# JEE (MAIN) JANUARY 2023 DATE-31/01/2023 (SHIFT-1)

## MATHEMATICS

**1.** Find the remainder when $5^{99}$ is divided by 11.

**Ans.** (9)

**Sol.** $5^{99} = 5.(22 + 3)^{49} = 5\ (22I + 3^{49})$

Now $3^{49} = 3^4 \,.\, (3^5)^9 = 81 \,.\, (242 + 1)^9 = 81.(11I_2 + 1)$

$\therefore\ 5^{99} = 11I_3 + 5 \times 81 = 11I_3 + 5 \times (77 + 4)$

$\Rightarrow 5^{99} = 11I_4 + 9$

$\therefore$ Remainder is 9

**2.** If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$, then sum of all diagonal elements of $(A + I)^{11}$.

**Ans.** **(4097)**

**Sol.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

$\Rightarrow A^2 = A$

so, $(A + I)^{11} = {}^{11}C_0 I + {}^{11}C_1 A + {}^{11}C_2\, A^2 + \ldots.. + {}^{11}C_{11} A^{11}$

$= I + A[{}^{11}C_1 + \ldots. + {}^{11}C_{11}] = I + A(2^{11} - 1)$

$\therefore\ tr(A + I)^{11} = 3 + (2^{11} - 1) \times 2 = 4097$

**3.** Consider two statements $S_1 : (p \to q) \vee (\sim q \vee \sim p)$ and $S_2 : (\sim p \vee q) \wedge (\sim q \vee p)$, then

(1) $S_1$ and $S_2$ are tautology

(2) $S_1$ is not tautology and $S_2$ is contradiction

(3) $S_1$ is not tautology and $S_2$ is not contradiction

(4) $S_1$ is tautology and $S_2$ is not contradiction

**Ans.** **(4)**

**Sol.** $S_1 : (p \to q) \vee (\sim q \vee p) \equiv (\sim p \vee q) \vee (\sim q \vee p) \equiv t$

$S_2 : (\sim p \vee q) \wedge (\sim q \vee p) \equiv (p \to q) \wedge (q \to p)$

$\Rightarrow S_2$ is neither tautology nor contradiction

**4.** Range of $f(x) = \dfrac{[x]}{1 + x^2}$ ; $x \in [2, 6)$ where $[\cdot]$ denotes greatest integer function

(1) $\left(\dfrac{2}{10}, \dfrac{2}{5}\right]$ (2) $\left(\dfrac{4}{26}, \dfrac{4}{17}\right]$ (3) $\left(\dfrac{5}{37}, \dfrac{5}{26}\right]$ (4) $\left(\dfrac{5}{37}, \dfrac{2}{5}\right]$

**Ans.** **(4)**

**Sol.** $x \in [2, 3)$, $f = \frac{2}{1+x^2} = \left(\frac{2}{10}, \frac{2}{5}\right]$

$x \in [3, 4)$ , $f = \frac{3}{1+x^2} = \left(\frac{3}{17}, \frac{3}{10}\right]$

$x \in [4, 5)$ , $f = \frac{4}{1+x^2} = \left(\frac{4}{26}, \frac{4}{17}\right]$

$x \in [5, 6)$ , $f = \frac{5}{1+x^2} = \left(\frac{5}{37}, \frac{5}{26}\right]$

So range is $\left(\frac{5}{37}, \frac{5}{26}\right] \cup \left(\frac{4}{26}, \frac{4}{17}\right] \cup \left(\frac{3}{17}, \frac{3}{10}\right] \cup \left(\frac{2}{10}, \frac{2}{5}\right]$

Range is $\left(\frac{5}{37}, \frac{2}{5}\right]$

**5.** If four consecutive positive terms are in GP whose sum is 126 & product is 1296 then find sum of all possible common ratio of these GP.

**Ans.** **(7)**

**Sol.** Let terms are $\frac{a}{r^3}$, $\frac{a}{r}$, ar, $ar^3$

product $\frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 = 1296.$

$a^4 = 1296$

$a = 6$ & $a = -6$

Sum $\frac{a}{r^3} + \frac{a}{r} + ar + ar^3 = 126$

Case I : $\frac{1}{r^3} + \frac{1}{r} + r + r^3 = 21$

$\frac{1+r^2+r^4+r^6}{r^3} = 21$

$1 + r^2 + r^4 + r^6 = 21r^3$

$\left(r^3 + \frac{1}{r^3}\right) + \left(r + \frac{1}{r}\right) - 21 = 0$

$\left(r + \frac{1}{r}\right)^3 - 3\left(r + \frac{1}{r}\right) + \left(r + \frac{1}{r}\right) - 21 = 0$

$\left(r + \frac{1}{r}\right)^3 - 2\left(r + \frac{1}{r}\right) - 21 = 0$

$t^3 - 2t - 21 = 0$

$(t - 3)(t^2 + 3t + 7) = 0$

$t = 3$

$r + \frac{1}{r} = 3$

$r^2 - 3r + 1 = 0$

$r = \frac{3 \pm \sqrt{5}}{2}$

Case II : $r = -6$

$t^3 - 2t + 21 = 0$

$(t + 3)(t^2 - 3t + 7) = 0$

$t = -3$

$r + \frac{1}{r} = -3$

$r^2 + 3r + 1 = 0$

$r = \frac{-3 \pm \sqrt{5}}{2}$

$\therefore$ there are two possible values of common ratio (i.e., $r^2$) and there sum is 7

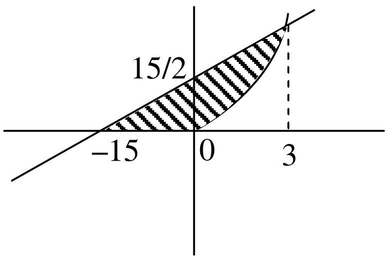
**6.** $f(x) = \frac{x + |x|}{2}$ and $g(x) = \begin{cases} x & ; \ x < 0 \\ x^2 & ; \ x \geq 0 \end{cases}$

Find area bounded by fog(x) with line $2y - x = 15$.

