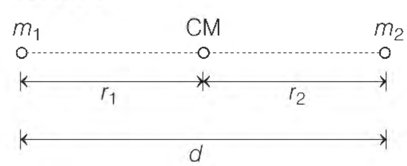
CHAPTER 07

Centre of Mass, Momentum and Impulse

Position of Centre of Mass



- **Two point masses**

$$r \propto \frac{1}{m} \quad \text{or} \quad \frac{r_1}{r_2} = \frac{m_2}{m_1}$$

$\therefore$
$$m_1 r_1 = m_2 r_2$$

$\Rightarrow$
$$r_1 = \frac{m_2}{m_1 + m_2} \cdot d$$

$\Rightarrow$
$$r_2 = \frac{m_1}{m_1 + m_2} \cdot d$$

- **More than two point masses**

(i) $$\mathbf{r}_{CM} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

(ii) $$X_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$\Rightarrow$
$$Y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$\Rightarrow$
$$Z_{CM} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}$$

- **More than two rigid bodies**

  (i) Centre of mass of symmetrical rigid body (like sphere, disc, cube, etc.) lies at its geometric centre.

  (ii) For two or more than two rigid bodies, we can use

$$\mathbf{r}_{CM} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

$$\Rightarrow \quad X_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\Rightarrow \quad Y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

and

$$Z_{CM} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}$$

  (iii) If three dimensional rigid body has uniform density, then mass in above formulae can be replaced by volume (V).

  For example, $$\mathbf{r}_{CM} = \frac{V_1 \mathbf{r}_1 + V_2 \mathbf{r}_2}{V_1 + V_2}$$

  (iv) In case of two dimensional body, mass can be replaced by area (A).

  For example, $$\mathbf{r}_{CM} = \frac{A_1 \mathbf{r}_1 + A_2 \mathbf{r}_2}{A_1 + A_2}$$

  (v) If some portion is removed from the body, then mass can be replaced by area (A).

  For example, $$\mathbf{r}_{CM} = \frac{A_1 \mathbf{r}_1 - A_2 \mathbf{r}_2}{A_1 - A_2}$$ (In case of two dimensional body)

  Here, A_1 = area of whole body (without removing),
  $\mathbf{r}_1$ = position vector of its centre of mass,
  A_2 = area of removed portion and
  $\mathbf{r}_2$ = position vector of centre of mass of removed portion.

Other Formulae of Centre of Mass

- $\mathbf{F}_{CM} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_T$
- $\mathbf{p}_{CM} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_T$
- $\mathbf{v}_{CM} = \dfrac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} = \dfrac{\mathbf{p}_{CM}}{M_{CM}} = \dfrac{\mathbf{p}_T}{M_T}$
- $\mathbf{a}_{CM} = \dfrac{m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2}{m_1 + m_2} = \dfrac{\mathbf{F}_{CM}}{M_{CM}} = \dfrac{\mathbf{F}_T}{M_T}$

Note *Here, T stands for total.*

Conservation of Linear Momentum

- **For a single mass or single body**

  If net force acting on the body is zero, then

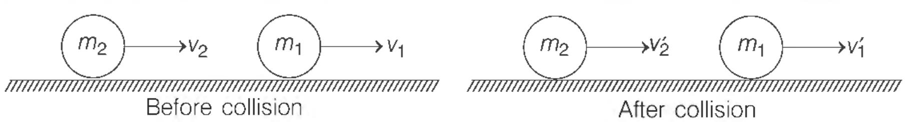
  $\mathbf{p}$ = constant or $\mathbf{v}$ = constant (if mass = constant)

- **For a system of particles or system of rigid bodies**

  If net external force acting on a system of particles or system of rigid bodies is zero, then $\mathbf{p}_{CM}$ = constant or $\mathbf{v}_{CM}$ = constant.

Collision

- **Head on elastic collision** In this case, linear momentum and kinetic energy both are conserved. After solving two conservation equations, we get



$$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2 \quad \text{...(i)}$$

and

$$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2 + \left(\frac{2m_1}{m_1 + m_2}\right) v_1 \quad \text{...(ii)}$$

In the above two formulae, following are three special cases :

(i) If $m_1 = m_2$, then $v_1' = v_2$ and $v_2' = v_1$ i.e. in case of equal masses bodies will exchange their velocities.

