^2x + 2\sin x.\cos x = t^2$

$\Rightarrow$ $\sin x.\cos x = \frac{t^2 - 1}{2}$

Now put sinx + cosx = t and $\sin x.\cos x = \frac{t^2 - 1}{2}$ in (i), we get $t = 1 + \frac{t^2 - 1}{2}$

$\Rightarrow$ $t^2 - 2t + 1 = 0$

$\Rightarrow$ t = 1

$\because$ t = sinx + cosx

$\Rightarrow$ sinx + cosx = 1 ....(ii)

divide both sides of equation (ii) by $\sqrt{2}$, we get

$\Rightarrow$ $\sin x.\frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$

$\Rightarrow$ $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

(i) if we take positive sign, we get $x = 2n\pi + \frac{\pi}{2}, n \in I$

(ii) if we take negative sign, we get
$x = 2n\pi, n \in I$

Problems for Self Practise - 14 :

(1) Solve sin2x + 5sinx + 1 + 5cosx = 0

(2) Solve 3cosx + 3sinx + sin3x – cos3x = 0

(3) Solve $(1 - \sin 2x)(\cos x - \sin x) = 1 - 2\sin^2 x$.

Answers : (1) $n\pi - \frac{\pi}{4}, n \in I$ (2) $n\pi - \frac{\pi}{4}, n \in I$

(3) $2n\pi + \frac{\pi}{2}, n \in I$ or $2n\pi, n \in I$ or $n\pi + \frac{\pi}{4}, n \in I$

17.3.5 Trigonometric equations which can be solved by the use of boundness of the trigonometric ratios sinx and cosx.

SOLVED EXAMPLE

Example 39 : Solve $\sin x\left(\cos\frac{x}{4}-2\sin x\right)+\left(1+\sin\frac{x}{4}-2\cos x\right)\cos x=0$

Solution : $\because$ $\sin x\left(\cos\frac{x}{4}-2\sin x\right)+\left(1+\sin\frac{x}{4}-2\cos x\right)\cos x=0$ ....(i)

$\Rightarrow$ $\sin x.\cos\frac{x}{4}-2\sin^2x+\cos x+\sin\frac{x}{4}.\cos x-2\cos^2x=0$

$\Rightarrow$ $\left(\sin x.\cos\frac{x}{4}+\sin\frac{x}{4}.\cos x\right)-2(\sin^2x+\cos^2x)+\cos x=0$

$\Rightarrow$ $\sin\frac{5x}{4}+\cos x=2$ ....(ii)

Now equation (ii) will be true if $\sin\frac{5x}{4}=1$ and $\cos x=1$

$\Rightarrow$ $\frac{5x}{4}=2n\pi+\frac{\pi}{2}$, $n\in I$ and $x=2m\pi$, $m\in I$

$\Rightarrow$ $x=\frac{(8n+2)\pi}{5}$, $n\in I$ ....(iii)

and $x=2m\pi$, $m\in I$ ....(iv)

Now to find general solution of equation (i)

$\frac{(8n+2)\pi}{5}=2m\pi\Rightarrow 8n+2=10m\Rightarrow n=\frac{5m-1}{4}$

if	m = 1	then	n = 1
if	m = 5	then	n = 6
	....	....	....
	....	....	....
if	$m=4p-3$, $p\in I$	then	$n=5p-4$, $p\in I$

$\therefore$ general solution of given equation can be obtained by substituting either $m=4p-3$ in equation (iv) or $n=5p-4$ in equation (iii)

$\therefore$ general solution of equation (i) is $(8p-6)\pi$, $p\in I$

Problems for Self Practise - 15:

(1) Solve $\sin 3x+\cos 2x=-2$

(2) If $x^2-4x+5-\sin y=0$, $y\in[0,2\pi)$, then -

(A) $x=1, y=0$ (B) $x=1, y=\pi/2$ (C) $x=2, y=0$ (D) $x=2, y=\pi/2$

(3) If $\sin x+\cos x=\sqrt{y+\frac{1}{y}}$, $y>0$, $x\in[0,\pi]$, then find the least positive value of x satisfying the given condition.

Answers : (1) $(4p-3)\frac{\pi}{2}$, $p\in I$ (2) (D) (3) $x=\frac{\pi}{4}$

18. TRIGONOMETRIC INEQUALITIES :

Solutions of elementary trigonometric inequalities are obtained from graphs in intervel $[0, 2\pi)$ if the period is 2π. Then we add $2n\pi$ to the solution and if the period is π. Then we add $n\pi$ to the solution.

SOLVED EXAMPLE

Example 40 : Find the solution set of inequality $\sin x > 1/2$.

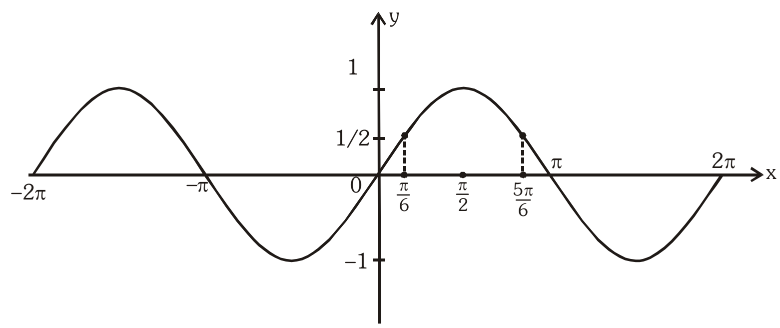
Solution : When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$.

From the graph of $y = \sin x$, it is obvious that between 0 and 2π,

$\sin x > \frac{1}{2}$ for $\pi/6 < x < 5\pi/6$

Hence, $\sin x > 1/2$

$\Rightarrow$ $2n\pi + \pi/6 < x < 2n\pi + 5\pi/6,\ n \in I$



Thus, the required solution set is $\bigcup_{n\in I}\left(2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right)$ **Ans.**

Example 41 : If the set of all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x+\sqrt{2}| < \sqrt{6}$ is $\left(\frac{a\pi}{24}, \frac{b\pi}{24}\right)$ then find the value of $\left|\frac{a-b}{3}\right|$.

Solution : $|4\sin x+\sqrt{2}| < \sqrt{6}$

$\Rightarrow$ $-\sqrt{6} < 4\sin x+\sqrt{2} < \sqrt{6} \Rightarrow -\sqrt{6}-\sqrt{2} < 4\sin x < \sqrt{6}-\sqrt{2}$

$\Rightarrow$ $\frac{-(\sqrt{6}+\sqrt{2})}{4} < \sin x < \frac{\sqrt{6}-\sqrt{2}}{4} \Rightarrow -\frac{5\pi}{12} < x < \frac{\pi}{12}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Comparing with $\frac{a\pi}{24} < x < \frac{b\pi}{24}$, we get, a = –10, b = 2

$\therefore \left|\frac{a-b}{3}\right| = \left|\frac{-10-2}{3}\right| = 4$ **Ans.**

19. HEIGHTS AND DISTANCES

19.1 introduction :

One of the important application of trigonometry is in finding the height and distance of the point which are not directly measurable. This is done with the help of trigonometric ratios.

19.2 Angles of Elevation and Depression :

Let OP be a horizontal line in the vertical plane in which an object R is given and let OR be joined.

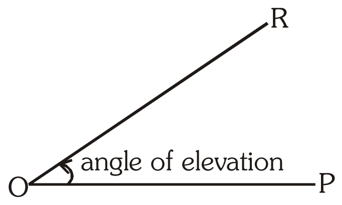


Fig. (a)

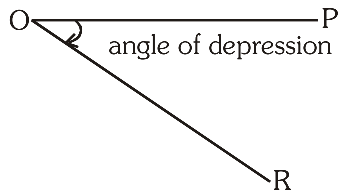


Fig. (b)

In Fig. (a), where the object R is above the horizontal line OP, the angle POR is called the angle of elevation of the object R as seen from the point O. In Fig. (b) where the object R is below the horizontal line OP, the angle POR is called the angle of depression of the object R as seen from the point O.

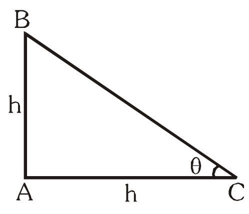
Remark :

Unless stated to the contrary, it is assumed that the height of the observer is neglected, and that the angles of elevation are measured from the ground.

SOLVED EXAMPLE

Example 42: Find the angle of elevation of the sum when the length of shadow of a vertical pole is equal to its height.

Solution : Let height of the pole AB = h and
length of the shadow of the Pole (AC) = h



In ΔABC $\tan\theta = \frac{AB}{AC} = \frac{h}{h} = 1$

$\Rightarrow$ $\tan\theta = 1$

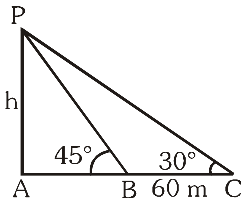
$\Rightarrow$ $\tan\theta = \tan 45°$

$\Rightarrow$ $\theta = 45°$

Example 43 : The shadow of the tower standing on a level ground is found to be 60 metres longer when the sun's altitude is 30° than when it is 45°. The height of the tower is-

(1) 60 m (2) $30(\sqrt{3}-1)$m (3) $60\sqrt{3}$ m (4) $30(\sqrt{3}+1)$ m.

Solution :



AC = h cot 30° = $\sqrt{3}$ h

AB = h cot 45° = h

$\therefore$ BC = AC – AB = h $(\sqrt{3}-1)$

$\Rightarrow$ 60 = h $(\sqrt{3}-1)$

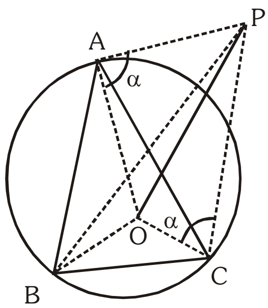
$\therefore$ $h = \dfrac{60}{\sqrt{3}-1} = \dfrac{60(\sqrt{3}+1)}{3-1}$ = 30 $(\sqrt{3}+1)$

Example 44 : The angle of elevation of the tower observed from each of the three point A,B,C on the ground, forming a triangle is the same angle α. If R is the circum - radius of the triangle ABC, then the height of the tower is-

(1) R sin α (2) R cos α (3) R cot α (4) R tan α

Solution : The tower makes equal angles at the vertices of the triangle, therefore foot of the tower is at the circumcentre.

From Δ OCP, OP is perpendicular to OC.



