# Hydrodynamics

**Hydrodynamics** : It Involves study of flow of liquid when it is moving with respect to container.

**Type of flow**

**Steady and unsteady flow**

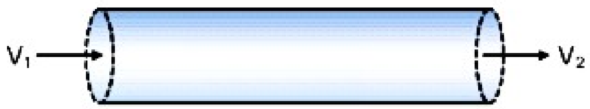
In a steady flow, the velocity, pressure and density at any point in the flow do not change with time, i.e.,

$$\frac{dv}{dt} = 0, \frac{dp}{dt} = 0 \text{ and } \frac{d\rho}{dt} = 0$$

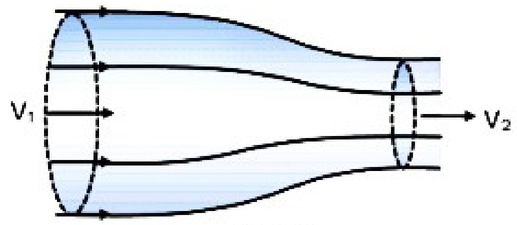
In an unsteady flow, the velocity at a point in the flow varies with time, i.e, $\frac{dv}{dt} \neq 0$.

**Uniform and Non-uniform flow**

A flow is said to be uniform at any instant of time if the velocity (both in magnitude and in direction) does not vary along the direction of flow. In a non-uniform flow, velocity varies in the direction of flow.



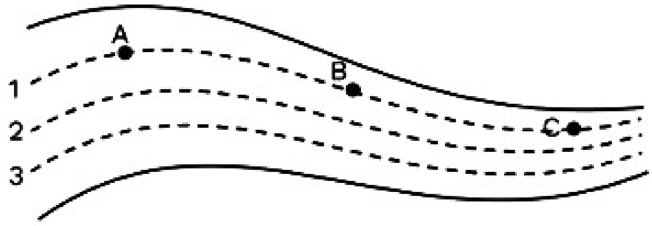

$V_2 = V_1$
Uniform flow




$V_2 > V_1$
Non-uniform flow

**Streamline flow**

If every point of a steadily flowing liquid follows exactly the same path that has been followed by the particles preceding it, the flow is said to be streamlined. The path is known as 'streamline'. In figure, the paths 1, 2, 3 are streamlines. If a liquid follows the path ABC, particles following it move along the same path.

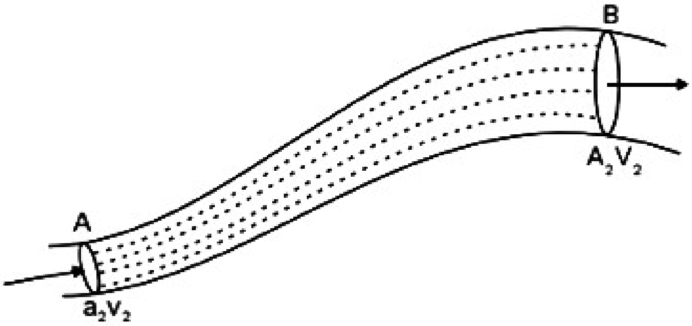


**Turbulent flow**

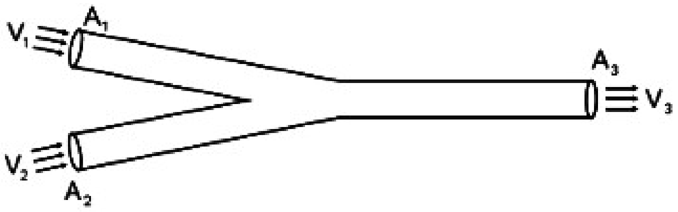
When the flow velocity becomes more than a certain critical value, the nature of flow becomes complicated. Random, irregular, local currents (called vortices) develop throughout the fluids. The resistance to the flow increases tremendously. This flow is called 'turbulent flow'.

**Continuity equation:**

The continuity equation is outcome of the law of mass conservation in the flow of an incompressible fluid.