(ii) If $m_1 >> m_2$ and $v_1 = 0$, then $v_1' \approx 0$ and $v_2' \approx -v_2$.

(iii) If $m_2 >> m_1$ and $v_1 = 0$, then $v_1' = 2v_2$ and $v_2' \approx v_2$.

- **Head on inelastic collision** In this type of collision, only linear momentum remains constant. Suppose two unknowns velocities are v_1' and v_2'. Make following two equations to solve them.

  (i) Conservation of linear momentum

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

  (ii) Definition of coefficient of restitution (e)

$$e = \frac{|\text{relative velocity of separation}|}{|\text{relative velocity of approach}|} = \frac{v_1' - v_2'}{v_2 - v_1}$$

- **Perfectly inelastic collision** In perfectly inelastic collision, the colliding bodies stick together after collision and move with a common velocity given by

$$\mathbf{v} = \frac{\text{total momentum of the system}}{\text{total mass}} = \frac{\mathbf{p}_{\text{total}}}{M_{\text{total}}}$$

- **Oblique collision (both elastic and inelastic)** Resolve the velocities along common normal and common tangent directions. Now,

  (i) velocity components along common tangent direction will remain unchanged.

(ii) along common normal direction, theory of head on collision (elastic as well as inelastic) can be used.

- If a body is dropped from a height h, then

  v_0 = velocity just before striking with ground = $\sqrt{2gh}$

  v_1 = velocity just after striking with ground = ev_0

  v_n = velocity just after striking n^{th} time with ground = $e^n v_0$

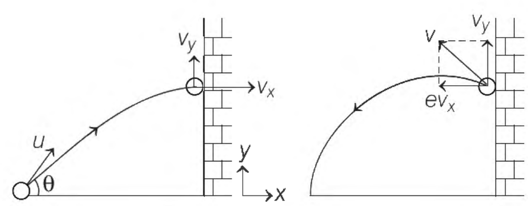
  h_n = height attained by the body after n^{th} collision = $\dfrac{v_n^2}{2g} = e^{2n} h$

  Here, e = coefficient of restitution.

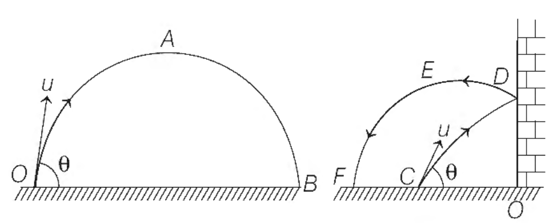
- Suppose a ball is projected with speed u at an angle θ with horizontal. It collides at some distance with a wall parallel to y-axis as shown in figure. Let v_x and v_y be the components of its velocity along x and y-directions at the time of impact with wall.

  Coefficient of restitution between the ball and the wall is e.

  Component of its velocity along y-direction (common tangent) v_y will remain unchanged while component of its velocity along x-direction (common normal) v_x will become ev_x in opposite direction.



Further, since v_y does not change due to collision, the time of flight (time taken by the ball to return to the same level) and maximum height attained by the ball will remain same as it would have been in the absence of collision with the wall. Thus,



$$t_{OAB} = t_{CD} + t_{DEF} = T = \frac{2u \sin\theta}{g}$$

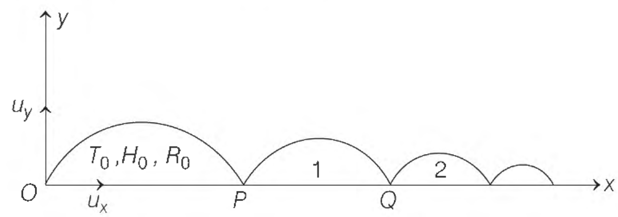
and $$h_A = h_E = \frac{u^2 \sin^2\theta}{2g}$$

