DPP - Daily Practice Problems

Date : ______ Start Time : ______ End Time : ______

MATHEMATICS CM04

SYLLABUS : Complex Numbers and Quadratic Equations

Max. Marks : 69 **Time : 60 min.**

GENERAL INSTRUCTIONS

- The Daily Practice Problem Sheet contains 20 Questions divided into 5 sections.
  Section I has **6** MCQs with ONLY 1 Correct Option, **3** marks for each correct answer and **–1** for each incorrect answer.
  Section II has **4** MCQs with ONE or MORE THAN ONE Correct options.
  For each question, marks will be awarded in one of the following categories:
  Full marks: **+4** If only the bubble(s) corresponding to all the correct option(s) is (are) darkened.
  Partial marks: **+1** For darkening a bubble corresponding to each correct option provided NO INCORRECT option is darkened.
  Zero marks: If none of the bubbles is darkened.
  Negative marks: **–2** In all other cases.
  Section III has **5** Single Digit Integer Answer Type Questions, **3** marks for each Correct Answer and **0** mark in all other cases.
  Section IV has Comprehension Type Questions having **4** MCQs with ONLY ONE corect option, **3** marks for each Correct Answer and **0** mark in all other cases.
  Section V has **1** Matching Type Question, **2** marks for the correct matching of each row and **0** mark in all other cases.
- You have to evaluate your Response Grids yourself with the help of Solutions.

Section I - Straight Objective Type

This section contains 6 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

1. A complex number z satisfies the equation $|z|^2 - 2iz + 2c(1+i) = 0$, where c is real. The values of c for which the above equation has no solution can be given by

(a) $c \in (-\infty, -1-\sqrt{2})$ (b) $c \in [-1-\sqrt{2}, -1+\sqrt{2}]$

(c) $c \in (-1-\sqrt{2}, \infty)$ (d) $c \in \mathbf{R}$

2. If $z_1 = a+ib$ and $z_2 = c+id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\mathrm{Re}(z_1\bar{z}_2) = 0$, then the pair of complex numbers $\omega_1 = a+ic$ and $\omega_2 = b+id$ do not satisfy

(a) $|\omega_1| = 1$ (b) $|\omega_2| = 1$

(c) $\mathrm{Re}(\omega_1\bar{\omega}_2) = 0$ (d) $\mathrm{In}\left(\omega_1\bar{\omega}_2\right) = 0$

RESPONSE GRID	1. ⓐⓑⓒⓓ 2. ⓐⓑⓒⓓ

Space for Rough Work

3. If A, G and H are the Arithmetic mean, Geometric mean and Harmonic mean between two unequal positive integers. Then the equation $Ax^2 - |G|x - H = 0$ does not have
(a) both roots fractions
(b) one negative fraction root
(c) exactly one positive root
(d) no root greater than 2

4. If a, b, c are positive rational numbers such that $a > b > c$ and the quadratic equation $(a+b-2c)x^2 + (b+c-2a)x + (c+a-2b) = 0$ has a root in the interval $(-1, 0)$, then
(a) $c + a > 2b$
(b) Both roots of the given equation are irrational
(c) The equation $ax^2 + 2bx + c = 0$ has both negative real roots
(d) The equation $cx^2 + 2ax + b = 0$ has both positive real roots

5. Let $[a]$ denote the greatest integer less than or equal to a. Given that the quadratic equation $x^2 + [a^2 - 5a + b + 4]x + b = 0$ has roots -5 and 1. Then the set of values of a is
(a) $\left(-1, \frac{5-3\sqrt{5}}{2}\right] \cup \left[\frac{5+3\sqrt{5}}{2}, 6\right)$
(b) $\left(\frac{5-3\sqrt{5}}{2}, \frac{5+3\sqrt{5}}{2}\right)$
(c) $(-\infty, -1] \cup [6, \infty)$
(d) $(-\infty, \infty)$

6. The point of intersection of the curves $\arg(z - 3i) = \frac{3\pi}{4}$ and $\arg(2z + 1 - 2i) = \frac{\pi}{4}$ is
(a) $\frac{1}{4}(3+9i)$ (b) $\frac{1}{4}(3-9i)$
(c) $\frac{1}{2}(3+2i)$ (d) None of these

Section II - Multiple Correct Answer Type

This section contains 4 multiple correct answer(s) type questions. Each question has 4 choices (a), (b), (c) and (d), out of which **ONE OR MORE** is/are correct.

