07

Differentiation

7.01 Introduction

In the previous class, we have learnt the differentiation by using first principle and derived some formulae given below:-

Standard Results

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \log_e a$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) $\frac{d}{dx}(\sin x) = \cos x$

(vi) $\frac{d}{dx}(\cos x) = -\sin x$

(vii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(viii) $\frac{d}{dx}(\cot x) = -\cos ec^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(x) $\frac{d}{dx}(\cos ecx) = -\cos ecx \cot x$

By using the above formulae the derivative of various other functions can be found.

7.02 Derivative of composite functions

Theorem : If the functions f and g are differentiable at any point c in the interval then $f \pm g, fg$ and f/g are also differentiable at point c and

(i) $D(f \pm g)(c) = f'(c) \pm g'(c)$

(ii) $D(fg)(c) = f'(c)g(c) + f(c)g'(c)$

(iii) $D\{f/g\}(c) = \frac{g(c)f'(c) - g'(c)f(c)}{[g(c)]^2}$, when $g(c) \neq 0$

Proof : Since functions f and g are differentiable at point $c \in [a, b]$ and $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$

and $\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c)$

(i) $$D(f \pm g)(c) = \lim_{x \to c} \frac{(f \pm g)(x) - (f \pm g)(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \pm \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = f'(c) \pm g'(c).$$

(ii) $$D(fg)(c)=\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c}$$

$$=\lim_{x\to c}\frac{g(x)\{f(x)-f(c)\}+f(c)\{g(x)-g(c)\}}{x-c}$$

$$=\lim_{x\to c}g(x).\lim_{x\to c}\frac{f(x)-f(c)}{x-c}+f(c)\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

$$=g(c)f'(c)+f(c)g'(c).$$

(iii) $$D(f/g)=\lim_{x\to c}\frac{(f/g)(x)-(f/g)(c)}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)/g(x)-f(c)/g(c)}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)g(c)-g(x)f(c)}{g(x)g(c)(x-c)}$$

$$=\lim_{x\to c}\frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-g(x)f(c)}{g(x)g(c)(x-c)}$$

$$=\lim_{x\to c}\frac{g(c)\{f(x)-f(c)\}-f(c)\{g(x)-g(c)\}}{g(x)g(c)(x-c)}$$

$$=\lim_{x\to c}\frac{1}{g(x)g(c)}.\left[g(c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}-f(c)\lim_{x\to c}\frac{g(x)-g(c)}{x-c}\right]$$

$$=\frac{g(c)f'(c)-f(c)g'(c)}{[g(c)]^2},\quad g(c)\neq 0.$$

7.03 Derivative of a function of funcitons or chain rule of derivative

Let $y=f(u)$ i.e. y is a function of u and $u=\phi(x)$ i.e. u itself is a function of x. Let there be small increment δx, $\delta y, \delta u$ corresponding to x, y and u then

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}.\frac{\delta u}{\delta x}$$

Now if $\delta x\to 0$ then $\delta u\to 0$ therefore

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta u\to 0}\frac{\delta y}{\delta u}.\lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$

or $$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$$

Illustrative Examples

Example 1. Differentiate the following functions with respect to x

(i) $\log_e \log_e x^2$ (ii) $e^{\sin x^2}$ (iii) $\tan\left(\log_e \sqrt{1+x^2}\right)$

Solution : (i) Let $y = \log_e \log_e x^2$

Let $\log_e x^2 = u,\ x^2 = v$

then $y = \log_e u,\ u = \log_e v,\ v = x^2$

$$\therefore \quad \frac{dy}{du} = \frac{1}{u}, \quad \frac{du}{dv} = \frac{1}{v}, \quad \frac{dv}{dx} = 2x$$

Now $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx} = \frac{1}{u}.\frac{1}{v}.2x = \frac{1}{\log_e x^2}.\frac{1}{x^2}.2x = \frac{2}{x\log_e x^2}$$

Alternate Method: Let $y = \log_e \log_e x^2$

$$\frac{dy}{dx} = \frac{d}{dx}\log_e \log_e x^2 = \frac{1}{\log x^2}\cdot\frac{d}{dx}\log_e x^2$$

$$= \frac{1}{\log x^2}\cdot\frac{1}{x^2}\cdot\frac{d}{dx}\left(x^2\right) = \frac{2x}{x^2\cdot\log x^2}\cdot\frac{2}{x\log x^2}$$

(ii) Let $y = e^{\sin x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin x^2}\right)$$

$$= e^{\sin x^2}\frac{d}{dx}\left(\sin x^2\right) = e^{\sin x^2}\left(\cos x^2\right)\frac{d}{dx}\left(x^2\right)$$

$$= e^{\sin x^2}(\cos x^2)(2x) = 2x\cos x^2.e^{\sin x^2}$$

(iii) Let $y = \tan\left(\log_e \sqrt{1+x^2}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\tan(\log_e \sqrt{1+x^2})\right\}$$

$$= \sec^2(\log_e \sqrt{1+x^2})\frac{d}{dx}(\log_e \sqrt{1+x^2})$$

$$= \sec^2(\log_e \sqrt{1+x^2})\frac{1}{\sqrt{1+x^2}}\frac{d}{dx}(\sqrt{1+x^2})$$

$$= \frac{1}{\sqrt{1+x^2}}.\sec^2(\log_e \sqrt{1+x^2}).\frac{1}{2\sqrt{1+x^2}}.\frac{d}{dx}(1+x^2)$$

$$= \frac{1}{\sqrt{1+x^2}} . \sec^2(\log_e \sqrt{1+x^2}) . \frac{1}{2\sqrt{1+x^2}}(0+2x)$$

$$= \frac{x}{(1+x^2)} . \sec^2(\log_e \sqrt{1+x^2}).$$

Example 2. Differentiate the following functions with respect to x

(i) $\frac{\sin(ax+b)}{\cos(cx+d)}$ (ii) $\cos x^3 . \sin^2(x^5)$ (iii) $\sec(\tan\sqrt{x})$

Solution : (i) Let $y = \frac{\sin(ax+b)}{\cos(cx+d)}$

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{\sin(ax+b)}{\cos(cx+d)}\right\}$$

$$= \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{\cos(cx+d).\cos(ax+b)\frac{d}{dx}(ax+b) - \sin(ax+b)\{-\sin(cx+d)\}\frac{d}{dx}(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{\cos(cx+d)\cos(ax+b)(a) + \sin(ax+b)\sin(cx+d)(c)}{\cos^2(cx+d)}.$$

(iii) Let $y = \cos x^3 \cdot \sin^2(x^5)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\cos x^3 \cdot \sin^2(x^5)\}$$

$$= \cos x^3 \frac{d}{dx}\sin^2(x^5) + \sin^2(x^5)\frac{d}{dx}\cos x^3$$

$$= \cos x^3 . 2\sin(x^5)\frac{d}{dx}\sin(x^5) + \sin^2(x^5)(-\sin x^3)\frac{d}{dx}(x^3)$$

$$= \cos x^3 . 2\sin(x^5)\cos(x^5).\frac{d}{dx}(x^5) - \sin^2(x^5)\sin x^3(3x^2)$$

$$= \cos x^3 . 2\sin(x^5)\cos(x^5).5x^4 - \sin^2(x^5)\sin x^3(3x^2)$$

$$= 10x^4\cos x^3.\sin(x^5)\cos(x^5) - 3x^2\sin^2(x^5)\sin x^3.$$

(iii) Let $y = \sec(\tan\sqrt{x})$

$$\frac{dy}{dx} = \frac{d}{dx}\sec(\tan\sqrt{x})$$

$$= \sec(\tan\sqrt{x}).\tan(\tan\sqrt{x}).\frac{d}{dx}(\tan\sqrt{x})$$

$$= \sec(\tan\sqrt{x}).\tan(\tan\sqrt{x})\sec^2\sqrt{x}\frac{d}{dx}(\sqrt{x})$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}.\frac{1}{2}x^{1/2-1}$$

$$= \frac{1}{2}\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}.\frac{1}{\sqrt{x}}.$$

Example 3. Diferentiate the following functions with respect to x

(i) $2\sqrt{\cot\left(x^2\right)}$ (ii) $\cos\left(\sqrt{x}\right)$

Solution : (i) Let $y = 2\sqrt{\cot(x^2)}$

$$\frac{dy}{dx} = 2\frac{d}{dx}(\sqrt{\cot x^2})$$

$$= 2.\frac{1}{2\sqrt{\cot x^2}}.\frac{d}{dx}(\cot x^2)$$

$$= \frac{1}{\sqrt{\cot x^2}}.\left\{-\cos ec^2(x^2)\right\}\frac{d}{dx}(x^2)$$

$$= -\frac{\operatorname{cosec}^2(x^2)}{\sqrt{\cot x^2}}.(2x) = -\frac{2x\sqrt{\tan x^2}}{\sin^2(x^2)}$$

$$= \frac{-2x\sqrt{\sin x^2}}{\sin^2(x^2)\sqrt{\cos x^2}} = \frac{-2x}{\sin(x^2)\sqrt{\sin x^2\cos x^2}}$$

$$= \frac{-2\sqrt{2}x}{\sin(x^2)\sqrt{2\sin x^2\cos x^2}} = \frac{-2\sqrt{2}x}{\sin(x^2)\sqrt{\sin(2x^2)}}.$$

(ii) Let $y = \cos\left(\sqrt{x}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x}) = -\sin\sqrt{x}\frac{d}{dx}(\sqrt{x}) = \frac{-\sin\sqrt{x}}{2\sqrt{x}}.$$

Exercise 7.1

Differentiate the following functions with respect to x

1. $\sin x^2$
2. $\tan(2x+3)$
3. $\sin\left\{\cos\left(x^2\right)\right\}$
4. $\dfrac{\sec x - 1}{\sec x + 1}$
5. $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$
6. $\sin x^{o}$
7. $\log_e \sqrt{\dfrac{1-\cos x}{1+\cos x}}$
8. $\sec x^{o}$
9. $\log \sqrt{\dfrac{1+\sin x}{1-\sin x}}$
10. $\log_e \left\{\dfrac{x+\sqrt{x^2+a^2}}{a}\right\}$
11. $\log_e \left\{\dfrac{x^2+x+1}{x^2-x+1}\right\}$
12. $\tan\left\{\log_e \sqrt{1+x^2}\right\}$
13. $a^{\tan 3x}$
14. $\log_e(\sec x + \tan x)$
15. $\sin^3 x . \sin 3x$

7.04 Derivatives of inverse trigonometrical functions

We know that inverse trigonometric functions are continuous in their domains. To differentiate these functions, we shall use the chain rule.

