

TRIGONOMETRIC RATIOS & IDENTITIES

1. RELATION BETWEEN SYSTEM OF MEASUREMENT OF ANGLES :

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

$$1 \text{ Radian} = \frac{180}{\pi} \text{ degree} \approx 57^{\circ}17'15'' \text{ (approximately)}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian} \approx 0.0175 \text{ radian}$$

2. BASIC TRIGONOMETRIC IDENTITIES :

(a) $\sin^2 \theta + \cos^2 \theta = 1$ or $\sin^2 \theta = 1 - \cos^2 \theta$ or $\cos^2 \theta = 1 - \sin^2 \theta$

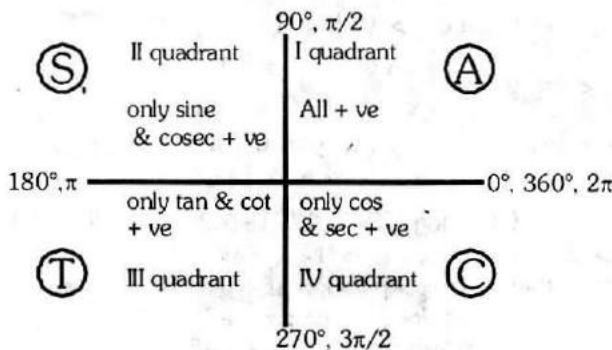
(b) $\sec^2 \theta - \tan^2 \theta = 1$ or $\sec^2 \theta = 1 + \tan^2 \theta$ or $\tan^2 \theta = \sec^2 \theta - 1$

(c) If $\sec \theta + \tan \theta = k \Rightarrow \sec \theta - \tan \theta = \frac{1}{k} \Rightarrow 2 \sec \theta = k + \frac{1}{k}$

(d) $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$ or $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$ or $\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$

(e) If $\operatorname{cosec} \theta + \cot \theta = k \Rightarrow \operatorname{cosec} \theta - \cot \theta = \frac{1}{k} \Rightarrow 2 \operatorname{cosec} \theta = k + \frac{1}{k}$

3. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :



4. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin \theta$, $\cos(2n\pi + \theta) = \cos \theta$, where $n \in \mathbb{I}$

(b) $\sin(-\theta) = -\sin \theta$

$\cos(-\theta) = \cos \theta$

$\sin(90^\circ - \theta) = \cos \theta$

$\cos(90^\circ - \theta) = \sin \theta$

$\sin(90^\circ + \theta) = \cos \theta$

$\cos(90^\circ + \theta) = -\sin \theta$

$\sin(180^\circ - \theta) = \sin \theta$

$\cos(180^\circ - \theta) = -\cos \theta$

$\sin(180^\circ + \theta) = -\sin \theta$

$\cos(180^\circ + \theta) = -\cos \theta$

$\sin(270^\circ - \theta) = -\cos \theta$

$\cos(270^\circ - \theta) = -\sin \theta$

$\sin(270^\circ + \theta) = -\cos \theta$

$\cos(270^\circ + \theta) = \sin \theta$

Note :

(i) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in \mathbb{I}$

(ii) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$; $\cos(2n+1)\frac{\pi}{2} = 0$ where $n \in \mathbb{I}$

5. IMPORTANT TRIGONOMETRIC FORMULAE :

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A + B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$

(ix) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$.

(x) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$.

(xi) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

(xii) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$$(xiii) \quad \sin C + \sin D = 2 \sin \left(\frac{C+D}{2} \right) \cos \left(\frac{C-D}{2} \right)$$

$$(xiv) \quad \sin C - \sin D = 2 \cos \left(\frac{C+D}{2} \right) \sin \left(\frac{C-D}{2} \right)$$

$$(xv) \quad \cos C + \cos D = 2 \cos \left(\frac{C+D}{2} \right) \cos \left(\frac{C-D}{2} \right)$$

$$(xvi) \quad \cos C - \cos D = 2 \sin \left(\frac{C+D}{2} \right) \sin \left(\frac{D-C}{2} \right)$$

$$(xvii) \quad \sin 2\theta = 2 \sin \theta \cos \theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$(xviii) \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$(xix) \quad 1 + \cos 2\theta = 2 \cos^2 \theta \text{ or } \cos \theta = \pm \sqrt{\frac{1 + \cos 2\theta}{2}}$$

$$(xx) \quad 1 - \cos 2\theta = 2 \sin^2 \theta \text{ or } \sin \theta = \pm \sqrt{\frac{1 - \cos 2\theta}{2}}$$