$\angle OCP = \alpha$

so $\tan\alpha = \dfrac{OP}{OA} \Rightarrow$ OP = OA tan α

OP = R tan α

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Allied angle

A-1. Find the radian measures corresponding to the following degree measures

(i) 75º (ii) 210° (iii) 860°

A-2. Find the degree measures corresponding to the following radian measures

(i) $\frac{5\pi}{6}$ (ii) 8π (iii) $-\frac{2\pi}{3}$ (iv) $\frac{11\pi}{12}$

A-3. Find the value of :

(i) $\sin^2\frac{\pi}{6} + \cos^2\frac{2\pi}{3} - \tan^2\frac{3\pi}{4}$ (ii) $2\sin^2\frac{7\pi}{6} - \text{cosec}\,\frac{5\pi}{6}\cos^2\frac{\pi}{3}$

(iii) $3\cos^2\frac{5\pi}{4} + \sec\frac{4\pi}{3} + 5\tan^2\frac{\pi}{3}$ (iv) $\cot^2\frac{\pi}{6} + \text{cosec}\,\frac{5\pi}{6} + 3\tan^2\frac{11\pi}{6}$

A-4. Find the value of :

(i) cos 570° (ii) sin 675° (iii) tan (–390°) (iv) cot 585°

A-5. Prove that

(i) $\dfrac{\cos(3\pi+\theta)\cos(2\pi-\theta)}{\sin(5\pi-\theta)\cos\left(\frac{\pi}{2}+\theta\right)} = \cot^2\theta.$

(ii) $\cos\theta + \sin(270° + \theta) - \sin(270° - \theta) + \cos(180° + \theta) = 0.$

(iii) $\cos\left(\frac{3\pi}{2}+\theta\right)\cos(2\pi+\theta)\left[\cot\left(\frac{3\pi}{2}-\theta\right)+\cot(2\pi+\theta)\right] = 1.$

A-6. If $\tan\theta = -5/12$, θ lies in the second quadrant, then find the value of $\dfrac{\sin(180^\circ-\theta)+\tan(90^\circ+\theta)}{-\sec(90^\circ+\theta)-\text{cosec}(-\theta)}$.

Section (B) : Addition/Subtraction of T-Ratio, sum into product/vice versa

B-1. Show that :

(i) sin 10° . cos 50° + cos 10° . sin 50° = $\sqrt{3}/2$

(ii) cos 100° . cos 200° – sin 100° . sin 200º = 1/2

B-2. Find value of :

(i) $\cos^2 15^\circ - \cos^2 75^\circ$ (ii) $\cos^2 45^\circ - \sin^2 15^\circ$

B-3. If cos25º + sin 25º = k then find value of cos20º in terms of k.

B-4. Show that : $\cos^2\alpha + \cos^2(\alpha+\beta) - 2\cos\alpha\cos\beta\cos(\alpha+\beta) = \sin^2\beta$.

B-5. Show that : $\sin\frac{9\theta}{2}\cos 3\theta - \sin 2\theta\cos\frac{\theta}{2} = \cos 5\theta\ \sin\frac{5\theta}{2}$.

B-6. If A + B = 45^0, prove that (1 + tanA)(1 + tan B) = 2 and hence deduce that $\tan 22\frac{1}{2}^0 = \sqrt{2}-1$

B-7. Eliminate θ from the relations $a\sec\theta - x\tan\theta = y$ and $b\sec\theta + y\tan\theta = x$

Section (C) : Multiple & sub-multiple angle formula

C-1. Prove that :

(i) $(\text{cosec}\,\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

(ii) $\frac{2\sin\theta\tan\theta(1-\tan\theta)+2\sin\theta\sec^2\theta}{(1+\tan\theta)^2} = \frac{2\sin\theta}{(1+\tan\theta)}$

(iii) $\frac{\cos A\,\text{cosec}A - \sin A\sec A}{\cos A + \sin A} = \text{cosec}\,A - \sec A$

(iv) $\frac{1}{\sec\alpha - \tan\alpha} - \frac{1}{\cos\alpha} = \frac{1}{\cos\alpha} - \frac{1}{\sec\alpha + \tan\alpha}$

C-2. Prove that

(i) $\frac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B} = \tan(A+B)$

(ii) $\cot(A + 15^o) - \tan(A - 15^o) = \frac{4\cos 2A}{1+2\sin 2A}$

C-3. Prove that $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$

C-4. Prove that

(i) $\left\{\frac{1-\cot^2\left(\frac{\alpha-\pi}{4}\right)}{1+\cot^2\left(\frac{\alpha-\pi}{4}\right)} + \cos\frac{\alpha}{2}\cot 4\alpha\right\}\sec\frac{9\alpha}{2} = \text{cosec}\,4\alpha$.

(ii) $\frac{1}{\tan 3\alpha - \tan\alpha} - \frac{1}{\cot 3\alpha - \cot\alpha} = \cot 2\alpha$.

(iii) $\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$

(iv) $\frac{\cos A + \sin A}{\cos A - \sin A} - \frac{\cos A - \sin A}{\cos A + \sin A} = 2\tan 2A)$

C-5. Prove that $\sin\theta = \frac{\sin 3\theta}{1+2\cos 2\theta}$ and hence deduce the value of sin 15°.

C-6. Prove that :

(i) $\frac{\tan 3x}{\tan x} = \frac{2\cos 2x+1}{2\cos 2x-1}$ (ii) $\frac{2\sin x}{\sin 3x} + \frac{\tan x}{\tan 3x} = 1$

C-7. Prove that :

$\sin\theta \sin(60^\circ+\theta)\sin(60^\circ-\theta) = \frac{1}{4}\sin 3\theta$ and hence deduce that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

C-8. Find the value of

(i) $4\cos 72^\circ \cos 36^\circ$ (ii) $\sin^2 72^\circ - \sin^2 54^\circ$ (iii) $\cos^2 48^\circ - \sin^2 12^\circ$

Section (D) : Trigonometric series and conditional identities

D-1. Prove that :

(i) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$

(ii) $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11} = \frac{1}{32}$

D-2. Prove that $\sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + ... + \sin^2 n\theta = \frac{n}{2} - \frac{\sin n\theta \cos(n+1)\theta}{2\sin\theta}$

D-3. Find exact value of $\frac{\sin 80^\circ \sin 65^\circ \sin 35^\circ}{\sin 20^\circ + \sin 50^\circ + \sin 110^\circ}$

D-4. If A + B + C = π than prove that

$\sin\left(\frac{A}{2}\right) + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + \sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$

D-5. If $x + y = \pi + z$, then prove that $\sin^2 x + \sin^2 y - \sin^2 z = 2\sin x \sin y \cos z$.

D-6. If A + B + C = 2S then prove that

$\cos(S-A) + \cos(S-B) + \cos(S-C) + \cos S = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

D-7. If A, B, C are angle of triangle then show that $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin A \sin C} + \frac{\cos C}{\sin A \sin B} = 2$

Section (E) : Range and graph of trigonometric function

E-1. Sketch the following graphs :

(i) $y = 3\sin 2x$ (ii) $y = 2\tan x$ (iii) $y = \sin\frac{x}{2}$

E-2. Find the extreme values of $\sin x \sin\left(\frac{2\pi}{3}+x\right)\sin\left(\frac{2\pi}{3}-x\right)$

E-3. Find the maximum and minimum values of following trigonometric functions

(i) cos 2x + 4 sin x (ii) $\dfrac{1}{3\sin x + 4\cos x + 6}$ (iii) $27^{\cos 2x}.81^{\sin 2x}$

E-4. Find the greatest and least value of y

(i) y = 10 cos²x – 6 sin x cos x + 2 sin²x (ii) y = 3 cos$\left(\theta + \dfrac{\pi}{3}\right)$ + 5 cos θ + 3

E-5. If the equation $\sin^4 x - 3\sin 2x + \dfrac{13}{4} - a = 0$ has atleast one solution then the number of integral value(s) of a is

E-6. The number of roots of the equation cot x = $\dfrac{\pi}{2}$ + x in $\left[-\pi, \dfrac{3\pi}{2}\right]$ is

Section (F) : Trigonometric Equations and Inequations

F-1. What are the general values of θ which satisfy the equations :

(i) cosθ = $\dfrac{1}{\sqrt{2}}$ (ii) cot (x – 1) = $\sqrt{3}$ (iii) tanθ = 2 (iv) cosecθ = 2.

(v) $2\cot^2\theta = \text{cosec}^2\theta$

F-2. Solve

(i) sin9θ = sin3θ (ii) cotθ + tanθ = 2cosecθ (iii) sin4θ = cos3θ

(iv) cotθ = tan10θ (v) cotθ – tanθ = 2. (vi) cosecθ = cotθ + $\sqrt{3}$.

(vii) tan2θ tan3θ = 1 (viii) tanθ + tan2θ + $\sqrt{3}$ tanθ tan2θ = $\sqrt{3}$.

F-3. Solve the following equations

(i) (2 sin x – cos x) (1 + cos x) = $\sin^2 x$ (ii) 6 – 10cosx = $3\sin^2 x$

(iii) $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$

F-4. Solve

(i) sinθ + sin3θ + sin5θ = 0.

(ii) cos θ + sin θ = cos 2θ + sin 2θ.

(iii) $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

(iv) $\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2\theta$, where n is constant and n ≠ 0, 1

F-5. Solve

(i) $\sqrt{3}$ sinθ – cosθ = 2 (ii) sin x + cos x = 1 + sin x. cos x.

(iii) 3 cos x + 4 sin x = 5

F-6. Find the values of α lying between 0 and π for which the inequality : $\tan\alpha > \tan^3\alpha$ is valid.

F-7. Solve $2\sin^2 x - \sin x - 1 > 0$

F-8. Solve : $\left(\sin x - \dfrac{1}{2}\right)\left(\cos x + \dfrac{\sqrt{3}}{2}\right) \geq 0$

Section (G) : Heights and Distances

G-1. Two pillars of equal height stand on either side of a roadway which is 60 m wide. At a point in the roadway between the pillars, the angle of elevation of the top of pillars are 60º and 30º. Then find height of pillars -

G-2. At a point on level ground, the angle of elevation of a vertical tower is found to be such that its tangent is $\frac{5}{12}$. On walking 192 metres towards the tower, the tangent of the angle of elevation is $\frac{3}{4}$. Find the height of the tower.

G-3. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height h. At a point on the plane, the angle of elevation of the bottom and the top of the flag staff are α and β respectively. Prove that the height of tower is $\frac{h\tan\alpha}{\tan\beta-\tan\alpha}$.

