2 Inverse Trigonometric Functions

❝ *Architects and construction engineers often use Inverse Trigonometric Functions to find the height and length of structures like a bridge by finding the angle using the inverse of sine. With this method, they also calculate the safest angle of the bridge where it will be most resilient to hold the load.*

Topic Notes

- *Basic Concepts*

BASIC CONCEPTS 1

TOPIC 1

INVERSE OF A FUNCTION

As we already know that the inverse of a function exists, now we shall go into the depths of it, trying to understand and study about the various restrictions that exist on domains and ranges of trigonometric functions. These restrictions, in fact, help to ensure that the inverse of various trigonometric functions exist and thereby, help us observe their behaviour through graphical representations

In Chapter 1, we have studied that the inverse of a function f, denoted by f^{-1}, exists if f is one-one and onto.

Let $f : X \to Y$ such that $f(x) = y$ be a one–one and onto function. Then, we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$.

Here, domain of g = range of f, and range of g = domain of f, g is called the inverse of f and is denoted by f^{-1}. Further, g is also one–one and onto and inverse of g is f.

Thus, $$g^{-1} = (f^{-1})^{-1} = f.$$

Similarly, $y = \sin^{-1} x$ is called the inverse of sine function and we read as sine inverse of x.

TOPIC 2

DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS

Here, we will talk of the inverse of those functions which are not one-one, onto or both. In particular, trigonometric functions are not one-one and onto on their natural domains and ranges. Here, are the domains and ranges of various trigonometric functions:

Function	Domain	Range
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R - \{x : x = (2n+1)\frac{\pi}{2}, n \in Z\}$	R
$\cot x$	$R - \{x : x = n\pi, n \in Z\}$	R
$\sec x$	$R - \{x : x = (2n+1)\frac{\pi}{2}, n \in Z\}$	$R - (-1, 1)$
$\text{cosec}\, x$	$R - \{x : x = n\pi, n \in Z\}$	$R - (-1, 1)$

Example 1.1: Simplify the following and write the domain and range of the result obtained:

$$y = \frac{2}{\sec x \sin 2x}$$

Ans. Given, $$y = \frac{2}{\sec x \sin 2x}$$

$$\Rightarrow \quad y = \frac{2\cos x}{2\sin x \cos x}$$

$$\Rightarrow \quad y = \frac{\cos x}{\sin x \cos x}$$

$$\Rightarrow \quad y = \frac{1}{\sin x}$$

$$\Rightarrow \quad y = \text{cosec}\, x$$

$\therefore$ Domain: $R - \{x : x = n\pi, n \in Z\}$

and Range: $R - (-1, 1)$

⚠ Caution

➥ *In such types of questions, do not try to find domain and range of the given function, always simplify until you get a single trigonometric term or any algebraic term, then only you will be able to find its domain and range.*

Principal and General Values

Condition	Principal Value	General Value
If $\sin\theta = \sin\alpha$	$\theta = n\pi, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\theta = n\pi + (-1)^n \alpha$, $n \in I$
If $\cos\theta = \cos\alpha$	$\theta = \alpha, 0 < \alpha < \pi$	$\theta = 2n \pm \alpha, n \in I$
If $\tan\theta = \tan\alpha$	$\theta = \alpha, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\theta = n\pi + \alpha, n \in I$

TOPIC 3

INVERSE OF TRIGONOMETRIC FUNCTIONS

All trigonometric functions are periodic functions so all trigonometric functions are many-one functions. If we restrict their domains, we can reduce a trigonometric function into one-one and onto functions. For instance, if we restrict the domain of sine function, from R to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then it becomes one-one and onto function with range [–1, 1].

Actually, sine function restricted to any of the intervals $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right], \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and so on, then it becomes one-one and onto function with range [–1, 1].

Thus, the inverse of sine function, denoted by $\sin^{-1}$ is a function with domain [–1, 1] and range could be any of the intervals $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and so on.