7. Let z_1, z_2, z_3 be complex number such that $|z_1| = |z_2| = |z_3| = 1$ and $\frac{z_1^2}{z_2 z_3} + \frac{z_2^2}{z_3 z_1} + \frac{z_3^2}{z_1 z_2} = -1$, then value of $|z_1 + z_2 + z_3|$ can be
(a) 2 (b) 3
(c) 4 (d) 1

8. Consider the quadratic equation $x^2 - 2px + p^2 - 1 = 0$ where p is parameter, then
(a) Both the roots of the equation are less than 4 if $p \in (-\infty, 3)$
(b) Both the roots of the equation are greater than -2 if $p \in (-\infty, -1)$
(c) Exactly one root of the equation lies in the interval $(-2, 4)$ if $p \in (-1, 3)$
(d) 1 lies between the roots of the equation if $p \in (0, 2)$

9. Equation $\frac{\pi^e}{x-e} + \frac{e^\pi}{x-\pi} + \frac{\pi^\pi + e^e}{x-\pi-e} = 0$ has
(a) one real root in (e, π) and other in $(\pi - e, e)$
(b) one real root in (e, π) and other in $(\pi, \pi + e)$
(c) Two real roots in $(\pi - e, \pi + e)$
(d) No real root

10. If $S = \sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}\right)$ then
(a) $S + \bar{S} = 0$ (b) $S\bar{S} = 1$
(c) $\sqrt{S} = \pm\frac{1}{\sqrt{2}}(1+i)$ (d) $S - \bar{S} = 0$

RESPONSE GRID					
	3. ⓐⓑⓒⓓ	4. ⓐⓑⓒⓓ	5. ⓐⓑⓒⓓ	6. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ
	8. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ		

Space for Rough Work

Section III - Integer Type

This section contains 5 questions. The answer to each of the questions is a single digit integer ranging from 0 to 9.

11. If $z^2 - z + 1 = 0$, and the value of

$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\ldots+\left(z^{24}+\frac{1}{z^{24}}\right)$$

is 8k, then k =

12. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d then find the value of $\dfrac{a+b+c+d}{605}$, when $a \neq b \neq c \neq d \neq 0$

13. If the roots of equation $ax^2 + bx + c = 0 \quad (a \neq 0)$ are α and β, and the roots of the equation $a^5x^2 + ba^2c^2x + c^5 = 0$ are 4 and 8 then the numerical value of $\alpha\beta$ is _________ .

14. If ω and ω^2 be the non-real cube roots of unity and

$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}=2\omega^2 \text{ and}$$

$$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}=2\omega, \text{ where } a, b, c \text{ are real}$$

then the value of $\dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1}$ is equal to :

15. If $z = \frac{1}{2}\left(\sqrt{3}-i\right)$, and the smallest value of positive integer n for which $(z^{89} + i^{97})^{94} = z^n$ is 2k, then k =

Section IV - Comprehension Type

Based upon the given paragraphs, 4 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

PARAGRAPH-1

Suppose z and w be two complex numbers such that $|z| \le 1$, $|w| \le 1$ and $|z + iw| = |z - iw| = 2$. Use the results $|z|^2 = z\bar{z}$ and $|z + w| \le |z| + |w|$, answer the following questions

16. Which of the following is true about $|z|$ and $|\omega|$

(a) $|z| = |w| = \dfrac{1}{2}$ (b) $|z| = \dfrac{1}{2}, |w| = \dfrac{3}{4}$

(c) $|z| = |w| = \dfrac{3}{4}$ (d) $|z| = |w| = 1$

17. Which of the following is true for z and ω

(a) $\text{Re}(z) = \text{Re}(w)$ (b) $I_m(z) = I_m(w)$

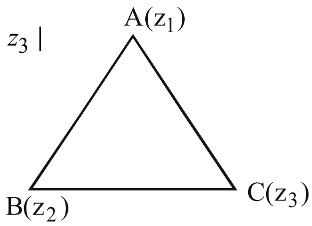
(c) $\text{Re}(z) = I_m(w)$ (d) $I_m(z) = \text{Re}(w)$

PARAGRAPH-2

Suppose z_1, z_2 and z_3 represent the vertices A, B and C of an equilateral triangle ABC on the Argand plane.