Illustrative Examples

Example 4. Differentiate the function $\sin^{-1} x$ for all $x \in (-1, 1)$

Solution : Let $y = \sin^{-1} x$

$\Rightarrow \quad x = \sin y$

Differentiate both sides with respect to x

$$1 = \cos y \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(\sin^{-1} x)} \qquad \ldots (1)$$

here $\dfrac{dy}{dx}$, exists only when $\cos y \neq 0$

$\Rightarrow \quad \cos(\sin^{-1} x) \neq 0$

$\Rightarrow \quad \sin^{-1} x \neq \dfrac{-\pi}{2}$ or $\dfrac{\pi}{2} \quad \Rightarrow x \neq -1, 1 \quad \Rightarrow x \in (-1, 1)$

from (1) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\sin^2 y}} = \dfrac{1}{\sqrt{1-x^2}} \qquad \because \sin y = x$

Note : Derivatives of remaining inverse trigonometric functions can be derived in the similar manner.

(i) $\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$

(iii) $\frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$ (iv) $\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}}$

(v) $\frac{d}{dx}\left(\cos ec^{-1}x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Example 5. Find $\frac{dy}{dx}$:

(i) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (ii) $y=\sin^{-1}\sqrt{\cos x}$

(iii) $y = \sqrt{\cos^{-1}\sqrt{x}}$ (iv) $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Solution : (i) Given $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Here putting $x = \tan\theta$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x \qquad [\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1}x]$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{1+x^2}$$

(ii) Given $y = \sin^{-1}\left(\sqrt{\cos x}\right)$

Let $\sqrt{\cos x} = u,$ then

$$y = \sin^{-1}u$$

$$\therefore \quad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\because \quad u = \sqrt{\cos x}$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2\sqrt{\cos x}}\frac{d}{dx}\left(\cos x\right) = \frac{-\sin x}{2\sqrt{\cos x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\left\{\frac{-\sin x}{2\sqrt{\cos x}}\right\} \qquad \text{[using (1) and (2)]}$$

putting the value of *u*,

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\cos x}}\left\{\frac{-\sin x}{2\sqrt{\cos x}}\right\}=\frac{-\sin x}{2\sqrt{1-\cos x}\sqrt{\cos x}}$$

(iii) $y=\sqrt{\cot^{-1}\sqrt{x}}$

Let $\sqrt{x}=u$ and $\cot^{-1}\sqrt{x}=\cot^{-1}u=t$, then

$y=\sqrt{t},\ t=\cot^{-1}u$ and $u=\sqrt{x}$

$\therefore$ $\frac{dy}{dt}=\frac{1}{2\sqrt{t}},\ \frac{dt}{du}=\frac{-1}{1+u^2},\ \frac{du}{dx}=\frac{1}{2\sqrt{x}}$

$\Rightarrow$ $\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{du}.\frac{du}{dx}$

$$=\left(\frac{1}{2\sqrt{t}}\right).\left(\frac{-1}{1+u^2}\right)\left(\frac{1}{2\sqrt{x}}\right)=\frac{-1}{4\sqrt{t}\sqrt{x}\left(1+u^2\right)}$$

$$=\frac{-1}{4\sqrt{\left(\cot^{-1}u\right)}\left(\sqrt{x}\right)\left(1+u^2\right)} \qquad [\because t=\cot^{-1}u]$$

$\therefore$ $$\frac{dy}{dx}=\frac{-1}{4\sqrt{x}(1+x)\sqrt{\cot^{-1}\sqrt{x}}} \qquad [\because u=\sqrt{x}]$$

(iv) Given that $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Let $x=\tan\theta$

$\therefore$ $y=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)=\tan^{-1}\left(\tan 3\theta\right)$

$\because$ $x\in\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

$\Rightarrow$ $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$ $\Rightarrow$ $-\frac{1}{\sqrt{3}}<\tan\theta<\frac{1}{\sqrt{3}}$

$\Rightarrow$ $-\frac{\pi}{6}<\theta<\frac{\pi}{6}$ $\Rightarrow$ $-\frac{3\pi}{6}<3\theta<\frac{3\pi}{6}$

$\Rightarrow$ $-\frac{\pi}{2}<3\theta<\frac{\pi}{2}$

$\Rightarrow \quad y = \tan^{-1}(\tan 3\theta) \quad \left(\because -\dfrac{\pi}{2} < 3\theta < \dfrac{\pi}{2}\right)$

$\Rightarrow \quad y = 3\theta \ \Rightarrow y = 3\tan^{-1} x$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{3}{1+x^2}$

Example 6. Differentiate the following with respect to x

(i) $\tan^{-1}\left(\sin e^x\right)$ (ii) $\sin^{-1}\left(\sqrt{\sin x^2}\right)$ (iii) $\sin^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$

Solution : (i) Let $y = \tan^{-1}(\sin e^x)$

Here putting $\sin e^x = u,\ e^x = v$

$$y = \tan^{-1}(u), \quad u = \sin v, \quad v = e^x$$

$$\Rightarrow \quad \frac{dy}{du} = \frac{1}{1+u^2}, \quad \frac{du}{dv} = \cos v, \quad \frac{dv}{dx} = e^x$$

now,
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx} = \frac{1}{1+u^2}.\cos v.e^x$$

putting the values of u and v

$$\frac{dy}{dx} = \frac{1}{1+\sin^2 e^x}.\cos(e^x).e^x = \frac{e^x \cos e^x}{1+\sin^2 e^x}$$

(ii) Let $y = \sin^{-1}(\sqrt{\sin x^2})$

Here $\sqrt{\sin x^2} = u, \quad \sin x^2 = v, \quad x^2 = \omega$

$$y = \sin^{-1} u, \quad u = \sqrt{v}, \quad v = \sin\omega, \quad \omega = x^2$$

$$\Rightarrow \quad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}, \quad \frac{du}{dv} = \frac{1}{2\sqrt{v}}, \quad \frac{dv}{d\omega} = \cos\omega, \quad \frac{d\omega}{dx} = 2x$$

now,
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{d\omega}\cdot\frac{d\omega}{dx} = \frac{1}{\sqrt{1-u^2}}.\frac{1}{2\sqrt{v}}.\cos\omega.2x$$

Putting the values of u, v and ω

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin x^2}}.\frac{1}{2\sqrt{\sin x^2}}.(\cos x^2)(2x) = \frac{x\cos x^2}{\sqrt{\left(\sin x^2\right)\left(1-\sin x^2\right)}}$$

(iii) Let $y = \sin^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$

here put $\dfrac{a+b\cos x}{b+a\cos x}=u$

$$y=\sin^{-1}u, \qquad u=\frac{a+b\cos x}{b+a\cos x}$$

$$\frac{dy}{du}=\frac{1}{\sqrt{1-u^2}}\,;\ \frac{du}{dx}=\frac{(b+a\cos x)\dfrac{d}{dx}(a+b\cos x)-(a+b\cos x)\dfrac{d}{dx}(b+a\cos x)}{(b+a\cos x)^2}$$

$$\Rightarrow \quad \frac{dy}{du}=\frac{b+a\cos x}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}$$

$$\frac{du}{dx}=\frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}=\frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$\Rightarrow \quad \frac{du}{dx}=\frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\frac{b+a\cos x}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}\cdot\frac{\left(a^2-b^2\right)\sin x}{\left(b+a\cos x\right)^2}$$

$$=\frac{-(b^2-a^2)\sin x}{(b+a\cos x)\sqrt{(b^2-a^2)\sin^2 x}}=\frac{-\sqrt{(b^2-a^2)}}{(b+a\cos x)}$$

Example 7. Differentiate the following functions with respect to x

(i) $\tan^{-1}(\sec x+\tan x)$ (ii) $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (iii) $\tan^{-1}\left(\sqrt{\dfrac{a-x}{a+x}}\right)$ (iv) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Solution : (i) Let $y=\tan^{-1}(\sec x+\tan x)$

$$=\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)=\tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right\}$$

$$=\tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$\therefore \quad y=(\pi/4)+(x/2).$$

Differentiating with respect to x

$$\frac{dy}{dx} = 0 + \frac{1}{2} = \frac{1}{2}$$

(ii) Let $$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

here putting $x = \tan\theta$

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\cos 2\theta) = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} \pm 2\theta\right)\right\}$$

$$= \frac{\pi}{2} \pm 2\theta = \frac{\pi}{2} \pm 2\tan^{-1} x$$

Differentiating with respect to x

$$\frac{dy}{dx} = 0 \pm \frac{2}{1+x^2} = \pm\frac{2}{1+x^2}.$$

(iii) Let $$y = \tan^{-1}\left(\sqrt{\frac{a-x}{a+x}}\right)$$

here putting $x = a\cos 2\theta$

$\therefore$ $$y = \tan^{-1}\left(\sqrt{\frac{a-a\cos 2\theta}{a+a\cos 2\theta}}\right) = \tan^{-1}\left(\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right) = \tan^{-1}(\tan\theta) = \theta$$

$$= \frac{1}{2}\cos^{-1}(x/a) \qquad [\because x = a\cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1}(x/a)]$$

Differentiating with respect to x

$$\frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{1-x^2/a^2}}.\frac{1}{a} = -\frac{1}{2\sqrt{a^2-x^2}}.$$

(iv) Let $$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

here putting $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$$

$$= \tan^{-1}\left(\tan(\theta/2)\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x \qquad [\because x = \tan\theta]$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{1}{2\left(1+x^2\right)}.$$

Example 8. Differentiate the following functions with respect to x

(i) $\tan^{-1}\left(\frac{3a^2x - x^3}{a\left(a^2-3x^2\right)}\right)$ (ii) $\tan^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

(iii) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ (iv) $\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

Solution : (i) Let $y = \tan^{-1}\left(\frac{3a^2x - x^3}{a\left(a^2-3x^2\right)}\right)$

Here putting $x = a\tan\theta$

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}\left(\tan 3\theta\right) = 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \quad \frac{dy}{dx} = 3\frac{1}{1+x^2/a^2}\left(\frac{1}{a}\right) = \frac{3a}{x^2+a^2}.$$

(ii) Let $y = \tan^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

Here put $x = \cos\theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos(\theta/2)+\sqrt{2}\sin(\theta/2)}{\sqrt{2}\cos(\theta/2)-\sqrt{2}\sin(\theta/2)}\right) = \tan^{-1}\left(\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right) = \frac{\pi}{4}+\frac{\theta}{2} = \frac{\pi}{4}+\frac{1}{2}\cdot\cos^{-1}x. \qquad [\because \cos\theta = x]$$

Differentiating with respect to x

$$\frac{dy}{dx} = 0+\frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right) = \frac{-1}{2\sqrt{1-x^2}}$$

(iii) Let $$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$$

Here put $x^2=\cos\theta$

$$y=\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos(\theta/2)+\sqrt{2}\sin(\theta/2)}{\sqrt{2}\cos(\theta/2)-\sqrt{2}\sin(\theta/2)}\right)$$

$$=\tan^{-1}\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)=\frac{\pi}{4}+\frac{\theta}{2}=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$[\because x^2=\cos\theta]$

