Inverse Trigonometry

INVERSE TRIGONOMETRIC FUNCTIONS Section - 1

1.1 Definition :

If $\sin x = 1/2$, we can write one value of $x = \pi/6$.

If $\sin x = 1/3$, i.e. x is not a well known angle, then we can write $x = \sin^{-1}(1/3)$.

Similarly, $\cos x = t \Rightarrow x = \cos^{-1} t$.

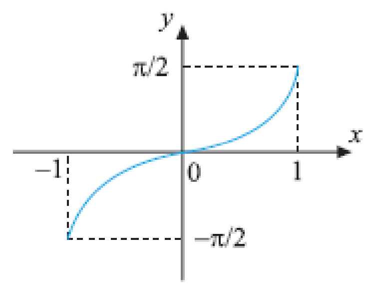
$\tan x = t \Rightarrow x = \tan^{-1} t$.

1.2 Principal value branches of Inverse Trignometric Functions

(i) $y = \sin^{-1} x \Rightarrow x = \sin y$

In $x = \sin y$, for one value of x, y can take infinite values.

But if $y = \sin^{-1} x$ is a function, then y should possess only one value of y for every value of x. This means we should restrict the values which y can possess. The restricted set of values which y can possess is its principal value branch.



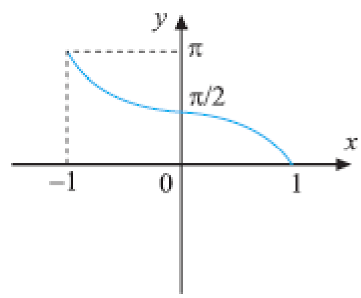
Here $-1 \le \sin y \le 1 \Rightarrow \frac{-\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow$ Domain: $x \in [-1.1]$

Range: $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Principal value branch of $\sin^{-1} x \equiv \frac{-\pi}{2} \le y \le \frac{\pi}{2}$

(ii) $y = \cos^{-1} x \Rightarrow x = \cos y$



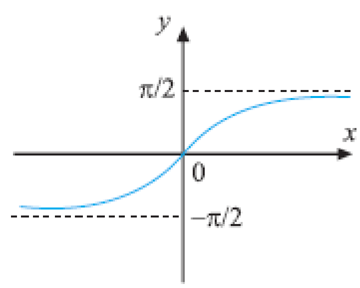
Here $-1 \le \cos y \le 1 \Rightarrow 0 \le y \le \pi$

Domain: $x \in [-1, 1]$

Range: $y \in [0, \pi]$

Principal value branch of $\cos^{-1} x \equiv 0 \le y \le \pi$

(iii) $y = \tan^{-1} x \Rightarrow x = \tan y$



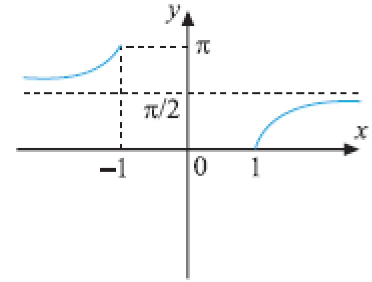
Here $-\infty < \tan y < \infty \Rightarrow \frac{-\pi}{2} < y < \frac{\pi}{2}$

Domain: $x \in R$

Range: $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Principal value branch of $\tan^{-1} x \equiv \frac{-\pi}{2} < y < \frac{\pi}{2}$

(iv) $y \equiv \sec^{-1} x \Rightarrow x = \sec y$



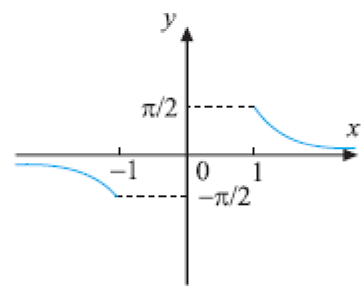
Here $|x| \geq 1$ and $0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}$

Domain: $x \in (-\infty - 1] \cup [1, \infty)$

Range: $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Principal value branch of $\sec^{-1} x \equiv 0 \leq y \leq \pi,\ y \neq \pi/2$

(v) $y = \text{cosec}^{-1} x \Rightarrow x = \text{cosec}\, y$



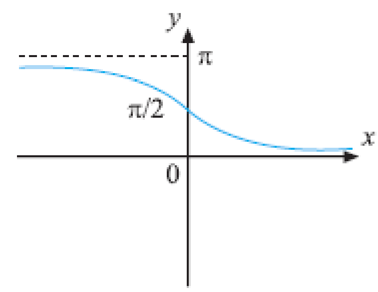
Here $|x| \geq 1$ and $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}\ y \neq 0$

Domain: $x \in (-\infty, -1] \cup [1, \infty)$

Range: $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Principal value branch of $\text{cosec}^{-1} x \equiv \frac{-\pi}{2} \leq y \leq \frac{\pi}{2},\ y \neq 0$

(vi) $y = \cot^{-1} x \Rightarrow x = \cot y$



Here $-\infty < \cot y < \infty \Rightarrow 0 < y < \pi$

Domain: $x \in R$

Range: $y \in (0, \pi)$

Principal value branch of $\cot^{-1} x \equiv y \in (0, \pi)$

Summary table of Inverse Trigonometric Functions :

S.No.	Functions	Domain	Range
1.	$y = \sin^{-1} x$	$x \in [-1, 1]$	$y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$x \in [-1, 1]$	$y \in [0, \pi]$
3.	$y = \tan^{-1} x$	$x \in R$	$y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \text{cosec}^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
5.	$y = \sec^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
6.	$y = \cot^{-1} x$	$x \in R$	$y \in (0, \pi)$

- Note the similarity in principal value branch of

  $y = \sin^{-1} x, y = \text{cosec}^{-1} x,\ y = \tan^{-1} x$ and $y = \cos^{-1} x,\ y = \sec^{-1} x,\ y = \cot^{-1} x.$
- Interval for allowed values of y is known as principal value branch of that inverse function.

Illustrating the Concept :

Find the principal value of

(i) $\sin^{-1}\left(\frac{1}{2}\right)$ **(ii)** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **(iii)** $\cos^{-1}\frac{\sqrt{3}}{2}$

(iv) $\cos^{-1}\left(-\frac{1}{2}\right)$ **(v)** $\sec^{-1}\frac{2}{\sqrt{3}}$ **(vi)** $\sec^{-1}(-2)$.

(i) We know that $\sin^{-1} x$ denotes an angle in the interval $[-\pi/2, \pi/2]$ whose sine is x for $x \in [-1, 1]$.

$$\therefore \quad \sin^{-1}\left(\frac{1}{2}\right) = \text{An angle in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ whose sine is } \frac{1}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

(ii) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \text{An angle in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ whose sine is } \frac{1}{\sqrt{2}}$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

(iii) For any $x \in [-1, 1]$, $\cos^{-1} x$ represents an angle in $[0, \pi]$ whose cosine is x.

$$\therefore \quad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \text{An angle in } [0, \pi] \text{ whose cosine is } \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

(iv) $\cos^{-1}\left(-\frac{1}{2}\right) = \text{An angle in } [0, \pi] \text{ whose cosine is } -\frac{1}{2}$

$$\Rightarrow \quad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

(v) For any $x \in R - (-1, 1)$, $\sec^{-1} x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is x.

$$\therefore \quad \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \text{An angle in } [0, \pi] - \left\{\frac{\pi}{2}\right\} \text{ whose secant is } \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$$

(vi) $\sec^{-1}(-2) = \text{An angle in } [0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is (-2)

$$\Rightarrow \quad \sec^{-1}(-2) = \frac{2\pi}{3}.$$

Illustration - 1

$tan^{-1}1 + cos^{-1}\left(-\frac{1}{2}\right) + sin^{-1}\left(-\frac{1}{2}\right)$ *is equal to:*

(A) $\frac{5\pi}{12}$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{3\pi}{4}$ **(D)** $\frac{5\pi}{6}$

SOLUTION : (C)

As $\tan^{-1} 1 = \frac{\pi}{4}$, $\cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$ and $\sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$

$$\Rightarrow \quad \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi}{4}$$

1.3 Properties of Inverse Trigonometric Functions

(A)

$\sin^{-1}(-x) = -\sin^{-1}x$	for all	$x \in [-1, 1]$
$\cos^{-1}(-x) = \pi - \cos^{-1}x$	for all	$x \in [-1, 1]$
$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$	for all	$x \in (-\infty, -1] \cup [1, \infty)$
$\sec^{-1}(-x) = \pi - \sec^{-1}x$	for all	$x \in (-\infty, -1] \cup [1, \infty)$
$\tan^{-1}(-x) = -\tan^{-1}x$	for all	$x \in R$
$\cot^{-1}(-x) = \pi - \cot^{-1}x$	for all	$x \in R$

Proof :

I. Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

Let $\sin^{-1}(-x) = \theta$ . . . **(i)**

Taking sine on both side

Then, $-x = \sin\theta$

$\Rightarrow \quad x = -\sin\theta \quad \Rightarrow \quad x = \sin(-\theta)$

$\Rightarrow \quad -\theta = \sin^{-1} x$ [as $x \in [-1,1]$ and $-\theta \in [-\pi/2, \pi/2]$ for all $\theta \in [-\pi/2, \pi/2]$]

$\Rightarrow \quad \theta = -\sin^{-1} x$ . . . **(ii)**

From (i) and (ii), we get :

$\sin^{-1}(-x) = -\sin^{-1} x$

II. Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$.

Let $\cos^{-1}(-x) = \theta$ . . . **(i)**

Then, $-x = \cos\theta$

$\Rightarrow \quad x = -\cos\theta$

$\Rightarrow \quad x = \cos(\pi - \theta)$

$\Rightarrow \quad \cos^{-1} x = \pi - \theta$ [as $x \in [-1,1]$ and $\pi - \theta \in [0, \pi]$, for all $\theta \in [0, \pi]$]

$\Rightarrow \quad \theta = \pi - \cos^{-1} x$ . . . **(ii)**

From (i) and (ii), we get :

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

Similarly, we can prove other results.