Then $AB = BC = CA$

$\Rightarrow |z_2 - z_1| = |z_3 - z_2| = |z_1 - z_3|$



Also $\angle CAB = \dfrac{\pi}{3}$

$\Rightarrow \arg \dfrac{z_3 - z_1}{z_2 - z_1} = \pm\dfrac{\pi}{3}$

Now solve the following questions :

18. If a and b are two real numbers lying between 0 and 1 such that $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle then

(a) $a = 2 + \sqrt{3}$ (b) $b = 4 - \sqrt{3}$

(c) $a = b = 2 - \sqrt{3}$ (d) $a = 2,\ b = \sqrt{3}$

RESPONSE GRID				
11. ⓪①②③④⑤⑥⑦⑧⑨	12. ⓪①②③④⑤⑥⑦⑧⑨		
13. ⓪①②③④⑤⑥⑦⑧⑨	14. ⓪①②③④⑤⑥⑦⑧⑨		
15. ⓪①②③④⑤⑥⑦⑧⑨	16. ⓐⓑⓒⓓ	17. ⓐⓑⓒⓓ	18. ⓐⓑⓒⓓ

Space for Rough Work

19. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2 =$

(a) z_0^2 (b) $3z_0^2$

(c) $9z_0^2$ (d) 0

Section V - Matrix-Match Type

This section contains 1 question. It contains statements given in two columns, which have to be matched. Statements in column I are labelled as A, B, C and D whereas statements in column II are labelled as p, q, r and s. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct matches are A-p, A-r, B-p, B-s, C-r, C-s and D-q, then the correctly bubbled matrix will look like the following:

	p	q	r	s
A	p	q	r	s
B	p	q	r	s
C	p	q	r	s
D	p	q	r	s

20. **Column-I** **Column-II**

(A) The roots of cubic equation $(z + \alpha\beta)^3 = \alpha^3$ $(\alpha \neq 0, \alpha \in R)$ represent the vertices of a triangle of area equal to — p. $|\tan \alpha|$

(B) If α is a complex number then the radius of the circle $\left|\dfrac{z-\alpha}{z-\overline{\alpha}}\right| = 2$ is equal to — q. $\dfrac{3\sqrt{3}}{4}|\alpha|^2$

(C) If $\arg z = \alpha$ and $|z-1| = 1$ then $\left|\dfrac{z-2}{z}\right|$ is equal to — r. $\dfrac{2}{3}|\alpha - \overline{\alpha}|$

(D) Let A and B represent complex numbers z_1 and z_2, which are roots of the equation $z^2 + pz + q = 0$. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin then $\dfrac{p^2}{q}$ is equal to — s. $4\cos^2\dfrac{\alpha}{2}$

RESPONSE GRID	19. ⓐⓑⓒⓓ 20. A - ⓟⓠⓡⓢ; B - ⓟⓠⓡⓢ; C - ⓟⓠⓡⓢ; D - ⓟⓠⓡⓢ

DAILY PRACTICE PROBLEM DPP CM04 - MATHEMATICS			
Total Questions	20	Total Marks	69
Attempted		Correct	
Incorrect		Net Score	
Cut-off Score	22	Qualifying Score	33

$$\textbf{Net Score} = \sum_{i=\mathrm{I}}^{\mathrm{V}} \left[\left(\text{correct}_i \times MM_i \right) - \left(In_i - NM_i \right) \right]$$

Space for Rough Work

1. **(a)** Let $z = x + iy$, then the equation is

$x^2 + y^2 - 2i(x+iy) + 2c(1+i) = 0$

$\Rightarrow (x^2 + y^2 + 2y + 2c) + i(2c - 2x) = 0$

$\Rightarrow x^2 + y^2 + 2y + 2c = 0$ and $x = c$

$\Rightarrow c^2 + y^2 + 2y + 2c = 0$

$\Rightarrow y = -1 \pm \sqrt{1 - 2c - c^2}$

$\because y \in \mathbf{R} \Rightarrow 1 - 2c - c^2 \ge 0$

$\Rightarrow c^2 + 2c - 1 \le 0 \Rightarrow -1 - \sqrt{2} \le c \le -1 + \sqrt{2}$

$\therefore$ The equation has a solution, if

$c \in [-1-\sqrt{2}, -1+\sqrt{2}]$ and the solution is given by

$z = c + i(-1 \pm \sqrt{1 - 2c - c^2})$

The equation has no solution, if

$c \in (-\infty, -1-\sqrt{2}) \cup (-1+\sqrt{2}, \infty)$

2. **(d)** $|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1$ ...(1)

and Re $(z_1 \bar{z}_2) = 0 \Rightarrow$ Re$\{(a + ib)(c - id)\} = 0$

$\Rightarrow ac + bd = 0$ ...(2)

Now from (1) and (2), $a^2 + b^2 = 1$

$\Rightarrow a^2 + \dfrac{a^2c^2}{d^2} = 1 \Rightarrow a^2 = d^2$ ...(3)

Also $c^2 + d^2 = 1 \Rightarrow c^2 + \dfrac{a^2c^2}{b^2} = 1$

$\Rightarrow b^2 = c^2$ ...(4)

$|\omega_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$ [From (1) and (4)]

and $|\omega_2| = \sqrt{b^2 + d^2} = \sqrt{c^2 + d^2} = 1$ [from (1) and (4)]

Further Re $(\omega_1 \bar{\omega}_2) =$ Re$\{(a + ic)(b - id)\}$

$= ab + cd = ab + \left(-\dfrac{ac}{b}\right)c$ [From (2)]

$= \dfrac{ab^2 - ac^2}{b} = 0$ [from (4)].