Corresponding to each such interval, we get a branch of the function $\sin^{-1}$. The branch with range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called the principal value branch, whereas other intervals give different branches of the sine function.

Thus, $\sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

This definition gives

(1) $\sin(\sin^{-1} x) = x$, if $-1 \le x \le 1$

(2) $\sin^{-1}(\sin x) = x$, if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

In other words, if $y = \sin^{-1}x$, then $\sin y = x$.

Example 1.2: $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = ?$

(a) $\frac{7\pi}{6}$ (b) $\frac{5\pi}{6}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$ **[NCERT]**

Ans. *(b)* $\frac{5\pi}{6}$

Explanation: Since $\frac{7\pi}{6} \notin [0, \pi]$, we cannot conclude that $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{7\pi}{6}$

So, we write $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$

Thus, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

$$= \cos^{-1}\left\{\cos\left(\pi + \frac{\pi}{6}\right)\right\}$$

$$= \cos^{-1}\left[-\cos\frac{\pi}{6}\right]$$

$$= \pi - \cos^{-1}\left[\cos\frac{\pi}{6}\right]$$

$$= \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

⚠ Caution

➥ *In such type of questions, inverse trigonometric properties are applicable only in its domain. In the above question, you cannot write $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{7\pi}{6}$ as $\frac{7\pi}{6} \notin [0, \pi]$. So, first simplify using formula to make the domain defined, and then apply inverse trigonometric property.*

Example 1.3: $\sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right] = ?$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) 1

Ans. *(d) 1*

Explanation:

$$\sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$$

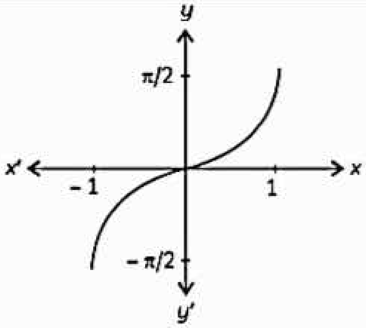
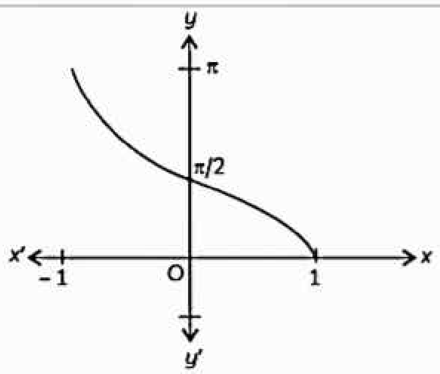
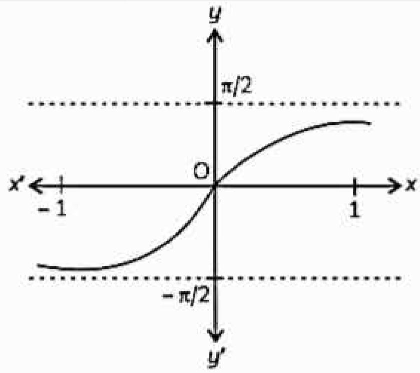
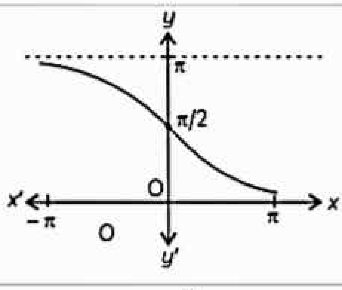
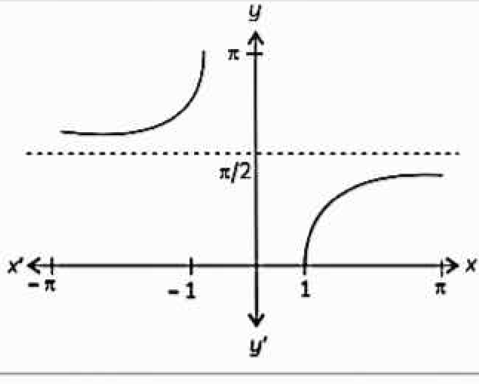
$$= \sin\frac{\pi}{2}$$