(B) (i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x,$ for all $x \in (-\infty, -1] \cup [1, \infty]$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & , \text{ for } x > 0 \\ -\pi + \cot^{-1} x & , \text{ for } x < 0 \end{cases}$

(C) (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for all $x \in [-1, 1]$

(ii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$ for all $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ for all $x \in R$

(D) (i) $\underset{(0 \le x \le 1)}{\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}} = \underset{(|x| < 1)}{\tan^{-1}\frac{x}{\sqrt{1-x^2}}} = \underset{(0 > x \le 1)}{\cot^{-1}\frac{\sqrt{1-x^2}}{x}} = \underset{(0 \le x < 1)}{\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)} = \underset{(|x| \le 1) - \{0\}}{\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)}$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

$(0 \le x \le 1)$ $\quad (0 < x \le 1)$ $\quad (|x| < 1)$ $\quad (|x| \le 1)$ $\quad (0 \le x < 1)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

$(x \in R)$ $\quad (0 \le x < \infty)$ $\quad (x > 0)$ $\quad (x \ge 0)$ $\quad (x \in R - \{0\})$

(E) (i) $\sin(\sin^{-1} x) = x \;\forall\; x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x \;\forall\; x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x \;\forall\; x \in R$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x \;\forall\; x \in (-\infty, -1] \cup [1, \infty)$

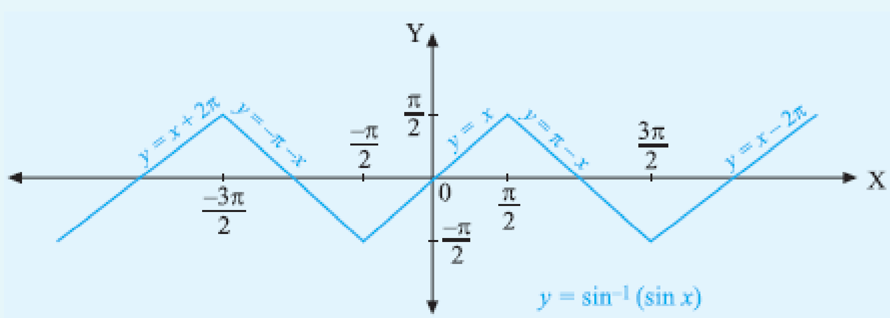
(v) $\sec(\sec^{-1} x) = x \;\forall\; x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot(\cot^{-1} x) = x \;\forall\; x \in R$

(F) **(i)** $\sin^{-1}(\sin x) = x$ if $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

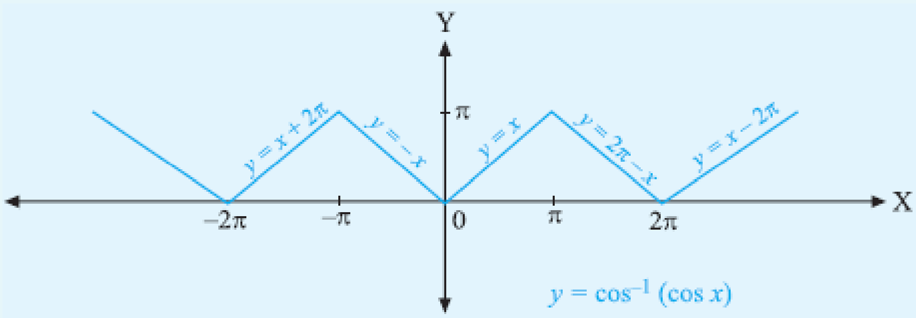
But, if $x \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then: $\sin^{-1}(\sin x) = \begin{cases} -\pi - x & ; \frac{-3\pi}{2} \le x \le \frac{-\pi}{2} \\ x & ; \frac{-\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x & ; \frac{\pi}{2} \le x \le \frac{3\pi}{2} \end{cases}$

Now, As $y = \sin^{-1}(\sin x)$ is periodic function with period of 2π, to obtain the graph of $y = f(x)$ we draw the graph for one interval of length 2π and repeat for entire values of x.



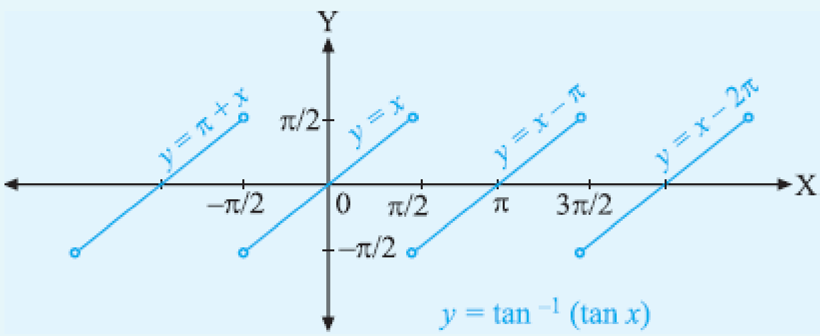
(ii) $\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$

But if $x \notin [0, \pi]$ then $\cos^{-1}(\cos x) = \begin{cases} x & ; 0 \le x \le \pi \\ 2\pi - x & ; \pi \le x \le 2\pi \end{cases}$



(iii) $\tan^{-1}(\tan x) = x \text{ if } \frac{-\pi}{2} < x < \frac{\pi}{2}$

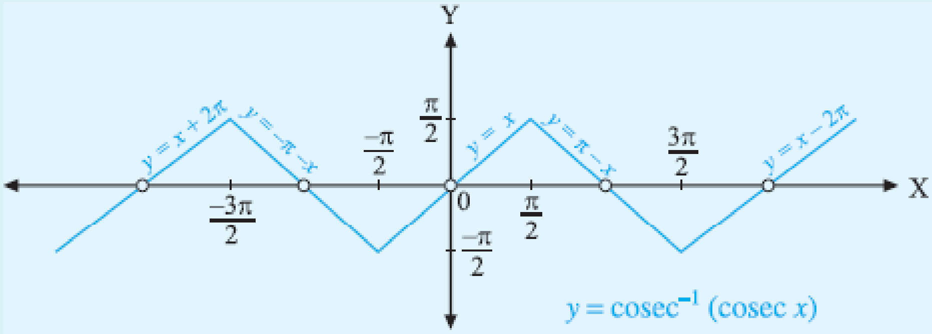
But if $x \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ then $\tan^{-1}(\tan x) = \begin{cases} x & ; \quad \frac{-\pi}{2} < x < \frac{\pi}{2} \\ x - \pi & ; \quad \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$



(iv) $y = \text{cosec}^{-1}(\text{cosec } x) = \lambda \text{ if } \frac{-\pi}{2} \le x \le \frac{\pi}{2} - \{0\}$

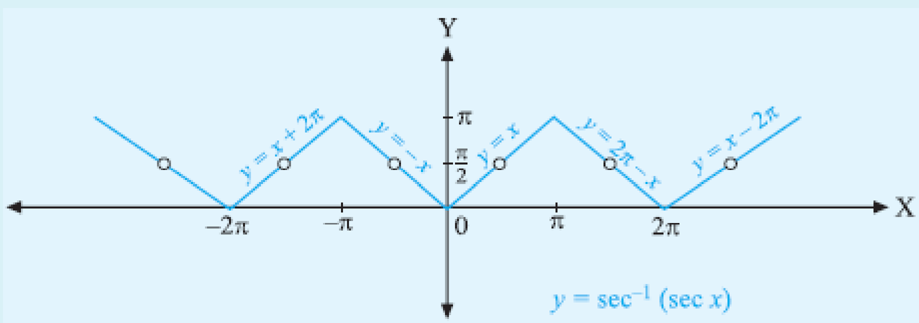
But if $x \notin \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

then $\text{cosec}^{-1}(\text{cosec } x) = \begin{cases} x & ; \quad -\frac{\pi}{2} \le x < 0 \quad \text{or} \quad 0 < x \le \frac{\pi}{2} \\ \pi - x & ; \quad -\frac{\pi}{2} \le \pi - x < 0 \quad \text{or} \quad 0 < \pi - x \le \frac{\pi}{2} \end{cases}$



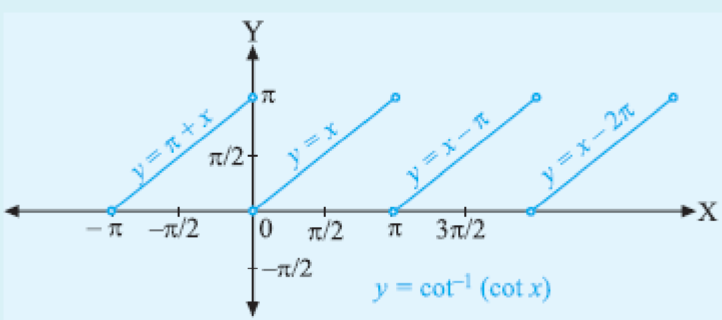
(v) $y = \sec^{-1}(\sec x) = x$ if $0 \le x \le \pi - \left(\frac{\pi}{2}\right)$

But if $x \notin 0 \le x \le \pi - \left(\frac{\pi}{2}\right)$ then $\sec^{-1}(\sec x) = \begin{cases} x & ; \quad x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\ 2\pi - x & ; \quad 2\pi - x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \end{cases}$



(vi) $y = \cot^{-1}(\cot x) = x$ if $0 < x < \pi$

But if $x \notin 0 < x < \pi$ then $\cot^{-1}(\cot x) = \begin{cases} x + \pi & ; \quad -\pi < x < 0 \\ x & ; \quad 0 < x < \pi \\ x - \pi & ; \quad \pi < x < 2\pi \end{cases}$



Note : Yor are advised to learn these difintions as standard results which help in calculus.

