CHAPTER 06

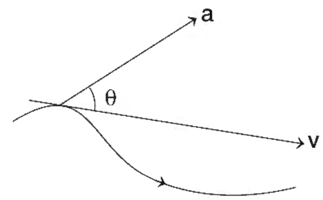
Circular Motion

Terms Related to Circular Motion

- A particle in circular motion may have two types of velocities :
  (i) linear velocity v and (ii) angular velocity ω
  These two velocities are related by the equation
  $$v = R\omega \qquad \text{(where, } R = \text{radius of circular path)}$$
- Acceleration of particle in circular motion may have two components:
  (i) tangential component (a_t) and
  (ii) normal or radial component (a_n)
  As the name suggests, tangential component is tangential to the circular path, given by
  $$a_t = \text{rate of change of speed} = \frac{dv}{dt} = \frac{d\,|\mathbf{v}|}{dt} = R\alpha$$
  where, α = angular acceleration = rate of change of angular velocity $= \frac{d\omega}{dt}$
  The normal or radial component also known as centripetal acceleration is towards centre and is given by
  $$a_n = R\omega^2 = \frac{v^2}{R}$$
- Net acceleration of particle is resultant of two perpendicular components a_n and a_t. Hence,
  $$a = \sqrt{a_n^2 + a_t^2}$$
- Tangential component a_t is responsible for change of speed of the particle. This can be positive, negative or zero, depending upon the situation whether the speed of particle is increasing, decreasing or constant.
  Normal component is responsible for change in direction of velocity. This component can never be equal to zero in circular motion.

- In general, in any curvilinear motion, direction of instantaneous velocity is tangential to the path, while acceleration may have any direction.

  If we resolve the acceleration in two normal directions, one parallel to velocity and another perpendicular to velocity, the first component is a_t while the other is a_n.



Thus, a_t = component of **a** parallel to **v**

$$= a\cos\theta = \frac{\mathbf{a}\cdot\mathbf{v}}{v}$$

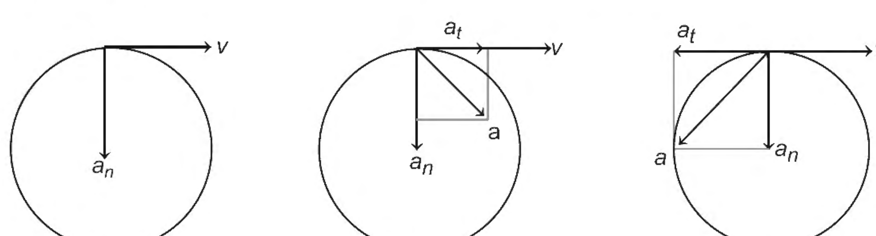
$$= \frac{dv}{dt} = \frac{d|\mathbf{v}|}{dt} = \text{rate of change of speed.}$$

and a_n = component of **a** perpendicular to **v**

$$= \sqrt{a^2 - a_t^2} = v^2/R$$

Here, v is the speed of particle at that instant and R is called the radius of curvature to the curvilinear path at that point.

- In $a_t = a\cos\theta$, if θ is acute, a_t will be positive and speed will increase. If θ is obtuse, a_t will be negative and speed will decrease. If θ is 90°, a_t is zero and speed will remain constant.
- Now, depending upon the value of a_t, circular motion may be of three types :
  (i) Uniform circular motion in which speed remains constant or $a_t = 0$.
  (ii) Circular motion of increasing speed in which a_t is positive.
  (iii) Circular motion of decreasing speed in which a_t is negative.

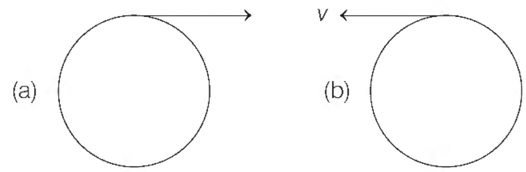


$a_t = 0$
$a = a_n$ | $a = \sqrt{a_n^2 + a_t^2}$ | $a = \sqrt{a_n^2 + a_t^2}$

v = constant | v is increasing | v is decreasing

- Circular motion is a two dimensional motion (motion in a plane).

  Linear velocity vector and linear acceleration vector lie in the plane of circle. Angular velocity vector and angular acceleration vector are perpendicular to the plane of the circle given by right hand screw law.

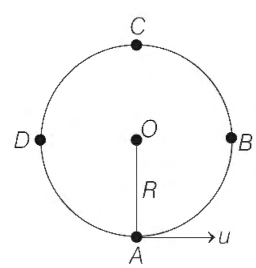


In Fig. (a), for example, if speed is increasing, then angular velocity vector and angular acceleration vector both are perpendicular to paper inwards.

In Fig. (b), if speed is decreasing, then angular velocity vector is perpendicular to paper inwards while angular acceleration vector is perpendicular to paper outwards.

Vertical Circular Motion

- Suppose a bob of mass m is suspended from a light string of length R as shown. If velocity at bottommost point of bob is u, then depending upon the value of u following three cases are possible :



(i) If $u \geq \sqrt{5gR}$, bob will complete the circle.