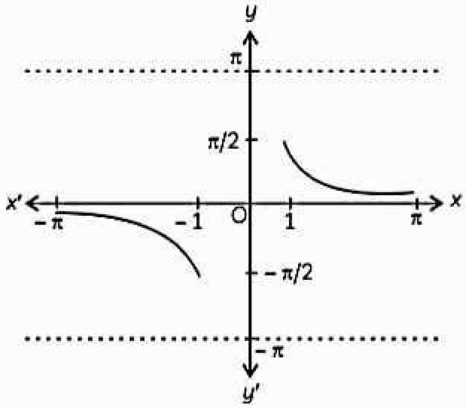
$$= 1$$

Graphs of Inverse Trigonometric Functions

The graph of an inverse trigonometric function can be obtained from the graph of the trigonometric function by interchanging x and y axes, *i.e.*, if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse of sine function.

Graphs of Inverse Trigonometric Functions in Principal Value Branch

Function	Domain	Principal Value Branch	Graph
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	
$y = \sec^{-1} x$	$(-\infty, -1) \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	

Function	Domain	Principal Value Branch	Graph
$y = \text{cosec}^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$	

Important

- *In the graph of an inverse trigonometric function, if we draw a vertical line, then it cuts the graph at many points which shows that it is not a function. That's why we restrict the domain for inverse trigonometric functions.*
- *Whenever no branch of an inverse trigonometric function is mentioned, we mean the principal value branch of that function.*
- *The value of an inverse trigonometric function which lies in the range of principal branch is called the principal value of that inverse trigonometric function.*
- *$\sin^{-1} x$ should not be confused with $(\sin x)^{-1}$. In fact, $(\sin x)^{-1} = 1/\sin x$.*

Similar concept is applied on other trigonometric functions.

Domains and Ranges of the Principle Value Branches of the Inverse Trigonometric Functions

Function	Domain	Range
$\sin^{-1} x$ (or arc sin x)	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ When $x < 0$, $\sin^{-1}x \in \left[-\frac{\pi}{2}, 0\right]$
$\cos^{-1} x$ (or arc cos x)	$[-1, 1]$	$[0, \pi]$ When $x < 0$, $\cos^{-1} x \in \left[-\frac{\pi}{2}, \pi\right]$
$\tan^{-1} x$ (or arc tan x)	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ When $x < 0$, $\tan^{-1} x \in \left(-\frac{\pi}{2}, 0\right)$
$\text{cosec}^{-1} x$ (or arc cosec x)	R − (− 1, 1)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ When $x < 0$, $\text{cosec}^{-1}x \in \left[-\frac{\pi}{2}, 0\right)$
$\sec^{-1} x$ (or arc sec x)	R − (− 1, 1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$ When $x < 0$, $\sec^{-1} x \in \left[\frac{\pi}{2}, \pi\right]$
$\cot^{-1} x$ (or arc cot x)	R	$(0, \pi)$ When $x < 0$, $\cot^{-1} x \in \left(\frac{\pi}{2}, \pi\right)$

Example 1.4: Find the principal value of the following:

(A) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (B) $\text{cosec}^{-1} 2$

(C) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Ans. (A) Let $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$\cos y = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$

$$\Rightarrow \quad y = \frac{\pi}{6}$$

Principal value branch of $\cos^{-1}$ is $[0, \pi]$.

Hence, principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

(B) Let $y = \text{cosec}^{-1}2$

$$\Rightarrow \quad \text{cosec}\, y = 2 = \text{cosec}\,\frac{\pi}{6}$$

$$\Rightarrow \quad y = \frac{\pi}{6}$$

Principal value branch of cosec^{-1} is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}.$$

Hence, principal value of cosec^{-1} is $\frac{\pi}{6}$.