Illustrating the Concept :

Evaluate the following :

(i) $\sin^{-1} \sin 4\pi/3$ **(ii)** $\cos^{-1} \cos 5\pi/4$ **(iii)** $\tan^{-1} \tan 2\pi/3$

(i) $4\pi/3$ does not lie in the principal value branch of $\sin^{-1}x$. Hence $\sin^{-1} \sin 4\pi/3 \neq 4\pi/3$.

$$\sin^{-1}\sin 4\pi/3 = \sin^{-1}\sin(\pi + \pi/3) = \sin^{-1}(\sin(-\pi/3))$$

$$= -\sin^{-1}\sin\pi/3 = -\pi/3 \quad \Rightarrow \quad \sin^{-1}\sin 4\pi/3 = -\pi/3.$$

(ii) $$\cos^{-1}\cos 5\pi/4 = \cos^{-1}\cos(\pi + \pi/4)$$

$$= \cos^{-1}(-\cos\pi/4) = \pi - \cos^{-1}\cos\pi/4 = \pi - \pi/4 = 3\pi/4.$$

(iii) $$\tan^{-1}\tan 2\pi/3 = \tan^{-1}\tan(\pi - \pi/3)$$
$$= \tan^{-1}(-\tan\pi/3) = -\tan^{-1}\tan\pi/3 = -\pi/3.$$

Illustration - 2 *The value of* $\sin[\cot^{-1}\{\cos(\tan^{-1} x)\}]$ is $A_1\,(x > 0)$ *and the value of* $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$ is $A_2\,(x > 0)$ *then relation between* A_1 *and* A_2 *is :*

(A) $A_1 = A_2$ **(B)** $A_1 \neq A_2$ **(C)** $A_1 = 2A_2$ **(D)** $A_1 = -A_2$

SOLUTION : (A)

Consider A_1

We have, $\cos(\tan^{-1} x)$

Here $\tan^{-1} x$ is an angle whose cosine is taken so,

$$\therefore \quad \sin[\cot^{-1}\{\cos(\tan^{-1} x)\}] \qquad \left[As\ \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$= \sin\left\{\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \sin\left\{\sin^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}} \qquad \text{...... (i)}$$

Consider A_2

$$\text{We have, } \sin\left(\cot^{-1} x\right) = \sin\left\{\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \quad \cos[\tan^{-1}[\sin(\cot^{-1} x)]] = \cos\left\{\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right\}$$

$$= \cos\left\{\cos^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} = \sqrt{\frac{1+x^2}{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}} \qquad \text{...... (ii)}$$

From (i) and (ii), $A_1 = A_2$

Illustration - 3 *The simplest form of*

$$tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}, -\pi < x < \pi, \text{ is :}$$

(A) $-x/2$ **(B)** $x/2$ **(C)** x **(D)** *None of these*

SOLUTION : (AB)

We have, $\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\} = \tan^{-1}\left\{\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right\} = \tan^{-1}\left\{\sqrt{\tan^2\frac{x}{2}}\right\} = \tan^{-1}\left(\left|\tan\frac{x}{2}\right|\right)$

$$= \begin{cases} \tan^{-1}\left(-\tan\frac{x}{2}\right) & \text{if } -\pi < x < 0 \\ \tan^{-1}\left(\tan\frac{x}{2}\right) & \text{if } 0 \le x < \pi \end{cases} = \begin{cases} \tan^{-1}\left\{\tan\left(\frac{-x}{2}\right)\right\} = -\frac{x}{2} & \text{if } -\pi < x < 0 \\ \tan^{-1}\left\{\tan\frac{x}{2}\right\} = \frac{x}{2} & \text{if } 0 < x < \pi \end{cases}$$

Illustration - 4 *The simplest form of*

$$tan^{-1}\left(\frac{\cos x}{1+\sin x}\right), \frac{-\pi}{2} < x < \frac{\pi}{2}, \text{ is :}$$

(A) $\frac{\pi}{2} - x$ **(B)** $\frac{\pi}{4} - x$ **(C)** $\frac{\pi}{4} - \frac{\pi}{2}$ **(D)** *None of these*

SOLUTION : (C)

We have, $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2} + \cos\frac{x}{2}}\right\}$

$$= \tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right\} = \tan^{-1}\left\{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}-\frac{x}{2} \qquad \left[\text{As} -\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow -\frac{\pi}{4}<-\frac{\pi}{2}<\frac{\pi}{4} \Rightarrow 0<\frac{\pi}{4}-\frac{x}{2}<\frac{\pi}{2}\right]$$

Aliternate solution :

We have, $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}\right\}$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\} = \tan^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \frac{\pi}{4}-\frac{x}{2}$$

Illustration - 5 *The simplest form of*

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), \; -\frac{\pi}{4} < x < \frac{\pi}{4}, \text{ is :}$$

(A) $\frac{\pi}{4} - x$ (B) $\frac{\pi}{2}-\frac{x}{2}$ (C) $\pi - \frac{x}{2}$ (D) $\frac{\pi}{4}+\frac{x}{2}$

SOLUTION : (A)

we have, $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-x\right)\right\} = \frac{\pi}{4}-x$

$$\left[\text{as} -\frac{\pi}{4}<x<\frac{\pi}{4} \Rightarrow 0<\frac{\pi}{4}-x<\frac{\pi}{2}\right]$$

Illustration - 6 *The simplest form of*

$$\tan^{-1}\left\{\frac{x}{\sqrt{a^2 - x^2}}\right\}, \; -a < x < a, \text{ is :}$$

(A) $\sin^{-1}\frac{x}{a}$ **(B)** $\frac{1}{a}\sin^{-1}\frac{x}{a}$ **(C)** $\sin a$ **(D)** *None of these*

SOLUTION : (A)

IMPORTANT NOTE :

In order to simplify trigonometrical expressions involving inverse trigonometrical functions, following substitutions are very helpful :

Experssion	Substitution
$a^2 + x^2$	$x = a \tan\theta$ or, $x = a \cot\theta$
$a^2 - x^2$	$x = a \sin\theta$ or, $x = a \cos\theta$
$x^2 - a^2$	$x = a \sec\theta$ or, $x = a \operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or, $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2\cos 2\theta$

Substituiting $x = a\sin\theta$, we have $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\} = \tan^{-1}\left\{\frac{a\sin\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\right\}$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{a\cos\theta}\right\} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a} \quad \left[\text{as } x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}\right]$$

Illustration - 7 *The simplest form of*

$$\tan^{-1}\left\{\sqrt{\frac{a-x}{a+x}}\right\}, -a < x < a$$

(A) $\cos^{-1}\frac{x}{a}$ **(B)** $\frac{1}{2}\cos^{-1}\frac{x}{a}$ **(C)** $\frac{1}{2}\sin^{-1}\frac{x}{a}$ **(D)** *None of these*

SOLUTION : (B)

Substituting $x = a\cos\theta$, we have $\tan^{-1}\sqrt{\frac{a-x}{a+x}} = \tan^{-1}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}} = \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

$$= \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} = \tan^{-1}\left(\left|\tan\frac{\theta}{2}\right|\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) \qquad \left[as\ -a < x < a \Rightarrow 0 < \theta < \pi \Rightarrow 0 < \frac{\theta}{2} < \frac{\pi}{2}\right]$$

$$= \frac{\theta}{2} = \frac{1}{2}\cos^{-1}\frac{x}{a} \qquad \left[as\ x = a\cos\theta \Rightarrow \cos\theta = \frac{x}{a} \Rightarrow \theta = \cos^{-1}\frac{x}{a}\right]$$

Illustration - 8 *The value of* $\sin^{-1}(\sin 10) + \sin^{-1}(\sin 5)$ *is :*

(A) $\pi - 15$ **(B)** $\pi + 15$ **(C)** $\pi + 5$ **(D)** $\pi - 5$

SOLUTION : (D)

We know that $\sin^{-1}(\sin\theta) = \theta$, if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Here, $\theta = 10$ radians which is in radian but does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

But, $3\pi - \theta$ i.e. $3\pi - 10$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Also, $\sin(3\pi - 10) = \sin 10$.

$\therefore$ $\sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = 3\pi - 10$.

Here, $\theta = 5$ radians.

Clearly, it does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

But, $2\pi - 5$ and $5 - 2\pi$ both lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that

$$\sin(5 - 2\pi) = \sin(-(2\pi - 5)) = -\sin(2\pi - 5) = -(-\sin 5) = \sin 5$$

$\therefore \quad \sin^{-1}(\sin 5) = \sin^{-1}(\sin(5 - 2\pi)) = 5 - 2\pi.$

Thus $\sin^{-1}(\sin 10) = \sin^{-1}(\sin 5)$

$= (3\pi - 10) + (5 - 2\pi) = \pi - 5$

Illustration - 9

The value of $\cos^{-1}(\cos 10) - \tan^{-1}\{\tan(-6)\}$ *is :*

(A) $\pi - 4$ **(B)** $2\pi - 4$ **(C)** $\pi + 16$ **(D)** $2\pi - 16$

SOLUTION : (B)

We know that $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$.

Here, $\theta = 10$ radians.

Clearly, it does not lie between 0 and π. However, $(4\pi - 10)$ lies between 0 and π such that

$\cos(4\pi - 10) = \cos 10.$

$\therefore \quad \cos^{-1}(\cos 10) = \cos^{-1}(\cos(4\pi - 10)) = 4\pi - 10$

We know that $\tan^{-1}(\tan\theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Here, $\theta = -6$ radians which does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Also, we find that $2\pi - 6$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that

$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$

$\therefore \quad \tan^{-1}\{\tan(-6)\} = \tan^{-1}\{\tan(2\pi - 6)\} = 2\pi - 6$

Thus $\quad \cos^{-1}(\cos 10) - \tan^{-1}\{\tan(-6)\}$

$= (4\pi - 10) - (2\pi - 6) = 2\pi - 4$

Illustration - 10 *The simplest form of*

$$tan^{-1}\left\{\frac{\sqrt{1+cos\ x}+\sqrt{1-cos\ x}}{\sqrt{1+cos\ x}-\sqrt{1-cos\ x}}\right\},\ 0<\ x<\ \frac{\pi}{2},\ is:$$

(A) $\frac{\pi}{2}+\frac{x}{2}$ **(B)** $\frac{\pi}{2}-\frac{x}{2}$ **(C)** $\frac{\pi}{4}+\frac{x}{2}$ **(D)** *None of these*

SOLUTION : (C)

We have, $\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right\}$

$=\tan^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\}$ $\left[\text{as } 0<\frac{x}{2}<\frac{\pi}{4}\ \therefore \cos\frac{x}{2}>0,\ \sin\frac{x}{2}>0\right]$

$=\tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\}=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$

$=\frac{\pi}{4}+\frac{x}{2}$ $\left[\text{as } 0<x<\frac{\pi}{2}\ \therefore \frac{\pi}{4}<\frac{\pi}{4}+\frac{x}{2}<\frac{\pi}{2}\right]$

Illustration - 11 *The simplest form of*

$$cot^{-1}\left\{\frac{\sqrt{1+sin\ x}+\sqrt{1-sin\ x}}{\sqrt{1+sin\ x}-\sqrt{1-sin\ x}}\right\},0<x<\frac{\pi}{2},\ is:$$