Further, $CO + OF \le$ Range or OB

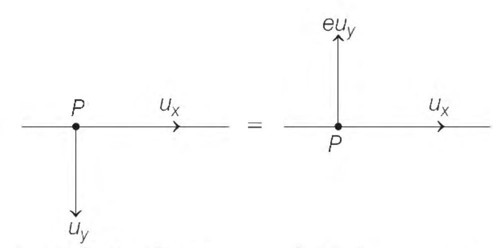
If collision is elastic, then $CO + OF = \text{Range} = \dfrac{u^2 \sin 2\theta}{g}$

and if it is inelastic, $CO + OF <$ Range

- In the projectile motion as shown in figure.



$$\Rightarrow \quad T = \frac{2u_y}{g} \Rightarrow T \propto u_y$$
$$H = \frac{u_y^2}{2g} \Rightarrow H \propto u_y^2$$
$$R = u_x T = u_x\left(\frac{2u_y}{g}\right) \Rightarrow R \propto u_x u_y$$



As shown in above figure, vertical component of velocity just after collision becomes eu_y or e times, while horizontal component remains unchanged.

Hence, the next time T will become e times (as $T \propto u_y$), H will become e^2 times (as $H \propto u_y^2$) and R will also become e times (as $R \propto u_x u_y$).

Thus, if T_0, H_0 and R_0 are the initial values, then after first collision,

$$T_1 = eT_0 , H_1 = e^2 H_0 \quad \text{and} \quad R_1 = eR_0$$

Similarly after n-collisions,

$$T_n = e^n T_0 , H_n = e^{2n} H_0 \quad \text{and} \quad R_n = e^n R_0$$

Linear Impulse

- When a large force acts for a short interval of time, then product of force and time is called linear impulse. It is a vector quantity denoted by **J**. This is equal to the change in linear momentum. Thus,

  Linear impulse

$$\mathbf{J} = \mathbf{F} \cdot \Delta t = \Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = m(\mathbf{v}_f - \mathbf{v}_i)$$

- In one dimensional motion, we can write

$$J = F\Delta t = \Delta p = p_f - p_i = m(v_f - v_i)$$

- In this case, we will choose a sign convention and all vector quantities are substituted with proper signs.

- If F - t graph is given, then linear impulse and therefore, change in linear momentum can also be obtained by area under F-t graph with projection along t-axis.
- If $\mathbf{F}$ is a function of time, then linear impulse and therefore, change in linear momentum can be obtained by integration of force in the given time interval.

Variable Mass

- A thrust force will act when mass of a system either increases or decreases. This force is given by $\mathbf{F}_t = \mathbf{v}_r\left(\pm\frac{dm}{dt}\right)$

  Here, $\mathbf{v}_r$ is relative velocity of mass dm which either enters or leaves the system on which thrust force has to be applied.
- Magnitude of thrust force is given by, $F_t = \left|\mathbf{v}_r\left(\pm\frac{dm}{dt}\right)\right|$
- Direction of $\mathbf{F}_t$ is parallel to $\mathbf{v}_r$, if mass of system is increasing or $\frac{dm}{dt}$ is positive. Direction of $\mathbf{F}_t$ is antiparallel to $\mathbf{v}_r$, if mass of system is decreasing or $\frac{dm}{dt}$ is negative.

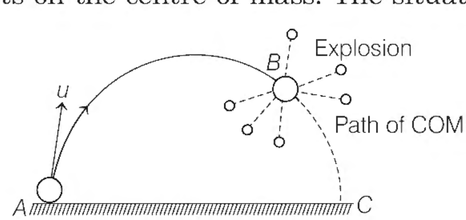
  Based on this fact, velocity of rocket at time t is given by

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

  Here, u = initial velocity of rocket
  v_r = exhaust velocity of gases (Assumed constant)
  m_0 = initial mass of rocket (with gases)
  and m = mass of rocket at time t.

  Value of g has been assumed constant in above equation.
- If mass is just dropped from a moving body, then the mass which is dropped acquires the same velocity as that of the moving body.

  Hence, $\mathbf{v}_r = 0$ or no thrust force will act in this case although mass is decreasing.
- If two or more than two particles are in motion freely under gravity, then absolute acceleration of each particle is g (downwards). So, relative acceleration between any two particles is zero but acceleration of their centre of mass is again g (downwards).
- If a projectile explodes in air in different parts, the path of the centre of mass remains unchanged. This is because during explosion no other external force (except gravity) acts on the centre of mass. The situation is as shown in figure.