(C) Let $y = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$$\Rightarrow \quad \sec y = \frac{2}{\sqrt{3}} = \sec\frac{\pi}{6}$$

$$\Rightarrow \quad y = \frac{\pi}{6}$$

Principal value branch of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

Hence, principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

Example 1.5: Find the value of $\tan^{-1} 1 + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$.

Ans. Let $x = \tan^{-1} 1$

$$\Rightarrow \quad \tan x = 1 = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad x = \frac{\pi}{4}$$

$\left[\text{Since, principal value branch of } \tan^{-1} \text{ is} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Let $y = \cos^{-1}\left(\frac{1}{2}\right)$

$$\Rightarrow \quad \cos y = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad y = \frac{\pi}{3}$$

[Since, principal value branch of $\cos^{-1}$ is $[0, \pi]$]

Let $z = \sin^{-1}\left(\frac{1}{2}\right)$

$$\Rightarrow \quad \sin z = \frac{1}{2} = \sin\frac{\pi}{6}$$

$$\Rightarrow \quad z = \frac{\pi}{6}$$

$\left[\text{Since, principal value branch of } \tan^{-1} \text{ is} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Now, $\tan^{-1}1 + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

$$= \frac{\pi}{4} + \frac{\pi}{3} + \frac{\pi}{6}$$

$$= \frac{3\pi + 4\pi + 2\pi}{12}$$

$$= \frac{9\pi}{12} = \frac{3\pi}{4}$$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The value of the expression $\sec^{-1}(2) + \sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-\sqrt{3})$ is:

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{-\pi}{3}$ (d) $\frac{\pi}{6}$

[Delhi Gov. 2022]

Ans. *(d)* $\frac{\pi}{6}$

Explanation:

$$\sec^{-1}(2) + \sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-\sqrt{3})$$

$$= \frac{\pi}{3}+\frac{\pi}{6}-\frac{\pi}{3} = \frac{\pi}{6}$$

2. $\cos\left[\frac{\pi}{3} - \cos^{-1}\left(\frac{1}{2}\right)\right]$ **is equal to:**

(a) 0 (b) 1

(c) − 1 (d) 2

3. If $\tan^{-1} x = y$, then:

(a) $-1 < y < 1$ (b) $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$

(c) $\frac{-\pi}{2} < y < \frac{\pi}{2}$ (d) $y \in \left\{\frac{-\pi}{2}, \frac{\pi}{2}\right\}$

[CBSE Term-1 SQP 2021]

4. The domain of the function $\cos^{-1}(2x - 1)$ is:

(a) [0, 1] (b) [−1, 1]

(c) (−1, 1) (d) [0, π]

[NCERT Exemplar]

Ans. *(a) [0, 1]*

Explanation: We know that, the domain of $\cos^{-1}$ function is [−1, 1]

$\therefore$ $-1 \le (2x - 1) \le 1$

$\Rightarrow$ $-1 + 1 \le (2x - 1) + 1 \le 1 + 1$

$\Rightarrow$ $0 \le 2x \le 2 \Rightarrow 0 \le x \le 1$

So, domain of $\cos^{-1}(2x - 1)$ is [0, 1].

5. The principal value of $\tan^{-1}(-\sqrt{3})$ is:

(a) $-\frac{\pi}{3}$ (b) $-\frac{\pi}{6}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

Ans. *(a)* $-\frac{\pi}{3}$

Explanation: Let

$$\tan^{-1}(-\sqrt{3}) = \theta$$

$\Rightarrow$ $\tan\theta = -\sqrt{3}$

$\Rightarrow$ $\tan\theta = -\tan\frac{\pi}{3}$

$\Rightarrow$ $\tan\theta = \tan\left(-\frac{\pi}{3}\right)$

The range of the principal value of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$

$\therefore$ The principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$.