(A) $\frac{x}{2}$ **(B)** $-\frac{x}{2}$ **(C)** x **(D)** $-x$

SOLUTION : (A)

We have, $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}=\cot^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$

$$\left[\text{as}\left(\cos\frac{x}{2}\pm\sin\frac{x}{2}\right)^2=\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\cos^2\frac{\pi}{2}\pm2\sin\frac{x}{2}\cos\frac{x}{2}=1\pm\sin x\right]$$

$$=\cot^{-1}\left\{\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right\} \qquad \left[\text{as}\sqrt{x^2}=|x|\right]$$

$$=\cot^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\} \qquad \left[\text{as }0<\frac{x}{2}<\frac{\pi}{4}\ \therefore\cos\frac{x}{2}>\sin\frac{x}{2}\right]$$

$$=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2} \qquad \left[\text{as }0<\frac{x}{2}<\frac{\pi}{4}\right]$$

Illustration - 12 *The simplest form of*

$$\tan^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right\},\ 0<x<1,\ \textit{is}:$$

(A) $\frac{1}{2}\cos^{-1}x$ **(B)** $\pi-\frac{1}{2}\cos^{-1}x$ **(C)** $\frac{\pi}{2}-\cos^{-1}x$ **(D)** $\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$

SOLUTION : (D)

Putting $x = \cos 2\theta$, we have $\tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right\}$

$$=\tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right\}=\tan^{-1}\left\{\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right\}$$

$$\left[\text{as }0<x<1\Rightarrow0<\cos2\theta<1\Rightarrow0<2\theta<\frac{\pi}{2}\Rightarrow0<\theta<\frac{\pi}{4}\ \textit{and}\ \cos\theta>0,\ \sin\theta>0\right]$$

$$=\tan^{-1}\left\{\frac{1-\tan\theta}{1+\tan\theta}\right\}=\tan^{-1}\left\{\frac{1-\tan\theta}{1+\tan\theta}\right\}=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\}$$

$$=\frac{\pi}{4}-\theta \qquad \left[\text{as }0<\theta<\frac{\pi}{4}\ \Rightarrow\ 0<\frac{\pi}{4}-\theta<\frac{\pi}{4}\right]$$

$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$ $\left[\text{as } \cos 2\theta = x \therefore\ 2\theta = \cos^{-1} x \Rightarrow \theta = \frac{1}{2}\cos^{-1} x\right]$

Illustration - 13 *The simplest form of*

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\} = -1 < x < 1, \text{ is :}$$

(A) $\frac{\pi}{2} + \cos^{-1} x^2$ **(B)** $\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$ **(C)** $\frac{\pi}{4} - \frac{1}{2}\cos^{-1} x^2$ **(D)** *None of these*

SOLUTION : (B)

Putting $x^2 = \cos 2\theta$, we have $\tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right\}$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right\} = \tan^{-1}\left\{\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right\} = \tan^{-1}\left\{\frac{1+\tan\theta}{1-\tan\theta}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}$$

$= \frac{\pi}{4} + \theta$ $\left[\text{as } -1 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow 0 < 2\theta < \frac{\pi}{2} \Rightarrow 0 < \theta < \frac{\pi}{4}\right]$

$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$ $\left[as\ x^2 = \cos 2\theta \Rightarrow 2\theta = \cos^{-1} x^2\right]$

Illustration - 14 *The simplest form of*

$$\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \text{ when } -\frac{\pi}{4} < x < \frac{\pi}{4}, \text{ is :}$$

(A) $x + \frac{\pi}{4}$ **(B)** $x - \frac{\pi}{4}$ **(C)** x **(D)** *None of theses*

SOLUTION : (A)

$$\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sin^{-1}\left(\sin x \cos\frac{\pi}{4} + \cos x \cos\frac{\pi}{4}\right)$$

$$= \sin^{-1}\left\{\sin\left(x + \frac{\pi}{4}\right)\right\} = x + \frac{\pi}{4} \qquad \left[\text{as } \frac{-\pi}{4} < x < \frac{\pi}{4} \Rightarrow 0 < x + \frac{\pi}{4} < \frac{\pi}{2}\right]$$

Illustration - 15 *The simplest form of*

$$\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right) \text{ when } \frac{\pi}{4} < x < \frac{5\pi}{4}, \text{ is:}$$

(A) $x + \frac{\pi}{4}$ **(B)** $x - \frac{\pi}{4}$ **(C)** $-x$ **(D)** x

SOLUTION : (B)

$$\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \cos^{-1}\left(\sin x \sin\frac{\pi}{4} + \cos x \cos\frac{\pi}{4}\right)$$

$$= \cos^{-1}\left\{\cos\left(x - \frac{\pi}{4}\right)\right\} = x - \frac{\pi}{4} \qquad \left[\text{as } \frac{\pi}{4} < x < \frac{5\pi}{4} \Rightarrow 0 < x - \frac{\pi}{4} < \pi\right]$$

Illustration - 16 *The value of* $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$, *is :*

(A) 5 **(B)** 10 **(C)** 15 **(D)** 20

SOLUTION : (C)

We have, $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$

$$= \{\sec(\tan^{-1} 2)\}^2 + \{\operatorname{cosec}(\cot^{-1} 3)\}^2$$

$$= \left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\operatorname{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$= \{\sec(\sec^{-1}\sqrt{5})\}^2 + \{\operatorname{cosec}(\operatorname{cosec}^{-1}\sqrt{10})\}^2 = (\sqrt{5})^2 + (\sqrt{10})^2 = 15$$

Illustration - 17

If $a > b > c > 0$, *then* $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)$ *is equal to :*

(A) 0 **(B)** π **(C)** $-\pi$ **(D)** *None of these*

SOLUTION : (B)

We know that

$$\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1}x & , \text{ for } x>0\\ -\pi+\cot^{-1}x & , \text{ for } x<0\end{cases}$$

$$\Rightarrow \quad \cot^{-1}x=\begin{cases}\tan^{-1}\frac{1}{x} & , \text{ for } x>0\\ \pi+\tan^{-1}\frac{1}{x} & , \text{ for } x<0\end{cases}$$

$$\therefore \quad \cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

$$= \tan^{-1}\left(\frac{a-b}{1+ab}\right)+\tan^{-1}\left(\frac{b-c}{1+bc}\right)+\pi+\tan^{-1}\left(\frac{c-a}{1+ca}\right)$$

$$= \tan^{-1}a-\tan^{-1}b+\tan^{-1}b-\tan^{-1}c+\pi+\tan^{-1}c-\tan^{-1}a$$

$$= \pi.$$

IMPORTANT RESULTS

Section - 2

(A) (i)
$$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } xy<1\\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } x>0,\ y>0 \text{ and } xy>1\\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if } x<0,\ y<0 \text{ and } xy>1\end{cases}$$

(ii)
$$\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\dfrac{x-y}{1+xy} & \text{if } xy>-1\\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if } x>0,\ y<0 \text{ and } xy<-1\\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if } x<0,\ y>0 \text{ and } xy<-1\end{cases}$$

(B) (i)
$$\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy<0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & \text{if } 0<x,\ y\le 1 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & \text{if } -1\le x,\ y<0 \text{ and } x^2+y^2>1\end{cases}$$

(ii)
$$\sin^{-1}x-\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy>0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & \text{if } 0<x\le 1,\ -1<y\le 0 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & \text{if } -1\le x<0,\ 0<y\le 1 \text{ and } x^2+y^2>1\end{cases}$$

(C) (i)
$$\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & \text{if } -1\le x,\ y\le 1 \text{ and } x+y\ge 0\\ 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & \text{if } -1\le x,\ y\le 1 \text{ and } x+y\le 0\end{cases}$$

(ii) $$\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} & \text{if}\quad -1\le x,\ y\le 1 \text{ and } x\le y\\ -\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} & \text{if}\quad -1\le y\le 0,\ 0<x\le 1 \text{ and } x\ge y\end{cases}$$

Illustrating the Concept :

(i) *Prove that :* $\tan^{-1}\dfrac{2}{11}+\tan^{-1}\dfrac{7}{24}=\tan^{-1}\dfrac{1}{2}$

We have, $\tan^{-1}\dfrac{2}{11}+\tan^{-1}\dfrac{7}{24}$

$$=\tan^{-1}\left\{\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right\}\qquad\left[\text{As }\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),\ \text{if}\ xy<1\right]$$

$$=\tan^{-1}\left\{\frac{48+77}{264-14}\right\}=\tan^{-1}\left(\frac{125}{250}\right)=\tan^{-1}\left(\frac{1}{2}\right)$$

(ii) *Prove that :* $\underset{(x\,\in R)}{\tan^{-1}x}+\underset{(|x|<1)}{\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)}=\underset{\left(|x|<\frac{1}{\sqrt{3}}\right)}{\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)}$

We have, $\tan^{-1}x+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$$=\tan^{-1}\left\{\frac{x+\frac{2x}{1-x^2}}{1-\frac{2x^2}{1-x^2}}\right\}\qquad\left[\text{As }\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),\ \text{if}\ xy<1\right]$$

$$=\tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\left(|x|<\frac{1}{\sqrt{3}}\right)$$

Illustration - 18

The value of $sin^{-1}\frac{12}{13}+cos^{-1}\frac{4}{5}+tan^{-1}\frac{63}{16}$ *is :*

(A) 0 **(B)** $-\pi$ **(C)** π **(D)** *None of these*

SOLUTION : (C)

We have, $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}$

$$=\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{13}$$ $\left[\text{As } \sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5} \text{ and } \cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}\right]$

$$=\pi+\tan^{-1}\left\{\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right\}+\tan^{-1}\frac{63}{16}$$ $\left[\text{As } \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy>1\right]$

$$=\pi+\tan^{-1}\left(\frac{63}{-16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16}$$ $[\text{As } \tan^{-1}(-x)=-\tan^{-1}x]$

$$=\pi$$

Illustration - 19

The value of $cos^{-1}\frac{12}{13}+sin^{-1}\frac{3}{5}$ *is :*

(A) $\sin^{-1}\frac{56}{65}$ **(B)** $\sin^{-1}\frac{12}{13}$ **(C)** 0 **(D)** *None of these*

SOLUTION : (A)