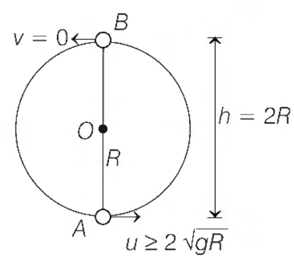
(ii) If $\sqrt{2gR} < u < \sqrt{5gR}$, string will slack between B and C. At the time of slacking, tension in the string will become zero ($T = 0$) and velocity is non-zero ($v \neq 0$). After slacking motion of bob is projectile.

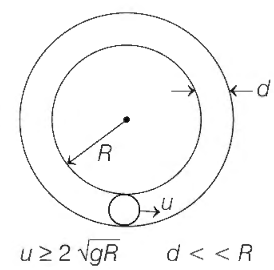
(iii) If $u \leq \sqrt{2gR}$, bob will oscillate between BAD. In this case, $v = 0$ but $T \neq 0$.

The above three conditions have been derived for a particle moving in a vertical circle attached to a string.

- The same conditions apply, if a particle moves inside a smooth spherical shell of radius R. The only difference is that the tension is replaced by the normal reaction N.
- If $u = \sqrt{5gR}$, bob will just complete the circle. In this case, velocity at topmost point is $v = \sqrt{gR}$. Tension in this critical case is zero at topmost point and 6 mg at bottommost point.
- If $u = \sqrt{2gR}$, bob will just reach to the point B. At that point, velocity and tension both will become zero.
- At height h from bottom, velocity of bob will be $v = \sqrt{u^2 - 2gh}$.

- Velocity of bob becomes zero at height $h_1 = \frac{u^2}{2g}$ (in case of oscillation) and tension in string becomes zero at height $h_2 = \frac{u^2 + gR}{3g}$.
- If a particle of mass m is connected to a light rod and whirled in a vertical circle of radius R, then to complete the circle, the minimum velocity of the particle at the bottommost point is not $\sqrt{5\,gR}$. This is because in this case, velocity of the particle at the topmost point can be zero also.



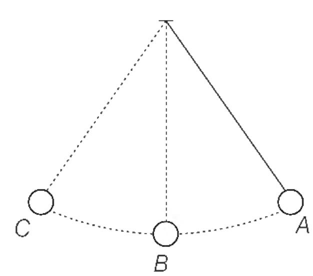


The minimum value of u in this case is $2\sqrt{gR}$ or $\sqrt{4gR}$.

Same is the case when a particle is compelled to move inside a smooth vertical tube.

- Oscillation of a pendulum is the part of a vertical circular motion. At points A and C, velocity is zero, therefore centripetal force or acceleration (also called radial acceleration) will be zero. Only tangential force or acceleration is present. From A to B or C to B, speed of the bob increases.

  Therefore, tangential force or acceleration is parallel to velocity. From B to A or B to C, speed of the bob decreases. Hence, tangential force or acceleration is antiparallel to velocity.



Dynamics of Circular Motion

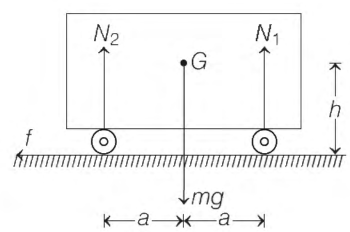
- In circular motion, forces are normally resolved in tangential and radial directions. In tangential direction, net force should be ma_t and in radial direction, net force should be ma_r or $\frac{mv^2}{R}$ or $mR\omega^2$. In uniform circular motion, $a_t = 0$. Hence, in tangential direction, net force should be zero.
- If plane of uniform circular motion is horizontal, then one of the tangent is vertical also. So, we resolve the forces in horizontal and vertical directions.

In vertical tangential direction, net force is zero and in horizontal radial direction towards centre, net force should be $\frac{mv^2}{R}$ or $mR\omega^2$.

- In vertical circular motion, speed of particle continuously keeps on changing. As the particle moves upwards, speed or kinetic energy decreases. Since, speed is continuously changing, so a_t is never zero.
- During the motion only two forces tension (T) and weight (mg) are acting. Tension is already in radial direction towards centre. So, we will have to resolve only mg.

Condition of Toppling of a Vehicle on Circular Track

- While moving in a circular track, normal reaction on the outer wheels (N_1) is more than the normal reaction on inner wheels (N_2).



or $$N_1 > N_2$$

This can be proved as below.

$$N_1 + N_2 = mg \quad \text{...(i)}$$

and $$f = \frac{mv^2}{r} \quad \text{...(ii)}$$

For rotational equilibrium of car, net torque about centre of gravity should be zero.

or $$N_1(a) = N_2(a) + f(h) \quad \text{...(iii)}$$

$\Rightarrow$ $$N_2 = N_1 - \left(\frac{h}{a}\right) f = N_1 - \left(\frac{mv^2}{r}\right)\left(\frac{h}{a}\right) \quad \text{...(iv)}$$

or $$N_2 < N_1$$

From Eq. (iv), we see that N_2 decreases as v increases.