$$(xxi) \quad \tan \theta = \frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1 + \cos 2\theta} = \pm \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}}$$

$$(xxii) \quad \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$(xxiii) \quad \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$

$$(xxiv) \quad \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.$$

$$(xxv) \quad \tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$(xxvi) \quad \sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B) = \cos^2 B - \cos^2 A.$$

$$(xxvii) \quad \cos^2 A - \sin^2 B = \cos(A+B) \cdot \cos(A-B).$$

$$(xxviii) \sin(A + B + C)$$

$$= \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B \\ - \sin A \sin B \sin C$$

$$= \Sigma \sin A \cos B \cos C - \Pi \sin A$$

$$= \cos A \cos B \cos C [\tan A + \tan B + \tan C - \tan A \tan B \tan C]$$

$$(xxix) \cos(A + B + C)$$

$$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C \\ - \cos A \sin B \sin C$$

$$= \Pi \cos A - \Sigma \sin A \sin B \cos C$$

$$= \cos A \cos B \cos C [1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$$

$$(xxx) \tan(A + B + C)$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$$

$$(xxxi) \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n-1)\beta)$$

$$= \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

$$(xxxii) \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (n-1)\beta)$$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

6. VALUES OF SOME T-RATIOS FOR ANGLES 18° , 36° , 15° , 22.5° , 67.5° etc.

$$(a) \sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ = \sin \frac{\pi}{10} \quad \cos 18^\circ = \frac{\sqrt{5}+1}{4}$$

$$(b) \cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ = \cos \frac{\pi}{5}$$

$$(c) \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ = \sin \frac{\pi}{12}$$

$$(d) \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ = \cos \frac{\pi}{12}$$

$$(e) \tan \frac{\pi}{12} = 2 - \sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot \frac{5\pi}{12}$$

$$(f) \tan \frac{5\pi}{12} = 2 + \sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot \frac{\pi}{12}$$

$$(g) \tan(22.5^\circ) = \sqrt{2} - 1 = \cot(67.5^\circ) = \cot \frac{3\pi}{8} = \tan \frac{\pi}{8}$$

$$(h) \tan(67.5^\circ) = \sqrt{2} + 1 = \cot(22.5^\circ)$$

7. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(a) $a \cos \theta + b \sin \theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

(b) Minimum value of $a^2 \tan^2 \theta + b^2 \cot^2 \theta = 2ab$, where $a, b > 0$

(c) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \leq a \cos(\alpha+\theta) + b \cos(\beta+\theta) \leq \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(d) Minimum value of $a^2 \cos^2 \theta + b^2 \sec^2 \theta$ is either $2ab$ or $a^2 + b^2$, if for some real θ equation $a \cos \theta = b \sec \theta$ is true or not true ($a, b > 0$)

(e) Minimum value of $a^2 \sin^2 \theta + b^2 \csc^2 \theta$ is either $2ab$ or $a^2 + b^2$, if for some real θ equation $a \sin \theta = b \csc \theta$ is true or not true ($a, b > 0$)

8. IMPORTANT RESULTS :

$$(a) \sin \theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{4} \sin 3\theta$$

$$(b) \cos \theta \cos(60^\circ - \theta) \cos(60^\circ + \theta) = \frac{1}{4} \cos 3\theta$$

$$(c) \tan \theta \tan (60^\circ - \theta) \tan (60^\circ + \theta) = \tan 3\theta$$

$$(d) \cot \theta \cot (60^\circ - \theta) \cot (60^\circ + \theta) = \cot 3\theta$$

$$(e) (i) \sin^2 \theta + \sin^2 (60^\circ + \theta) + \sin^2 (60^\circ - \theta) = \frac{3}{2}$$

$$(ii) \cos^2 \theta + \cos^2 (60^\circ + \theta) + \cos^2 (60^\circ - \theta) = \frac{3}{2}$$

$$(f) (i) \text{ If } \tan A + \tan B + \tan C = \tan A \tan B \tan C, \\ \text{then } A + B + C = n\pi, n \in I$$

$$(ii) \text{ If } \tan A \tan B + \tan B \tan C + \tan C \tan A = 1, \\ \text{then } A + B + C = (2n + 1) \frac{\pi}{2}, n \in I$$

$$(g) \cos \theta \cos 2\theta \cos 4\theta \dots \cos (2^{n-1} \theta) = \frac{\sin(2^n \theta)}{2^n \sin \theta}$$