6. The principal value of $\left[\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})\right]$ is:

(a) π (b) $-\frac{\pi}{2}$

(c) 0 (d) $2\sqrt{3}$

[CBSE Term-1 2021]

Ans. *(b)* $-\frac{\pi}{2}$

Explanation: $\left[\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})\right]$

$$= \tan^{-1}\sqrt{3} - (\pi - \cot^{-1}\sqrt{3})$$

$$= \frac{\pi}{3} - \pi + \frac{\pi}{6} = -\frac{\pi}{2}$$

7. Which of the following is the principal value branch of $\cos^{-1} x$?

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) (0, π)

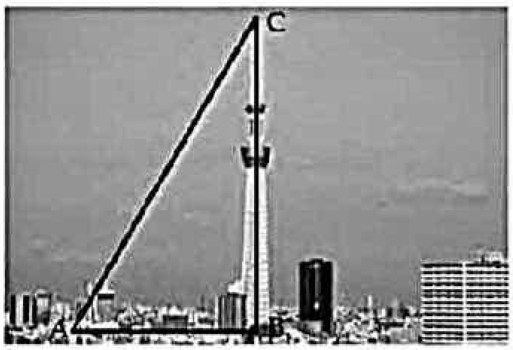
(c) [0, π] (d) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$

[NCERT Exemplar]

Ans. *(c) [0, π]*

Explanation: [0, π] is the principal value branch of $\cos^{-1} x$.

8. A man was quite fascinated to see so many high rise buildings around him. The distance between the man standing at A and the top of the building C is 80 m and the height of the tall building BC is $40\sqrt{3}$ *m*.



The principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is:

(a) $\frac{\pi}{3}$ (b) $\frac{2}{\sqrt{3}}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

Ans. *(a)* $\frac{\pi}{3}$

Explanation: Let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$, then

$\sin y = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \sin y = \sin\frac{\pi}{3}$

The range of the principal value of $\sin^{-1}$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore$ The principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{3}$.

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

9. *All the trigonometric functions are periodic (i.e., they repeat their values after fixed intervals). So they are neither one-one nor onto and hence are not bijections. But restricting their domains and co-domains they can be made bijections. Having made the trigonometric function one-one and onto, the inverse trigonometric functions exist.*

(A) The function $f : D \to [-1, 1]$ defined by $f(x) = \cos x$ is one-one and onto if D is

(a) $[0, \pi]$ (b) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(c) $\left[0,\frac{\pi}{2}\right]$ (d) $\left[-\frac{\pi}{2},0\right]$

(B) The domain of the function $\tan^{-1} x$, is

(a) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (b) $(0, \pi)$

(c) $[-1, 1]$ (d) R

(C) The range of the function $\text{cosec}^{-1} x$, is

(a) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (b) $R - (-1, 1)$

(c) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\}$ (d) R

(D) For $x \in [-1, 1]$, $\sin^{-1}(-x)$ is equal to

(a) $\sin^{-1}(x)$ (b) $-\sin^{-1} x$

(c) $-\frac{1}{\sin^{-1}(x)} - \{0\}$ (d) $\cos^{-1}(x)$

(E) $\cos(\cos^{-1} x) = x$ for

(a) $x \in [-1, 1]$ (b) $x \in R$

(c) $x \in R - (-1, 1)$ (d) $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Ans. (A) *(a) $[0, \pi]$*

Explanation: As for $f(x) = \cos x$

At $x \to 0$, $\cos x$ is 1

At $x \to \pi$, $\cos x$ is -1

So, in $[0, \pi]$ the function is one-one and onto.

(C) *(c)* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right] - \{0\}$

10. *All the trigonometric functions are periodic (i.e., they repeat their values after fixed intervals). So they are neither one-one onto. But restricting their domains and co-domains they can made one-one and onto. For example, $f : D \to [-1, 1]$ given by $f(x) = \sin x$ is one-one and onto, where D can be any one of the intervals $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \left[-\frac{\pi}{2},\frac{\pi}{2}\right], \left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ and so on.*

So, $f^{-1} : [-1, 1] \to D$ exists.

If we choose $D = \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then inverse function:

$$\sin^{-1}(x) : [-1, 1] \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

is called the principal value branch of the inverse sine function.