Diferentiating with respect to x

$$\frac{dy}{dx}=0+\frac{1}{2}\left\{-\frac{1}{\sqrt{1-x^4}}.2x\right\}=\frac{-x}{\sqrt{1-x^4}}$$

(iv) Let $$y=\tan^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$$

$$=\tan^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$=\tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\}$$

$$=\tan^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)=\tan^{-1}\left(\cot\frac{x}{2}\right)=\tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\}=\frac{\pi}{2}-\frac{x}{2}$$

$\therefore$ $$\frac{dy}{dx}=0-\frac{1}{2}=-\frac{1}{2}.$$

Exercise 7.2

Differentiate the following functions with respect to x

1. (a) $\sin^{-1}\{2x\sqrt{1-x^2}\},\ -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ (b) $\sin^{-1}(3x-4x^3)x\in\left(-\frac{1}{2},\frac{1}{2}\right)$

2. (a) $\cos^{-1}\left(\frac{2x}{1-x^2}\right),\ x\in(-1,1)$ (b) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ x\in(0,1)$

3. (a) $\cos^{-1}(4x^3-3x),\ x\in\left(\frac{1}{2},1\right)$ (b) $\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$ (Hint : $x=\cos\theta$)

4. (a) $\sec^{-1}\left(\frac{1}{2x^2-1}\right);x\in\left(0,\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ x\in(0,\ \infty)$

5. (a) $\sin^{-1}\left(\frac{1+x^2}{1-x^2}\right)+\cos^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ (b) $\cos^{-1}(2x)+2\cos^{-1}\left(\sqrt{1-4x^2}\right)$

   (Hint : $\sin^{-1}\theta+\cos^{-1}\theta=\pi/2$) (Hint: $2x=\cos\theta$)

6. (a) $\tan^{-1}\left(\frac{a+x}{1-ax}\right)$ (Hint : $x=\tan\theta,\ a=\tan\alpha$) (b) $\tan^{-1}\left(\frac{2^{x+1}}{1-4^x}\right)$ (Hint : $2^x=\tan\theta$)

7. (a) $\sin\left\{2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right\}$ (Hint : $x=\cos\theta$) (b) $\cot^{-1}\left(\sqrt{1+x^2}+x\right)$ (Hint : $x=\tan\theta$)

7.05 Derivative of implicit functions

When a relationship between x and y is expressed in such a way that it is easy to solve for y and write $y=f(x)$, we say that y is given as an explicit function of x. Whereas if x cannot be expressed in terms of y (or y in terms of x) then it is called as Implicit function.

For example :

(i) Equation $x-2y-4=0$ can be expressed as $\left(y=\frac{1}{2}(x-4)\right)$. Thus this function is Explicit function.

(ii) Equation $x^3+y^3+3axy=c$ cannot be expressed independently as x in terms of y or y in terms of x, so this function is Implicit function.

Illustrative Examples

Example 9. Evaluate $\frac{dy}{dx}$

(i) $x^3+x^2y+xy^2+y^3=81$ (ii) $\sin^2 y+\cos xy=\pi$

(iii) $\sin^2 x+\cos^2 x=1$ (iv) $2x+3y=\sin x$

Solution : (i) Given $x^3 + x^2 y + xy^2 + y^3 = 81$

Differentiating with respect to x

$$3x^2 + x^2\frac{dy}{dx} + y(2x) + x\left(2y\frac{dy}{dx}\right) + y^2 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(x^2 + 2xy + 3y^2\right)\frac{dy}{dx} = -\left(3x^2 + 2xy + y^2\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}.$$

(ii) $\because \quad \sin^2 y + \cos xy = \pi$

Differentiating with respect to x

$$2\sin y\frac{d}{dx}(\sin y) + (-\sin xy)\frac{d}{dx}(xy) = 0$$

$$\Rightarrow \quad 2\sin y\cos y\frac{dy}{dx} - \sin(xy)\left\{x\frac{dy}{dx} + y\right\} = 0$$

$$\Rightarrow \quad (2\sin y\cos y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$\Rightarrow$

$$\frac{dy}{dx} = \frac{y\sin xy}{2\sin y\cos y - x\sin xy} = \frac{y\sin xy}{\sin 2y - x\sin xy}.$$

(iii) $\because \quad \sin^2 x + \cos^2 y = 1$

Differentiating with respect to x

$$2\sin x\frac{d}{dx}(\sin x) + 2\cos y\frac{d}{dx}(\cos y) = 0$$

$$\Rightarrow \quad 2\sin x\cos x + 2\cos y(-\sin y)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sin 2x - \sin 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

(iv) $\because \quad 2x + 3y = \sin x$

Differentiating with respect to x

$$2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x - 2}{3}.$$

Example 10. Find $\frac{dy}{dx}$:

(i) $xy + y^2 = \tan x + y$ (ii) $ax + by^2 = \cos y$

Solution : (i) $\because$ $xy + y^2 = \tan x + y$

Differentiating with respect to x

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \quad (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

7.06 Logarithimic Differentiation

When the function is of the form $[f(x)]^{g(x)}$, where $f(x)$, $g(x) > 0$, then to find its derivative we take logarithm both the sides and get the results. This method is called as logarithmic differentiation. This method is applicable even if the function is algebraic.

Working method : Let $y = u^v$, where u and v, are the function of x

taking log both the sides $\log_e y = \log_e u^v$

$$\Rightarrow \quad \log_e y = v \log_e u$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = v.\frac{1}{u}\frac{du}{dx} + \log_e u.\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\frac{v}{u}\frac{du}{dx} + \log_e u\frac{dv}{dx}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = u^v\left\{\frac{v}{u}\frac{du}{dx} + \log_e u\frac{dv}{dx}\right\}$$

Illustrative Examples

Example 11. Differentiate the following functions with respect to x

(i) x^x (ii) $(\sin x)^x$ (iii) $x^{\log_e x}$ (iv) $x^{\sin x}$

Solution : (i) Let $y = x^x$

taking log both the sides

$$\log_e y = \log_e (x^x)$$

$$\Rightarrow \quad \log_e y = x \log_e x$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log_e x$$

$$\Rightarrow \qquad \frac{dy}{dx} = y\{1+\log_e x\} = x^x\{1+\log_e x\} = x^x \log_e ex$$

(ii) Let $\qquad y = (\sin x)^x$

taking log both the sides $\qquad \log_e y = x\log_e \sin x$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{\sin x}.\cos x + 1.\log_e \sin x$$

$$\Rightarrow \qquad \frac{dy}{dx} = y\left\{x\cot x + \log_e \sin x\right\}$$

$$\Rightarrow \qquad \frac{dy}{dx} = (\sin x)^x\left\{x\cot x + \log_e \sin x\right\}$$

(iii) Let $\qquad y = x^{\log_e x}$

taking log both the sides

$$\log_e y = \log_e x.\log_e x$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}.\log_e x + \frac{1}{x}.\log_e x$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2y}{x}\log_e x = \frac{2x^{\log_e x}}{x}.\log_e x = 2x^{(\log_e x - 1)}.\log_e x$$

(iv) Let $\qquad y = x^{\sin x}$

taking log both the sides

$$\log_e y = \sin x \log_e x$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \sin x.\frac{1}{x} + (\cos x)\log_e x$$

$$\Rightarrow \qquad \frac{dy}{dx} = y\left\{\frac{\sin x}{x} + \cos x.\log_e x\right\}$$

$$= x^{\sin x}\left\{\frac{\sin x}{x} + \cos x.\log_e x\right\}$$

$$= x^{\sin x - 1}.\sin x + x^{\sin x}.\cos x.\log_e x.$$

Example 12. Differentiate the following functions with respect to x

(i) $\cos x.\cos 2x.\cos 3x$ (ii) $(\log x)^{\cos x}$ (iii) $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution : (i) Let $y = \cos x.\cos 2x.\cos 3x$

taking log both sides

$$\log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}(-2\sin 2x) + \frac{1}{\cos 3x}(-3\sin 3x)$$

$$\frac{dy}{dx} = -y\{\tan x + 2\tan 2x + 3\tan 3x\}$$

$$= -\cos x.\cos 2x.\cos 3x\{\tan x + 2\tan 2x + 3\tan 3x\}$$

(ii) Let $y = (\log x)^{\cos x}$

taking log both sides

$$\log y = \cos x \log(\log x)$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}\{\log(\log x)\} + \log(\log x)\frac{d}{dx}(\cos x)$$

$$= \cos x.\frac{1}{\log x}.\frac{1}{x} - \sin x.\log(\log x)$$

$$\frac{dy}{dx} = y\left\{\frac{\cos x}{x\log x} - \sin x.\log(\log x)\right\}$$

$$= (\log x)^{\cos x}\left\{\frac{\cos x}{x\log x} - \sin x\log(\log x)\right\}$$

(iii) Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

taking log both the sides

$$\log y = \frac{1}{2}\{\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\}$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)} - \frac{1}{(x-5)}\right]$$

$$\frac{dy}{dx}=\frac{y}{2}\left[\frac{1}{(x-1)}+\frac{1}{(x-2)}-\frac{1}{(x-3)}-\frac{1}{(x-4)}-\frac{1}{(x-5)}\right]$$

$$=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{(x-1)}+\frac{1}{(x-2)}-\frac{1}{(x-3)}-\frac{1}{(x-4)}-\frac{1}{(x-5)}\right]$$

Example 13. Determine $\frac{dy}{dx}$

(i) $x^y = y^x$ (ii) $y=\sqrt{x+\sqrt{x+\sqrt{x+.....\infty}}}$ (iii) $(\cos x)^y = (\sin y)^x$ (iv) $x^y.y^x = \kappa$

Solution : (i) Here $x^y = y^x$

taking log both the sides

$$y\log x = x\log y$$

Differentiating with respect to x

$$y.\frac{1}{x}+\log x.\frac{dy}{dx}=x.\frac{1}{y}\frac{dy}{dx}+\log y.1$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\log x-\frac{x}{y}\right\}=\log y-\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y(x\log y-y)}{x(y\log x-x)}.$$

(ii) Here $y=\sqrt{x+\sqrt{x+\sqrt{x+.....\infty}}}$

$$\therefore \quad y=\sqrt{x+y}$$

$$\Rightarrow \quad y^2=x+y$$

Differentiating with respect to x

$$2y\frac{dy}{dx}=1+\frac{dy}{dx}$$

$$(2y-1)\frac{dy}{dx}=1$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{2y-1}$$

(iii) Here $(\cos x)^y = (\sin y)^x$

taking log both the sides

$$y\log(\cos x)=x\log(\sin y)$$

Differentiating with respect to x

$$y.\frac{1}{\cos x}(-\sin x)+\log(\cos x)\frac{dy}{dx}=x.\frac{\cos y}{\sin y}\frac{dy}{dx}+\log(\sin y).1$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{\log(\sin y)+y\tan x}{\log(\cos x)-x\cot y}$$