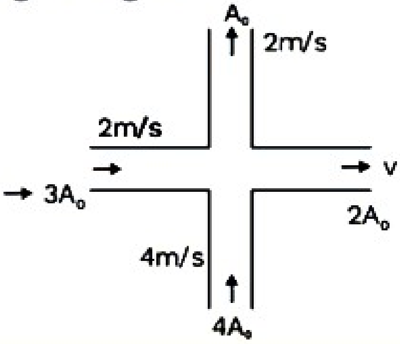



av = constant




$A_1V_1 + A_2V_2 = A_3V_3$

**Q. Find value of velocity V in given figure:**




**Sol.** Inlet flow rate = Outlet flow rate

$(3A_0 \times 2) + (4A_0 \times 4) = (A_0 \times 2) + (2A_0V)$

$22A_0 = 2A_0 + 2A_0V$

$V = 10$ m/s

## BERNOULLI'S EQUATION:

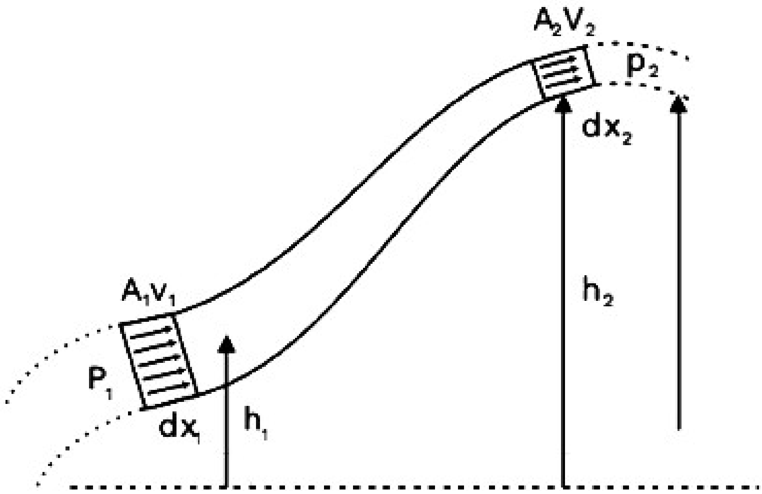
According to Bernoulli's theorem, sum total of three type of energies, kinetic energy, potential energy and pressure energy remains constant along a streamline in a steady flow of an ideal fluid. That is,

$$\boxed{P + \rho gh + \frac{1}{2}\rho v^2 = \text{constant}}$$

In other words, there is no dissipation of energy due to internal friction between adjacent layers of the fluid and density of this fluid remains constant.

### Proof of Bernoulli's Equation:

Figure shows flow of an ideal fluid through a tube of varying cross section and height




Using equation of continuity

$\Rightarrow A_1V_1 = A_2V_2$

$\Rightarrow A_1dx_1 = A_2dx_2$

Work done by pressure = $(A_1P_1)dx_1 - (A_2P_2)dx_2$

Work done by gravity = $A_2dx_2\rho gh_2 - A_1dx_2\rho gh_1$

Change in Kinetic energy = $\frac{1}{2}m_2v_2^2 - \frac{1}{2}mv_1^2$

$$= \frac{1}{2}A_2dx_2\rho V_2^2 - \frac{1}{2}A_1dx_1\rho v_1^2$$

Work done by pressure and gravity = Change in kinetic energy

$$\Rightarrow (A_1P_1dx_1 - A_2P_2dx_2) - (A_2dx_2\rho gh_2 - A_1dx_1\rho gh_1) = \frac{1}{2}A_2dx_2\rho V_2^2 - \frac{1}{2}A_1dx_1\rho v_1^2$$

$$\Rightarrow (P_1 - P_2) - (\rho gh_2 - \rho gh_1) = \frac{1}{2}\rho V_2^2 - \frac{1}{2}\rho V_1^2$$