Also, $\text{Im}(\omega_1 \overline{\omega_2}) = bc - ad = bc - a\left(-\dfrac{ac}{b}\right)$

$= \dfrac{(a^2 + b^2)c}{b} = \dfrac{c}{b} = \pm 1 \ne 0$

$\therefore |\omega_1| = 1, |\omega_2| = 1$ and Re$(\omega_1 \overline{\omega}_2) = 0$

3. **(a)** If $f(x) = Ax^2 - |G|\,x - H$, then $f(0) = -H < 0$ and $f(-1) = A + |G| - H > 0$. So, $f(x) = 0$ has one root in $(-1, 0)$ hence the equation has a negative fraction root. Also,
$f(2) = 4A - 2|G| - H = 2(A - |G|) + (A - H) + A > 0$.
So, $f(x) = 0$ has one root in $(0, 2)$, hence the equation has a positive root, which cannot exceed 2.

4. **(c)** $a > b > c$ ...(1)

and given equation is

$(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ ...(2)

$\because$ Equation (2) has a root in the interval $(-1, 0)$

$\therefore f(-1)f(0) < 0$

$\Rightarrow (2a - b - c)(c + a - 2b) < 0$ ...(3)

From (1), $a > b \Rightarrow a - b > 0$ and

$a > c \Rightarrow a - c > 0 \therefore 2a - b - c > 0$ ...(4)

From (3) and (4), $c + a - 2b < 0$ or $c + a < 2b$. Option (a) is wrong. Again, the sum of coefficients of the equation $= 0$, that is one root is 1 and the other root is $\dfrac{c + a - 2b}{a + b - 2c}$, which is a rational number as a, b, c are rational. Hence, both the roots of the equation are rational.

$\Rightarrow$ (b) is wrong. Further, the discriminate of equation $ax^2 + 2bx + c = 0$ is $D = 4b^2 - 4ac$.

As deduced earlier, $c + a < 2b$

$\Rightarrow 4b^2 > (c + a)^2$

$\Rightarrow 4b^2 > c^2 + a^2 + 2ac$

$\Rightarrow 4b^2 - 4ac > c^2 + a^2 - 2ac$

$= (c - a)^2 \Rightarrow 4b^2 - 4ac > 0 \Rightarrow D > 0$. Also, each of a,b,c are positive.

$\therefore$ The equation $ax^2 + 2bx + c = 0$ has real and negative roots. So (c) is correct.

5. **(a)** Since -5 and 1 are the roots. Product of roots $= -5 \times 1 = b \Rightarrow b = -5$ and

Sum of roots $= -5 + 1 = -[a^2 - 5a + b + 4]$

$\Rightarrow [a^2-5a-1]=4 \Rightarrow 4 \le a^2-5a-1<5$

$[\because [x]=n \Rightarrow n \le x < n+1]$

$\Rightarrow a^2-5a-5 \ge 0$ and $a^2-5a-6<0$

$$\Rightarrow a \le \frac{5-\sqrt{45}}{2} \text{ or } a \ge \frac{5+\sqrt{45}}{2} \text{ and } -1<a<6$$

$$\Rightarrow -1<a \le \frac{5-3\sqrt{5}}{2} \text{ or } \frac{5+3\sqrt{5}}{2} \le a < 6$$

$$\Rightarrow a \in \left(-1, \frac{5-3\sqrt{5}}{2}\right] \cup \left[\frac{5+3\sqrt{5}}{2}, 6\right)$$

6. **(d)** Let $z = x+iy$, then $\arg(z-3i) = \arg(x+iy-3i) = \frac{3\pi}{4}$

$\Rightarrow x<0, y-3>0 \quad (\because \frac{3\pi}{4}$ is in II quadrant)

and $\frac{y-3}{x} = \tan\frac{3\pi}{4} = -1$

$\Rightarrow y=-x+3 \ \forall\, x<0$ and $y>3$ ...(1)

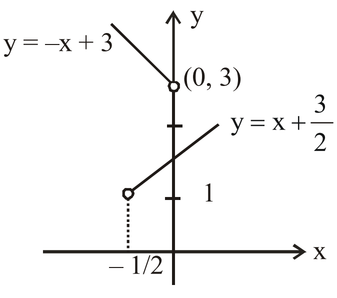
and $\arg(2z+1-2i) = \arg[(2x+1)+i(2y-2)] = \frac{\pi}{4}$

$\Rightarrow 2x+1>0, 2y-2>0 \quad (\because \frac{\pi}{4}$ is in I quadrant)

and $\frac{2y-2}{2x+1} = \tan\frac{\pi}{4} = 1 \Rightarrow 2y-2=2x+1$

$\Rightarrow y = x+\frac{3}{2} \ \forall\, x>-\frac{1}{2}, y>1$ .....(2)

From equations (1) and (2), we get graph



It is clear from the graph that two lines do not intersect.