(iv) Here $\qquad x^y.y^x=k$

taking log both the sides

$$\log x^y+\log y^x=\log k$$

$$\Rightarrow \qquad y\log x+x\log y=\log k$$

Differentiating with respect to x

$$y.\frac{1}{x}+\log x\frac{dy}{dx}+x.\frac{1}{y}\frac{dy}{dx}+\log y.1=0$$

$$\Rightarrow \qquad \left(\log x+\frac{x}{y}\right)\frac{dy}{dx}=-\left(\log y+\frac{y}{x}\right)$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{-y(x\log y+y)}{x(y\log x+x)}.$$

Example 14. Find $\frac{dy}{dx}$ **:**

(i) $x^a.y^b=(x+y)^{a+b}$ (ii) $\sqrt{x^2+y^2}=\log(x^2-y^2)$

(iii) $x\sqrt{1+y}+y\sqrt{1+x}=0$ (iv) $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

Solution : (i) Here $\qquad x^a.y^b=(x+y)^{a+b}$

taking log both the sides

$$\log x^a+\log y^b=(a+b)\log(x+y)$$

$$\Rightarrow \qquad a\log x+b\log y=(a+b)\log(x+y)$$

Differentiating with respect to x

$$a.\frac{1}{x}+b.\frac{1}{y}\frac{dy}{dx}=(a+b).\frac{1}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \qquad \left(\frac{b}{y}-\frac{a+b}{x+y}\right)\frac{dy}{dx}=\frac{a+b}{x+y}-\frac{a}{x}$$

$$\Rightarrow \qquad \left\{\frac{b(x+y)-y(a+b)}{y(x+y)}\right\}\frac{dy}{dx}=\frac{x(a+b)-a(x+y)}{x(x+y)}$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{y}{x}.$$

(ii) Here $$\sqrt{x^2+y^2} = \log(x^2-y^2)$$

Differentiating with respect to x

$$\frac{1}{2\sqrt{x^2+y^2}}\left(2x+2y\frac{dy}{dx}\right) = \frac{1}{\left(x^2-y^2\right)}\left(2x-2y\frac{dy}{dx}\right)$$

$$\left\{\frac{y}{\sqrt{x^2+y^2}}+\frac{2y}{x^2-y^2}\right\}\frac{dy}{dx} = \frac{2x}{x^2-y^2}-\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{dy}{dx} = \frac{2x\sqrt{x^2+y^2}-x\left(x^2-y^2\right)}{y\left(x^2-y^2\right)+2y\sqrt{x^2+y^2}}.$$

(iii) Here $$x\sqrt{1+y}+y\sqrt{1+x} = 0$$

$$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$$

squaring both the sides

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow \quad x^2-y^2+x^2y-xy^2 = 0$$

$$\Rightarrow \quad (x-y)(x+y)+xy(x-y) = 0$$

$$\Rightarrow \quad (x-y)(x+y+xy) = 0$$

If $x-y=0$ or $x=y$, which does not satisfy the given equation, $x-y\neq 0$

$\therefore \quad x+y+xy=0$

Differentiating with respect to x

$$1+\frac{dy}{dx}+1.y+x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1+x)\frac{dy}{dx} = -(1+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{1+y}{1+x}\right)$$

(iv) Here $$\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$$

Here putting $$x=\sin\theta,\ y=\sin\phi$$

$$\sqrt{1-\sin^2\theta}+\sqrt{1-\sin^2\phi} = a(\sin\theta-\sin\phi)$$

$$\Rightarrow \quad \cos\theta+\cos\phi = a(\sin\theta-\sin\phi)$$

$$\Rightarrow \quad 2\cos\frac{\theta+\phi}{2}.\cos\frac{\theta-\phi}{2} = 2a\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

$$\Rightarrow \qquad \cot\frac{\theta-\phi}{2}=a$$

$$\Rightarrow \qquad \frac{\theta-\phi}{2}=\cot^{-1}(a)$$

$$\Rightarrow \qquad \theta-\phi=2\cot^{-1}(a)$$

$$\Rightarrow \qquad \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$$

Differentiating with respect to x

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0$$

$$\Rightarrow \qquad \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

Example 15. Find $\frac{dy}{dx}$:

(i) $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+...\infty}}}$ (ii) $y=(\sin x)^{(\sin x)^{...\infty}}$ (iii) $y=e^{x+e^{x+e^{x+...\infty}}}$

Solution : (i) Here $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+...\infty}}}$

or $y=\sqrt{\log x+y}$

Squaring both the sides $y^2=\log x+y$

Differentiating both the sides with respect to x

$$2y\cdot\frac{dy}{dx}=\frac{1}{x}+\frac{dy}{dx}$$

$$\Rightarrow \qquad (2y-1)\frac{dy}{dx}=\frac{1}{x}$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{1}{x(2y-1)}.$$

(ii) Here $y=(\sin x)^{(\sin x)^{...\infty}}$

$$=(\sin x)^y$$

taking log both the sides

$$\log y=y\log(\sin x)$$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx}=y.\frac{1}{\sin x}.\cos x+\log(\sin x).\frac{dy}{dx}$$

$$\left\{\frac{1}{y}-\log(\sin x)\right\}\frac{dy}{dx}=y\cot x$$

$$\frac{dy}{dx}=\frac{y^2\cot x}{1-y\log(\sin x)}.$$

(iii) Here $y=e^x+e^z+e^x+\ldots\infty=e^{x+y}$

taking log both the sides

$$\log y=(x+y)\log e$$

$\Rightarrow$ $\log y=x+y$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx}$$

$$\left(\frac{1}{y}-1\right)\frac{dy}{dx}=1$$

$\Rightarrow$ $$\frac{dy}{dx}=\frac{y}{1-y}.$$

Exercise 7.3

Find $\dfrac{dy}{dx}$:

1. (a) $2x+3y=\sin y$ (b) $x^2+xy+y^2=200$
2. (a) $\sqrt{x}+\sqrt{y}=\sqrt{a}$ (b) $\tan(x+y)+\tan(x-y)=4$
3. (a) $\sin x+2\cos^2 y+xy=0$ (b) $x\sqrt{y}+y\sqrt{x}=1$
4. (a) $(x^2+y^2)^2=xy$ (b) $\sin(xy)+\dfrac{x}{y}=x^2-y$
5. (a) $x^3+y^3=3axy$ (b) $x^y+y^x=a^b$
6. (a) $y=x^y$ (b) $x^a.y^b=(x-y)^{a+b}$
7. (a) $y=e^x+e^{x^2}+....+e^{x^5}$ (b) $y=\sqrt{e^{\sqrt{x}}},x>0$
8. (a) $y=\dfrac{\cos x}{\log x},x>0$ (b) $y=\sqrt{x}^{\sqrt{x}^{\sqrt{x}^{\cdots\infty}}}$
9. (a) $y\sqrt{1-x^2}=\sin^{-1}x$ (b) $y\sqrt{1+x}=\sqrt{1-x}$
10. (a) $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}$ (b) $y^x+x^y+x^x=a^b$

7.07 Derivative of parametric functions

When x and y are represented in terms of other variable like $x = f(t),\ y = \phi(t)$ then variable t is said to be parameter and equations of such type are known as parametric equations. The below given formula is also used to find $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ where } \frac{dx}{dt} \neq 0.$$

Illustrative Examples

Example 16. Find $\frac{dy}{dx}$, when

(i) $x = 2at^2,\ y = at^4$ (ii) $x = \sin t,\ y = \cos 2t$ (iii) $x = 4t,\ y = \frac{4}{t}$

Solution : (i) Here $x = 2at^2 \Rightarrow \frac{dx}{dt} = 4at$

and $y = at^4 \Rightarrow \frac{dy}{dt} = 4at^3$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2.$$

(ii) Here $x = \sin t \Rightarrow \frac{dx}{dt} = \cos t$

and $y = \cos 2t \Rightarrow \frac{dy}{dt} = -2\sin 2t$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{2\sin 2t}{\cos t} = \frac{-2.2\sin t\cos t}{\cos t} = -4\sin t$$

(iii) Here $x = 4t \Rightarrow \frac{dx}{dt} = 4$

and $y = \frac{4}{t} \Rightarrow \frac{dy}{dt} = -\frac{4}{t^2}$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4/t^2}{4} = -\frac{1}{t^2}$$

Example 17. Find $\frac{dy}{dx}$, when

(i) $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$, $y = \cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$ (ii) $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$

(iii) $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$, $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Solution: (i) $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$, $y = \cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$

Here putting $t = \tan\theta$

$$x = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta \qquad \Rightarrow \frac{dx}{d\theta} = 2$$

and $$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta \qquad \Rightarrow \frac{dy}{d\theta} = 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2}{2} = 1.$$

(ii) Here $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$

differentiating with respect to t

$$\frac{dx}{dt} = \frac{(1+t^3)(3a) - 3at(0+3t^2)}{(1+t^3)^2} = \frac{3a - 6at^3}{(1+t^3)^2}$$

and $$\frac{dy}{dt} = \frac{(1+t^3)(6at) - 3at^2(0+3t^2)}{(1+t^3)^2} = \frac{6at - 3at^4}{(1+t^3)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6at - 3at^4}{3a - 6at^3} = \frac{t(2-t^3)}{1-2t^3}$$

(iii) Here $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$, $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

differentiating with respect to θ

$$\frac{dx}{d\theta} = e^{\theta}.\left(\theta + \frac{1}{\theta}\right) + e^{\theta}\left(1 - \frac{1}{\theta^2}\right) = e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)$$

$$\frac{dy}{d\theta} = -e^{-\theta}\left(\theta - \frac{1}{\theta}\right) + e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) = e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\theta^2 + 1 - \theta^3 + \theta\right)}{e^{\theta}\left(\theta^2 - 1 + \theta^3 + \theta\right)}.$$

Example 18. If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then prove that $x\frac{dy}{dx} + y = 0$

Solution : Given $t - \frac{1}{t} = x^2 + y^2$ and $t^2 + \frac{1}{t^2} = x^4 + y^4$

$\because$ $$\left(t - \frac{1}{t}\right)^2 = t^2 + \frac{1}{t^2} - 2$$

$\Rightarrow$ $$(x^2 + y^2)^2 = x^4 + y^4 - 2$$

$\Rightarrow$ $$x^4 + y^4 + 2x^2y^2 = x^4 + y^4 - 2$$

$\therefore$ $$x^2y^2 = -1$$

differentiating with respect to x

$$x^2.2y\frac{dy}{dx} + 2x.y^2 = 0$$

$\Rightarrow$ $$2xy\left(x\frac{dy}{dx} + y\right) = 0$$

$\Rightarrow$ $$x\frac{dy}{dx} + y = 0.$$

Exercise 7.4

Find $\frac{dy}{dx}$, when

1. (a) $x = a\sec t,\ y = b\tan t$ (b) $x = \log t + \sin t,\ y = e^t + \cos t$
2. (a) $x = \log t,\ y = e^t + \cos t$ (b) $x = a\cos\theta,\ y = b\sin\theta$
3. (a) $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$ (b) $x = \theta - \sin\theta,\ y = a(1 + \cos\theta)$
4. (a) $x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$ (b) $x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t$
5. (a) $x = \sqrt{\sin 2\theta},\ y = \sqrt{\cos 2\theta}$ (b) $x = a\cos^3 t,\ y = a\sin^3 t$
6. If $x^3 + y^3 = t - \frac{1}{t}$ and $x^6 + y^6 = t^2 + \frac{1}{t^2}$, then prove that $x^4y^2\frac{dy}{dx} = 1$

7.08 Second Order Derivative

Let $y = f(x)$

then $\frac{dy}{dx} = f'(x)$ (1)

Now if $f'(x)$ is derivable then we can differentiate equation (1) with respect to x. Then left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is known as second order derivative of $f(x)$ and is written as $\frac{d^2y}{dx^2}$, or $f''(x)$.