$$\Rightarrow P_1 - P_2 + \rho gh_1 - \rho gh_2 = \frac{1}{2}\rho V_2^2 - \frac{1}{2}\rho V_1^2$$

$$\Rightarrow P_1 + \rho gh_1 + \frac{1}{2}\rho V_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho V_2^2 = \text{constant}$$

$$\boxed{P + \rho gh + \frac{1}{2}\rho v^2 = \text{constant}}$$ .......**Bernoulli's equation**

$$\underbrace{P}_{\downarrow} + \underbrace{\rho gh}_{\downarrow} + \underbrace{\frac{1}{2}\rho v^2}_{\downarrow} = \text{constant}$$

| | | |
|---|---|---|
| Pressure | Potential | Kinetic |
| Energy | energy | energy |
| Per | Per | Per |
| Unit | unit | unit |
| Volume | volume | volume |

$$\boxed{\frac{P}{\rho g} + h + \frac{1}{2}\frac{V^2}{g} = \text{constant}}$$ .......**Bernoulli's equation**

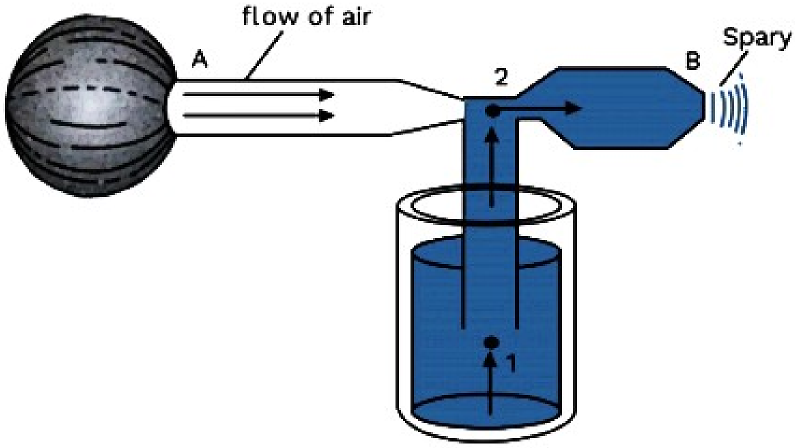
$$\underbrace{\frac{P}{\rho g}}_{\downarrow} + \underbrace{h}_{\downarrow} + \underbrace{\frac{1}{2}\frac{V^2}{g}}_{\downarrow} = \text{constant}$$

| | | |
|---|---|---|
| Pressure | Gravitational | Velocity |
| head | head | head |

**Application of Bernoulli's theorem:**

**Atomizer or sprayer**:

An atomiser is usually used in perfume and deodorant bottles. When the rubber balloon is pressed, the air passes with a large velocity over the tube dipping in the liquid to be sprayed. Due to this, pressure over the tube dipping in the liquid decreases. It makes the liquid rise up in the tube. Due to the applied pressure, the air rushing out with a large velocity from the balloon blows away the liquid coming out of the nozzle in the form of a fine spray.

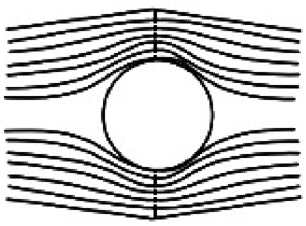



**Blowing off of the roofs of houses during a storm:**

In a storm, cyclone or hurricane, sometimes the light roofs of thatched houses are blown off. This is because, due to the high velocity wind blowing over the roof, the pressure above the roof decreases. As the atmospheric pressure below the roof is greater than pressure above the roof, the roof gets lifted and is blown away by the wind.

**Dynamic Lift On Spinning Ball:**

**(a) (a) Ball moving without spin:** In the figure, we can see the streamlines drawn around a ball which is moving relative to the fluid but is not spinning. The symmetric figure of the streamlines shows that the velocity of the fluid (air) is same at top and bottom of the ball. Pressure is also same at both points and hence there is no net upward or downward force on the ball.