In critical case, $N_2 = 0$

and $$N_1 = mg \quad \text{[From Eq. (i)]}$$

$\therefore$ $$N_1(a) = f(h) \quad \text{[From Eq. (iii)]}$$

or $$(mg)(a) = \left(\frac{mv^2}{r}\right)(h)$$

or $$v = \frac{gra}{h}$$

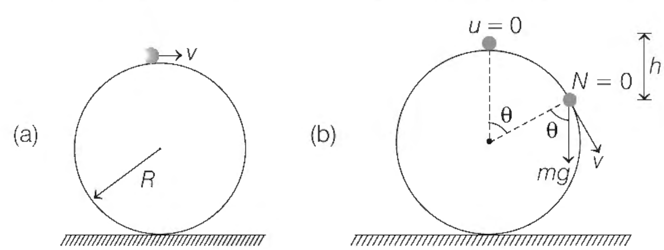
Now, if $v > \sqrt{\frac{gra}{h}}$, $N_2 < 0$ and the car topples outwards.

Therefore, for a safe turn without toppling, $v \le \sqrt{\frac{gra}{h}}$.

- From the above discussion, we can conclude that while taking a turn on a level road there are two critical speeds, one is the maximum speed for sliding $(= \sqrt{\mu rg})$ and another is maximum speed for toppling $\left(= \sqrt{\frac{gra}{h}}\right)$. One should keep ones car's speed less than both for neither to slide nor to overturn.

Motion of a Ball over a Smooth Solid Sphere

Suppose a small ball of mass m is given a velocity v over the top of a smooth sphere of radius R. The equation of motion for the ball at the topmost point will be



$$mg - N = \frac{mv^2}{R}$$

or
$$N = mg - \frac{mv^2}{R}$$

From this equation, we see that the value of N decreases as v increases. Minimum value of N can be zero.

Hence,
$$0 = mg - \frac{mv_{max}^2}{R}$$

or
$$v_{max} = \sqrt{Rg}$$

So, ball will lose contact with the sphere right from the beginning if velocity of the ball at topmost point $v > \sqrt{Rg}$.

If $v < \sqrt{Rg}$, it will lose contact after moving certain distance over the sphere.

Now, let us find the angle θ, where the ball loses contact with the sphere if velocity at topmost point is just zero. [see Fig. (b)].

$$h = R(1 - \cos\theta) \quad \text{...(i)}$$

$$v^2 = 2gh \quad \text{...(ii)}$$

$$mg\cos\theta = \frac{mv^2}{R} \quad (\text{as } N = 0) \quad \text{...(iii)}$$

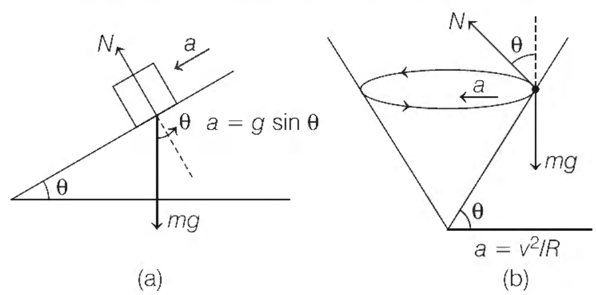
Solving Eqs. (i), (ii) and (iii), we get

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ$$

Thus, the ball can move on the sphere maximum upto $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

Other Cases of Circular Motion

- In the following two figures, surface is smooth. So, only two forces N and mg are acting. But directions of acceleration are different.



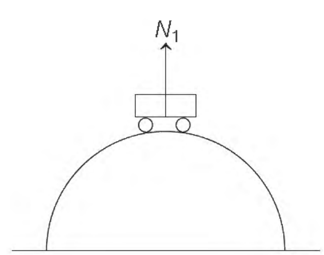
Net force perpendicular to acceleration should be zero. So, in the first figure,

$$N = mg\cos\theta$$

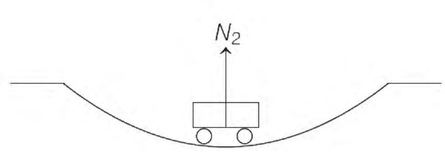
and in the second figure, $N\cos\theta = mg$

- (i) When a vehicle is moving over a convex bridge, then at the maximum height, reaction (N_1) is

$$N_1 = mg - \frac{mv^2}{r}$$

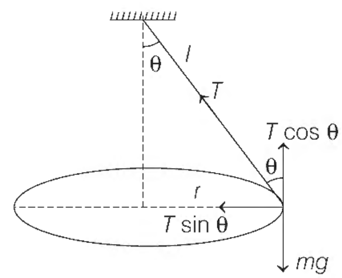


(ii) When a vehicle is moving over a concave bridge, then at the lowest point, reaction (N_2) is



$$N_2 = mg + \frac{mv^2}{r}$$

- **Conical Pendulum** If a simple pendulum is fixed at one end and the bob is rotating in a horizontal circle, then it is called a conical pendulum.



From the figure,

$$T \sin\theta = mr\omega^2$$

$$T \cos\theta = mg$$

$$r = l \sin\theta$$

and

$$\text{time period} = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$