Derivatives of higher orders can also be found like this

Illustrative Examples

Example 19. Find the second order derivative of the following functions

(i) x^{20} (ii) $x^3 \log x$ (iii) $e^{6x}.\cos 3x$

(iv) $\log(\log x)$ (v) $\sin(\log x)$ (vi) $\tan^{-1} x$.

Solution : (i) Let $y = x^{20}$

$\Rightarrow$ $\frac{dy}{dx} = 20x^{19}$

$\Rightarrow$ $\frac{d^2y}{dx^2} = 20.19x^{18} = 380x^{18}$.

(ii) Let $y = x^3 \log x$

$\Rightarrow$ $\frac{dy}{dx} = x^3.\frac{1}{x} + \log x.3x^2 = x^2 + 3x^2 \log x$

$\therefore$ $\frac{d^2y}{dx^2} = 2x + 3\left\{x^2.\frac{1}{x} + \log x.2x\right\}$

$= 2x + 3(x + 2x\log x) = 5x + 6x\log x = x(5 + 6\log x)$.

(iii) Let $y = e^{6x} \cos 3x$

$\Rightarrow$ $\frac{dy}{dx} = e^{6x}(-\sin 3x).3 + \cos 3x.e^{6x}.6$

$= 6e^{6x}.\cos 3x - 3e^{6x}.\sin 3x$

$\therefore$

$\frac{d^2y}{dx^2} = 6\{e^{6x}(-\sin 3x).3 + \cos 3x.e^{6x}.6\} - 3\{e^{6x}.\cos 3x.3 + \sin 3x.e^{6x}.6\}$

$= -18e^{6x}\sin 3x + 36e^{6x}\cos 3x - 9e^{6x}\cos 3x - 18e^{6x}\sin 3x$

$= 9e^{6x}(3\cos 3x - 4\sin 3x)$.

(iv) Let $y = \log(\log x)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\log x}.\frac{1}{x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{\log x}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\frac{d}{dx}\left(\frac{1}{\log x}\right)$$

$$= -\frac{1}{x^2 \log x} + \frac{1}{x}\left\{\frac{\log x.(0) - 1.\frac{1}{x}}{(\log x)^2}\right\} = -\frac{1}{x^2 \log x} + \frac{1}{x}\left(-\frac{1}{x(\log x)^2}\right)$$

$$= -\frac{1}{x^2 \log x} - \frac{1}{x^2 (\log x)^2} = -\frac{1}{x^2 \log x}\left(1 + \frac{1}{\log x}\right).$$

(v) Let $y = \sin(\log x)$

$$\Rightarrow \quad \frac{dy}{dx} = \cos(\log x).\frac{1}{x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \cos(\log x)\left(-\frac{1}{x^2}\right) + \frac{1}{x}\{-\sin(\log x)\}.\frac{1}{x}$$

$$= -\frac{\cos(\log x)}{x^2} - \frac{\sin(\log x)}{x^2} = -\frac{1}{x^2}\{\cos(\log x) + \sin(\log x)\}.$$

(vi) Let $y = \tan^{-1} x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{(1+x^2)(0) - 1.(0+2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

Example 20. If $y = (x + \sqrt{x^2 - 1})^m$, then prove that

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2 y = 0.$$

Solution : Given $y = (x + \sqrt{x^2 - 1})^m$

Differentiating with respect to x

$$\frac{dy}{dx} = m\left(x + \sqrt{x^2 - 1}\right)^{m-1}\left\{1 + \frac{2x}{2\sqrt{x^2 - 1}}\right\}$$

$$= m(x + \sqrt{x^2 - 1})^{m-1}\frac{(\sqrt{x^2 - 1} + x)}{\sqrt{x^2 - 1}} = \frac{m(x + \sqrt{x^2 - 1})^m}{\sqrt{x^2 - 1}} = \frac{my}{\sqrt{x^2 - 1}}$$

squaring both the sides

$$\left(x^2-1\right)\left(\frac{dy}{dx}\right)^2=m^2y^2$$

again differentiating with respect to x

$$\left(x^2-1\right)\cdot 2\left(\frac{dy}{dx}\right).\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2=m^2 2y\frac{dy}{dx}$$

dividing by $2\frac{dy}{dx}$

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0.$$

Example 21. If $x^3+y^3+3ax^2=0$

$$\frac{d^2y}{dx^2}+\frac{2a^2x^2}{y^5}=0.$$

Solution : Here $\qquad x^3+y^3+3ax^2=0 \qquad (1)$

Differentiating with respect to x

$$3x^2+3y^2\frac{dy}{dx}+3a.2x=0$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\left(\frac{x^2+2ax}{y^2}\right) \qquad (2)$$

Again differentiating with respect to x

$$\frac{d^2y}{dx^2}=-\left[\frac{y^2(2x+2a)-(x^2+2ax)2y\frac{dy}{dx}}{(y^2)^2}\right]$$

Substituting the value of $\frac{dy}{dx}$ from (2)

$$\frac{d^2y}{dx^2}=-\frac{1}{y^3}\left\{y\left(2x+2a\right)+\left(x^2+2ax\right)2.\frac{\left(x^2+2ax\right)}{y^2}\right\}$$

$$=-\frac{2}{y^5}\left\{y^3(x+a)+x^4+4a^2x^2+4ax^3\right\}$$

from equation (1) putting $\qquad y^3=-\left(3ax^2+x^3\right)$

$$\frac{d^2y}{dx^2}=-\frac{2}{y^5}\left\{-\left(3ax^2+x^3\right)(x+a)+x^4+4a^2x^2+4ax^3\right\}$$

$$=-\frac{2}{y^5}\left\{-3ax^3-x^4-3a^2x^2-ax^3+x^4+4a^2x^2+4ax^3\right\}$$

$$= -\frac{2}{y^5}\left(a^2x^2\right)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$$

Example 22. If $y = \sin\left(a\sin^{-1}x\right)$, then prove that

$$\left(1-x^2\right)y_2 - xy_1 + a^2y = 0$$

Solution : Here $\qquad y = \sin(a\sin^{-1}x)$

Differentiating with respect to x

$$y_1 = \cos\left(a\sin^{-1}x\right).\frac{a}{\sqrt{1-x^2}}$$

Squaring both the sides $(1-x^2)y_1^2 = a^2\cos^2(a\sin^{-1}x) = a^2\{1-\sin^2(a\sin^{-1}x)\}$

$$\Rightarrow \qquad (1-x^2)y_1^2 = a^2(1-y^2)$$

Again differentiating with respect to x

$$(1-x^2)2y_1y_2 - 2xy_1^2 = a^2(0-2yy_1)$$

Dividing by $2y_1$,

$$(1-x^2)y_2 - xy_1 + a^2y = 0.$$

Exercise 7.5

1. Find $\frac{d^2y}{dx^2}$, when

   (a) $y = x^3 + \tan x$ (b) $y = x^2 + 3x + 2$ (c) $y = x\cos x$

   (d) $y = 2\sin x + 3\cos x$ (e) $y = e^{-x}\cos x$ (f) $y = a\sin x - b\cos x$

2. If $y = a\sin x + b\cos x$, then prove that

$$\frac{d^2y}{dx^2} + y = 0.$$

3. If $y = \sec x + \tan x$, then prove that

$$\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}.$$

4. If $y = a\cos nx + b\sin nx$, then prove that

$$\frac{d^2y}{dx^2} + n^2y = 0.$$

5. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$

6. If $x^3 + y^3 - 3axy = 0$, then prove that

$$\frac{d^2y}{dx^2} = \frac{2a^2xy}{(ax-y^2)^3}.$$

7. If $y = \sin^{-1} x$, then prove that : $\left(1-x^2\right)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0.$

8. If $y = (\sin^{-1} x)^2$,then prove that : $\left(1-x^2\right)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0.$

7.09 Rolle's theorem

If a real valued function f is defined in the interval $[a, b]$, such that

(i) f is continuous in the closed interval in $[a, b]$

(ii) f is derivable in the open interval in (a, b)

(iii) $f(a) = f(b)$

then in the open interval (a, b) there exists a point c such that $f'(c) = 0$

7.10 Geometrical meaning of Rolle's Theorem

We can define Rolle's Theorem under two conditions:

Case I: when the function f is constant then

$$f(x) = c, \quad \forall\, x \in [a, b]$$

The graph of the function will be paralel to x-axis. Thus for every point in the open interval (a, b) $f'(x) = 0$ (see fig : 7.01)

Case II: When function f is not constant then

as per Rolle's theorem let f be continuous in a closed interval $[a, b]$ and derivable in the open interval (a, b), then f is derivable.That means tangents can be drawn at $x \in (a,b)$ to the curve $y=f(x)$ .Also $f(a) = f(b)$, it is clear from this that the value of the function $f(x)$ will either increase or decrease (see fig 7.02), under both the conditions there exists a point which will always be parallel to x-axis i.e. at that point $f'(x) = 0$, i.e. at these points the slope of the line will be zero