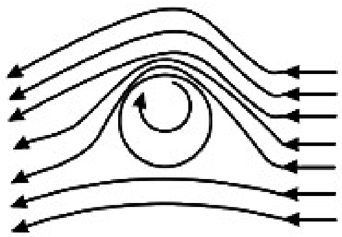
(a)

**(b) Ball moving with spin:** For a spinning ball, air is dragged along with the spin of the ball. Rougher the surface of the ball, more is the air dragged by the surface.

If a ball is moving as well as spinning, then the streamlines are as shown in the figure.

For a forward moving ball, the air moves backwards relatively and hence the velocity of air at top of the ball is more as compared to the velocity of air at bottom of the ball. As a result, the pressure is top is less than the pressure at bottom of the ball and a net upward force acts on the ball.

This lift force is known as dynamic lift and this effect is termed as **Magnus effect**.

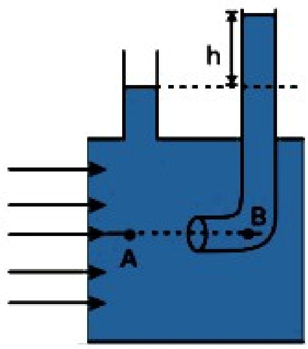


(b)

**Pitot Tube:**

Pitot tube is used to measure the speed of flow of a fluid. Let v be the velocity of liquid at A and $P_A$ and $P_B$ are pressures at A and B. Then $P_B - P_A = h\rho g$, where $\rho$ is density of liquid rise in vertical tube.

Applying Bernoulli's theorem at A and B, we have




$$P_A + \frac{1}{2}\rho v^2 = P_B + 0$$

$$\Rightarrow v^2 = \frac{2(P_B - P_A)}{\rho}$$

$$\Rightarrow v = \sqrt{\frac{2(P_B - P_A)}{\rho}}$$

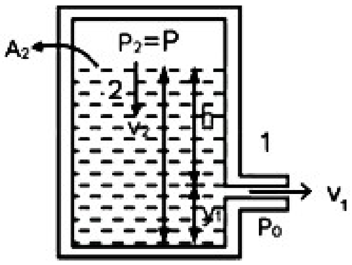
$$\Rightarrow v = \sqrt{2gh}$$

**Speed of efflux (Torricelli's law)**

According to equation of continuity

$A_1V_1 = A_2V_2$

$\Rightarrow V_2 = \frac{A_1V_1}{A_2}$




If $A_2 >> A_1$, then $v_2 \to 0$

Now by applying Bernoulli equation at points 1 and 2

$$P_0 + \frac{1}{2}\rho V_1^2 + \rho g y_1 = P + \rho g y_2$$

$$v_1 = \sqrt{2gh + \frac{2(P - P_0)}{\rho}} \qquad [\because \quad h = y_2 - y_1]$$

**Case-I** when $P >> P_0$ [In rocket propulsion]

$$V_1 = \sqrt{2\frac{(P - P_0)}{\rho}}$$

**Case II** when $P = P_0$ [tank is open to atmosphere]

$$V_1 = \sqrt{2gh}$$

**Venturimeter:**

It is a gauge put on a pipe two measure flow speed of liquid.

Consider that a liquid having a density $\rho$ is flowing in a pipe of cross-sectional area $A_1$.

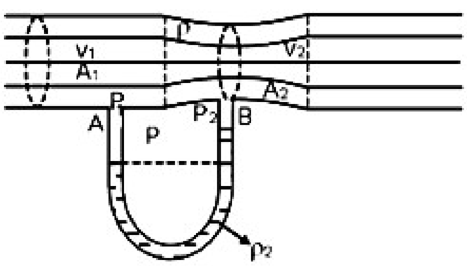
Also, $A_2$ is the cross sectional area at the throat and a manometer is attached as shown in the figure. Let $v_1$ and $P_1$ be the velocity of the flow and pressure at point A and $P_2$, $v_2$ be the corresponding quantities at point B.