We have, $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}$

$$=\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{3}{5}$$ $\left[\text{As } \cos^{-1}\frac{12}{13}=\sin^{-1}\frac{5}{13}\right]$

$$=\sin^{-1}\left\{\frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2}\right\}=\sin^{-1}\left\{\frac{5}{13}\times\frac{4}{5}+\frac{3}{5}\times\frac{12}{13}\right\}=\sin^{-1}\frac{56}{65}$$

Illustration - 20 *The value of* $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}$ *is :*

(A) $\cos^{-1}\frac{60}{85}$ **(B)** $\cos^{-1}\frac{24}{85}$ **(C)** $\cos^{-1}\frac{84}{85}$ **(D)** *None of these*

SOLUTION : (C)

We have, $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}$

$$=\cos^{-1}\frac{4}{5}-\cos^{-1}\frac{15}{17} \qquad \left[\text{As } \sin^{-1}\frac{3}{5}=\cos^{-1}\frac{4}{5}, \sin^{-1}\frac{8}{17}=\cos^{-1}\frac{15}{17}\right]$$

$$=\cos^{-1}\left\{\frac{4}{5}\times\frac{15}{17}+\sqrt{1-\left(\frac{4}{5}\right)^2}\times\sqrt{1-\left(\frac{15}{17}\right)^2}\right\}=\cos^{-1}\left\{\frac{4}{5}\times\frac{15}{17}+\frac{3}{5}\times\frac{8}{17}\right\}=\cos^{-1}\left\{\frac{60}{85}+\frac{24}{85}\right\}=\cos^{-1}\frac{84}{85}$$

Illustration - 21 *The value of* $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$ *is :*

(A) 0 **(B)** 1 **(C)** π **(D)** $\pi/4$

SOLUTION : (D)

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$$

$$=\left\{\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}\right\}+\tan^{-1}\frac{1}{8}$$

$$=\tan^{-1}\left\{\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right\}+\tan^{-1}\frac{1}{8} \qquad \left[\text{As } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy<1\right]$$

$$=\tan^{-1}\frac{7}{9}+\tan^{-1}\frac{1}{8}=\tan^{-1}\left\{\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right\}=\tan^{-1}\left(\frac{65}{65}\right)=\tan^{-1}1=\frac{\pi}{4}$$

Illustration - 22 *The value of* $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}$ *is :*

(A) $\pi/4$ **(B)** π **(C)** 1 **(D)** 0

SOLUTION : (A)

$$\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad \left[\text{As } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy<1\right]$$

$$=\tan^{-1}\frac{6}{17}+\tan^{-1}\frac{11}{23}=\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)=\tan^{-1}\left(\frac{325}{325}\right)=\tan^{-1}1=\frac{\pi}{4}$$

Illustration - 23 *The value of* $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}$ *is equal to :*

(A) $\sin^{-1}\frac{15}{17}$ **(B)** $\sin^{-1}\frac{77}{85}$ **(C)** 0 **(D)** *None of these*

SOLUTION : (B)

$$\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}$$

$$[\text{Using, } \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, \text{ if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1]$$

$$=\sin^{-1}\left\{\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2}+\frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right\}=\sin^{-1}\left\{\frac{3}{5}\times\frac{15}{17}+\frac{8}{17}\times\frac{4}{5}\right\}=\sin^{-1}\frac{77}{85}$$

Illustration - 24 *The value of* $\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}$ *is equal to :*

(A) $\cos^{-1}\frac{33}{65}$ **(B)** $\cos^{-1}\frac{56}{65}$ **(C)** 1 **(D)** *None of these*

SOLUTION : (A)

$$\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}$$

$$= \sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13} \qquad \left[\text{As}\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}\right]$$

[Using, $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$, if $-1\le x,\ y\le 1$ and $x^2+y^2\le 1$]

$$= \sin^{-1}\left\{\frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2}\right\}$$

$$= \sin^{-1}\left\{\frac{3}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{4}{5}\right\}=\sin^{-1}\frac{56}{65}=\cos^{-1}\sqrt{1-\left(\frac{56}{65}\right)^2}=\cos^{-1}\frac{33}{65}$$

MORE ILLUSTRATIONS — Section - 3

Illustration - 25 *Sketch the graph for :*

(i) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ **(ii)** $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ **(iii)** $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(iv) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ **(v)** $\sin^{-1}(3x-4x^3)$ **(vi)** $\cos^{-1}(4x^3-3x)$.

SOLUTION :

As we know, all the above mentioned six curves are non-periodic, but have restricted domain and range.

So, we shall first define each curve for its domain and range and then sketch these curves.

(i) Sketch for $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Here, for domain $\left|\frac{2x}{1+x^2}\right|\le 1$

$\Rightarrow \quad 2|x| \le 1 + x^2$ [as $1+x^2 > 0$ for all x]

$\Rightarrow \quad |x|^2 - 2|x| + 1 \ge 0$ [as $x^2 = |x^2|$]

$\Rightarrow \quad (|x| - 1)^2 \ge 0 \quad \Rightarrow \quad x \in R.$

For Range : $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$\Rightarrow \quad y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ $\left[as;\ y = \sin^{-1}\theta \Rightarrow y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right]$

Defining the curve :

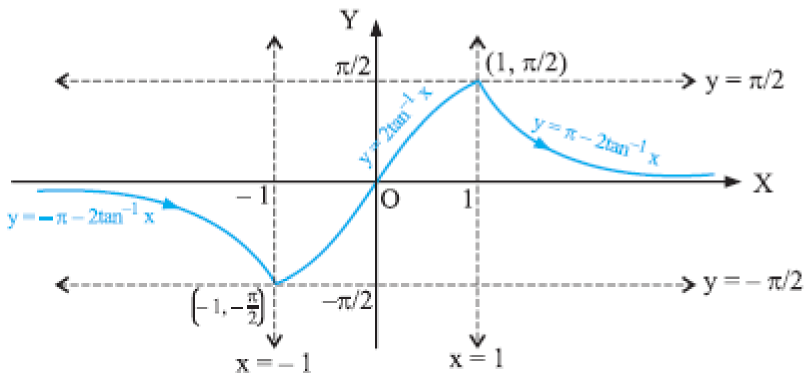
Let, $x = \tan\theta$

$$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta) = \begin{cases} \pi - 2\theta & ; \ 2\theta > \dfrac{\pi}{2} \\ 2\theta & ; \ -\dfrac{\pi}{2} \le 2\theta \le \dfrac{\pi}{2} \\ -\pi - 2\theta & ; \ 2\theta < -\dfrac{\pi}{2} \end{cases}$$ [see property 1.3 F(i)]

or $$y = \begin{cases} \pi - 2\tan^{-1}x & ; \ \tan^{-1}x > \dfrac{\pi}{4} \\ 2\tan^{-1}x & ; \ -\dfrac{\pi}{4} \le \tan^{-1}x \le \dfrac{\pi}{4} \\ -\pi - 2\tan^{-1}x & ; \ \tan^{-1}x < -\dfrac{\pi}{4} \end{cases}$$ [as $\tan\theta = x \Rightarrow \theta = \tan^{-1}x$]

or $$y = \begin{cases} \pi - 2\tan^{-1}x & ; \ x > 1 \\ 2\tan^{-1}x & ; \ -1 \le x \le 1 \\ -\pi - 2\tan^{-1}x & ; \ x < -1 \end{cases}$$ . . . (i)

Thus, $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is defined for $x \in R$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the graph for Eq. (i) could be shown in Fig.



Thus, the graph for $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

(ii) Sketch for $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Here, for domain $\left|\frac{1-x^2}{1+x^2}\right| \le 1$

$\Rightarrow$ $\quad | 1 - x^2 | \le 1 + x^2$ $\quad$ [as $1+x^2 > 0, \forall x \in R$]

which is true for all x ; as $1 + x^2 \ge 1 - x^2$

$\therefore \quad x \in R$

For range : $\quad y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \quad \Rightarrow \quad y \in [0, \pi)$

Define the curve :

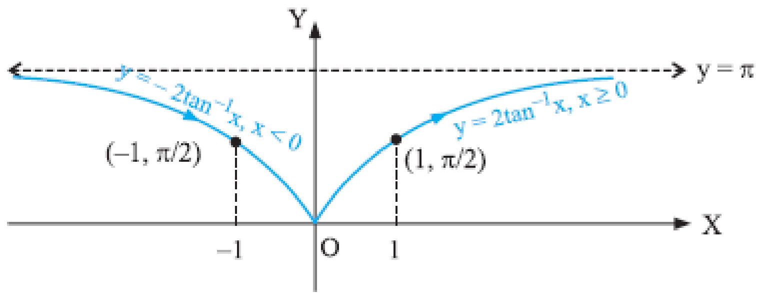
Let, $\quad x = \tan\theta$

$$\therefore \quad y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$

$$= \begin{cases} 2\theta & ; \quad 2\theta \ge 0 \\ -2\theta & ; \quad 2\theta < 0 \end{cases}$$ [see property 1.3 F(ii)]

$$\Rightarrow \begin{cases} 2\tan^{-1}x & ; \quad \tan^{-1}x \geq 0 \\ -2\tan^{-1}x & ; \quad \tan^{-1}x < 0 \end{cases}$$ [as $\tan\theta = x \Rightarrow \theta = \tan^{-1}x$]

So, the graph of $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & ; \quad x \geq 0 \\ -2\tan^{-1}x & ; \quad x < 0 \end{cases}$ is shown as :



Thus, the graph for $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & ; \quad x \geq 0 \\ -2\tan^{-1}x & ; \quad x < 0 \end{cases}$.