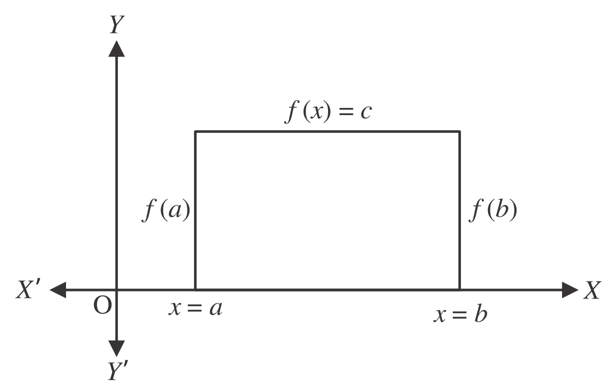
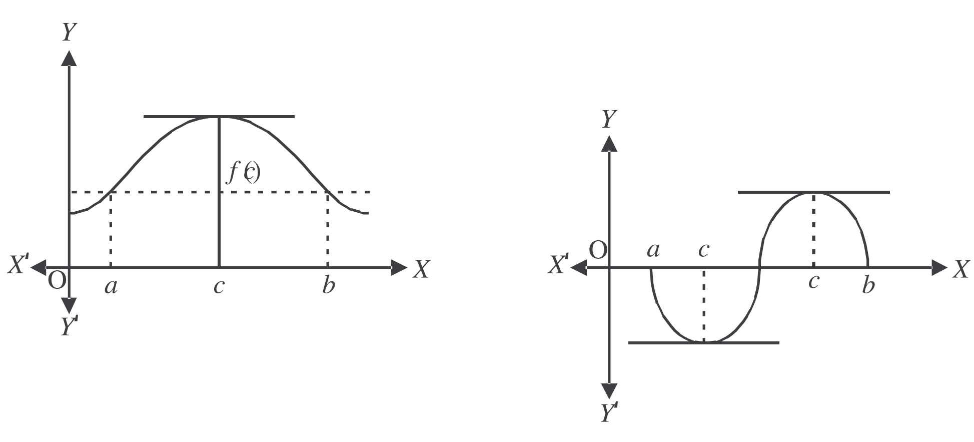


Fig. 7.01



(a) (b)

Fig 7.02

7.11 Lagrange's mean value theorem

If a real valued function f is defined in the closed interval $[a, b]$ such that

(i) f is continuous in $[a, b]$

(ii) f is differentiable in (a, b)

then there exists a point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Note : Mean value yheorem is the extension of Rolle's theorem.

7.12 Geometrical meaning of Lagrange's mean value theorem (LMVT)

The graph of function $y = f(x)$ is shown in fig. 7.03. Also $f'(c)$ is the slope of $y = f(x)$ at point $(c, f(c))$. It is clear from the fig 7.03 that $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the line drawn from the points $(a, f(a))$ and $(b, f(b))$. According to the LMVT there exists a point c in (a, b) such that the tangent drawn at point $(c, f(c))$ is parallel to the line drawn from the points $(a, f(a))$ and $(b, f(b))$.

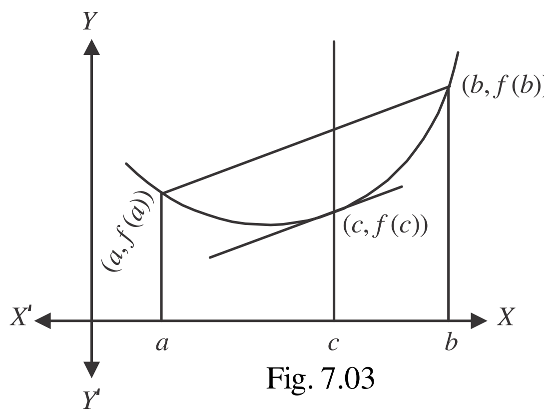


Fig. 7.03

7.13 Other form of Lagrange's mean value theorem

If we take $b = a + \hbar,\ \hbar > 0, c = a + \theta\hbar,\ 0 < \theta < 1$ and $c \in (a, b) \Rightarrow a + \theta\hbar \in (a, a + \hbar)$, in Lagrange's mean value theorem then it takes the form as shown below-

If the real valued function f is defined in the interval $[a, a + h]$

(i) f is continuous in the closed interval $[a, a + h]$

(ii) f is differentiable in the open interval $(a, a + h)$ then there exists a real number θ in the interval (0, 1) such that $f(a + h) = f(a) + hf'(a + \theta h)$

Note: For this theorem $f(a) = f(b)$ is not necessary. If $f(a) = f(b)$ then this theorem changes into Rolle's theorem.

Illustrative Examples

Example 23. Verify the Rolle's theorem for the following functions

(i) $f(x) = \sqrt{4 - x^2};\quad x \in [-2, 2]$ (ii) $f(x) = e^x \sin x;\quad x \in [0, \pi]$

Solution : (i) Clearly the function $f(x) = \sqrt{4 - x^2}$ is continuous in the interval $[-2, 2]$ and $f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$, which is defined at every point of the interval $(-2, 2)$ i.e. $f(x)$, is derivable in the interval $(-2, 2)$

$\because\quad f(-2) = 0 = f(2)$

$\Rightarrow\quad f(-2) = f(2)$

function $f(x)$, satisfies all the three conditions

Hence $\quad f'(c) = 0 \Rightarrow \dfrac{-c}{\sqrt{4 - c^2}} = 0$

$\Rightarrow\quad c = 0 \quad \because c \in (-2, 2)$

Thus Rolle's theorem is verified

(ii) $$f(x) = e^x \sin x, \; x \in [0, \pi]$$

Clearly the function $f(x)$, is continuous in the interval $[0, \pi]$ and $f'(x) = e^x \cos x + e^x \sin x$, which is defined at every point of the interval $(0, \pi)$ i.e. $f(x)$, is derivable in the interval $(0, \pi)$

$\because$ $$f(0) = 0 = f(\pi)$$

function $f(x)$, satisfies all the three conditions

Hence $$f'(c) = 0 \Rightarrow e^c \cos c + e^c \sin c = 0$$

$$\Rightarrow \quad e^c(\cos c + \sin c) = 0$$

$$\Rightarrow \quad \cos c + \sin c = 0$$

$$\Rightarrow \quad c = \frac{3\pi}{4} \qquad \because c \in (0, \pi)$$

Thus Rolle's theorem is verified

Example 24. Verify Rolle's theorem for the following functions

(i) $f(x) = 3 + (x-2)^{2/3}; \quad x \in [1, 3]$ (ii) $f(x) = \sin\frac{1}{x}; \quad x \in [-1, 1]$

Solution : (i) $$f(x) = 3 + (x-2)^{2/3}; \quad x \in [1, 3]$$

Clearly $f(x)$, is continuous in the interval at $[1, 3]$

$f'(x) = \dfrac{2}{3(x-2)^{1/3}}$, is infinite in the interval at $x = 2 \in (1, 3)$, $f(x)$ is not derivable.

Thus Rolle's theorem is not verified for $f(x)$ in the interval $[1, 3]$

(ii) $$f(x) = \sin\frac{1}{x}; \quad x \in [-1, 1]$$

$\because$ Function $f(x) = \sin\frac{1}{x}$ is not continuous at $x = 0$ $0 \in [-1, 1]$ thus $f(x), [-1, 1]$ is not continuous, Rolle's theorem is not verified for $f(x) = \sin\frac{1}{x}$ in the interval $[-1, 1]$.

Example 25. Examine the applicability of Lagrange's mean value theorem for following functions:

(i) $f(x) = |x|; \quad x \in [-1, 1]$ (ii) $f(x) = \frac{1}{x}; \quad x \in [-1, 1]$

(iii) $f(x) = x - \frac{1}{x}; \quad x \in [1, 3]$ (iv) $f(x) = x - 2\sin x; \quad x \in [-\pi, \pi]$

Solution : (i) $\because$ $f(x) = |x|$ is continuous everywhere hence it is continuous in the interval $[-1, 1]$ also $f(x) = |x|$ is not derivable at $x = 0$ therefore function $f(x)$, is not derivable in the interval $(-1, 1)$. Thus LMVT is not verified for $f(x)$ in the interval $[-1, 1]$

(ii) $\because f(x) = \frac{1}{x}; \quad x = 0 \in [-1, 1]$ is not continuous so $f(x)$ is also not continuous in the interval [–1, 1], thus LMVT is not verified.

(iii) Here $f(x) = x - \frac{1}{x}; \quad x \in [1, 3],$ which is continuous at [1, 3] and $f'(x) = 1 + \frac{1}{x^2},$ which exists and finite in the interval (1, 3) thus $f(x)$ is derivable in the interval (1, 3). Hence function $f(x)$, satisfies the conditions of Lagrange's MVT

Now $$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow \quad 1 + \frac{1}{c^2} = \frac{3 - \frac{1}{3} - \left(1 - \frac{1}{1}\right)}{2}$$

$$\Rightarrow \quad 1 + \frac{1}{c^2} = \frac{4}{3}$$

$$\Rightarrow \quad \frac{1}{c^2} = \frac{1}{3}$$

$$\Rightarrow \quad c = \pm\sqrt{3}$$

$$\Rightarrow \quad x = \sqrt{3} \in (1, 3)$$

Thus LMVT is verified.

(iv) Here $f(x) = x - 2\sin x; \quad x \in [-\pi, \pi]$ clearly $f(x)$, is continuous and derivable in the interval $[-\pi, \pi]$ thus $f(x)$ satisifes both the conditions of MVT in the interval $[-\pi, \pi]$, hence their exists a point c in the interval $(-\pi, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$$

$$\Rightarrow \quad 1 - 2\cos c = \frac{\pi - (-\pi)}{2\pi} = \frac{2\pi}{2\pi} = 1$$

$$\Rightarrow \quad \cos c = 0$$

$$\Rightarrow \quad c = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \qquad \because c = \pm\frac{\pi}{2} \in (-\pi, \pi)$$

Thus LMVT is satisfied.

Exercise 7.6

1. Verify Rolle's theorem for the functions given below:

(a) $f(x) = e^x(\sin x - \cos x); \quad x \in [\pi/4, 5\pi/4]$ (b) $f(x) = (x-a)^m (x-b)^n; \quad x \in [a, b], m, n \in N$

(c) $f(x) = |x|; \quad x \in [-1, 1]$ (d) $f(x) = x^2 + 2x - 8; \quad x \in [-4, 2]$

(e) $f(x) = \begin{cases} x^2 + 1 & ; \quad 0 \le x \le 1 \\ 3 - x & ; \quad 1 < x \le 2 \end{cases}$ (f) $f(x) = [x]; \quad x \in [-2, 2]$

2. Verify Rolle's theorem for the functions given below :

(a) $f(x) = x^2 + 5x + 6; \quad x \in [-3, -2]$ (b) $f(x) = e^{-x}\sin x; \quad x \in [0, \pi]$

(c) $f(x) = \sqrt{x(1-x)}; \quad x \in [0, 1]$ (d) $f(x) = \cos 2x; \quad x \in [0, \pi]$

3. Verify Lagrange's mean value theorem for the functions given below:

(a) $f(x) = x + \frac{1}{x}; \quad x \in [1, 3]$ (b) $f(x) = \frac{x^2 - 4}{x - 1}; \quad x \in [0, 2]$

(c) $f(x) = x^2 - 3x + 2; \quad x \in [-2, 3]$ (d) $f(x) = \frac{1}{4x - 1}; \quad x \in [1, 4]$

Miscellaneous Examples

Example 26. Find the differential coefficient of the function with respect to x

(a) $\cos x^{\circ}$ (b) $\sin \log(1 + x^2)$ (c) $\log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