G-4. From the top of a cliff 25 m high the angle of elevation of a tower is found to be equal to the angle of depression of the foot of the tower. Then find height of the tower -

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PART-II : OBJECTIVE QUESTIONS

Section (A) : Allied angle

A-1. $\dfrac{\tan\left(x-\frac{5\pi}{2}\right)\cdot\cos\left(\frac{7\pi}{2}+x\right)-\sin^3\left(\frac{3\pi}{2}-x\right)}{\cos\left(x-\frac{\pi}{2}\right)\cdot\tan\left(\frac{3\pi}{2}+x\right)}$ when simplified reduces to:

(A) $\sin x\cos x$ (B) $-\sin^2 x$ (C) $-\sin x\cos x$ (D) $\sin^2 x$

A-2. The expression $3\left[\sin^4\left(\frac{\pi}{2}+\alpha\right)+\sin^4(5\pi+\alpha)\right]-2\left[\sin^6\left(\frac{3\pi}{2}-\alpha\right)+\sin^6(3\pi+\alpha)\right]$ is equal to

(A) 0 (B) 1 (C) 3 (D) $\sin 4\alpha + \sin 6\alpha$

A-3 If $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ then θ is equal to -

(A) 30° (B) 150° (C) 210° (D) 120º

A-4. The value of tan 1° tan 2° tan 3° ... tan 89° is

(A) 1 (B) 0 (C) ∞ (D) $\frac{1}{2}$

A-5 If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then xy + yz + zx is equal to

(A) – 1 (B) 0 (C) 1 (D) 2

A-6 If 0° < x < 90° & cosx = $\frac{3}{\sqrt{10}}$, then the value of $\log_{10}$ sin x + $\log_{10}$ cos x + $\log_{10}$tan x is

(A) 0 (B) 1 (C) – 1 (D) 2

A-7 Find the value of $\tan\frac{11\pi}{3}-2\sin\frac{9\pi}{3}-\frac{3}{4}\operatorname{cosec}^2\frac{\pi}{4}+4\cos^2\frac{17\pi}{6}$ is

(A) $\frac{3-2\sqrt{3}}{2}$ (B) $\frac{3+2\sqrt{3}}{2}$ (C) $\frac{\sqrt{3}+1}{2}$ (D) $\frac{\sqrt{3}-1}{2}$

Section (B) : Addition/Subtraction of T-Ratio, sum into product/vice versa

B-1. The value of $\frac{\cos 66^\circ\ \cos 6^\circ-\sin 6^\circ \cos 24^\circ}{\sin 21^\circ\ \cos 39^\circ-\sin 39^\circ\ \sin 69^\circ}$ is

(A) – 1 (B) 1 (C) 2 (D) 0

B-2. If tan A and tan B are the roots of the quadratic equation $x^2 - ax + b = 0$, then the value of $\sin^2(A + B)$.

(A) $\frac{a^2}{a^2+(1-b)^2}$ (B) $\frac{a^2}{a^2+b^2}$ (C) $\frac{a^2}{(b+c)^2}$ (D) $\frac{a^2}{b^2(1-a)^2}$

B-3. If tan A – tan B = x and cot B – cot A = y, then cot (A – B) is equal to

(A) $\frac{1}{y}-\frac{1}{x}$ (B) $\frac{1}{x}-\frac{1}{y}$ (C) $\frac{1}{x}+\frac{1}{y}$ (D) $\frac{1}{x+y}$

B-4. If tan 25° = x, then $\frac{\tan 155^\circ-\tan 115^\circ}{1+\tan 155^\circ \tan 115^\circ}$ is equal to

(A) $\frac{1-x^2}{2x}$ (B) $\frac{1+x^2}{2x}$ (C) $\frac{1+x^2}{1-x^2}$ (D) $\frac{1-x^2}{1+x^2}$

B-5. If A + B = 225°, then the value of $\left(\frac{\cot A}{1+\cot A}\right).\left(\frac{\cot B}{1+\cot B}\right)$ is

(A) 2 (B) $\frac{1}{2}$ (C) 3 (D) $\frac{1}{3}$

B-6. If sin2A = λ sin2B, then value of $\frac{\tan(A+B)}{\tan(A-B)}$ =

(A) $\frac{\lambda+1}{\lambda-1}$ (B) $\frac{\lambda-1}{\lambda+1}$ (C) $\frac{\lambda}{2}$ (D) $\frac{1}{\lambda}$

B-7. If cosec A + cotA = 11/2 then tanA is equal to

(A) $\frac{111}{44}$ (B) $\frac{44}{117}$ (C) $\frac{44}{125}$ (D) $\frac{117}{125}$

B-8. If $\alpha = \frac{\pi}{19}$ then value of $\frac{\sin 23\alpha - \sin 3\alpha}{\sin 16\alpha + \sin 4\alpha}$ is equal to :-

(A) 1 (B) $\frac{1}{2}$ (C) –1 (D) 2

B-9. Value of expression $\cos^2 73^\circ + \cos^2 47^\circ + \cos 73^\circ \cos 47^\circ$ is equal to :-

(A) $\frac{1}{4}$ (B) 2 (C) $\frac{3}{4}$ (D) 1

B-10. If $a \cos\theta + b \sin\theta = 3$ & $a \sin\theta - b \cos\theta = 4$ then $a^2 + b^2$ has the value equal to :-
(A) 25 (B) 14 (C) 7 (D) 10

Section (C) : Multiple & sub-multiple angle formula

C-1. If A lies in the third quadrant and 3 tan A – 4 = 0, then 5 sin 2A + 3sinA + 4 cosA is equal to

(A) 0 (B) $-\frac{24}{5}$ (C) $\frac{24}{5}$ (D) $\frac{48}{5}$

C-2. If cos A = 3/4, then the value of $16\cos^2 (A/2) - 32 \sin (A/2) \sin (5A/2)$ is
(A) – 4 (B) – 3 (C) 3 (D) 4

C-3. $\frac{\sin 5\theta + \sin 2\theta - \sin\theta}{\cos 5\theta + 2\cos 3\theta + 2\cos^2\theta + \cos\theta}$ is equal to -

(A) $\tan\theta$ (B) $\cos\theta$ (C) $\cot\theta$ (D) $\sin\theta$

C-4. If $\tan^2\theta = 2\tan^2\phi + 1$, then the value of $\cos 2\theta + \sin^2\phi$ is
(A) 1 (B) 2 (C) – 1 (D) Independent of ϕ

C-5. The value of tan 3A – tan 2A – tan A is equal to
(A) tan 3A tan 2A tan A (B) – tan 3A tan 2A tan A
(C) tan A tan 2A – tan 2A tan 3A – tan 3A tan A (D) none of these

C-6. The value of $\frac{1}{\sin 20^\circ} - \frac{1}{\sqrt{3}\cos 20^\circ}$ is

(A) $\frac{2\sqrt{3}}{3}$ (B) $\frac{4\sqrt{3}}{3}$ (C) $\sqrt{3}$ (D) 1

C-7. The value of the expression $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$ is

(A) $\frac{1}{8}$ (B) $\frac{1}{16}$ (C) $\frac{1}{4}$ (D) 0

C-8. $\sin 67\frac{1}{2}^\circ + \cos 67\frac{1}{2}^\circ$ is equal to

(A) $\frac{1}{2}\sqrt{4+2\sqrt{2}}$ (B) $\frac{1}{2}\sqrt{4-2\sqrt{2}}$ (C) $\frac{1}{4}\left(\sqrt{4+2\sqrt{2}}\right)$ (D) $\frac{1}{4}\left(\sqrt{4-2\sqrt{2}}\right)$

C-9. Find exact value of sin78° – sin66° – sin42° + sin6° is :-

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) $-\frac{1}{2}$ (D) 1

C-10. If A = cos 6° cos 42° and B = sec 66° sec 78°, then

(A) A = 8B (B) A = $\frac{1}{4}$ B (C) A = $\frac{1}{16}$ B (D) 3A = 2B

Section (D) : Trigonometric series and conditional identities

D-1. If A + B + C = $\frac{3\pi}{2}$, then cos 2A + cos 2B + cos2C is equal to-

(A) 1 – 4cosA cosB cosC (B) 4 sinA sin B sinC
(C) 1 + 2cosA cosB cosC (D) 1 – 4 sinA sinB sinC

D-2. In any triangle ABC, sin A – cos B = cos C, then angle B (where B > C) is

(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$

D-3. The value of $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7}$ is

(A) 1/2 (B) – 1/2 (C) 0 (D) 1

D-4. The value of $\cos\frac{\pi}{10}\cos\frac{2\pi}{10}\cos\frac{4\pi}{10}\cos\frac{8\pi}{10}\cos\frac{16\pi}{10}$ is :

(A) $\frac{\sqrt{10+2\sqrt{5}}}{64}$ (B) $-\frac{\cos(\pi/10)}{16}$ (C) $\frac{\cos(\pi/10)}{16}$ (D) $-\frac{\sqrt{10+2\sqrt{5}}}{16}$

D-5. The value of $\cos\frac{\pi}{19} + \cos\frac{3\pi}{19} + \cos\frac{5\pi}{19} + ... + \cos\frac{17\pi}{19}$ is equal to:

(A) 1/2 (B) 0 (C) 1 (D) 2

Section (E) : Range and graph of trigonometric function

E-1. **STATEMENT-1 :** sin 2 > sin 3

STATEMENT-2 : If x, y $\in \left(\frac{\pi}{2}, \pi\right)$, x < y, then sin x > sin y

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true

E-2. If $f(\theta) = \sin^4\theta + \cos^2\theta$, then range of $f(\theta)$ is

(A) $\left[\frac{1}{2}, 1\right]$ (B) $\left[\frac{1}{2}, \frac{3}{4}\right]$ (C) $\left[\frac{3}{4}, 1\right]$ (D) None of these

E-3. Find the maximum value of $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$ -

(A) 1 (B) 2 (C) 3 (D) 4

E-4. The difference between maximum and minimum value of the expression $y = 1 + 2\sin x + 3\cos^2 x$ is

(A) $\frac{16}{3}$ (B) $\frac{13}{3}$ (C) 7 (D) 8

E-5. If cosx (sin x + cosx) = k then find maximum value of k for which equation have a solution:-

(A) 2 (B) $\sqrt{2}$ (C) 1 (D) $\frac{\sqrt{2}+1}{2}$

Section (F) : Trigonometric Equations and inequations

F-1. The most general solution of $\tan\theta = -1$ and $\sin\theta = \frac{1}{\sqrt{2}}$ is :

(A) $n\pi + \frac{7\pi}{4}, n \in I$ (B) $n\pi + (-1)^n \frac{3\pi}{4}, n \in I$ (C) $2n\pi + \frac{3\pi}{4}, n \in I$ (D) $2n\pi + \frac{7\pi}{4}, n \in I$

F-2. The solution set of the equation $4\sin\theta.\cos\theta - 2\cos\theta - 2\sqrt{3}\sin\theta + \sqrt{3} = 0$ in the interval $(0, 2\pi)$ is

(A) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ (B) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ (C) $\left\{\frac{3\pi}{4}, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$ (D) $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}\right\}$

F-3. Number of values of x in $(0, 2\pi)$ for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x.\tan 2x} = 1$ is

(A) 0 (B) 1 (C) 2 (D) 3

F-4. Number of values of θ satisfying for $\sin^2\theta - \cos\theta = \frac{1}{4}$ for $0 \le \theta \le 2\pi$ is :-

(A) 1 (B) 2 (C) 3 (D) 4

F-5. If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is -

(A) $2n\pi \pm \frac{\pi}{3}, n \in I$ (B) $2n\pi \pm \frac{\pi}{6}, n \in I$ (C) $n\pi \pm \frac{\pi}{3}, n \in I$ (D) $n\pi \pm \frac{\pi}{4}, n \in I$

F-6. Total number of solutions of equation sinx . tan4x = cosx belonging to $(0, 2\pi)$ are :

(A) 4 (B) 7 (C) 8 (D) 10

F-7. If $x \in [0, \pi]$, the number of solutions of the equation sin 7x + sin 4x + sin x = 0 is:

(A) 3 (B) 8 (C) 6 (D) 4

F-8. $\frac{\cos 3\theta}{2\cos 2\theta - 1} = \frac{1}{2}$ if

(A) $\theta = n\pi + \frac{\pi}{3}, n \in I$ (B) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in I$ (C) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in I$ (D) $\theta = n\pi + \frac{\pi}{6}, n \in I$