Using Bernoulli's theorem:

$$\frac{P_1}{\rho} + gh_1 + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + gh_2 + \frac{1}{2}v_2^2, \text{we get}$$

$$\Rightarrow \frac{P_1}{\rho} + gh + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + gh + \frac{1}{2}v_2^2 \qquad (\text{Since } h_1 = h_2 = h)$$

$$\Rightarrow (P_1 - P_2) = \frac{1}{2}\rho(v_2^2 - v_1^2) \qquad ....(1)$$




According to continuity equation, $A_1v_1 = A_2v_2$

$$\Rightarrow v_2 = \frac{A_1}{A_2}v_1$$

$$(P_1 - P_2) = \frac{1}{2}\rho\left[\left(\frac{A_1}{A_2}\right)^2 v_1^2 - v_1^2\right] = \frac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

Since $A_1 > A_2$, therefore $P_1 > P_2$

$$\Rightarrow \quad v_1^2 = \frac{2(P_1 - P_2)}{\rho\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]} = \frac{2A_2^2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}$$

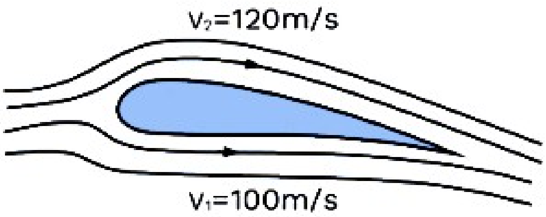
Where $(P_1 - P_2) = \rho_m gh$ and h is the difference in heights of the liquid levels in the two tubes

$$v_1 = \sqrt{\frac{2\rho_m gh}{\rho\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]}}$$

The volume flow rate of the liquid per second is given by $R = A_1V_1$

**Q.** **Calculate the value of lift force on this wing.**

$\rho_{air}$ **= 1.6 kg/m³**




**Area of wing = 20 m²**

**Sol.** Using Bernoulli's theorem between point 1 and 2, we get

$$P_1 + \frac{1}{2}\rho(100)^2 = P_2 + \frac{1}{2}\rho(120)^2$$

$$\Rightarrow \quad (P_1 - P_2) = \frac{1}{2}\rho \times \left[(120)^2 - (100)^2\right]$$

$$\Rightarrow \quad (P_1 - P_2) = \frac{1}{2}\rho(4400)$$

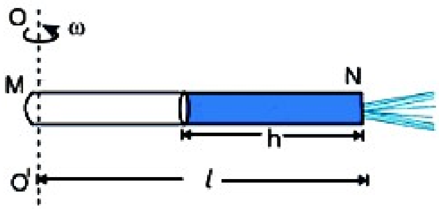
$$\Rightarrow \quad (P_1 - P_2) = \frac{1}{2} \times 1.6 \times 4400 = 3520 \text{ Pa}$$

Force $= (P_1 - P_2) \times$ Area

Force $= (3520 \times 20) = 7040$ N

$= 70.4$ kN

**Q.** **A horizontal tube filled with an ideal fluid has a length l and is rotating with constant angular speed ω about an axis passing as shown. The free end of the tube has an orifice. What is the relation between the velocity of fluid relative to the tube and the length of the fluid column h in the tube?**

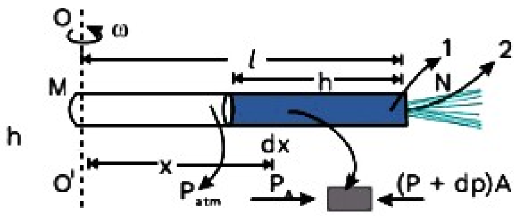



**Sol.** Applying Bernoulli's theorem at '1' and '2' just inside orifice and just outside orifice respectively. We have

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \qquad \text{........ (i)}$$