$\therefore$ No point of intersection.

Caution : *It is most likely that the students after getting two straight lines, solve them to get the point of intersection* $\left(\frac{3}{4}, \frac{9}{4}\right)$. *Clearly the principal values of arguments must be considered.*

7. **(a, d)**

We have $z_1^3+z_2^3+z_3^3=-z_1z_2z_3$.

$\Rightarrow -4z_1z_2z_3 = z_1^3+z_2^3+z_3^3-3z_1z_2z_3$

$= (z_1+z_2+z_3)(z_1^2+z_2^2+z_3^2-z_2z_3-z_3z_1-z_1z_2)$

$= (z_1+z_2+z_3)[(z_1+z_2+z_3)^2-3(z_2z_3+z_3z_1+z_1z_2)]$

$\Rightarrow z^3-3z(z_2z_3+z_3z_1+z_1z_2)+4z_1z_2z_3=0$

where $z = z_1+z_2+z_3$

$$\Rightarrow z^3 = z_1z_2z_3\left[3z\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)-4\right]$$

$$= z_1z_2z_3\left[3z\left(\bar{z}_1+\bar{z}_2+\bar{z}_3\right)-4\right] \quad [\because |z_1|=|z_2|=|z_3|=1]$$

$$= z_1z_2z_3\left[3z\bar{z}-4\right]$$

$\Rightarrow |z|^3 = |z_1|\,|z_2|\,|z_3|\ |3|z|^2-4|$

$\Rightarrow |z|^3-|3|z|^2-4|=0$

If $|z| \ge 2/\sqrt{3}$, we get

$|z|^3-|3|z|^2+4|=0$

$\Rightarrow (|z|-2)(|z|^2-|z|-2)=0$

$\Rightarrow (|z|-2)^2(|z|+1)=0$

$\Rightarrow |z|-2=0$ or $|z|=2$

If $|z|<2/3$, we get

$|z|^3+3|z|^2-4=0$

$\Rightarrow (|z|-1)(|z|^2+4|z|+4)=0$

$\Rightarrow |z|-1=0 \Rightarrow |z|=1$

8. **(a,d)** Discriminant $D = 4p^2-4(p^2-1)=4>0$

$\because$ Roots of the equation are real and distinct

Now both the roots are less than 4 if

$D \ge 0$, $f(4)>0$ and $4>-\frac{-2p}{2}$

$\Rightarrow 16-8p+p^2-1>0$ and

$4>p \Rightarrow (p-3)(p-5)>0$ and $p<4$

$\Rightarrow p<3$ or $p>5$ and $p<4 \Rightarrow p \in (-\infty, 3)$

Again both the roots are greater than

-2 if $D \ge 0$, $f(-2)>0$ and $-2<-\frac{-2p}{2}$

$\Rightarrow (4+4p+p^2+1)>0$ and

$3<p \Rightarrow (p+3)(p+1)>0$ and $p>-3$

$\Rightarrow p<-3$ or $p>-1$ and $p>-3 \Rightarrow p \in (-1, \infty)$

Further exactly one root lies in the interval $(-2, 4)$ if

$D>0$ and $f(-2)f(4)<0$

$\Rightarrow (p+3)(p+1)(p-3)(p-5)<0$

$\Rightarrow\ p \in (-3, -1) \cup (3, 5)$

Finally, 1 lies between the roots if $D > 0$ and $f(1) < 0$

$\Rightarrow\ 1 - 2p + p^2 - 1 < 0 \Rightarrow p(p-2) < 0$

$\Rightarrow\ 0 < p < 2 \Rightarrow\ p \in (0, 2)$

Alternatively :

$x^2 - 2px + p^2 - 1 = 0 \Rightarrow (x-p)^2 = 1$

$\therefore x = p \pm 1$

Both the roots are less than 4 if $p + 1 < 4$ and $p - 1 < 4 \Rightarrow p < 3$

Both the roots are greater than –2 if $p + 1 > -2$ and $p - 1 > -2 \Rightarrow p > -1$