F-9. If $\sin\theta + 7\cos\theta = 5$, then $\tan(\theta/2)$ is a root of the equation

(A) $x^2 - 6x + 1 = 0$ (B) $6x^2 - x - 1 = 0$ (C) $6x^2 + x + 1 = 0$ (D) $x^2 - x + 6 = 0$

F-10. The solution of inequality $\cos 2x \le \cos x$ is

(A) $x \in \left[2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$ (B) $x \in \left[2n\pi - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right]$

(C) $x \in \left[2n\pi, 2n\pi + \frac{2\pi}{3}\right]$ (D) $x \in \left[2n\pi - \frac{2\pi}{3}, 2n\pi\right]$

F-11. Complete set of values of x in the interval $[0, 2\pi]$ for which $4\sin^2x - 8\sin x + 3 \le 0$

(A) $\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$ (B) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ (C) $\left[0, \frac{2\pi}{3}\right]$ (D) None of these

F-12. Which of the following set of values of x satisfy the inequation $\tan^2x - (1 + \sqrt{3})\tan x + \sqrt{3} < 0$

(A) $\left(\frac{(4n+1)\pi}{4}, \frac{(3n+1)\pi}{3}\right)$, $(n \in Z)$ (B) $\left(\frac{(2n+1)\pi}{4}, \frac{(2n+1)\pi}{3}\right)$, $(n \in Z)$

(C) $\left(\frac{(4n+1)\pi}{4}, \frac{(4n+1)\pi}{3}\right)$, $(n \in Z)$ (D) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$

Section (G) : Heights and Distances

G-1. A tower subtends an angle of 30° at a point on the same level as its foot, and at a second point h m above the first, the depression of the foot of tower is 60°. The height of the tower is.

(A) h m (B) 3h m (C) $\sqrt{3}$ h m (D) $\frac{h}{3}$ m.

G-2. Let α be the solution of $16^{\sin^2\theta} + 16^{\cos^2\theta} = 10$ in $(0, \pi/4)$. If the shadow of a vertical pole is $\frac{1}{\sqrt{3}}$ of its height, then then the altitude of the sun is-

(A) α (B) $\frac{\alpha}{2}$ (C) 2α (D) $\frac{\alpha}{3}$

G-3. A vertical lamp-post of height 9 metres stands at the corner of a rectangular field. The angle of elevation of its top from the farthest corner is 30°, while from another corner it is 45°. The area of the field is-

(A) $81\sqrt{2}$ m^2 (B) $9\sqrt{2}$ m^2 (C) $81\sqrt{3}$ m^2 (D) $9\sqrt{3}$ m^2

G-4. A round ballon of radius r subtends an angle α at the eye of the observer, while the angle of elevation of its centre is β. The height of the centre of ballon is-

(A) $r \operatorname{cosec} \alpha \sin \frac{\beta}{2}$ (B) $r \sin \alpha \operatorname{cosec} \frac{\beta}{2}$ (C) $r \sin \frac{\alpha}{2} \operatorname{cosec} \beta$ (D) $r \operatorname{cosec} \frac{\alpha}{2} \sin \beta$

G-5. A man on the top of a vertical tower observes a car moving at a uniform speed coming directly towards it. If it takes 12 minutes for the angle of depression to change from 30º to 45º, then the car will reach the tower in

(A) 17 minutes 23 seconds (B) 16 minutes 23 seconds
(C) 16 minutes 18 seconds (D) 18 minutes 22 seconds

PART-III : MATCH THE COLUMN

1. If α and β are distinct roots of the equation, $a \cos \theta + b \sin \theta = c$ such that $\alpha - \beta \neq 2n\pi$ then match the entries of column-I with the entries of column-II

	Column – I		Column–II
(A)	$\sin\alpha + \sin \beta =$	(P)	$\frac{2b}{a+b}$
(B)	$\sin\alpha \cdot \sin \beta =$	(Q)	$\frac{c-a}{c+a}$
(C)	$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} =$	(R)	$\frac{2bc}{a^2+b^2}$
(D)	$\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} =$	(S)	$\frac{c^2-a^2}{a^2+b^2}$

2.

	Column–I		Column–II
(A)	If for some real x, the equation $x + \frac{1}{x} = 2 \cos \theta$ holds, then $\cos \theta$ is equal to	(p)	2
(B)	If $\cos \theta + \sec \theta = 2$, then $\cos^{2020} \theta + \sec^{2020}\theta$ is equal to	(q)	1
(C)	Maximum value of $\sin^4\theta + \cos^4\theta$ is	(r)	0
(D)	Least value of $3 \sin^2\theta + 2 \cos^2\theta$ is	(s)	– 1

3.

	Column - I		Column - II
(A)	$\tan 9° - \tan 27° - \tan 63° + \tan 81°$	(p)	1
(B)	$\operatorname{cosec} 10° - \sqrt{3} \sec 10°$	(q)	2
(C)	$2\sqrt{2} \sin 10° \left[\frac{\sec 5°}{2} + \frac{\cos 40°}{\sin 5°} - 2 \sin 35°\right]$	(r)	3
(D)	$\sqrt{3}$ (cot 70º + 4 cos 70º)	(s)	4

Exercise # 2

PART-I : OBJECTIVE

1. If $\frac{\sin A}{\sin B} = \frac{\sqrt{3}}{2}$ and $\frac{\cos A}{\cos B} = \frac{\sqrt{5}}{2}$, $0 < A, B < \pi/2$, then tan A + tan B is equal to

(A) $\sqrt{3}/\sqrt{5}$ (B) $\sqrt{5}/\sqrt{3}$ (C) 1 (D) $(\sqrt{5}+\sqrt{3})/\sqrt{5}$

2. If $\cos\alpha = \frac{2\cos\beta - 1}{2-\cos\beta}$ then $\tan\frac{\alpha}{2}\cdot\cot\frac{\beta}{2}$ has the value equal to {where $\alpha, \beta \in (0, \pi)$}

(A) 2 (B) $\sqrt{2}$ (C) 3 (D) $\sqrt{3}$

3. $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha =$

(A) $\tan\alpha$ (B) $\cot\alpha$ (C) $\cot 16\alpha$ (D) $16\cot\alpha$

4. The value of $(\cos^4 1^\circ + \cos^4 2^\circ + \cos^4 3^\circ + ... + \cos^4 179^\circ) - (\sin^4 1^\circ + \sin^4 2^\circ + \sin^4 3^\circ + ... + \sin^4 179^\circ)$ equals to :-

(A) $2\cos 1^\circ$ (B) -1 (C) $2\sin 1^\circ$ (D) 0

5. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

(A) $\frac{\pi}{3}$ & $\frac{\pi}{6}$ (B) $\frac{\pi}{8}$ & $\frac{3\pi}{8}$ (C) $\frac{\pi}{4}$ & $\frac{\pi}{4}$ (D) $\frac{\pi}{5}$ & $\frac{3\pi}{10}$

6. If $\cos\alpha + \cos\beta = a$, $\sin\alpha + \sin\beta = b$ and $\alpha - \beta = 2\theta$, then $\frac{\cos 3\theta}{\cos\theta} =$

(A) $a^2 + b^2 - 2$ (B) $a^2 + b^2 - 3$ (C) $3 - a^2 - b^2$ (D) $(a^2 + b^2)/4$

7. If $x + y = 3 - \cos 4\theta$ and $x - y = 4\sin 2\theta$ then

(A) $x^4 + y^4 = 9$ (B) $\sqrt{x}+\sqrt{y} = 16$ (C) $x^3 + y^3 = 2(x^2 + y^2)$ (D) $\sqrt{x}+\sqrt{y} = 2$

8. In triangle ABC, the minimum value of $\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2}$ is equal to

(A) 3 (B) 4 (C) 5 (D) 6

9. The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2\cos 2x + a_3\sin^2 x = 0$ for all x is

(A) 0 (B) 1 (C) 2 (D) infinite

10. $\sum_{k=1}^{88}(-1)^{K+1}\frac{1}{\sin^2(k+1)^\circ - \sin^2 1^\circ}$ is equal to

(A) $\tan 2^\circ$ (B) $\cot 2^\circ$ (C) $\frac{\sin 2^\circ}{\cot 2^\circ}$ (D) $\frac{\cot 2^\circ}{\sin 2^\circ}$

11. Equation $k\cos x - 3\sin x = k + 1$ possess a solution iff

(A) $k \in (-\infty, 4]$ (B) $k \in [4, \infty)$ (C) $k \in (-\infty, 6]$ (D) $k \in (-\infty, 6) \cup (8, \infty)$

12. Number of solution of the equation $\tan^2\alpha + 2\sqrt{3}\tan\alpha = 1$ in $[0, 2\pi]$ is :-

(A) 3 (B) 5 (C) 4 (D) 2

13. The general solution of the equation $\tan x + \tan\left(x+\frac{\pi}{3}\right) + \tan\left(x+\frac{2\pi}{3}\right) = 3$ is

(A) $\frac{n\pi}{4}+\frac{\pi}{12}$, $n \in I$ (B) $\frac{n\pi}{3}+\frac{\pi}{6}$, $n \in I$ (C) $\frac{n\pi}{3}+\frac{\pi}{12}$, $n \in I$ (D) $\frac{n\pi}{3}+\frac{\pi}{4}$, $n \in I$

14. Number of solution of equation $(\sin x + \cos x)^{1+\sin 2x} = 2$, in $0 \le x \le \pi$ is :-

(A) 1 (B) 2 (C) 3 (D) 4

15. The number of solution(s) of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$, $0 \le x \le \pi/2$, is/are -

(A) 0 (B) 1 (C) infinite (D) none of these

16. Value of x and y which satisfy $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$ is :-

(A) $x = n\pi$, $y = \frac{1}{2}$, $n \in I$

(B) $x = \frac{n\pi}{2}$, $y = \frac{1}{2}$, $n \in I$

(C) $x = (2n+1)\frac{\pi}{2}$, $y = -\frac{1}{2}$, $n \in I$

(D) $x = 2n\pi \pm \frac{\pi}{3}$, $y = \frac{1}{2}$, $n \in I$

17. General solution of equation : $\tan^2\theta + \sec^2\theta + 3 = 2(\sqrt{2}\sec\theta + \tan\theta)$ is :-

(A) $n\pi + \frac{\pi}{4}$ (B) $2n\pi + \frac{\pi}{4}$ (C) $2n\pi + \frac{\pi}{3}$ (D) $2n\pi \pm \frac{\pi}{6}$

18. The solution of inequality $4^{\tan x} - 3.2^{\tan x} + 2 \le 0$ is

(A) $x \in \left[n\pi, n\pi + \frac{\pi}{4}\right]$; $n \in I$

(B) $x \in \left[n\pi, n\pi - \frac{\pi}{4}\right]$; $n \in I$

(C) $x \in \left[n\pi, n\pi + \frac{\pi}{6}\right]$; $n \in I$

(D) $x \in \left[n\pi, n\pi - \frac{\pi}{6}\right]$; $n \in I$

PART-II : NUMERICAL QUESTIONS

1. If $3\sin\alpha = 7\sin\beta$, then find the value of $\dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$.

2. If $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$ and $\cos 3\theta = \frac{1}{2}\left(a^k + \frac{1}{a^k}\right)$ then number of natural numbers 'k' less than 50 is (given $a \in R$)

3. If three angles A, B, C are such that
$\cos A + \cos B + \cos C = 0$ and if
$\cos A \cos B \cos C = \lambda(\cos 3A + \cos 3B + \cos 3C)$, then value of 3λ is :

4. Given $2y\cos\theta = x\sin\theta$ and $2x\sec\theta - y\,\mathrm{cosec}\,\theta = 3$, then find the value of $(x^2 + 4y^2)$

5. Let $f_K(x) = \sum_{n=1}^{K} \frac{\cos(3n+1) - \cos(3n+2)x}{1+2\cos(2n+1)x}$ then find $2f_{10}\left(\frac{\pi}{4}\right)+5$

6. If $7\theta = \pi$ then find exact value of $\cos\theta + \cos^2 15\theta + 2\cos^3 8\theta$

7. If $2\tan^2 x - 5\sec x - 1 = 0$ has 7 different roots in $\left[0, \frac{n\pi}{2}\right]$, $n \in N$, then find sum of all posible values of n.