Here $P_2$ is equal to atmospheric pressure ($P_{atm}$). The velocity of liquid at point 1 is much smaller than velocity of liquid at point '2', hence we can neglect $v_1$. Now Eq. (i) becomes

$$P_1 + 0 = P_{atm} + \frac{1}{2}\rho v_2^2 \Rightarrow P_1 - P_{atm} = \frac{1}{2}\rho v^2 \qquad \text{... (ii)}$$




Pressure in rotating fluid is given by

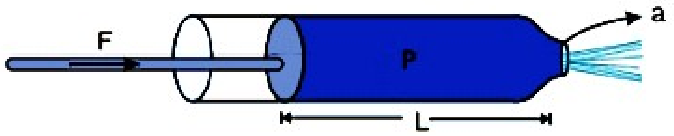
$$\int_{P_{atm}}^{P_1} dp = \rho\omega^2 \int_{l-h}^{l} x\,dx$$

$$= P_1 - P_{atm} = \frac{1}{2}\rho\omega^2(2hl - h^2) \qquad \text{... (iii)}$$

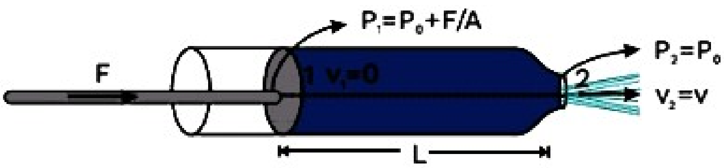
From Eqs. (ii) and (iii), $\frac{1}{2}\rho v^2 = \frac{1}{2}\rho\omega^2(2hl - h^2)$

$$\Rightarrow v = \omega h\sqrt{\left(\frac{2l}{h} - 1\right)}$$

**Q.** **Find the total work done in order to squeeze all water from a horizontally located cylinder as shown in figure during the time t by means of a contstant force acting on the piston? The volume of water in the cylinder is equal to V, the cross sectional area of the orifice is a, which is considerably less than the piston area. The friction and viscosity are negligibly small.**




**Sol.** If v the velocity at point 2, then applying Bernoulli's theorem at points '1' and '2' we have




$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Here $P_1 = P_0 + \frac{F}{A}$ and $P_2 = P_0$

Also $v_1$ is very much smaller than $v_2$ and can be neglected, Hence we have,

$$P_0 + \frac{F}{A} = P_0 + \frac{1}{2}\rho v^2 \qquad \text{...... (i)}$$

Force on piston, $F = P.A = \frac{1}{2}\rho v^2 A$

Volume of liquid coming through orifice per unit time = av

Volume of liquid coming through during time t, V = avt

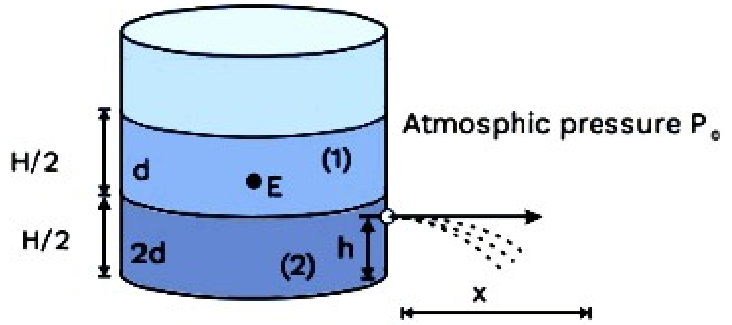
Velocity of liquid through orifice, $v = \left(\frac{V}{at}\right)$

From Eqs. (i) and (ii) $F = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 A$

Thus work done is $W = FL = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 AL = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 V$

Hence, $W = \frac{1}{2}\rho\frac{V^3}{a^2t^2}$

**Q.** **A container with large uniform cross-sectional area A placed at rest on a horizontal surface, holds two immiscible non-viscous and compressible liquids of density d and 2d, each having a height h/2 as shown. The liquid with lesser density is open to the atmospheric pressure $P_0$. A tiny hole of area (s<<A) is punched on the vertical side of the container at a height (h<H/2). Determine**