Similarly, principal value branch of other invese trigonometric functions are defined.

(A) Find the principal value of $\{\tan^{-1}(1) - \cot^{-1}(-1)\}$.

(B) Find the principal value of $\sec^2\{\tan^{-1}(2)\}$.

Ans. (A) $\{\tan^{-1}(1) - \cot^{-1}(-1)\} = \tan^{-1}(1) - \{\pi - \cot^{-1}(1)\}$

$$= \frac{\pi}{4} - \left(\pi - \frac{\pi}{4}\right) = -\frac{\pi}{2}$$

(B) $\sec^2\{\tan^{-1}(2)\} = 1 + \tan^2\{\tan^{-1}(2)\}$

$$= 1 + \{\tan(\tan^{-1}(2)\}^2$$

$$= 1 + 2^2 = 5$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

11. Write the value of $\tan^{-1}\left[2\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

[CBSE 2013]

Ans. $\tan^{-1}\left[2\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)\right] = \tan^{-1}\left[2\sin\left(\frac{\pi}{6}\right)\right]$

$$= \tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$= \tan^{-1}1 = \frac{\pi}{4}$$

12. What is the principal value of $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$? [DIKSHA]

Ans. $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$

$$= \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

$[\because \sin^{-1}(-\theta) = -\sin^{-1}\theta]$

$$= \frac{\pi}{3} + 2\times\frac{\pi}{6}$$

$\left[\because \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \text{ and } \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}\right]$

$$= \frac{\pi}{3} + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

13. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

[CBSE 2014]

14. Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

[NCERT Exemplar]

Ans. Since, $\frac{2\pi}{3} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore\ \tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{3}\right)\right)$

$$= \tan^{-1}\left(-\tan\left(\frac{\pi}{3}\right)\right)$$

$[\because \tan(\pi - x) = -\tan x]$

$$= \tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)$$

$$= -\frac{\pi}{3}$$

15. Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right), x > 1$ in the simplest form.

Ans. Let $x = \sec\theta$, then $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$\therefore \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \cot^{-1}(\cot\theta) = \theta = \sec^{-1}x$$

16. Evaluate: $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$.

Ans. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

$= \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right]$ $[\because \sin^{-1}(-\theta) = -\sin\theta]$

$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$ $\left[\because \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}\right]$

$$= \sin\left[\frac{\pi}{2}\right] = 1$$

17. Find the value of $\cot\left[\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right]$

Ans. $\cot\left[\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right] = \cot\left[\frac{\pi}{2} - 2\times\frac{\pi}{6}\right]$

$\left[\because \cot^{-1}\sqrt{3} = \frac{\pi}{6}\right]$

$$= \cot\left[\frac{\pi}{2} - \frac{\pi}{3}\right]$$

$$= \cot\left[\frac{3\pi - 2\pi}{6}\right]$$

$$= \cot\frac{\pi}{6} = \sqrt{3}$$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

18. Prove that:

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x, \frac{1}{\sqrt{2}} \le x \le 1.$$

[CBSE 2020]

Ans. Let $x = \cos\theta$

$\Rightarrow \quad \theta = \cos^{-1}x$

$\therefore \quad \text{L.H.S.} = \sin^{-1}\left(2\cos\theta\sqrt{1-\cos^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}(\sin 2\theta) = 2\theta$

$= 2\cos^{-1}x = \text{R.H.S}$

Hence, Proved.

19. Which is greater, $\tan 1$ or $\tan^{-1} 1$?

20. Prove that $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. **[CBSE 2018]**

21. Find the value of:

$$\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

[NCERT Exemplar]

Ans. Let $x = \tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Since, $\frac{5\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\frac{13\pi}{6} \notin [0, \pi]$

$\therefore x = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{6}\right)\right] + \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right]$

$\Rightarrow x = \tan^{-1}\left[-\tan\left(\frac{\pi}{6}\right)\right] + \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right]$

$\left[\because \tan\left(\pi - \frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right); \cos\left(2\pi + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)\right]$