8. Greatest integral values of a for which the equation cos 2x + a sin x = 2a – 7 possesses a solution.

9. If M and m denote maximum and minimum value of $\sqrt{49\cos^2\theta + \sin^2\theta} + \sqrt{49\sin^2\theta + \cos^2\theta}$ then find the value of (M + m)

10. In any triangle ABC, which is not right angled $\sum$ cosA .cosecB.cosecC is equal to

11. Value of x satisfying equation $\log_{\frac{-x^2-6x}{10}}(\sin 3x + \sin x) = \log_{\frac{-x^2-6x}{10}}(\sin 2x)$ is $-\frac{m\pi}{n}$ where m, n are coprime then $\frac{m}{n}$ is equal to

12. Number of values of x in $[0, 6\pi]$ satisfying trignometric equation $2\sin(11x) + \cos(3x) + \sqrt{3}\sin(3x) = 0$ is

13. The sum of all solution of the equation $2(\cos x + \cos 2x) + \sin 2x(1 + 2\cos x) = 2\sin x$ $(-\pi \le x \le \pi)$ is $\frac{-a\pi}{b}$ then $\frac{a}{b}$ is equal to

14. Arithmatic mean of all solution of trignometric equation $\tan^2 x \cdot \tan^2 3x \cdot \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x$ in $[0, \pi]$ is $\frac{k\pi}{m}$ find $\frac{k}{m}$.

15. Number of solutions of equation $\tan^2 x - 3\tan x + \cot^2 x - 3\cot x + 4 = 0$ in $[-50\pi, 50\pi]$ is :-

16. Sum of all the solutions of the inequation $\sqrt{2-\cos^7 x} \le \sin^{19} x + \cos^{21} x$ in the interval $[-2\pi, 2\pi]$ is :-

17. Find the number of values of θ, $0 \le \theta \le 2\pi$ such that the graph of $f(x) = \left(2\sin^2\frac{\theta}{2}\right)x^2 + \left(\cot\frac{\theta}{2}\right)x - 1$ and $g(x) = \left(2\cos^2\frac{\theta}{2}\right)x^2 - \left(\tan\frac{\theta}{2}\right)x + \cot^2\theta$ has exactly one point in common.

18. If sum of all the solutions of the equation cot x+ cosec x + sec x = tan x in $[0, 2\pi]$ is $\frac{k\pi}{\ell}$ then find $\frac{k}{\ell}$.

19. Number of value (s) of $x \in [0, 4\pi]$ satisfying the equation cos x – tanx = cot x is

20. In $(-3\pi, 3\pi)$, find the number of solutions of the equation $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta.\tan 2\theta.\tan 3\theta$

21. If $0 \le x \le 4\pi$, $0 \le y \le 4\pi$ and cos x . sin y = 1, then find the possible number of values of the ordered pair (x, y)

22. Find the number of values of θ satisfying the equation $\sin 3\theta = 4\sin\theta.\ \sin 2\theta.\ \sin 4\theta$ in $0 \le \theta \le 2\pi$

23. Consider the equation for $0 \le \theta \le 2\pi$; $\left(\sin 2\theta + \sqrt{3}\cos 2\theta\right)^2 - 5 = \cos\left(\frac{\pi}{6} - 2\theta\right)$. If greatest value of θ is $\frac{k\pi}{p}$ (k, p are coprime), then find $\frac{k}{p}$.

24. Find the number of solutions of $\sin\theta + 2\sin 2\theta + 3\sin 3\theta + 4\sin 4\theta = 10$ in$(0, 6\pi)$.

25. Find the values of x satisfying the equation $2 \sin x = 3x^2 + 2x + 3$.

26. Number of solution of sinx cosx – 4cosx + 6 sinx –25 > 0 in $[0, 4\pi]$ is :-

27. Number of solution of the equation $(1+\sin\theta)^{1/8} + \left(\frac{1}{\sin^8\theta} + \frac{1}{\sin^7\theta}\right)^{1/8} = 2^{9/8}(\sin\theta)^{1/8}$ in $[0, 2\pi]$ is :-

PART - III : ONE OR MORE THAN ONE CORRECT

1. The value of $\frac{(\cos 17° - \sin 17°)}{(\cos 17° + \sin 17°)}$ is

(A) tan 332° (B) tan 28° (C) cot 242° (D) cot 62°

2. If $\frac{\sin^4 x}{5} + \frac{\cos^4 x}{4} = \frac{1}{9}$, then which of the following is/are TRUE ?

(A) $\cot^2 x = \frac{4}{5}$ (B) $\tan^2 x = \frac{4}{5}$

(C) $\frac{64}{\cos^6 x} + \frac{125}{\sin^6 x} = 1458$ (D) $\frac{125}{\cos^6 x} + \frac{64}{\sin^6 x} = 1458$

3. If $\frac{1+\cos 2x}{\sin 2x} + 3\left(1 + (\tan x)\tan\frac{x}{2}\right)\sin x = 4$, then the value of tanx can be equal to

(A) 1 (B) $\frac{1}{2}$ (C) 3 (D) $\frac{1}{3}$

4. If sin x + sin y = a and cos x + cos y = b, then which of the following may be true.

(A) $\sin(x+y) = \frac{2ab}{a^2+b^2}$ (B) $\tan\frac{x-y}{2} = \sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

(C) $\tan\frac{x-y}{2} = -\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$ (D) $\cos(x+y) = \frac{2ab}{a^2+b^2}$

5. Which of the following is correct ?

(A) sin 2° > sin 2 (B) sin 2° < sin 2

(C) cos 2° > cos 2 (D) cos 2° < cos 2

6. If cos x + cos y = a, cos 2x + cos 2y = b, cos 3x + cos 3y = c, then

(A) $\cos^2 x + \cos^2 y = 1 + \frac{b}{2}$ (B) $\cos x \cdot \cos y = \frac{a^2}{2} - \left(\frac{b+2}{4}\right)$

(C) $2a^3 + c = 3a(1 + b)$ (D) a + b + c = 3abc

7. If $P_n = \cos^n\theta + \sin^n\theta$ and $Q_n = \cos^n\theta - \sin^n\theta$, then which of the following is/are true.

(A) $P_n - P_{n-2} = -\sin^2\theta\cos^2\theta\, P_{n-4}$ (B) $Q_n - Q_{n-2} = -\sin^2\theta\cos^2\theta\, Q_{n-4}$

(C) $P_4 = 1 - 2\sin^2\theta\cos^2\theta$ (D) $Q_4 = \cos^2\theta - \sin^2\theta$

8. If 2 cos θ + sin θ = 1, then the value of 4 cos θ + 3 sin θ is equal to

(A) 3 (B) –5 (C) $\frac{7}{5}$ (D) – 4

9. If $\cot x = \frac{4}{3}$, $x \in \left(\pi, \frac{3\pi}{2}\right)$ and $\tan y = \frac{-5}{12}$, $y \in \left(\frac{3\pi}{2}, 2\pi\right)$ then which of the following is(are) correct?

(A) $\sin(x + y) = \frac{-56}{65}$ (B) $\cos(x - y) = \frac{-33}{65}$ (C) $\sin 3x = \frac{-117}{125}$ (D) $\cos 2y = \frac{119}{169}$

10. If $\tan^2\alpha + 2\tan\alpha.\tan2\beta = \tan^2\beta + 2\tan\beta.\tan2\alpha$, then

(A) $\tan^2\alpha + 2\tan\alpha.\tan2\beta = 0$ (B) $\tan\alpha + \tan\beta = 0$

(C) $\tan^2\beta + 2\tan\beta.\tan2\alpha = 1$ (D) $\tan\alpha = \tan\beta$

11. If A, B, C are angle of ΔABC and tan A tan C = 3, tan B tan C = 6 then :-

(A) $A = \frac{\pi}{4}$ (B) tan (A + B) = –3 (C) $\tan(B - A) = \frac{1}{3}$ (D) cot (C–A) = 2

12. If the sides of a right angled triangle are {cos2α + cos2β + 2cos(α + β)} and {sin2α + sin2β + 2sin(α + β)}, then the length of the hypotenuse is:

(A) 2[1+cos(α – β)] (B) 2[1 – cos(α + β)] (C) $4\cos^2\frac{\alpha - \beta}{2}$ (D) $4\sin^2\frac{\alpha + \beta}{2}$

13. (a + 2) sin α + (2a – 1) cos α = (2a + 1) if tan α =

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$ (C) $\frac{2a}{a^2+1}$ (D) $\frac{2a}{a^2-1}$

14. Let $S = \sec^2\theta\sec^2\phi + 4\sec^2\theta\,\mathrm{cosec}^2\phi + 9\,\mathrm{cosec}^2\theta$ where $\theta, \phi \in \left(0, \frac{\pi}{2}\right)$. Which of the following statements is/are true?

(A) The minimum value of S is 36

(B) The minimum value of S is 18

(C) The minimum value of S occurs at $\theta = \frac{\pi}{4}$, $\phi = \tan^{-1}\sqrt{2}$

(D) The minimum value of S occurs at $\phi = \frac{\pi}{4}$, $\theta = \tan^{-1}\sqrt{2}$

15. If $\tan x = \frac{2b}{a-c}$, $(a \neq c)$

$y = a\cos^2 x + 2b\sin x\cos x + c\sin^2 x$

$z = a\sin^2 x - 2b\sin x\cos x + c\cos^2 x$, then

(A) $y = z$ (B) $y + z = a + c$ (C) $y - z = a - c$ (D) $y - z = (a-c)^2 + 4b^2$

16. If $a = \frac{5}{\sin^4 x + \cos^4 x - \frac{1}{2}\sin 2x + 1}$, then a can be

(A) 2 (B) 3 (C) 4 (D) 5

17. The equation $\sin^6 x + \cos^6 x = a^2$ has real solution if

(A) $a \in (-1, 1)$ (B) $a \in \left(-1, -\frac{1}{2}\right)$ (C) $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ (D) $a \in \left(\frac{1}{2}, 1\right)$

18. If $2\sec^2\alpha - \sec^4\alpha - 2\operatorname{cosec}^2\alpha + \operatorname{cosec}^4\alpha = 15/4$, then $\tan\alpha$ is equal to

(A) $1/\sqrt{2}$ (B) $1/2$ (C) $1/2\sqrt{2}$ (D) $-1/\sqrt{2}$

19. Let $y = \frac{\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x + \cos 6x + \cos 7x}{\sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x + \sin 6x + \sin 7x}$, then which of the following hold good?