(d) $\log(x + \sqrt{x^2 + a^2})$ (e) $\log_7(\log x)$

Solution : (a) Let $y = \cos x^{\circ}$

$\because$ $180^{\circ} = \pi$ radian

$$x^{\circ} = \frac{\pi}{180}x \text{ radian}$$

$$y = \cos\left(\frac{\pi x}{180}\right)$$

differentiating with respect ot x

$$\frac{dy}{dx} = -\sin\left(\frac{\pi x}{180}\right)\frac{d}{dx}\left(\frac{\pi x}{180}\right) = \frac{-\pi}{180}\sin\left(\frac{\pi x}{180}\right) = \frac{-\pi}{180}\sin x^{\circ}.$$

(b) Let $y = \sin \log\left(1 + x^2\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \cos \log\left(1 + x^2\right)\frac{d}{dx}\left\{\log\left(1 + x^2\right)\right\}$$

$$= \cos\left(\log\left(1+x^2\right)\right).\frac{1}{\left(1+x^2\right)}\frac{d}{dx}\left(1+x^2\right)$$

$$\frac{1}{\left(1+x^2\right)}\cos\left(\log(1+x^2)\right)(0+2x) = \frac{2x}{1+x^2}\cos\log(1+x^2)$$

(c) Let $y = \log\tan\left(\pi/4 + x/2\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\tan(\pi/4+x/2)}\frac{d}{dx}\{\tan(\pi/4+x/2)\}$$

$$= \frac{1}{\tan(\pi/4+x/2)}\sec^2(\pi/4+x/2)\frac{d}{dx}(\pi/4+x/2)$$

$$= \frac{1}{2\sin(\pi/4+x/2)\cos(\pi/4+x/2)}$$

$$= \frac{1}{\sin 2(\pi/4+x/2)} = \frac{1}{\sin(\pi/2+x)} = \frac{1}{\cos x} = \sec x.$$

(d) Let $y = \log(x+\sqrt{x^2+a^2})$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{(x+\sqrt{x^2+a^2})}\frac{d}{dx}(x+\sqrt{x^2+a^2})$$

$$= \frac{1}{(x+\sqrt{x^2+a^2})}.\left(1+\frac{2x}{2\sqrt{x^2+a^2}}\right)$$

$$= \frac{1}{\left(x+\sqrt{x^2+a^2}\right)}\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{\left(x^2+a^2\right)}}\right) = \frac{1}{\sqrt{x^2+a^2}}.$$

(e) Let $y = \log_7(\log x) = \frac{1}{\log_e 7}\{\log_e(\log x)\}$, (Base change formula)

defined for $x > 1$,

$$\frac{dy}{dx} = \frac{1}{(\log 7)}\frac{d}{dx}\{\log(\log x)\}$$

$$= \frac{1}{\log_7}.\frac{1}{\log x}\frac{d}{dx}(\log x) = \frac{1}{x\log 7.\log x}.$$

Example 27. Differentiate the following functions with respect to x

(a) $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ (b) $\tan^{-1}\left(\frac{x^{1/3}+a^{1/3}}{1-(ax)^{1/3}}\right)$ (c) $\sin^{-1}(x\sqrt{1-x}-\sqrt{x}\cdot\sqrt{1-x^2})$.

Solution : (a)
$$y=\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$$

$$=\sin^{-1}\left(\frac{2^x\cdot 2}{1+(2^x)^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \qquad [\text{let } 2^x=\tan\theta\ ]$$

$$=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}(2^x)$$

$$\Rightarrow \quad \frac{dy}{dx}=2\cdot\frac{1}{1+(2^x)^2}\frac{d}{dx}(2^x)=\frac{2}{(1+4^x)}.2^x\log 2=\frac{2^{x+1}\log 2}{1+4^x}.$$

(b) Let
$$y=\tan^{-1}\left(\frac{x^{1/3}+a^{1/3}}{1-(ax)^{1/3}}\right)$$

(Using formula, $\tan^{-1}\left(\frac{A+B}{1-AB}\right)=\tan^{-1}A+\tan^{-1}B$)

$$y=\tan^{-1}(x^{1/3})+\tan^{-1}(a^{1/3})$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{1+(x^{1/3})}\frac{d}{dx}(x^{2/3})+0$$

$$=\frac{(1/3)x^{-2/3}}{1+x^{2/3}}=\frac{1}{3x^{2/3}(1+x^{2/3})}.$$

(c) Let
$$y=\sin^{-1}(x\sqrt{1-x}-\sqrt{x}\cdot\sqrt{1-x^2})$$

(Using $\sin^{-1}A-\sin^{-1}B=\sin^{-1}(A\sqrt{1-B^2}-B\sqrt{1-A^2})$)

$$y=\sin^{-1}(x)-\sin^{-1}(\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{d}{dx}(\sqrt{x})$$

$$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

Example 28. Find $\frac{dy}{dx}$ when $x=(t+1/t)^a$ and $y=a^{t+1/t}$, where a is a constant

Solution : It is clear that both x and y are functions of t and $t\neq 0$ defined for all real numbers

Now $$\frac{dx}{dt}=a\left(t+\frac{1}{t}\right)^{a-1}\frac{d}{dt}\left(t+\frac{1}{t}\right)=a\left(t+\frac{1}{t}\right)^{a-1}\left(1-\frac{1}{t^2}\right).$$

Here $\frac{dx}{dt}\neq 0$ If $1-\frac{1}{t^2}\neq 0 \quad\Rightarrow\quad t\neq\pm 1$

and $$\frac{dy}{dt}=\frac{d}{dt}(a^{t+1/t})=a^{t+1/t}.\log a\frac{d}{dt}(t+1/t)=a^{t+1/t}\cdot\log a\left(1-\frac{1}{t^2}\right)$$

now for, $t\neq\pm 1$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a^{(t+1/t)}\left(1-\frac{1}{t^2}\right)\log a}{a(t+1/t)^{a-1}(1-1/t^2)}=\frac{a^{(t+1/t)}\log a}{a(t+1/t)^{a-1}}.$$

Example 29. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$ then prove that $p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}$.

Solution : Given $$p^2=a^2\cos^2\theta+b^2\sin^2\theta \tag{1}$$

differentiating with respect to θ

$$2P\frac{dp}{d\theta}=-2a^2\cos\theta\sin\theta+2b^2\sin\theta\cos\theta$$

$$=\left(b^2-a^2\right)\sin 2\theta \tag{2}$$

again differentiating with respect to θ

$$2p\frac{d^2p}{d\theta^2}+2\left(\frac{dp}{d\theta}\right)^2=2\left(b^2-a^2\right)\cos 2\theta$$

multiplying both sides with p^2

$$p^3\frac{d^2p}{d\theta^2}+p^2\left(\frac{dp}{d\theta}\right)^2=p^2\left(b^2-a^2\right)\cos 2\theta$$

adding p^4 both the sides

$$p^4+p^3\frac{d^2p}{d\theta^2}+\left(p\frac{dp}{d\theta}\right)^2=p^4+p^2\left(b^2-a^2\right)\cos 2\theta$$

putting the value from (2)

$$p^4+p^3\frac{d^2p}{d\theta^2}+\frac{\left(b^2-a^2\right)^2}{4}\cdot\sin^2 2\theta=p^4+p^2(b^2-a^2)\cos 2\theta$$

$$\Rightarrow p^4 + p^3 \frac{d^2p}{d\theta^2} + (b^2 - a^2)\sin^2\theta\cos^2\theta = p^2\{p^2 + (b^2 - a^2)(\cos^2\theta - \sin^2\theta)\}$$

$$= p^2\{(a^2\cos^2\theta + b^2\sin^2\theta) + (b^2 - a^2)(\cos^2\theta - \sin^2\theta)\} \text{ [from equation (1)]}$$

$$= p^2(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$= (a^2\cos^2\theta + b^2\sin^2\theta)(b^2\cos^2\theta + a^2\sin^2\theta) \qquad \text{[from (1)]}$$

$$\Rightarrow p^4 + p^3 \frac{d^2p}{d\theta^2} = a^2b^2(\sin^4\theta + \cos^4\theta) + a^4\sin^2\theta\cos^2\theta + b^4\sin^2\theta\cos^2\theta - (b^2 - a^2)^2\sin^2\theta\cos^2\theta$$

$$= a^2b^2(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta) = a^2b^2(\sin^2\theta + \cos^2\theta) = a^2b^2$$

$$\Rightarrow \quad p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$$

Example 30. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$ then prove that $y^2y_2 - xy_1 + y = 0$

Solution : From given equation, $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$

$$x^2 + y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2 = a^2 + b^2$$

differentiating with respect to x

$$\Rightarrow \quad 2x + 2yy_1 = 0$$

$$\Rightarrow \quad y_1 = -\frac{x}{y} \qquad (1)$$

again differentiating with respect to x

$$y_2 = -\left\{\frac{y.1 - xy_1}{y^2}\right\} = -\left\{\frac{y + x\cdot x/y}{y^2}\right\} \qquad \text{[from (1)]}$$

$$= -\frac{y^2 + x^2}{y^3} \qquad (2)$$

$$\Rightarrow \quad y^2y_2 - xy_1 + y = y^2\left(-\frac{y^2 + x^2}{y^3}\right) - x\left(\frac{-x}{y}\right) + y$$

$$= \frac{1}{y}\{-y^2 - x^2 + x^2 + y^2\} = 0.$$

Example 31. Verify Rolle's theorem for the functions given below:

(i) $f(x)=\log\left\{\dfrac{x^2+ab}{x(a+b)}\right\}; \quad x\in[a,b], x\neq 0$ (ii) $f(x)=\tan x; \quad x\in[0,\pi]$

Solution : (i) $$f(x)=\log\left\{\frac{x^2+ab}{x(a+b)}\right\}; \quad x\in[a,b], x\neq 0$$

$$=\log(x^2+ab)-\log x-\log(a+b)$$

clearly, $f(x)$ is continuous in $[a, b]$ and logarithmic functions are derivable thus $f(x)$ is derivable in the interval (a, b) as $f'(x)=\dfrac{2x}{x^2+ab}-\dfrac{1}{x}=\dfrac{x^2-ab}{x(x^2+ab)}$

now $$f(a)=\log\left\{\frac{a^2+ab}{a(a+b)}\right\}=\log 1=0$$

$\Rightarrow$ $$f(b)=\log\left\{\frac{b^2+ab}{b(a+b}\right\}=\log 1=0$$

$\Rightarrow$ $$f(a)=f(b)$$

$f(x)$, satisfies all the three conditions of Rolle's theorem

$$\therefore \quad f'(c)=0$$

$\Rightarrow$ $$\frac{c^2-ab}{c(c^2+ab)}=0$$

$\Rightarrow$ $$c=\sqrt{ab}\in(a,b)$$

Thus Rolle's theorem is verified.