$\Rightarrow x = -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right]$

$[\because \tan^{-1}(-x) = -\tan^{-1}x]$

$\Rightarrow x = -\frac{\pi}{6} + \frac{\pi}{6} = 0$

Thus, $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = 0$

⚠ Caution

➥ *In such types of questions, do not directly cancel trigonometric function with its inverse. First check whether the domain of trigonometric term is in principal range or not, if not, simplify the trigonometric term and change its domain using formulae, then do cancellation of function and its inverse.*

22. Write the value of:

$\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$. **[CBSE 2013]**

23. Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$ **[NCERT Exemplar]**

Ans. We have,

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$

$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\sqrt{3} + \tan^{-1}\left[-\sin\left(\frac{\pi}{2}\right)\right]$

$\left[\because \tan^{-1}(-x) = -\tan^{-1}x, \cot^{-1}x = \tan^{-1}\frac{1}{x}, \sin(-x) = -\sin x\right]$

$= -\frac{\pi}{6} + \frac{\pi}{3} + \tan^{-1}(-1)$

$= \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$

Thus, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] = -\frac{\pi}{12}$

⚠ Caution

➥ *In such types of questions, first simplify each term independently and always try to convert all given inverse trigonometric functions to $\tan^{-1}$ or $\sin^{-1}$ or $\cos^{-1}$ to apply formula.*

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

24. Prove that: $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **[CBSE 2012]**

Ans. L.H.S. $= \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$$= \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right)$$

$$= \tan^{-1}\left(\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}\right)$$

[Dividing numerator and denominator by $\cos\frac{x}{2}$]

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) = \frac{\pi}{4} - \frac{x}{2}$$

= R.H.S.

Hence, proved.

25. Show that: $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$.

26. Prove that:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2},$$

$x \in \left(0, \frac{p}{4}\right)$ **[CBSE 2014, 11]**

Ans. Consider $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$

$\left[\text{For } x \in \left(0, \frac{\pi}{4}\right), 1 \pm \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} \pm 2\sin\frac{x}{2}\cos\frac{x}{2} = \left(\cos\frac{x}{2} \pm \sin\frac{x}{2}\right)^2\right]$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$= \cot^{-1}\left[\frac{\cos\frac{x}{2} + \sin\frac{x}{2} + \cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2} - \cos\frac{x}{2} + \sin\frac{x}{2}}\right]$$

$$= \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right]$$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2}$$

Hence, proved.

⚠ Caution

- *In such types of questions do not apply any inverse trigonometric formulae, first simplify the terms then apply the formulae.*

27. Prove that:

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$$

[NCERT Exemplar]

28. Prove that:

$$\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}.$$

[CBSE 2010]

TOPPER'S CORNER

VERY SHORT ANSWER Type Questions (VSA)

[**1** mark]

1. Find the value of $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$.

Ans.

$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$

$\tan^{-1}\sqrt{3} - (\pi - \cot^{-1}\sqrt{3})$

$\tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3}$

$\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3} - \pi$

$\frac{\pi}{2} - \pi$

$\frac{\pi - 2\pi}{2} = \frac{-\pi}{2}$ Ans.

As we know that $\tan^{-1}x + \cot^{-1}x = \pi/2$

[CBSE Topper 2018]

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

2. Prove that: $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$.

Ans.

$3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

R.H.S. $\sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta$

$\sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$\sin^{-1}(\sin 3\theta)$

3θ

$3\sin^{-1}x$

RHS = LHS

Hence Proved

$\frac{-1}{2} \le x \le \frac{1}{2}$

$\frac{-1}{2} \le \sin\theta \le \frac{1}{2}$

$\sin^{-1}\left(\frac{-1}{2}\right) \le \theta \le \sin^{-1}\left(\frac{1}{2}\right)$

$\frac{-\pi}{3} \le \theta \le \frac{\pi}{3}$

$\left[\frac{-\pi}{3}, \frac{\pi}{3}\right] \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

[CBSE Topper 2018]