Exactly one root lies in (–2, 4) if $-2 < p + 1 < 4$ or $-2 < p - 1 < 4$ but not both

$\Rightarrow p \in (-3, -1) \cup (3, 5)$

One root is less than 1 and other greater than 1 if $p + 1 < 1 < p - 1$ or $p - 1 < 1 < p + 1 \Rightarrow 0 < p < 2$

NOTE : *The alternate method is easier than the general method, so if the roots of quadratic in terms of parameter come out to be free of radical the alternative method is better.*

9. **(b,c)** The given equation is,

$$\pi^e(x-\pi)(x-\pi-e) + e^\pi(x-e)(x-\pi-e) + (\pi^\pi + e^e)(x-e)(x-\pi) = 0$$

Let $f(x) = \pi^e(x-\pi)(x-\pi-e) + e^\pi(x-e)(x-\pi-e) + (\pi^\pi + e^e)(x-e)(x-\pi)$

Then $f(e) = \pi^e(e-\pi)(-\pi) > 0$ $[\because\ e < \pi\,]$

and $f(\pi) = e^\pi(\pi - e)(-e) < 0$

$\therefore$ Equation $f(x) = 0$ has a real root in (e, π).

Again $f(\pi + e) = (\pi^\pi + e^e)(\pi)(e) > 0$.

$\therefore$ Equation $f(x) = 0$ has a real root in $(\pi, e + \pi)$.

$\therefore$ $f(x) = 0$ has a real roots in (e, π)

and other in $(\pi, \pi + e)$

Also, $\pi - e < e$

$\therefore$ Equation $f(x) = 0$ has two real roots in $(\pi - e, \pi + e)$.

10. **(a,b,c)**

Put $\omega = \cos\frac{2\pi}{11} + i\sin\frac{2\pi}{11}$,

so that for $1 \le k \le 10$

$$\sin\frac{2\pi}{11} - i\cos\frac{2\pi k}{11}$$

$$= -i\left(\cos\frac{2\pi k}{11} + i\sin\frac{2\pi k}{11}\right)$$

$= -i\omega^k$ [De Moivre's theorem]

Thus,

$$S = -i\sum_{k=1}^{10}\omega^k = -\frac{i\omega(1-\omega^{10})}{1-\omega} = \frac{i\omega(1-\omega^{11})}{1-\omega}$$

But $\omega^{11} = \cos 2\pi + i\sin 2\pi = 1 + i0 = 1$

$\therefore\quad S = i$

$\Rightarrow\quad S + \overline{S} = 0,\ S\overline{S} = 1$

and $\sqrt{S} = \pm\frac{1}{\sqrt{2}}(1+i)$

11. **(6)**

Solving $z^2 - z + 1 = 0 \Rightarrow z = \frac{1 \pm i\sqrt{3}}{2}$

Taking $z = \frac{1 + i\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\Rightarrow z^n = \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$, $n = 1, 2, \ldots\ldots\ldots, 24$

$\therefore\quad z^n + \frac{1}{z^n} = 2\cos\frac{n\pi}{3}$

$$\therefore\quad \left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \ldots\ldots\ldots\ldots\ldots\ldots + \left(z^{24} + \frac{1}{z^{24}}\right)^2$$

$$= 2^2\cos^2\frac{\pi}{3} + 2^2\cos^2\frac{2\pi}{3} + 2^2\cos^2\frac{3\pi}{3} + 2^2\cos^2\frac{24\pi}{3}$$

$$= 2\left[\left(1 + \cos\frac{2\pi}{3}\right) + \left(1 + \cos\frac{4\pi}{3}\right) + \left(1 + \cos\frac{6\pi}{3}\right) + \ldots\ldots\ldots\ldots\ldots + \left(1 + \cos\frac{48\pi}{3}\right)\right]$$

$$= 2\left[24 + \frac{\cos\left\{\frac{2\pi}{3} + \frac{23\pi}{3}\right\}\sin\frac{24\pi}{3}}{\sin\frac{\pi}{3}}\right] = 2(24+0) = 48$$

Using the formula,

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots +$

$$\cos\{\alpha + (n-1)\beta\} = \frac{\cos\left\{\alpha + \frac{(n-1)\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

12. (2) Roots of $x^2 - 10cx - 11d = 0$ are a and $b \Rightarrow a + b = 10c$ and $ab = -11d$