(iii) Sketch for $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Here, for domain $\frac{2x}{1-x^2} \in R$ except ; $1 - x^2 = 0$

i.e., $x \neq \pm 1$

or $x \in R - \{1, -1\}$

For range $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$\Rightarrow \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$ $\left[\text{as } y = \tan^{-1}\theta \Rightarrow y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Defining the curve

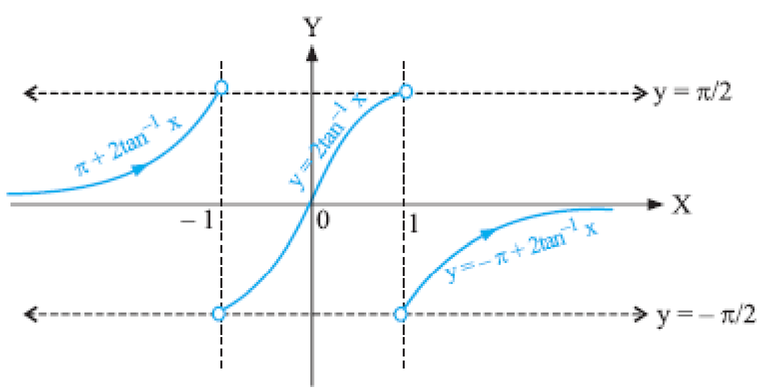
Let $x = \tan\theta$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = \begin{cases} \pi + 2\theta & ; \quad 2\theta < \frac{-\pi}{2} \\ 2\theta & ; \quad -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \\ -\pi + 2\theta & ; \quad 2\theta > \frac{\pi}{2} \end{cases} \text{ [see property 1.3 F(iii)]}$$

$$= \begin{cases} \pi + 2\tan^{-1}x & ; \quad \tan^{-1}x < -\frac{\pi}{4} \\ 2\tan^{-1}x & ; \quad -\frac{\pi}{4} < \tan^{-1}x < \frac{\pi}{4} \\ -\pi + 2\tan^{-1}x & ; \quad \tan^{-1}x > \frac{\pi}{4} \end{cases} \quad [\text{as } \tan\theta = x \;\Rightarrow\; \theta = \tan^{-1}x]$$

$$= \begin{cases} \pi + 2\tan^{-1}x & ; \quad x < -1 \\ 2\tan^{-1}x & ; \quad -1 < x < 1 \\ -\pi + 2\tan^{-1}x & ; \quad x > 1 \end{cases}$$

So, the graph of ; $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} \pi + 2\tan^{-1}x & ; \quad x < -1 \\ 2\tan^{-1}x & ; \quad -1 < x < 1 \\ -\pi + 2\tan^{-1}x & ; \quad x > 1 \end{cases}$ is shown as ;



Thus, the graph for $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} \pi + 2\tan^{-1}x & ; \quad x < -1 \\ 2\tan^{-1}x & ; \quad -1 < x < 1 \\ -\pi + 2\tan^{-1}x & ; \quad x > 1 \end{cases}$

(iv) Sketch for $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$. Here, for domain $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$\Rightarrow \quad x \in R$ except $1-3x^2=0 \quad \Rightarrow \quad x \neq \pm\frac{1}{\sqrt{3}} \quad \therefore \quad x \in R-\left\{\pm\frac{1}{\sqrt{3}}\right\}$

For range : $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) \Rightarrow y \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\left[\text{as } y=\tan^{-1}\theta \Rightarrow y \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Defining the curve :

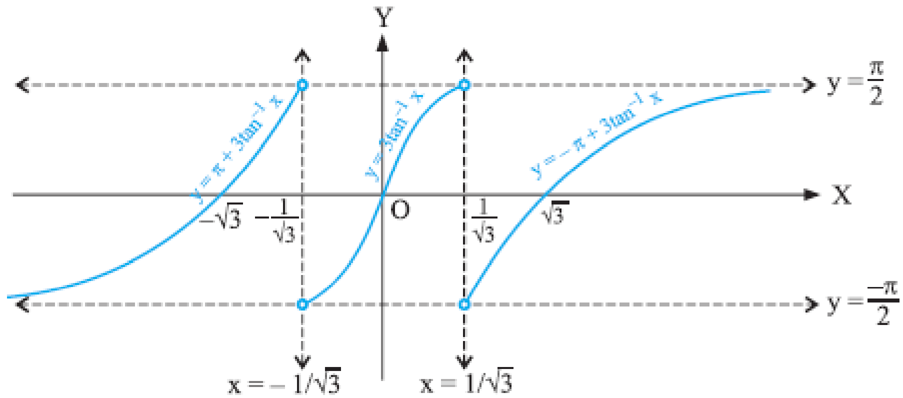
Let $x=\tan\theta$

$$\Rightarrow \quad y=\tan^{-1}(\tan 3\theta)=\begin{cases}\pi+3\theta & ; \quad 3\theta<-\frac{\pi}{2} \\ 3\theta & ; \quad -\frac{\pi}{2}<3\theta<\frac{\pi}{2} \\ -\pi+3\theta & ; \quad 3\theta>\frac{\pi}{2}\end{cases}=\begin{cases}\pi+3\tan^{-1}x & ; \quad \tan^{-1}x<-\frac{\pi}{6} \\ 3\tan^{-1}x & ; \quad -\frac{\pi}{6}<\tan^{-1}x<\frac{\pi}{6} \\ -\pi+3\tan^{-1}x & ; \quad \tan^{-1}x>\frac{\pi}{6}\end{cases}$$

$$=\begin{cases}\pi+3\tan^{-1}x & ; \quad x<-\frac{1}{\sqrt{3}} \\ 3\tan^{-1}x & ; \quad -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}} \\ -\pi+3\tan^{-1}x & ; \quad x>\frac{1}{\sqrt{3}}\end{cases}$$

So, the graph of ;

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=\begin{cases}\pi+3\tan^{-1}x & ; \quad x<-\frac{1}{\sqrt{3}} \\ 3\tan^{-1}x & ; \quad -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}} \\ -\pi+3\tan^{-1}x & ; \quad x>\frac{1}{\sqrt{3}}\end{cases}$$



(v) Sketch curve $y = \sin^{-1}(3x - 4x^3)$

For domain $y = \sin^{-1}(3x - 4x^3) \Rightarrow x \in [-1, 1]$

For range $y = \sin^{-1}(3x - 4x^3) \Rightarrow y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Defining the curve :

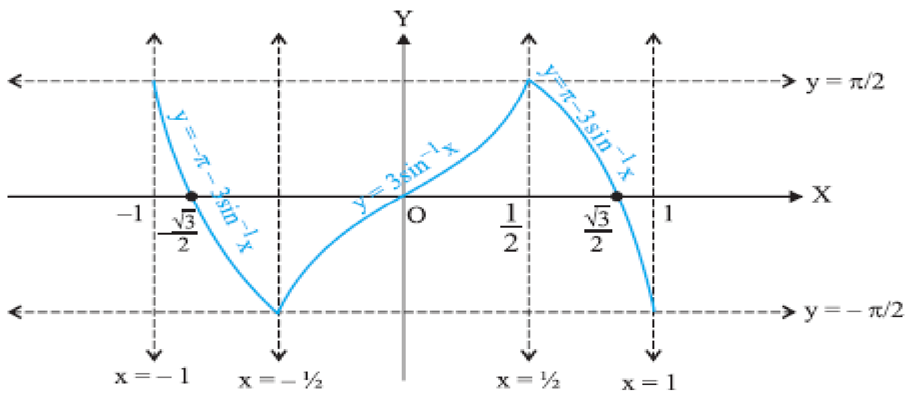
Let $x = \sin\theta$,

$$\Rightarrow \quad y = \sin^{-1}(\sin 3\theta) = \begin{cases} \pi - 3\theta & ; \frac{\pi}{2} \le 3\theta \le \frac{3\pi}{2} \\ 3\theta & ; -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2} \\ -\pi - 3\theta & ; -\frac{3\pi}{2} \le 3\theta \le -\frac{\pi}{2} \end{cases}$$

$$= \begin{cases} \pi - 3\sin^{-1} x & ; \frac{\pi}{6} \le \sin^{-1} x \le \frac{\pi}{2} \\ 3\sin^{-1} x & ; -\frac{\pi}{6} \le \sin^{-1} x \le \frac{\pi}{6} \\ -\pi - 3\sin^{-1} x & ; -\frac{\pi}{2} \le \sin^{-1} x \le -\frac{\pi}{6} \end{cases}$$

$$\therefore \quad y = \sin^{-1}(3x - 4x^3) = \begin{cases} \pi - 3\sin^{-1} x & ; \frac{1}{2} \le x \le 1 \\ 3\sin^{-1} x & ; -\frac{1}{2} \le x \le \frac{1}{2} \\ -\pi - 3\sin^{-1} x & ; -1 \le x \le -\frac{1}{2} \end{cases}$$

So the graph of $y = \sin^{-1}(3x - 4x^3) = \begin{cases} \pi - 3\sin^{-1}x & ; \quad \frac{1}{2} \le x \le 1 \\ 3\sin^{-1}x & ; \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ -\pi - 3\sin^{-1}x & ; \quad -1 \le x \le -\frac{1}{2} \end{cases}$



(vi) Sketch the curve $y = \cos^{-1}(4x^3 - 3x)$

Here, domain $\in [-1, 1]$

range $y \in [0, \pi]$

Defining the curve :

Let $x = \cos\theta$

$$\Rightarrow \quad y = \cos^{-1}(\cos 3\theta) = \begin{cases} 2\pi - 3\theta & ; \quad \pi \le 3\theta \le 2\pi \\ 3\theta & ; \quad 0 \le 3\theta \le \pi \\ -2\pi + 3\theta & ; \quad 2\pi \le 3\theta \le 3\pi \end{cases} = \begin{cases} 2\pi - 3\cos^{-1}x & ; \quad \frac{\pi}{3} \le \cos^{-1}x \le \frac{2\pi}{3} \\ 3\cos^{-1}x & ; \quad 0 \le \cos^{-1}x \le \frac{\pi}{3} \\ -2\pi + 3\cos^{-1}x & ; \quad \frac{2\pi}{3} \le \cos^{-1}x \le \pi \end{cases}$$

$$= \begin{cases} 2\pi - 3\cos^{-1} x & ; \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x & ; \quad \frac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1} x & ; \quad -1 \le x \le -\frac{1}{2} \end{cases}$$

[as If $0 \le \theta \le \frac{\pi}{3} \Rightarrow \cos\frac{\pi}{3} \le \cos\theta \le \cos 0$ or $\frac{1}{2} \le \cos\theta \le 1$. Here, the interval changed since, $\cos x$ is decreasing in $[0, \pi]$]

So, the graph of ; $y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 2\pi - 3\cos^{-1} x & ; \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x & ; \quad \frac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1} x & ; \quad -1 \le x \le -\frac{1}{2} \end{cases}$ is shown as;