(ii) $\because$ $f(x)=\tan x, x=\pi/2$ is not continuous as $\pi/2\in[0,\pi]$ i.e $f(x)$, is not continuous in the interval $[0, \pi]$, thus for $f(x)=\tan x; \quad x\in=[0,\pi]$ Rolle's theorem is not verified.

Miscellaneous Exercise–7

Differentiate the following functions with respect to x (Q 1-8)

1. $\sin^{-1}(x\sqrt{x}); \quad 0\le x\le 1$

2. $\dfrac{\cos^{-1} x/2}{\sqrt{2x+7}}; \quad -2<x<2$

3. $\cot^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}; \quad 0<x<\dfrac{\pi}{2}$

4. $x^3.e^x.\sin x$

5. $\log\left(\dfrac{x}{a^x}\right)$

6. $(x\log x)^{\log x}$

7. $x^{x^2-3}+(x-3)^{x^2}; \quad x>3$

8. $\sin^{-1} x+\sin^{-1}\sqrt{1-x^2}$

9. If $\log x = \tan^{-1}\left(\frac{y-x^2}{x^2}\right)$, then find $\frac{dy}{dx}$

10. If $y = 12(1-\cos t)$, $x = 10(t-\sin t)$, then find $\frac{dy}{dx}$

11. If $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \tan^{-1} a$, then prove that $\frac{dy}{dx} = \frac{y}{x}$

12. If $\sin y = x\sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

13. If $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, then find $\frac{dy}{dx}$.

14. If $y = \sin(\sin x)$, then show that

$$\frac{d^2y}{dx^2} + \tan x.\frac{dy}{dx} + y\cos^2 x = 0.$$

15. (a) If $y = e^{ax}\sin bx$, then show that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2+b^2\right)y = 0.$$

(b) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, then prove that

$$(1-x^2)y_2 = 3ay_1 - y = 0.$$

16. Verify Rolle's theorem for the functions

(a) $f(x) = (x-2)\sqrt{x}; \quad x \in [0, 2]$

(b) $f(x) = (x-1)(x-3); \quad x \in [1, 3]$

17. Examine the applicability of Lagrange's mean value theorem for the functions given below.

(a) $f(x) = (x-1)(x-2)(x-3); \quad x \in [0, 4]$

(b) $f(x) = \begin{cases} 1+x & ; \quad x < 2 \\ 5-x & ; \quad x \geq 2 \end{cases}; \quad x \in [1, 3]$

Important Points

1. If the functions f and g are differentiable at any point c in the interval $[a, b]$ then $f \pm g$, fg and f/g are also differentiable at point c and

(i) $D(f \pm g)(c) = f'(c) \pm g'(c)$

(ii) $D(fg)(c) = f'(c)g(c) + f(c)g'(c)$

(iii) $D(f/g)(c) = \frac{g(c)f'(c) - g'(c)f(c)}{[g(c)]^2}$; when $g(c) \neq 0$

2. If $y = f(u)$ and $u = \phi(x)$ then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

3. (i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$; (ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$; (iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$; (v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$; (vi) $\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

4. A real valued functions is continuous at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if its continuous on the whole of its domain. Every differentiable function is continuous, but the converse is not true.

5. For the function of the type $y = u^v$ solve it by taking log on both the sides.

6. $x = f(t),\ y = g(t)$ in this t is the parameter, we get $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ where $dx/dt \neq 0$

7. If $f'(x)$ is also a continuous function of x then it can be again differentiated.

8. **Rolle's Theroem:**

If Real valued function f is defiend in the interval [a, b], such that,

(i) f is continuous in the closed interval [a, b]

(ii) f is differentiable in the open interval (a, b)

(iii) $f(a) = f(b)$

then in the open interval (a, b) there exists a point c such that $f'(c) = 0$

9. **Lagrange's mean value theorem:**

If a real valued function f is defiend in the closed interval [a, b] such that

(i) f is continuous in [a, b]

(ii) f is differentiable in (a, b)

then there exists a point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

10. **Lagrange's Mean Value Theorem:**

If we take $\mathrm{b = a + h,\ h > 0,\ c = a + \theta h,\ 0 < \theta < 1}$ and $\mathrm{c \in (a,\ b) \Rightarrow a + \theta h \in (a,\ a + h)}$, in lagrange mean value theorem then it takes the form as shown below.

If the real valued function f is defiend in the interval [a, a + h] such that

(i) f is continuous in the closed interval $[a, a + \hbar]$

(ii) f is differentiable in the open interval $(a, a + \hbar)$ then there exists a real number θ in the interval (0, 1) such that $f(a + \hbar) = f(a) + \hbar f'(a + \theta \hbar)$

ANSWERS

Exericse 7.1

1. $2x\cos x^2$ 2. $2\sec^2(2x+3)$ 3. $-2x\sin x^2\cos\left(\cos x^2\right)$ 4. $\dfrac{2\sin x}{(1+\cos x)^2}$ 5. $\dfrac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}$

6. $\dfrac{\pi}{180}\cos x^{o}$ 7. $\cos ecx$ 8. $\dfrac{\pi}{180}\sec x^{o}\tan x^{o}$ 9. $\sec x$ 10. $\dfrac{1}{\sqrt{x^2+a^2}}$ 11. $\dfrac{2(1-x^2)}{1+x^2+x^4}$

12. $\left(\dfrac{x}{1+x^2}\right)\sec^2\left(\log\sqrt{1+x^2}\right)$ 13. $3.a^{\tan 3x}.\sec^2 3x.\log a$ 14. $\sec x$ 15. $3\sin^2 x.\sin 4x$

Exercise 7.2

1.(a) $\dfrac{2}{\sqrt{1-x^2}}$ (b) $\dfrac{3}{\sqrt{1-x^2}}$ 2. (a) $\dfrac{-2}{1+x^2}$ (b) $\dfrac{2}{1+x^2}$ 3. (a) $\dfrac{-3}{\sqrt{1-x^2}}$ (b) $\dfrac{-1}{2\sqrt{1-x^2}}$

4. (a) $\dfrac{-2}{\sqrt{1-x^2}}$ (b) $\dfrac{2}{1+x^2}$ 5. (a) 0 (b) $\dfrac{2}{\sqrt{1-4x^2}}$ 6. (a) $\dfrac{1}{1+x^2}$ (b) $\dfrac{2^{x+1}.\log 2}{1+4^x}$

7. (a) $\dfrac{-x}{\sqrt{1-x^2}}$ (b) $-\dfrac{1}{2\left(1+x^2\right)}$

Exercise 7.3

1. (a) $\dfrac{2}{\cos y-3}$ (b) $\dfrac{-(2x+y)}{x+2y}$ 2. (a) $-\sqrt{\dfrac{y}{x}}$ (b) $\dfrac{\sec^2(x+y)+\sec^2(x-y)}{\sec^2(x-y)-\sec^2(x+y)}$

3. (a) $\dfrac{\cos x+y}{2\sin 2y-x}$ (b) $\dfrac{-y}{x}\left(\dfrac{\sqrt{y}+2\sqrt{x}}{\sqrt{x}+2\sqrt{y}}\right)$ 4. (a) $\dfrac{4x^3+4xy^2-y}{x-4x^2y-4y^3}$ (b) $\dfrac{y\left\{2xy-1-y^2\cos(xy)\right\}}{\left\{y^2x\cos(xy)-x+y^2\right\}}$

5. (a) $\dfrac{ay-x^2}{y^2-ax}$ (b) $-\left\{\dfrac{yx^{y-1}+y^x\log y}{xy^{x-1}+x^y\log x}\right\}$ 6. (a) $\dfrac{y^2}{x(1-y\log x)}$ (b) $\dfrac{y}{x}$

7. (a) $e^x+2xe^{x^2}+3x^2e^{x^3}+4x^3e^{x^4}+5x^4e^{x^5}$ (b) $\dfrac{e^{\sqrt{x}}}{4\sqrt{x.e^{\sqrt{x}}}}$

8. (a) $\dfrac{x\sin x\log x+\cos x}{x\left(\log x\right)^2}$ (b) $\dfrac{y^2}{x\left(2-y\log x\right)}$ 9. (a) $\dfrac{1+xy}{1+x^2}$ (b) $\dfrac{y}{x^2-1}$

10. (a) $\dfrac{\cos x}{2y-1}$ (b) $-\left\{\dfrac{y^x.\log y+y.x^{y-1}+x^x(1+\log x)}{x.y^{x-1}+x^y\log x}\right\}$

Exercise 7.4

1. (a) $\frac{b}{a}.\operatorname{cosec} t$ (b) $\frac{t(e^t - \sin t)}{1 + t\cos t}$ 2. (a) $t\left(e^t - \sin t\right)$ (b) $\frac{-b}{a}\cot\theta$

3. (a) $\frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$ (b) $-\cot\frac{\theta}{2}$ 4. (a) $\frac{\cos t(1 - 2\cos 2t)}{1 + 2\cos 2t}$ (b) $\tan t$

5. (a) $-(\tan 2\theta)^{3/2}$ (b) $-\tan t$

Exericse 7.5

1. (a) $6x + 2\sec^2 x\tan x$; (b) 2 ; (c) $-(x\cos x + 2\sin x)$; (d) $-2\sin x - 3\cos x$; (e) $2e^{-x}\sin x$;

(f) $-a\sin x + b\cos x$ 5. $\frac{4\sqrt{2}}{3a}$

Exericse 7.6

1. (a) valid (b) valid (c) invalid (d) valid (e) invalid (f) invalid
3. (a) valid (b) invalid (c) invalid (d) valid (e) valid (f) invalid

Miscellaneous Exercise – 7

1. $\frac{3}{2}\frac{\sqrt{x}}{\sqrt{1 - x^3}}$ 2. $-x\left\{\frac{2x + 7 + \sqrt{4 - x^2}\cos^{-1}x/2}{\sqrt{4 - x^2}(2x + 7)^{3/2}}\right\}$

3. $\frac{1}{2}$

4.. $x^3e^x\cos x + x^3e^x\sin x + 3x^2e^x\sin x$

5. $\frac{1}{x} - \log a$ 6. $(x\log x)^{\log x}.\left\{\frac{\log x(1 + \log x)}{x\log x} + \frac{\log(x.\log x)}{x}\right\}$

7. $x^{x^2-3}\left\{\frac{x^2 - 3}{x} + 2x\log x\right\} + (x - 3)^{x^2}\left\{\frac{x^2}{x - 3} + 2x\log(x - 3)\right\}$

8. 0

9. $2x\{1 + \tan(\log x)\} + x\sec^2(\log x)$

10. $\frac{6}{5}\cot\left(\frac{t}{2}\right)$

13. $(\sin x - \cos x)^{\sin x - \cos x}.(\cos x + \sin x)\{1 + \log(\sin x - \cos x)\}; \sin x > \cos x$