(A) The value of y when $x = \pi/8$ is not defined. (B) The value of y when $x = \pi/16$ is 1.

(C) The value of y when $x = \pi/32$ is $\sqrt{2} - 1$. (D) The value of y when $x = \pi/48$ is $2 + \sqrt{3}$.

20. In ΔABC if $\sin A \sin(B - C) = \sin C \sin(A - B)$, then (where $A \neq B \neq C$)

(A) tan A, tan B, tan C are in AP

(B) cot A, cot B, cot C are in AP

(C) cos2A, cos 2B, cos2C are in AP

(D) sin 2A, sin2B, sin 2C are in AP

21. If $x + y = z$, then $\cos^2 x + \cos^2 y + \cos^2 z - 2\cos x \cos y \cos z$ is equal to

(A) $\cos^2 z$ (B) $\sin^2 z$ (C) $\cos(x + y - z)$ (D) 1

22. The equation $\sin x + \cos(k + x) + \cos(k - x) = 2$ has real solution(s), then sin k can be :-

(A) $\frac{-3}{4}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

23. If $\sin x(3 - 2\cos 2x) = 6\sin^2 x - 1$, then $(\cos 2x + \sin x - 1)$ is equal to :-

(A) 1 (B) -1 (C) $\frac{3}{2}$ (D) $\frac{-1}{2}$

24. If the quardratic equations $x^2 + (\sin\theta)x + \operatorname{cosec}\theta = 0$ $(\theta \in (0, \pi))$ and $2x^2 + x + c = 0$ (where $c \in R$) have a common root, then:-

(A) c = 4 (B) c = 2

(C) sum of all values of θ is π (D) number of solution of θ is 4

25. If the equation $2(1 + a^2) = \sin 2\theta + 2a(\sin\theta + \cos\theta)$ has real solution, then which of the following statements is (are) true ?

(A) Sum of all possible value of 'a' is zero

(B) 'a' can take only two real values.

(C) Number of values of θ satisfying the equation in $[0, 4\pi]$ are 4.

(D) Number of values of θ satisfying the equation in $[0, 4\pi]$ are 2.

26. If (sin x + 2 cos x) (cos x + 2 sin x) = $\frac{9}{2}$, then x can be equal to :-

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) $\frac{5\pi}{4}$ (D) $\frac{-3\pi}{4}$

27. If the expression (cos 3θ + sin 3θ) + (2 sin 2θ – 3). (sinθ – cosθ) is positive, then θ equal to :-

(A) $\left(2n\pi-\frac{3\pi}{4}, 2n\pi+\frac{\pi}{4}\right), n\in I$ (B) $\left(2n\pi-\frac{\pi}{2}, 2n\pi+\frac{\pi}{6}\right), n\in I$

(C) $\left(2n\pi-\frac{\pi}{3}, 2n\pi+\frac{\pi}{3}\right), n\in I$ (D) $\left(2n\pi-\frac{\pi}{4}, 2n\pi+\frac{3\pi}{4}\right), n\in I$

28. The general solution of the equation cosx . cos6x = – 1, is :

(A) x = (2n + 1)π, n ∈ I (B) x = 2nπ, n ∈ I
(C) x = (2n – 1)π, n ∈ I (D) none of these

29. Which of the following set of values of x satisfy the inequation sin 3x < sin x.

(A) $\left(\frac{(8n-1)\pi}{4}, 2n\pi\right)$, n ∈ I (B) $\left(\frac{(8n-1)\pi}{4}, \frac{(8n+1)\pi}{4}\right)$, n ∈ I

(C) $\left(\frac{(8n+1)\pi}{4}, \frac{(8n+3)\pi}{4}\right)$, n ∈ I (D) $\left((2n+1)\pi, \frac{(8n+5)\pi}{4}\right)$, n ∈ I

30. Solution set of inequality $\sin^3 x \cos x > \cos^3 x \sin x$, where x ∈ (0, π), is

(A) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (B) $\left(\frac{3\pi}{4}, \pi\right)$ (C) $\left(0, \frac{\pi}{4}\right)$ (D) $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

31. $4\sin^4 x + \cos^4 x = 1$ if

(A) x = nπ ; (n ∈ I) (B) x = nπ ± $\frac{1}{2}\cos^{-1}\left(\frac{1}{5}\right)$; (n ∈ I)

(C) x = $\frac{n\pi}{2}$; (n ∈ I) (D) x = – nπ ; (n ∈ I)

PART - IV : COMPREHENSION

Comprehenssion # 1

Let p be the product of the sines of the angles of a triangle ABC and q is the product of the cosines of the angles.

1. In this triangle tan A + tan B + tan C is equal to

(A) p + q (B) p – q (C) $\frac{p}{q}$ (D) none of these

2. tan A tan B + tan B tan C + tan C tan A is equal to

(A) 1 + q (B) $\frac{1+q}{q}$ (C) 1 + p (D) $\frac{1+p}{p}$

3. $\tan^3 A + \tan^3 B + \tan^3 C$ is

(A) $\frac{p^3-3pq^2}{q^3}$ (B) $\frac{q^3}{p^3}$ (C) $\frac{p^3}{q^3}$ (D) $\frac{p^3-3pq}{q^3}$

Comprehension # 2

Let f(x) = $\sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k.

4. All real number k for which f(x) is constant for all values of x is :-

(A) $k = \frac{2}{3}$ (B) $k = \frac{-3}{2}$ (C) $k = \frac{5}{2}$ (D) $k = \frac{-3}{5}$

5. All real number k for which there exists a real number 'c' such that f(c) = 0, is :-

(A) $\left[-1, \frac{-1}{2}\right]$ (B) [–3, –1] (C) [–2, –1] (D) [0, 1]

6. If k = –0.7, then all solutions to the equation f(x) = 0, is

(A) $x = \frac{n\pi}{2} \pm \frac{\pi}{6}, n \in I$ (B) $x = n\pi \pm \frac{\pi}{6}, n \in I$ (C) $x = n\pi - \frac{\pi}{6}, n \in I$ (D) $x = n\pi, n \in I$

Comprehension # 3

To solve a trigonometric inequation of the type sin x ≥ a where |a| ≤ 1, we take a hill of length 2π in the sine curve and write the solution within that hill. For the general solution, we add $2n\pi$. For instance, to solve $\sin x \geq -\frac{1}{2}$, we take the hill $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ over which solution is $-\frac{\pi}{6} < x < \frac{7\pi}{6}$. The general solution is $2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}$, n is any integer. Again to solve an inequation of the type sin x ≤ a, where |a| ≤ 1, we take a hollow of length 2π in the sine curve. (since on a hill, sinx ≤ a is satisfied over two intervals). Similarly cos x ≥ a or cosx ≤ a, |a| ≤ 1 are solved.

7. Solution to the inequation $\sin^6 x + \cos^6 x < \frac{7}{16}$ must be

(A) $n\pi + \frac{\pi}{3} < x < n\pi + \frac{\pi}{2}$ (B) $2n\pi + \frac{\pi}{3} < x < 2n\pi + \frac{\pi}{2}$

(C) $\frac{n\pi}{2} + \frac{\pi}{6} < x < \frac{n\pi}{2} + \frac{\pi}{3}$ (D) none of these

8. Solution to inequality cos 2x + 5 cos x + 3 ≥ 0 over $[-\pi, \pi]$ is

(A) $[-\pi, \pi]$ (B) $\left[\frac{-5\pi}{6}, \frac{5\pi}{6}\right]$ (C) $[0, \pi]$ (D) $\left[\frac{-2\pi}{3}, \frac{2\pi}{3}\right]$

9. Over $[-\pi, \pi]$, the solution of $2\sin^2\left(x + \frac{\pi}{4}\right) + \sqrt{3}\cos 2x \geq 0$ is

(A) $[-\pi, \pi]$ (B) $\left[\frac{-5\pi}{6}, \frac{5\pi}{6}\right]$

(C) $[0, \pi]$ (D) $\left[-\pi, \frac{-7\pi}{12}\right] \cup \left[-\frac{\pi}{4}, \frac{5\pi}{12}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. The maximum value of the expression $\dfrac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is

[IIT-JEE-2010, Paper-1, (3, 0)/84]

2. The positive integer value of n > 3 satisfying the equation

$\dfrac{1}{\sin\left(\frac{\pi}{n}\right)} = \dfrac{1}{\sin\left(\frac{2\pi}{n}\right)} + \dfrac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

3. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for n = 0, ±1, ± 2 and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is **[IIT-JEE-2010, Paper-1, (3, 0)/84]**

4. Let P = $\{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and Q = $\{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then

[IIT-JEE 2011, Paper-1, (3, –1), 80]

(A) $P \subset Q$ and $Q - P \neq \varnothing$
(B) $Q \not\subset P$
(C) $P \not\subset Q$
(D) P = Q

5. Let $\theta, \phi \in [0, 2\pi]$ be such that $2\cos\theta(1 - \sin\phi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1$, $\tan(2\pi - \theta) > 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$. Then ϕ cannot satisfy **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

(A) $0 < \phi < \frac{\pi}{2}$
(B) $\frac{\pi}{2} < \phi < \frac{4\pi}{3}$
(C) $\frac{4\pi}{3} < \phi < \frac{3\pi}{2}$
(D) $\frac{3\pi}{2} < \phi < 2\pi$

6. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

(A) infinitely many solutions
(B) three solutions
(C) one solution
(D) no solution

7. The number of distinct solutions of the equation

$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is

[JEE (Advanced) 2015, P-1 (4, 0) /88]

8. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the quation $x^2 - 2x \sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

[JEE(Advanced)-2016, Paper-1, (3, –1)/62]

(A) $2(\sec\theta - \tan\theta)$ (B) $2\sec\theta$ (C) $-2\tan\theta$ (D) 0

9. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x + \text{cosec}x + 2(\tan x - \cot x) = 0$ in the set S is equal to -

[JEE(Advanced)-2016, Paper-1, (3, –1), 62]

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

10. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to **[IIT-JEE 2016, Paper-2, (3, –1), 62]**

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$ (C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$

11. Let α and β be nonzero real numbers such taht that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then which of the following is/are true? **[JEE(Advanced)-2017]**

(A) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$ (B) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$ (D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

12. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3}a\cos x + 2b\sin x = c,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is ______

[JEE(Advanced)-2018, 3(0)]

PART - II : AIEEE PROBLEMS (LAST 10 YEARS)

1. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60º. He moves away from the pole along the line BC to a point D such that CD = 7 m. From D the angle of elevation of the point A is 45º. Then the height of the pole is-

[AIEEE 2008 (3, –1), 105]

(1) $\frac{7\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}+1}\right)$ m (2) $\frac{7\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}-1}\right)$ m (3) $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$ m (4) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)$ m

2. Let A and B denote the statements **[AIEEE 2009 (4, –1), 144]**

A : $\cos\alpha + \cos\beta + \cos\gamma = 0$

B : $\sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$, then :

(1) A is false and B is true
(2) both A and B are true
(3) both A and B are false
(4) A is true and B is false

3. Let $\cos(\alpha+\beta) = \frac{4}{5}$ and let $\sin(\alpha-\beta) = \frac{5}{13}$, where $0 \le \alpha, \beta \le \frac{\pi}{4}$. Then $\tan 2\alpha$ =

[AIEEE 2010 (4, –1), 144 JEE Mains-19]

(1) $\frac{56}{33}$ (2) $\frac{19}{12}$ (3) $\frac{20}{7}$ (4) $\frac{25}{16}$

4. If $A = \sin^2 x + \cos^4 x$, then for all real x : **[AIEEE 2011 (4, –1), 120]**

(1) $\frac{3}{4} \le A \le 1$ (2) $\frac{13}{16} \le A \le 1$ (3) $1 \le A \le 2$ (4) $\frac{3}{4} \le A \le \frac{13}{16}$

5. In a ΔPQR, if 3 sin P + 4 cos Q = 6 and 4 sin Q + 3 cos P = 1, then the angle R is equal to :

[AIEEE-2012, (4, –1)/120]

(1) $\frac{5\pi}{6}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{4}$ (4) $\frac{3\pi}{4}$

6. ABCD is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, BC = p and CD = q, then AB is equal to : **[AIEEE - 2013, (4, –¼),360]**

(1) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (2) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$ (3) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (4) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

7. The expression $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A}$ can be written as : **[AIEEE - 2013, (4, –¼),360]**

(1) sinA cosA + 1
(2) secA cosecA + 1
(3) tanA + cotA
(4) secA + cosecA

8. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ where $x \in R$ and $k \geq 1$. Then $f_4(x) - f_6(x)$ equals

[JEE(Main) 2014, (4, – ¼), 120]

(1) $\frac{1}{4}$ (2) $\frac{1}{12}$ (3) $\frac{1}{6}$ (4) $\frac{1}{3}$

9. If the angles of elevation of the top of a tower from three collinear points A, B and C, on a line leading to the foot of the tower, are 30º, 45º and 60º respectively, then the ratio, AB : BC , is **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $\sqrt{3}:1$ (2) $\sqrt{3}:\sqrt{2}$ (3) $1:\sqrt{3}$ (4) 2 : 3

10. If $0 \leq x < 2\pi$, then the number of real values of x, which satisfy the equation
cosx + cos2x + cos3x + cos4x = 0, is :- **[JEE(Main)-2016, (4, – 1), 120]**

(1) 5 (2) 7 (3) 9 (4) 3

11. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of cos4x is :- **[JEE(Main)-2017, (4, – 1), 120]**

(1) $-\frac{7}{9}$ (2) $-\frac{3}{5}$ (3) $\frac{1}{3}$ (4) $\frac{2}{9}$

12. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If $\angle BPC = \beta$, then $\tan\beta$ is equal to

(1) $\frac{6}{7}$ (2) $\frac{1}{4}$ (3) $\frac{2}{9}$ (4) $\frac{4}{9}$

13. If sum of all the solutions of the equation $8\cos x \cdot \left(\cos\left(\frac{\pi}{6}+x\right).\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right) = 1$ in $[0, \pi]$ is $k\pi$, then k is equal to : **[JEE(Main) 2018]**

(1) $\frac{13}{9}$ (2) $\frac{8}{9}$ (3) $\frac{20}{9}$ (4) $\frac{2}{3}$

14. For any $\theta \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$, the expression $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals : **[JEE(Main)-Jan 19]**

(1) $13 - 4\cos^6\theta$ (2) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$
(3) $13 - 4\cos^2\theta + 6\cos^4\theta$ (4) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$

15. If $0 \leq x < \frac{\pi}{2}$, then the number of values of x for which sin x – sin2x + sin3x = 0, is

[JEE(Main)-Jan 19]

(1) 2 (2) 1 (3) 3 (4) 4

16. Let α and β be two real roots of the equation $(k + 1)\tan^2 x - \sqrt{2}\,.\,\lambda\tan x = (1 - k)$, where $k(\neq -1)$ and λ are real numbers. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is ; **[JEE(Main)-2020 (Jan)]**

(1) 5 (2) 10 (3) $5\sqrt{2}$ (4) $10\sqrt{2}$

Answers

Exercise # 1

PART - I

SECTION-(A)

A-1. (i) $\frac{5\pi}{12}$ (ii) $\frac{7\pi}{6}$

(iii) $\frac{43\pi}{9}$

A-2. (i) 150° (ii) 1440°

(iii) –120° (iv) 165°

A-3. (i) – 0.5 (ii) 0

(iii) 14.5 (iv) 6

A-4. (i) $\left(\frac{-\sqrt{3}}{2}\right)$ (ii) $-\frac{1}{\sqrt{2}}$

(iii) $-\frac{1}{\sqrt{3}}$ (iv) 1

A-6. $\frac{181}{338}$

SECTION-(B)

B-2. (i) $\sqrt{3}/2$ (ii) $\sqrt{3}/4$

B-3. $\frac{k}{\sqrt{2}}$

B-7. $x^2 + y^2 = a^2 + b^2$

SECTION-(C)

C-5. $\frac{\sqrt{3}-1}{2\sqrt{2}}$

C-8. (i) 1 (ii) $-\sqrt{5}/4$

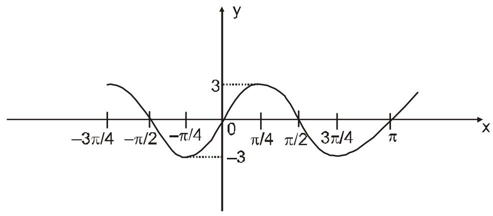
(iii) $\frac{\sqrt{5}+1}{8}$

SECTION-(D)

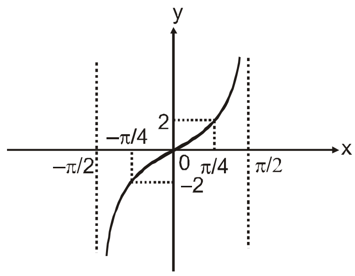
D-3. 1/4

SECTION-(E)

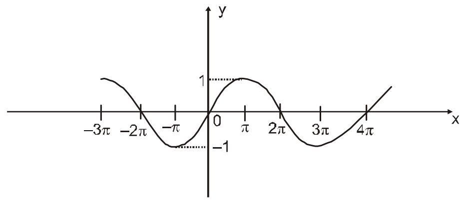
E-1. (i)



(ii)



(iii)



E-2. $-\frac{1}{4}, \frac{1}{4}$

E-3. (i) 3, –5 (ii) 1, $\frac{1}{11}$

(iii) 243, $\frac{1}{243}$

E-4. (i) $y_{max} = 11$; $y_{min} = 1$

(ii) $y_{max} = 10$; $y_{min} = -4$

E-5. 2 **E-6.** 3

SECTION-(F)

F-1. (i) $2n\pi \pm \frac{\pi}{4}, n \in I$

(ii) $n\pi + \frac{\pi}{6} + 1, n \in I$

(iii) $n\pi + \tan^{-1}(2), n \in I$

(iv) $n\pi + (-1)^n \frac{\pi}{6}, n \in I$

(v) $n\pi \pm \frac{\pi}{4}, n \in I$

F-2. (i) $\frac{m\pi}{3}$, $m \in I$ or $\frac{(2m+1)\pi}{12}$, $m \in I$

(ii) $2n\pi \pm \frac{\pi}{3}$, $n \in I$

(iii) $\left(2n+\frac{1}{2}\right)\frac{\pi}{7}$, $n \in I$ or $2n\pi - \frac{\pi}{2}$, $n \in I$

(iv) $\left(n+\frac{1}{2}\right)\frac{\pi}{11}$, $n \in I$

(v) $\left(n+\frac{1}{4}\right)\frac{\pi}{2}$, $n \in I$

(vi) $2n\pi + \frac{2\pi}{3}$, $n \in I$

(vii) $(2n+1)\frac{\pi}{10}$, $n \in I$

(viii) $\left(n+\frac{1}{3}\right)\frac{\pi}{3}$, $n \in I$

F-3. (i) $k\pi + (-1)^k \frac{\pi}{6}$, $(2k+1)\pi$, $k \in I$

(ii) $2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$, $n \in I$

(iii) $n\pi + \tan^{-1} 3$ or $n\pi + \tan^{-1} 4$, $n \in I$.

F-4. (i) $\frac{n\pi}{3}$, $n \in I$ or $\left(n \pm \frac{1}{3}\right)\pi$, $n \in I$

(ii) $2n\pi$, $n \in I$ or $\frac{2n\pi}{3} + \frac{\pi}{6}$, $n \in I$

(iii) $x = (2n+1)\frac{\pi}{4}$, $n \in I$

or $x = (2n+1)\frac{\pi}{2}$, $n \in I$

or $x = n\pi \pm \frac{\pi}{6}$, $n \in I$

(iv) $m\pi$, $m \in I$ or $\frac{m\pi}{n-1}$, $m \in I$

or $\left(m+\frac{1}{2}\right)\frac{\pi}{n}$, $m \in I$

F-5, (i) $2n\pi + \frac{2\pi}{3}$, $n \in I$

(ii) $2n\pi$, $(4n+1)\frac{\pi}{2}$, $n \in I$

(iii) $2n\pi + 2\tan^{-1}\frac{1}{2}$, $n \in I$

F-6. $\alpha \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

F-7. $\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{11\pi}{6}\right)$, $n \in I$

F-8. $\left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{7\pi}{6}\right]$, $n \in I$

SECTION-(G)

G-1. $15\sqrt{3}$ m **G-2.** 180 metres.
G-4. 50 m

PART - II

SECTION-(A)

A-1.	(D)	**A-2.**	(B)
A-3	(C)	**A-4.**	(A)
A-5	(B)	**A-6**	(C)
A-7	(A)		

SECTION-(B)

B-1.	(A)	**B-2.**	(A)
B-3.	(C)	**B-4.**	(A)
B-5.	(B)	**B-6.**	(A)
B-7.	(B)	**B-8.**	(C)
B-9.	(C)	**B-10.**	(A)

SECTION-(C)

C-1.	(A)	**C-2.**	(C)
C-3.	(A)	**C-4.**	(D)
C-5.	(A)	**C-6.**	(B)
C-7.	(B)	**C-8.**	(A)
C-9.	(C)	**C-10.**	(C)

SECTION-(D)

D-1.	(D)	**D-2.**	(A)
D-3.	(C)	**D-4.**	(B)
D-5.	(A)		

SECTION-(E)