Similarly c and d are the roots of

$x^2 - 10ax - 11b = 0 \Rightarrow c + d = 10a$ and $cd = -11b$

$\Rightarrow a + b + c + d = 10(a + c)$ and $abcd = 121bd$

$\Rightarrow b + d = 9(a + c)$ and $ac = 121$

Also we have $a^2 - 10ac - 11d = 0$ & $c^2 - 10ac - 11b = 0$

$\Rightarrow a^2 + c^2 - 20ac - 11(b + d) = 0$

$\Rightarrow (a+c)^2 - 22 \times 121 - 99(a+c) = 0 \Rightarrow a+c = 121$ or -22

For $a + c = -22$ we get $a = c$

$\therefore$ rejecting this value we have $a + c = 121$

$\therefore a + b + c + d = 10(a + c) = 1210$

13. (2) $ax^2 + bx + c = 0$ has roots α and β

$$\Rightarrow \alpha + \beta = -\frac{b}{a},\ \alpha\beta = \frac{c}{a}.$$

If the roots of equation $a^5x^2 + ba^2c^2x + c^5 = 0$ are γ and δ, then

$$\gamma + \delta = -\frac{b}{a}\left(\frac{c}{a}\right)^2 = (\alpha+\beta)\alpha^2\beta^2 = \alpha^3\beta^2 + \alpha^2\beta^3$$

Clearly roots are $\alpha^3\beta^2$ and $\alpha^2\beta^3$

$$\Rightarrow \alpha^5\beta^5 = 32 \Rightarrow \alpha\beta = 2$$

14. (2) The given relation can be rewritten as

$$\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = \frac{2}{\omega}$$

and $$\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} = \frac{2}{\omega^2}$$

$\Rightarrow$ ω and ω^2 are roots of $\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} = \frac{2}{x}$

$$\Rightarrow \frac{3x^2 + 2(a+b+c)x + bc + ca + ab}{(a+x)(b+x)(c+x)} = \frac{2}{x}$$

$$\Rightarrow x^3 - (bc + ca + ab)x - 2abc = 0 \quad \ldots(1)$$

Two roots of the equation (1) are ω and ω^2. Let the third root be α, then

$\alpha + \omega + \omega^2 = 0 \Rightarrow \alpha = -\omega - \omega^2 = 1.$

$\therefore$ $\alpha = 1$ will satisfy equation (1)

$$\Rightarrow \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$$

15. (5) We have $z = \frac{1}{2}\left(\sqrt{3} - i\right)$

$$= -\frac{1}{2}i\left(1 + i\sqrt{3}\right) = i\,\omega^2$$

where $\omega \neq 1$ is a cube of unity.

$\therefore$ $z^{89} = (i\omega^2)^{89} = i^{89}\,\omega^{178} = i\omega$

Also, $i^{97} = i^{96}\,i = i$

Thus, $(z^{89} + i^{97})^{94} = (i\omega + i)^{94} = [i(-\omega^2)]^{94} = -\omega^2$

Also, $z^n = i^n\,\omega^{2n}$

$\therefore$ The given equation becomes

$-\omega^2 = i^n\,\omega^{2n} \Rightarrow i^n\,\omega^{2n-2} = -1$

This is possible if n is of the type 4k + 2 and 2n –2 is a multiple of 3.

That is $2(4k + 2) - 2 = 8k + 2$ is multiple of 3.

The least value of k for which this is possible is 2. Therefore, n = 10.

16. (d) $\because |z + i\omega| \le |z| + |i\omega| = |z| + |i||\omega| \le 2$

$\therefore |z + i\omega| = 2 \Leftrightarrow |z| = |\omega| = 1.$

17. (d) Let $z = x + iy$ and $\omega = \alpha + i\beta$

Now $|z + i\omega| = 2 \Rightarrow (z + i\omega)(\bar{z} - i\bar{\omega}) = 4$

$\Rightarrow |z|^2 + |\omega|^2 + i\omega\bar{z} - i\bar{\omega}z = 4$

$\Rightarrow i\omega\bar{z} - i\bar{\omega}z = 2$ ...(1)

and $|z - i\bar{\omega}| = 2 \Rightarrow (z - i\bar{\omega})(\bar{z} + i\omega) = 4$

$\Rightarrow |z|^2 + |\omega|^2 + i\omega z - i\bar{\omega}\bar{z} = 4$

$\Rightarrow i\omega z - i\bar{\omega}\bar{z} = 2$ ...(2)

Add (1) and (2), $\Rightarrow i(\omega - \bar{\omega})(z + \bar{z}) = 4$

$\Rightarrow i(2i\beta)(2x) = 4 \Rightarrow \beta x = -1$ ...(3)

Subtract (1) from (2),

$\Rightarrow i(\omega + \bar{\omega})(z - \bar{z}) = 0 \Rightarrow \alpha y = 0$ ...(4)

From (4), either $\alpha = 0$ or $y = 0$.