$$(h) \cot A - \tan A = 2 \cot 2A$$

9. CONDITIONAL IDENTITIES :

If $A + B + C = 180^\circ$, then :

$$(a) \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$(b) \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$$

$$(c) \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$$

$$(d) \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

$$(e) \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

$$(f) \cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$$

$$(g) \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

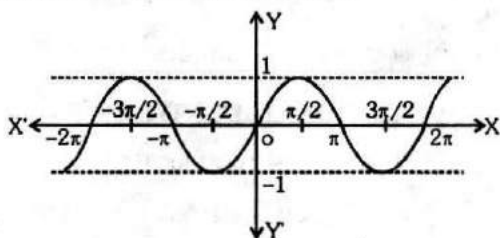
$$(h) \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

10. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

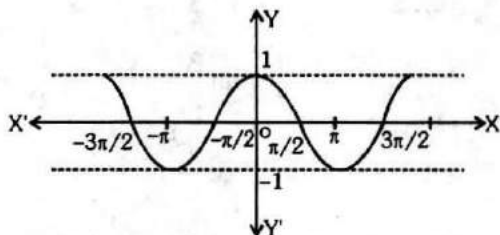
T-Ratio	Domain	Range	Period
$\sin x$	$\mathbb{R}$	$[-1, 1]$	2π
$\cos x$	$\mathbb{R}$	$[-1, 1]$	2π
$\tan x$	$\mathbb{R} - \{(2n+1)\pi/2 ; n \in \mathbb{I}\}$	$\mathbb{R}$	π
$\cot x$	$\mathbb{R} - \{n\pi : n \in \mathbb{I}\}$	$\mathbb{R}$	π
$\sec x$	$\mathbb{R} - \{(2n+1)\pi/2 : n \in \mathbb{I}\}$	$(-\infty, -1] \cup [1, \infty)$	2π
$\operatorname{cosec} x$	$\mathbb{R} - \{n\pi : n \in \mathbb{I}\}$	$(-\infty, -1] \cup [1, \infty)$	2π

11. GRAPH OF TRIGONOMETRIC FUNCTIONS :

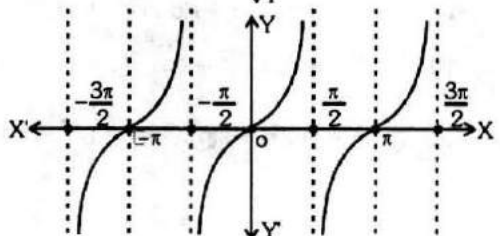
(a) $y = \sin x$



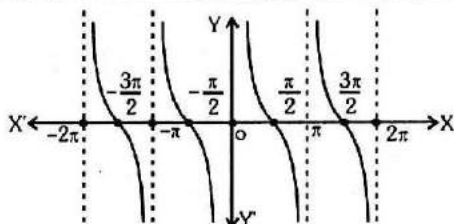
(b) $y = \cos x$



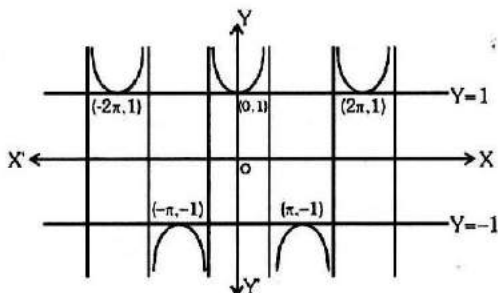
(c) $y = \tan x$



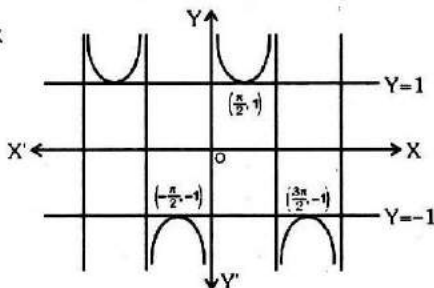
(d) $y = \cot x$



(e) $y = \sec x$



(f) $y = \csc x$



12. IMPORTANT NOTE :

(a) The sum of interior angles of a polygon of n -sides
 $= (n - 2) \times 180^\circ = (n - 2)\pi$.

(b) Each interior angle of a regular polygon of n sides

$$= \frac{(n - 2)}{n} \times 180^\circ = \frac{(n - 2)}{n} \pi.$$

(c) Sum of exterior angles of a polygon of any number of sides
 $= 360^\circ = 2\pi$.