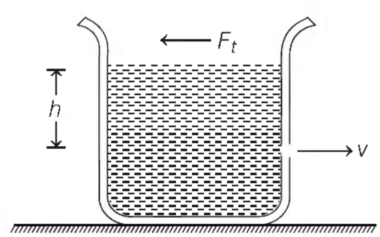
Path of COM is parabola, even though the different parts travel in different directions after explosion.

This situation continues till the first particle strikes the ground because after that force, behaviour of system of particles will change.

Centre of Mass, Frame of Reference or *C*-frame of Reference or Zero Momentum Frame

- A frame of reference carried by the centre of mass of an isolated system of particles (i.e. a system not subjected to any external forces) is called the centre of mass or C-frame of reference. In this frame of reference,
  (i) position vector of centre of mass is zero.
  (ii) velocity and hence, momentum of centre of mass is also zero.
- A liquid of density ρ is filled in a container as shown in figure. The liquid comes out from the container through a orifice of area a at a depth h below the free surface of the liquid with a velocity v. This exerts a thrust force in the container in the backward direction. This thrust force is given by

$$F_t = v_r\left(-\frac{dm}{dt}\right)$$



Here, $\quad v_r = v \quad$ (in forward direction)

and $\quad \left(-\frac{dm}{dt}\right) = \rho a v$

As $\left(\frac{dV}{dt}\right)$ = volume of liquid flowing per second = av

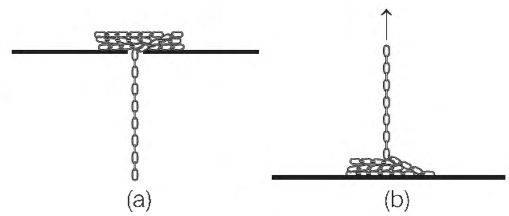
$$\therefore \quad \left(-\frac{dm}{dt}\right) = \rho\left(\frac{dV}{dt}\right) = \rho a v$$

$\therefore \; F_t = v(\rho a v)$ or $F_t = \rho a v^2$ (in backward direction)

Here, $\quad v = \sqrt{2gh}.$

- Suppose, a chain of mass per unit length λ begins to fall through a hole in the ceiling as shown in Fig. (a) or the end of the chain piled on the platform is lifted vertically as in Fig. (b).

  In both the cases, due to increase of mass in the portion of the chain which is moving with a velocity v at certain moment of time a thrust force acts on this part of the chain which is given by



$$F_t = v_r \left(\frac{dm}{dt}\right)$$

Here, $v_r = v$ and $\frac{dm}{dt} = \lambda v$

Here, v_r is upwards in case (a) and downwards in case (b). Thus,

$$F_t = \lambda v^2$$

The direction of F_t is upwards in case (a) and downwards in case (b).

- The net force on a system in a particular direction is zero (normally in horizontal direction). This can be done by giving the horizontal ground smooth.

  Since the system is at rest initially, so in this case individual bodies can move towards right or towards left, but centre of mass will remain stationary. Further, net force in horizontal direction is zero, hence total force towards right is equal to the total force towards left or,

$$\Sigma F_R = \Sigma F_L \quad \text{...(i)}$$

or $$\Sigma m_R a_R = \Sigma m_L a_L \quad \text{...(ii)}$$

Now, integrating a, we will get v and by further integrating v, we will get x.

$\therefore$ $$\Sigma m_R v_R = \Sigma m_L v_L \quad \text{...(iii)}$$

and $$\Sigma m_R x_R = \Sigma m_L x_L \quad \text{...(iv)}$$

- If a bomb or a projectile explodes in two or more than two parts, then it explodes due to its internal forces. Therefore, net force or net external force is zero. Hence, linear momentum of the system can be conserved just before and just after explosion.

  By this momentum conservation equation, we can find the velocity of some unknown parts. If explosion takes place in air, then during the explosion, the external force due to gravity (= weight) can be neglected, as the time of explosion is very short.

  So, impulse of this force is negligible and impulse is change in linear momentum. Hence, change in linear momentum is also negligible.