If $y = 0$, then $x^2 + y^2 = 1 \Rightarrow x = \pm 1 \Rightarrow z = 1$ or -1

If $\alpha = 0$, then $\alpha^2 + \beta^2 = 1 \Rightarrow \beta = \pm 1 \Rightarrow w = \pm i.$

So, $I_m(z) = \text{Re}(w) = 0$

18. **(c)** Using the result

$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$, we get

$a^2 - 1 + 2ai + 1 - b^2 + 2bi + 0 - a + b - i - abi = 0$

$\therefore\ a^2 - b^2 - a + b = 0$ and $2a + 2b - ab - 1 = 0$

$\Rightarrow\ a = b$ and $2a + 2b - ab - 1 = 0$

($\because\ a + b = 1$ does not give real solution)

$\therefore\ a = b$ and $a^2 - 4a + 1 = 0$

$a = b = 2 - \sqrt{3}$ $\qquad (\because\ a < 1, b < 1)$

19. **(b)** $z_0 = \dfrac{z_1 + z_2 + z_3}{3}$

$\Rightarrow\ z_1^2 + z_2^2 + z_3^2 - 2z_1 z_2 - 2z_2 z_3 - 2z_3 z_1 = 9z_0^2$

$\Rightarrow\ 3(z_1^2 + z_2^2 + z_3^2) = 9z_0^2 \Rightarrow\ z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

$(\because\ z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1)$

20. **A - q; B - r; C - p; D - s**

(A) $(z + \alpha\beta)^3 = \alpha^3 \Rightarrow z + \alpha\beta = \alpha, \omega\alpha, \omega^2\alpha$

$\Rightarrow z = \alpha - \alpha\beta,\ \omega\alpha - \alpha\beta,\ \omega^2\alpha - \alpha\beta$, say z_1, z_2, z_3 respectively

Now, $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| = \sqrt{3}\,|\alpha|$

So, the triangle is equilateral and has area

$= \dfrac{\sqrt{3}}{4} |z_1 - z_2|^2$

(B) $|z - \alpha|^2 = 4|z - \bar{\alpha}|^2$

$\Rightarrow z\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha} = 4(z\bar{z} - \alpha z - \bar{\alpha}\bar{z} + \alpha\bar{\alpha})$

$\Rightarrow 3z\bar{z} + (\alpha - 4\bar{\alpha})\bar{z} + (\bar{\alpha} - 4\alpha)z + 3\alpha\bar{\alpha} = 0$

or $z\bar{z} + \dfrac{\alpha - 4\bar{\alpha}}{3}\bar{z} + \dfrac{\bar{\alpha} - 4\alpha}{3}z + \alpha\bar{\alpha} = 0$

which is a circle of radius

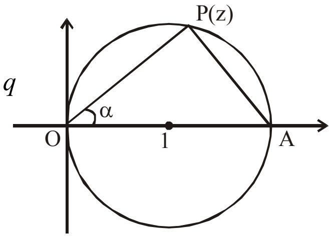
$= \sqrt{\left|\dfrac{\alpha - 4\bar{\alpha}}{3}\right|^2 - \alpha\bar{\alpha}} = \sqrt{-\dfrac{4}{9}(\alpha - \bar{\alpha})^2} = \dfrac{2}{3}|\alpha - \bar{\alpha}|$

(C) z lies on a circle of radius 1 and centre at (1, 0)

$\angle OPA = \pm\dfrac{\pi}{2} \Rightarrow \dfrac{2 - z}{0 - z} = \dfrac{|2 - z|}{|z|} e^{\pm i\frac{\pi}{2}}$

$\Rightarrow \dfrac{z - 2}{z} = \dfrac{AP}{OP}(\pm i) = \pm i \tan\alpha$

$\therefore \left|\dfrac{z - 2}{z}\right| = |\tan\alpha|$



(D) $z_1 + z_2 = -p$ and $z_1 z_2 = q$

Also,

$\dfrac{z_2}{z_1} = \cos\alpha \pm i\sin\alpha$

$\Rightarrow \dfrac{z_2 - z_1\cos\alpha}{z_1} = \pm i\sin\alpha$

or $z_2^2 - 2z_2 z_1 \cos\alpha + z_1^2\cos^2\alpha = -z_1^2\sin^2\alpha$

$\Rightarrow z_1^2 + z_2^2 = 2z_1 z_2 \cos\alpha$

or $(z_1 + z_2)^2 = 2z_1 z_2(1 + \cos\alpha) \Rightarrow \dfrac{p^2}{q} = 4\cos^2\dfrac{\alpha}{2}$