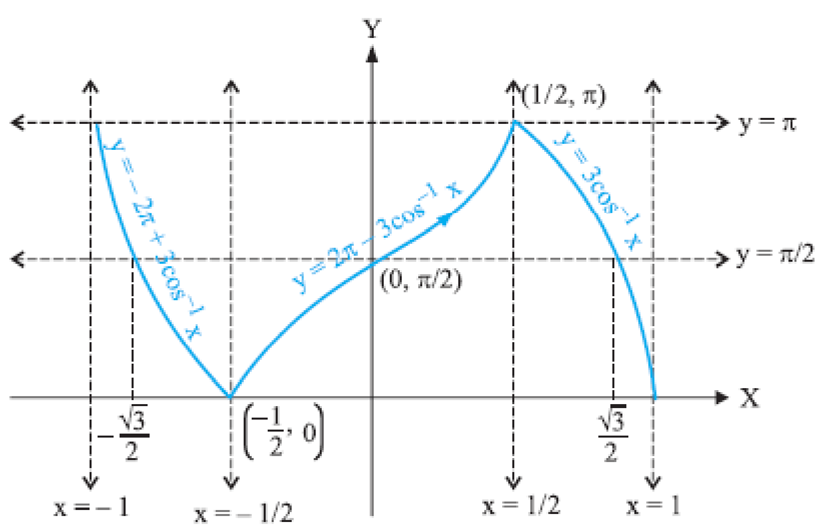


Illustration - 26

The value of $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$ *is :*

(A) $\tan^{-1}\frac{31}{17}$ **(B)** $\tan^{-1}\frac{4}{3}$ **(C)** 1 **(D)** 0

SOLUTION : (A)

$$2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left\{\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\}+\tan^{-1}\frac{1}{7} \qquad \left[\text{As } 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } -1<x<1\right]$$

$$=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\}=\tan^{-1}\frac{31}{17}$$

Illustration - 27

If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, *then find the value of x is.*

(A) 0 **(B)** 1 **(C)** 1/5 **(D)** 2/5

SOLUTION : (C)

We have, $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\sin^{-1}1$$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5}$$

$$\Rightarrow \cos^{-1}x=\cos^{-1}\frac{1}{5} \qquad \left[\text{As } \sin^{-1}\frac{1}{5}+\cos^{-1}\frac{1}{5}=\frac{\pi}{2}\right]$$

$$\Rightarrow x=\frac{1}{5}$$

Illustration - 28 *The value of x which satisfies* $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$ *is :*

(A) $x = 0$ **(B)** $x = 1$ **(C)** $x = \pm 1$ **(D)** $x = \pm\frac{1}{\sqrt{2}}$

SOLUTION : (D)

We have, $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$ $\Rightarrow$ $\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{x+2-x-1}{x+2+x+1}$

$\Rightarrow$ $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\tan^{-1}1$ $\Rightarrow$ $\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{1}{2x+3}$ $\Rightarrow$ $\frac{x-1}{x-2}=\frac{1}{2x+3}$

$\Rightarrow$ $\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}1-\tan^{-1}\frac{x+1}{x+2}$ $\Rightarrow$ $2x^2+x-3=x-2$

$\Rightarrow$ $2x^2-1=0$

$\Rightarrow$ $\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right)$ $\Rightarrow$ $x=\pm\frac{1}{\sqrt{2}}$

Illustration - 29 *The value of x which satisfies* $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$ *is :*

(A) $x = 1$ **(B)** $x=\frac{1}{3}$ **(C)** $x=\frac{1}{6}$ **(D)** *None of these*

SOLUTINON : (C) We have, $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\left\{\frac{2x+3x}{1-2x\times 3x}\right\}=\tan^{-1}1$, if $6x^2<1$

$\Rightarrow$ $\frac{5x}{1-6x^2}=1$, if $6x^2<1$

$\Rightarrow$ $6x^2+5x-1=0$ and $x^2<\frac{1}{6}$

$\Rightarrow$ $(6x-1)(x+1)=0$ and $-\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$

$\Rightarrow$ $x=-1,\frac{1}{6}$ and $-\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$ $\Rightarrow$ $x=\frac{1}{6}$

Illustration - 30 *The value of x which satisfies* $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\, x)$ *is :*

(A) $x = n\pi$ **(B)** $x = \frac{\pi}{3} + n\pi$ **(C)** $x = \frac{\pi}{4} + n\pi$ **(D)** *None of these*

SOLUTION : (C)

We have, $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\, x)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\,\text{cosec}\, x)$$

$$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = 2\,\text{cosec}\, x \quad \Rightarrow \quad \cos x \sin x = \sin^2 x \quad \Rightarrow \quad (\cos x - \sin x)\sin x = 0$$

$$\Rightarrow \quad \cos x = \sin x, (\sin x \neq 0) \quad \Rightarrow \quad \tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4} + n\pi$$

Illustration - 31 *The value of x which* $\tan^{-1}\sqrt{x^2 + x} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$ *is :*

(A) $x = 1$ **(B)** $x = 0, -1$ **(C)** $x = 2$ **(D)** *None of these*

SOLUTION : (B)

This equation holds, if

$$x^2 + x \geq 0 \quad \text{and} \quad 0 \leq x^2 + x + 1 \leq 1$$

Now, $x^2 + x \geq 0$ and $0 \leq x^2 + x + 1 \leq 1$

$$\Rightarrow \quad x^2 + x \geq 0 \quad \text{and} \quad x^2 + x + 1 \leq 0 \qquad [\text{as } x^2 + x + 1 > 0 \text{ for all } x]$$

$$\Rightarrow \quad x^2 + x \geq 0 \quad \text{and} \quad x^2 + x \leq 0$$

$$\Rightarrow \quad x^2 + x = 0$$

$$\Rightarrow \quad x = 0, -1$$

Clearly, both these values satisfy the given equation.

Hence, $x = 0, -1$ are the solutions of the given equation

Illustration - 32 *If* $x = cosec\,[tan^{-1}\{cos\,(cot^{-1}(sec\,(sin^{-1}a)))\}]$ *and*, $y = sec\,[cot^{-1}\{sin\,(tan^{-1}(cosec\,(cos^{-1}a)))\}]$. *Find relation between x and y.*

(A) $x = -y$ **(B)** $x = 2y$ **(C)** $x = 3y$ **(D)** $x = y$

SOLUTION : (D)

We have, $x = \text{cosec}\,[\tan^{-1}\{\cos(\cot^{-1}(\sec(\sin^{-1}a)))\}]$

$$= \text{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\left(\sec\left(\sec^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right\}\right] \qquad \left[\text{as } \sin^{-1}a = \sec^{-1}\frac{1}{\sqrt{1-a^2}}\right]$$

$$= \text{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right\}\right]$$

$$= \text{cosec}\left[\tan^{-1}\left\{\cos\left(\cos^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right\}\right] \qquad \left[\text{as } \cot^{-1}\frac{1}{\sqrt{1-a^2}} = \cos^{-1}\frac{1}{\sqrt{2-a^2}}\right]$$

$$= \text{cosec}\left(\tan^{-1}\frac{1}{\sqrt{2-a^2}}\right) = \text{cosec}\left(\text{cosec}^{-1}\sqrt{3-a^2}\right) = \sqrt{3-a^2}$$

and, $y = \sec\,[\cot^{-1}\{\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a)))\}]$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\left(\text{cosec}\left(\text{cosec}^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right\}\right] \qquad \left[\text{as } \cos^{-1}a = \text{cosec}^{-1}\frac{1}{\sqrt{1-a^2}}\right]$$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right\}\right]$$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right\}\right] \qquad \left[\text{as } \tan^{-1}\frac{1}{\sqrt{1-a^2}} = \sin^{-1}\frac{1}{\sqrt{2-a^2}}\right]$$

$$= \sec\left(\cot^{-1}\frac{1}{\sqrt{2-a}}\right) = \sec\left(\sec^{-1}\sqrt{3-a^2}\right) = \sqrt{3-a^2}$$

$$\therefore \quad x = y\sqrt{3-a^2}.$$

***Illustraing the Concept* :**

(i) *If* $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, *then prove that* $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$.

(ii) *If* $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, *then prove that* $x^2 + y^2 + z^2 + 2xyz = 1$.

(iii) *If* $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, *then prove that*

(a) $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

(b) $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$

(i) We have, $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{a} = \alpha$

$$\Rightarrow \quad \cos^{-1}\left\{\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right\}=\alpha \quad \Rightarrow \quad \frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$$

$$\Rightarrow \quad \frac{xy}{ab}-\cos\alpha=\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} \quad \Rightarrow \quad \left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\Rightarrow \quad \frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha+\cos^2\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=1-\cos^2\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=\sin^2\alpha$$

(ii) We have, $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \cos^{-1}(-z)$ [as $\cos^{-1}(-z) = \pi - \cos^{-1}z$]

$\Rightarrow \quad \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} = \cos^{-1}(-z)$

$\Rightarrow \quad xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow \quad (xy + z)^2 = (1 - x^2)(1 - y^2)$

$\Rightarrow \quad x^2 y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2 y^2 \quad \Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$

(iii) (a) Let $\sin^{-1} x = A$, $\sin^{-1} y = B$ and $\sin^{-1} z = C$. Then,

$x = \sin A$, $y = \sin B$ and $z = \sin C$

We have, $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

$\Rightarrow \quad A + B + C = \pi$

$\Rightarrow \quad \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

$\Rightarrow \quad 2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C = 4 \sin A \sin B \sin C$

$\Rightarrow \quad \sin A\sqrt{1-\sin^2 A} + \sin B\sqrt{1-\sin^2 B} + \sin C\sqrt{1-\sin^2 C} = 2 \sin A \sin B \sin C$

$\Rightarrow \quad x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

(b) We have, $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} z$

$\Rightarrow \quad \cos(\sin^{-1} x + \sin^{-1} y) = \cos(\pi - \sin^{-1} z)$

$\Rightarrow \quad \cos(\sin^{-1} x)\cos(\sin^{-1} y) - \sin(\sin^{-1} x)\sin(\sin^{-1} y) = -\cos(\sin^{-1} z)$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} - xy = -\sqrt{1-z^2} \qquad \left[\text{as } \cos(\sin^{-1} x) = \cos(\cos^{-1}\sqrt{1-x^2} = \sqrt{1-x^2}\right]$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)} = xy - \sqrt{1-z^2}$

$\Rightarrow \quad 1 - x^2 - y^2 + x^2y^2 = x^2y^2 + 1 - z^2 - 2xy\sqrt{1-z^2}$ [On squaring both sides]