**(a)** **the initial efflux speed of the liquid at the hole.**

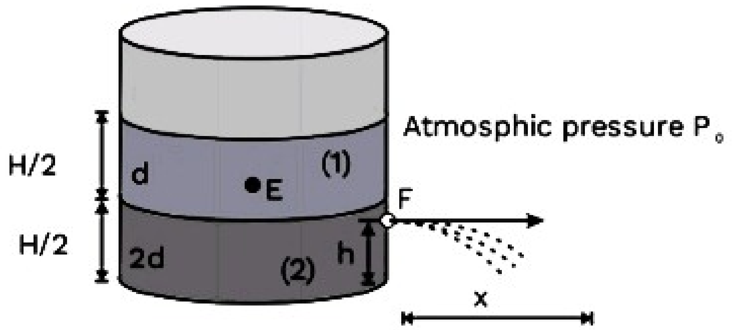
**(b)** **the horizontal distance x travelled by the liquid initially.**

**(c)** **the height $h_m$ at which the hole should be punched so that the liquid travels the maximum distance $x_m$ initially, Also calculate $x_m$ (neglect air resistance in the calculations).**

**Sol.** **(a)** **Method-1:** Applying Bernoulli's theorem at points E (a point on the interface of the liquids and on the top surface of the liquid of density 2d) and F,

i.e., pressure energy + kinetic energy + potential energy = constant

$$\Rightarrow \left(P_0 + \frac{H}{2}dg\right) + \left(\frac{1}{2}(2d) \times 0\right) + \left(\frac{H}{2}2dg\right) = P_0 + \frac{1}{2}(2d)v^2 + 2hdg$$




$$\Rightarrow \frac{3}{2}Hdg = dV^2 + 2hdg$$

$$\Rightarrow v^2 = \frac{3}{2}Hg - 2gh = \frac{(3H-4h)g}{2}$$

$\therefore v = \sqrt{\frac{(3H - 4h)}{2} g}$

**Method-2:** Let the equivalent depth of liquid (1) of density d in terms of density 2d which gives the same pressure as liquid (1) at interface, i.e.,

$$\frac{H}{2} dg = H'(2d)g \text{ which gives } H' = \frac{H}{4}$$

Hence, depth of the liquid above opening (in terms of depth of liquid of density 2d),

$$H' + \left(\frac{H}{2} - h\right) = \frac{H}{4} + \left(\frac{H}{2} - h\right)$$

$$H_{net} = \left(\frac{3H - 4h}{4}\right)$$

and velocity of efflux is given by

$$V = \sqrt{2gH_{net}}$$

$$V = \sqrt{\left(\frac{3H - 4h}{2}\right) g}$$

**(b)** Time taken by the liquid to fall though a vertical height h is given by

$$h = \frac{1}{2} gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}}$$

Therefore, horizontal distance traversed by liquid is

$$x = vt = \sqrt{\frac{(3H - 4h)g}{2}} \times \sqrt{\frac{2h}{g}}$$

$$= \sqrt{(3H - 4h)h}$$

**(c)** For the maximum horizontal distance x,

$$\frac{dx}{dh} = 0 \Rightarrow \frac{d}{dh}(3Hh - 4h^2)^{1/2} = 0$$

$$\text{or } \frac{1}{2}(3H - 4h^2)^{-1/2}(3H - 8h) = 0$$

This gives $3H - 8h = 0 \Rightarrow h = \frac{3H}{8}$

This is the height of opening for which the range is maximum, i.e.,

$$h_m = \frac{3H}{8}$$

The maximum horizontal distance is given by

$$x_m = \sqrt{(3H - 4h_m)h_m}$$

$$= \sqrt{\left\{\left(3H - 4\frac{3H}{8}\right)\frac{3H}{8}\right\}} = \frac{3}{4}H$$