E-1.	(A)	**E-2.**	(C)
E-3.	(D)	**E-4.**	(A)
E-5.	(D)		

SECTION-(F)

F-1.	(C)	**F-2.**	(D)
F-3.	(A)	**F-4.**	(B)
F-5.	(A)	**F-6.**	(D)
F-7.	(B)	**F-8.**	(B)
F-9.	(B)	**F-10.**	(B)
F-11.	(B)	**F-12.**	(A)

SECTION-(G)

G-1.	(D)	**G-2.**	(C)
G-3.	(A)	**G-4.**	(D)
G-5.	(B)		

PART - III

1. (A) $\rightarrow$ r; (B) $\rightarrow$ s; (C) $\rightarrow$ p; (D) $\rightarrow$ q

2. (A) $\rightarrow$ (q, s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p)

3. (A) $\rightarrow$ (s), (B) $\rightarrow$ (s), (C) $\rightarrow$ (s), (D) $\rightarrow$ (r)

Exercise # 2

PART - I

1.	(D)	**2.**	(D)
3.	(B)	**4.**	(B)
5.	(B)	**6.**	(B)
7.	(D)	**8.**	(B)
9.	(D)	**10.**	(D)
11.	(A)	**12.**	(C)
13.	(C)	**14.**	(A)
15.	(A)	**16.**	(A)
17.	(B)	**18.**	(A)

PART - II

1.	2.50	**2.**	25.00
3.	0.25	**4.**	4.00
5.	6.41	**6.**	0.25
7.	42.00	**8.**	6.00
9.	18.00	**10.**	2
11.	1.66 or 1.67	**12.**	42.00
13.	0.50	**14.**	0.46
15.	100.00	**16.**	0.00
17.	4.00	**18.**	2.50
19.	0	**20.**	17
21.	8	**22.**	15
23.	1.58	**24.**	0
25.	0	**26.**	0.00
27.	1.00		

PART - III

1.	(B,C,D)	**2.**	(A, C)
3.	(A, D)	**4.**	(A,B,C)
5.	(B,C)	**6.**	(A, B, C)
7.	(A,B,C,D)	**8.**	(A, C)
9.	(B, C, D)	**10.**	(B,C,D)
11.	(A,B,C,D)	**12.**	(A,C)
13.	(B,D)	**14.**	(A, C)
15.	(B,C)	**16.**	(B, C, D)
17.	(B,D)	**18.**	(A,D)
19.	(B, D)	**20.**	(B, C)
21.	(C,D)	**22.**	(B,C)
23.	(B,D)	**24.**	(A,C)
25.	(A,B,C)	**26.**	(A,C,D)
27.	(A,B)	**28.**	(A,C)
29.	(A,C,D)	**30.**	(A,B)
31.	(A,B,D)		

PART - II

1.	(C)	**2.**	(B)
3.	(D)	**4.**	(B)
5.	(A)	**6.**	(A)
7.	(C)	**8.**	(D)
9.	(D)		

Exercise # 3

PART - I

1.	2	**2.**	(n = 7)
3.	3	**4.**	(D)
5.	(A,C,D)	**6.**	(D)
7.	8	**8.**	(C)
9.	(C)	**10.**	(C)
11.	(B,C)	**12.**	0.5

PART - II

1.	(3)	**2.**	(2)
3.	(1)	**4.**	(1)
5.	(2)	**6.**	(1)
7.	(2)	**8.**	(2)
9.	(1)	**10.**	(2)
11.	(1)	**12.**	(3)
13.	(1)	**14.**	(1)
15.	(1)	**16.**	(2)

1. Let $\alpha = 4\sin^2 10^\circ + 4\sin^2 50^\circ \cdot \cos 20^\circ + \cos 80^\circ$ and $\beta = \cos^2\frac{\pi}{5} + \cos^2\frac{2\pi}{15} + \cos^2\frac{8\pi}{15}$ find $(\alpha + \beta)$

2. Simplify the expression $\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$

3. Let a, b, c, d be numbers in the interval $[0, \pi]$ such that

$\sin a + 7\sin b = 4(\sin c + 2\sin d)$,

$\cos a + 7\cos b = 4(\cos c + 2\cos d)$

Prove that $2\cos(a - d) = 7\cos(b - c)$.

4. Find the value of α for which the three element set $S = \{\sin\alpha, \sin 2\alpha, \sin 3\alpha\}$ is equal to the three element set $T = \{\cos\alpha, \cos 2\alpha, \cos 3\alpha\}$

5. If $a\sin\theta + b\cos\theta = a\operatorname{cosec}\theta - b\sec\theta = 1$, prove that $a^2 + b^2 = 1 + b^{2/3} - b^{4/3}$.

6. If $p(\sin\alpha - \cos\alpha\tan\theta)\sec\theta = q\tan\theta \cdot \sec(\alpha - \theta)$ then prove that $\theta = \frac{1}{2}\cot^{-1}\left(\frac{q + p\cos 2\alpha}{p\sin 2\alpha}\right)$.

7. If $\tan\alpha = \frac{p}{q}$ where $\alpha = 6\beta$, α being an acute angle, prove that;

$\frac{1}{2}(p\operatorname{cosec} 2\beta - q\sec 2\beta) = \sqrt{p^2 + q^2}$.

8. Prove that $\frac{\cos 2\theta}{\sin 6\theta} + \frac{\cos 6\theta}{\sin 18\theta} + \frac{\cos 18\theta}{\sin 54\theta} = \frac{1}{2}[\cot 2\theta - \cot 54\theta]$

9. If $\sin(\theta + \alpha) = a$ & $\sin(\theta + \beta) = b$ $(0 < \alpha, \beta, \theta < \pi/2)$ then find the value of $\cos 2(\alpha - \beta) - 4ab\cos(\alpha - \beta)$

10. Show that: $4\sin 27^\circ = (5 + \sqrt{5})^{1/2} - (3 - \sqrt{5})^{1/2}$

11. If $\tan\beta = \frac{\tan\alpha + \tan\gamma}{1 + \tan\alpha.\tan\gamma}$, prove that $\sin 2\beta = \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha.\sin 2\gamma}$.

12. Prove that in an acute angled triangle ABC, $\sum \tan A \tan B \geq 9$.

13. If $xy + yz + xz = 1$, then prove that $\frac{x}{1 - x^2} + \frac{y}{1 - y^2} + \frac{z}{1 - z^2} = \frac{4xyz}{(1 - x^2)(1 - y^2)(1 - z^2)}$.

14. Let a = $\frac{\pi}{7}$

(a) Show that $\sin^2 3a - \sin^2 a = \sin 2a \sin 3a$

(b) Show that cosec a = cosec 2a + cosec 4a

(c) Evaluate cos a – cos 2a + cos 3a

(d) Prove that cos a is a root of the equation $8x^3 + 4x^2 - 4x + 1 = 0$

(e) Evaluate tan a tan 2a tan 3a

(f) Evaluate $\tan^2 a + \tan^2 2a + \tan^2 3a$

(g) Evaluate $\tan^2 a \tan^2 2a + \tan^2 2a \tan^2 3a + \tan^2 3a \tan^2 a$

(h) Evaluate $\cot^2 a + \cot^2 2a + \cot^2 3a$

15. If cos 2x – cos x = sin 4x – sin x (where tan x $\neq$ 1), then find the value of cos 3x – sin 3x.

16. Prove that $\tan(-314^\circ) > \frac{61}{59}$.

17. Evaluate cos a cos 2a cos 3a..........cos 999a, where a = $\frac{2\pi}{1999}$

18. Prove that the average of the numbers

$2 \sin 2^\circ, 4 \sin 4^\circ, 6 \sin 6^\circ, ...180 \sin 180^\circ$ is $\cot 1^\circ$

19. If $\sqrt{2}\cos A = \cos B + \cos^3 B$, $\sqrt{2}\sin A = \sin B - \sin^3 B$, prove that $\sin(A - B) = \pm\frac{1}{3}$.

20. If A,B,C and D are angles of a quadrilateral and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2} = \frac{1}{4}$, prove that

A = B = C = D = $\pi/2$.

21. Solve $\tan 2\theta = \tan\frac{2}{\theta}$.

22. Find number of solution which are common to $\frac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2 + \sin x} = \frac{\cos x}{3}$ and $2\sin^2 x - 3\sin x + 1 = 0$ in

$x \in [0, 2\pi]$

23. Solve the system of equations :

$x + y = \frac{2\pi}{3}$, $\cos x + \cos y = \frac{3}{2}$ for $x, y \in [0, 6\pi]$

24. Solve the following system of simultaneous equations for x and y:

$4^{\sin x} + 3^{1/\cos y} = 11$

$5.16^{\sin x} - 2.3^{1/\cos y} = 2$ for $x, y \in [0, 2\pi]$

25. The least positive angle measured in degree satisfying the equation

$(\sin x + \sin 2x + \sin 3x)^3 = \sin^3 x + \sin^3 2x + \sin^3 3x$ is

26. Solve the following system of equations for x and y :

$$5^{(\operatorname{cosec}^2 x - 3\sec^2 y)} = 1,\ 2^{(2\operatorname{cosec} x + \sqrt{3}|\sec y|)} = 64$$

27. Find number of solution of equation $8 \sin x = \frac{\sqrt{3}}{\cos x} + \frac{1}{\sin x}$ in $x \in [0, 2\pi]$

28. Solve : $\sin 2x > \sqrt{2}\sin^2 x + (2-\sqrt{2})\cos^2 x$

29. Solve for x and y :

$x \cos^3 y + 3x \cos y \sin^2 y = 14$

$x \sin^3 y + 3x \cos^2 y \sin y = 13$

30. Solve for x, the equation $\sqrt{13 - 18\tan x} = 6 \tan x - 3$, where $-2\pi < x < 2\pi$.

Answers

1. 4 **2.** $\cos^2 x - \sin^2 x = \cos 2x$ **4.** $\frac{n\pi}{2} + \frac{\pi}{8}, n \in I$ **9.** $1 - 2a^2 - 2b^2$

14. (c) $\frac{1}{2}$; (e) $\sqrt{7}$; (f) 21; (g) 35; (h) 5 **15.** 1 **17.** $\frac{1}{2^{999}}$

21. $\frac{n\pi}{4} \pm \sqrt{1 + \frac{n^2\pi^2}{16}}$, $n \in I$ **22.** $x = \frac{\pi}{2}$ **23.** ϕ

24. $x = \frac{\pi}{6}, \frac{5\pi}{6}, y = \frac{\pi}{3}, \frac{5\pi}{3}$ **25.** $\frac{2\pi}{5}$

26. $x = n\pi + (-1)^n \frac{\pi}{6}$ and $y = m\pi \pm \frac{\pi}{6}$; where $m, n \in I$ **27.** 6.00

28. $n\pi + \frac{\pi}{8} < x < n\pi + \frac{\pi}{4}, n \in I$ **29.** $x = \pm 5\sqrt{5}$, $y = n\pi + \tan^{-1}\frac{1}{2}$, $n \in I$

30. $\alpha - 2\pi$; $\alpha - \pi$, α, $\alpha + \pi$, where $\tan \alpha = \frac{2}{3}$