$\Rightarrow \quad x^2 + y^2 - z^2 = 2xy\sqrt{1-z^2}$

$\Rightarrow \quad (x^2 + y^2 - z^2)^2 = 4x^2 y^2 (1 - z^2)$

$\Rightarrow \quad x^4 + y^4 + z^4 + 4x^2 y^2 z^2 = 2(x^2 y^2 + y^2 z^2 + z^2 x^2)$

Illustration - 33

The sum of roots of the equation $\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5}=\sin^{-1}x$ *is :*

(A) 1 **(B)** – 1 **(C)** 0 **(D)** *None of these*

SOLUTION : (C)

We have, $\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5}=\sin^{-1}x$

$$\Rightarrow \sin^{-1}\left\{\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right\}=\sin^{-1}x$$

$$\Rightarrow \frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}=x$$

$$\Rightarrow 3x\sqrt{25-16x^2}+4x\sqrt{25-9x^2}=25x$$

$$\Rightarrow x=0 \quad \text{or,} \quad 3\sqrt{25-16x^2}+4\sqrt{25-9x^2}=25$$

Now, $3\sqrt{25-16x^2}+4\sqrt{25-9x^2}=25$

$$\Rightarrow 4\sqrt{25-9x^2}=25-3\sqrt{25-16x^2}$$

$$\Rightarrow 16\,(25-9x^2)=625+9\,(25-16x^2)-150\sqrt{25-16x^2}$$

$$\Rightarrow 150\sqrt{25-16x^2}=450 \Rightarrow 25-16x^2=9 \Rightarrow x=\pm 1$$

Hence, $x = 0, 1, -1$ are roots of the given equation.

Illustration - 34

The value of x *which satisfies* $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ *is :*

(A) $x=1$ **(B)** $x=\frac{1}{2}$ **(C)** $x=0$ **(D)** *None of these*

SOLUTION : (C)

We have, $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

$$\Rightarrow \sin^{-1}(1-x)=\frac{\pi}{2}+2\sin^{-1}x$$

$\Rightarrow \quad 1 - x = \sin(\pi/2 + 2\sin^{-1} x)$

$\Rightarrow \quad 1 - x = \cos(2\sin,^{-1} x)$

$\Rightarrow \quad 1 - x = \cos\{\cos^{-1}(1 - 2x^2)\}$ $\quad$ [as $2\sin^{-1} x = \cos^{-1}(1 - 2x^2)$]

$\Rightarrow \quad 1 - x = (1 - 2x^2) \quad \Rightarrow \quad x = 2x^2$

$\Rightarrow \quad x(2x - 1) = 0 \quad \Rightarrow \quad x = 0, \frac{1}{2}$

For, $x = \frac{1}{2}$, we have LHS $= \sin^{-1}(1 - x) - 2\sin^{-1} x$

$$= \sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{2} = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6} \neq \text{R.H.S.}$$

So, $x = 1/2$ is not a root of the given equation.

Clearly, $x = 0$ satisfies the equation. Hence, $x = 0$ is a root of the given equation.

THINGS TO REMEMBER

1. Domain and Range (Principle value branch) of inverse Trignometric Functions

S. No.	Functions	Deomain	Range
1.	$y = \sin^{-1} x$	$x \in [-1, 1]$	$y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$x \in [-1, 1]$	$y \in [0, \pi]$
3.	$y = \tan^{-1} x$	$x \in R$	$y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \text{cosec}^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
5.	$y = \sec^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
6.	$y = \cot^{-1} x$	$x \in R$	$y \in (0, \pi)$

2. Property of inverse Tringonometric Function

(A)

$\sin^{-1}(-x) = -\sin^{-1}x$ for all $x \in [-1, 1]$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$ for all $x \in [-1, 1]$

$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$ for all $x \in (-\infty, -1] \cup [1, \infty)$

$\tan^{-1}(-x) = -\tan^{-1}x$ for all $x \in R$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$ for all $x \in R$

(B)

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & \text{for } x > 0 \\ -\pi + \cot^{-1}x, & \text{for } x < 0 \end{cases}$$

(C)

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for all $x \in [-1, 1]$

$\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$ for all $x \in (-\infty, -1] \cup [1, \infty)$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for all $x \in R$

(D)

(i) $$\sin^{-1}x = \underset{(0 \le x \le 1)}{\cos^{-1}\sqrt{1-x^2}} = \underset{(|x|<1)}{\tan^{-1}\frac{x}{\sqrt{1-x^2}}} = \underset{(0<x\le 1)}{\cot^{-1}\frac{\sqrt{1-x^2}}{x}} = \underset{(0 \le x < 1)}{\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)} = \underset{(|x| \le 1)-\{0\}}{\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)}$$

(ii) $$\cos^{-1}x = \underset{(0, \le x \le 1)}{\sin^{-1}\sqrt{1-x^2}} = \underset{(0 \le x < 1)}{\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)} = \underset{(|x|<1)}{\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)} = \underset{(|x| \le 1)}{\sec^{-1}\frac{1}{x}} = \underset{(0 \le x < 1)}{\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)}$$

(iii) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

$(x \in R)$ $\quad (0 \le x < \infty)$ $\quad (x > 0)$ $\quad (x \ge 0)$ $\quad (x \in R - \{0\})$

(E) (i) $\sin(\sin^{-1} x) = x \ \forall \ x \in [-1, 1]$ (ii) $\cos(\cos^{-1} x) = x \ \forall \ x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x \ \forall \ x \in R$ (iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x \ \forall \ x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec(\sec^{-1} x) = x \ \forall \ x \in (-\infty, -1] \cup [1, \infty)$ (vi) $\cot(\cot^{-1} x) = x \ \forall \ x \in R$

(F) (i)
$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x & ; \quad \dfrac{-3\pi}{2} \le x \le \dfrac{-\pi}{2} \\ x & ; \quad \dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x & ; \quad \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \end{cases}$$

(ii)
$$\cos^{-1}(\cos x) = \begin{cases} x & ; \quad 0 \le x \le \pi \\ 2\pi - x & ; \quad \pi \le x \le 2\pi \end{cases}$$

(iii)
$$\tan^{-1}(\tan x) = \begin{cases} x & ; \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2} \\ x - \pi & ; \quad \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$$

(iv)
$$\operatorname{cosec}^{-1}(\operatorname{cosec} x) = \begin{cases} x & ; \quad \dfrac{-\pi}{2} \le x < 0 \quad \text{or} \quad 0 < x \le \dfrac{\pi}{2} \\ \pi - x & ; \quad -\dfrac{\pi}{2} \le \pi - x < 0 \quad \text{or} \quad 0 < \pi - x \le \dfrac{\pi}{2} \end{cases}$$

(v)
$$\sec^{-1}(\sec x) = \begin{cases} x & ; \quad x \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right] \\ 2\pi - x & ; \quad 2\pi - x \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right] \end{cases}$$

(vi) $$\cot^{-1}(\cot x) = \begin{cases} x + \pi & ; \quad -\pi < x < 0 \\ x & ; \quad 0 < x < \pi \\ x - \pi & ; \quad \pi < x < 2\pi \end{cases}$$

3. Important Results

(A) (i) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if} \quad xy = 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if} \quad x > 0,\ y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & \text{if} \quad x < 0,\ y < 0 \text{ and } xy > 1 \end{cases}$$

(ii) $$\tan^{-1} x - \tan^{1} y = \begin{cases} \tan^{-1}\dfrac{x-y}{1-xy} & \text{if} \quad xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if} \quad x > 0,\ y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & \text{if} \quad x < 0,\ y > 0 \text{ and } xy < -1 \end{cases}$$

(B) (i) $$\sin^{-1} x + \sin^{1} y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\} & \text{if } -1 \le x,\ y \le 1 \text{ and } x^2 + y^2 \le 1 \\ & \text{or} \\ & \text{if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\} & \text{if } 0 < x,\ y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\} & \text{if } -1 \le x,\ y \le 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

(ii) $$\sin^{-1}x-\sin^{1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & \text{if } -1\le x, y\le 1 \text{ and } x^2+y^2\le 1 \\ & \text{or} \\ & \text{if } xy>0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & \text{if } 0<x\le 1, -1<y\le 0 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & \text{if } -1\le x\le 0, 0<y\le 1 \text{ and } x^2+y^2>1\end{cases}$$

(C) (i) $$\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & \text{if } -1\le x, y\le 1 \text{ and } x+y\ge 0 \\ 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & \text{if } -1\le x, y\le 1 \text{ and } x+y\le 0\end{cases}$$

(ii) $$\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & \text{if } -1\le x, y\le 1 \text{ and } x\le y \\ -\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & \text{if } -1\le y\le 0, 0<x\le 1 \text{ and } x\ge y\end{cases}$$

(D) (i) $$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)\begin{cases}\pi-2\tan^{-1}x & ; \quad x>1 \\ 2\tan^{-1}x & ; \quad -1\le x\le 1 \\ -\pi-2\tan^{-1}x & ; \quad x<-1\end{cases}$$

(ii) $$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\begin{cases}2\tan^{-1}x & ; \quad x\ge 0 \\ -2\tan^{-1}x & ; \quad x<0\end{cases}$$

(iii) $$y=\tan^{-1}\left(\frac{2x}{1+x^2}\right)\begin{cases}\pi+2\tan^{-1}x & ; \quad x<-1 \\ 2\tan^{-1}x & ; \quad -1<x<1 \\ -\pi+2\tan^{-1}x & ; \quad x>1\end{cases}$$

(iv) $$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} \pi + 3\tan^{-1} x & ; \quad x < -\dfrac{1}{\sqrt{3}} \\ 3\tan^{-1} x & ; \quad -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1} x & ; \quad x > \dfrac{1}{\sqrt{3}} \end{cases}$$

(v) $$y = \sin^{-1}(3x - 4x^3) = \begin{cases} \pi - 3\sin^{-1} x & ; \quad \dfrac{1}{2} \le x \le 1 \\ 3\sin^{-1} x & ; \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ -\pi - 3\sin^{-1} x & ; \quad -1 \le x \le -\dfrac{1}{2} \end{cases}$$

(vi) $$y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 2\pi - 3\cos^{-1} x & ; \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1} x & ; \quad \dfrac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1} x & ; \quad -1 \le x \le -\dfrac{1}{2} \end{cases}$$