(1) 72 (2) $\frac{123}{4}$ (3) $\frac{172}{5}$ (4) $\frac{72}{5}$

**Ans.** **(1)**

**Sol.**




$f(x) = \begin{cases} x & ; \ x \geq 0 \\ 0 & ; \ x < 0 \end{cases}$

$g(x) = \begin{cases} x & ; \ x < 0 \\ x^2 & ; \ x \geq 0 \end{cases}$

$f(g(x)) = \begin{cases} g(x) & ; \ g(x) \geq 0 \\ 0 & ; \ g(x) < 0 \end{cases}$

$\Rightarrow fog = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$2x^2 - x - 15$

$2x^2 - 6x + 5x - 15 = 0$

$(2x + 5)(x - 3) = 0$

$$\text{area} = \frac{1}{2} \times 15 \times \frac{15}{2} + \int_0^3 \left(\frac{x+15}{2} - x^2\right)dx$$

$$= \frac{225}{4} + \frac{1}{2}.\left[\frac{x^2}{2} + 15x - \frac{2x^3}{3}\right]_0^3$$

$$= \frac{225}{4} + \frac{1}{2}\left[\frac{9}{2} + 45 - 18\right] = \frac{225}{4} + \frac{1}{2}\left[\frac{63}{2}\right] = \frac{288}{4}$$

**7.** If $\sin^{-1}\alpha + \cos^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\frac{77}{36} = 0$, then $\sin^{-1}(\sin\alpha) + \cos^{-1}(\cos\alpha)$ is

(1) $\pi - \frac{16}{17}$ (2) $\frac{8}{17} - \pi$ (3) $\frac{16}{17}$ (4) $2\pi - \frac{8}{17}$

**Ans.** **(3)**

**Sol.** $\sin^{-1}\alpha = \tan^{-1}\left(\frac{77}{36}\right) - \tan^{-1}\left(\frac{3}{4}\right)$

$\Rightarrow \sin^{-1}\alpha = \tan^{-1}\frac{8}{15}$

$\Rightarrow \sin^{-1}\alpha = \sin^{-1}\frac{8}{17}$

$\Rightarrow \alpha = \frac{8}{17}$

$\therefore \sin^{-1}(\sin\alpha) + \cos^{-1}(\cos\alpha) = 2\alpha = \frac{16}{17}$

**8.** If $|\vec{a}| = \sqrt{14}$, $|\vec{b}| = \sqrt{6}$, $|\vec{a} \times \vec{b}| = \sqrt{46}$ find $(\vec{a}.\vec{b})^2$

(1) 38 (2) 48 (3) $\sqrt{15}$ (4) $2\sqrt{6}$

**Ans.** **(1)**

**Sol.** $(\vec{a} \times \vec{b})^2 + (\vec{a}.\vec{b})^2 = a^2b^2$

$46 + (\vec{a}.\vec{b})^2 = 14 \times 6$

$(\vec{a}.\vec{b})^2 = 84 - 46 = 38$

**9.** If $\sqrt{x^2 - 9} + \sqrt{x^2 - 4x + 3} = \sqrt{x^2 - \frac{25}{8}x + \frac{3}{8}}$ the number of solution are -

(1) 0 (2) 1 (3) 2 (4) 3

**Ans.** **(2)**

**Sol.** $\sqrt{(x+3)(x-3)}+\sqrt{(x-1)(x-3)}=\dfrac{\sqrt{(8x-1)(x-3)}}{2\sqrt{2}}$

$\Rightarrow \quad x = 3$

and $\sqrt{x+3}$ + $\sqrt{x-1}$ = $\sqrt{x-\dfrac{1}{8}}$ gives no sol :

**10.** $\displaystyle\int_{\pi/3}^{\pi/2} \frac{2}{\sin x(1+\cos x)}$ dx is equal to

(1) $\dfrac{1}{2}(\ell n\ 3) + \dfrac{1}{3}$ (2) $\ell n\ 3) + \dfrac{1}{3}$ (3) $\dfrac{1}{2}(\ell n\ 3)$ (4) $\dfrac{1}{3}$ $(\ell n\ 3) + \dfrac{1}{2}$

**Ans.** **(1)**

**Sol.** $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{2\sin\frac{x}{2}\cos^3\frac{x}{2}} = \int_{\pi/3}^{\pi/2} \frac{1}{2}\left(\frac{\sec^2\frac{x}{2}\times\sec^2\frac{x}{2}}{\tan\frac{x}{2}}\right) dx = \int_{\pi/3}^{\pi/2} \frac{1}{2}\left(\frac{1+\tan^2\frac{x}{2}}{\tan\frac{x}{2}}\right)\sec^2\frac{x}{2}\,dx$

$\tan\dfrac{x}{2} = t$

$= \displaystyle\int_{1/\sqrt{3}}^{1} \frac{1+t^2}{t}\,dt = \left|\ell n\,|t|+\frac{t^2}{2}\right|_{1/\sqrt{3}}^{1} = \ell n\left(\sqrt{3}\right) + \frac{1}{3} = \frac{1}{2}\ (\ell n\ 3) + \frac{1}{3}$

**11.** If $f(x) + \displaystyle\int_3^x \frac{f(t)\,.\,dt}{t} = \sqrt{x+1}$ ; then find 12f(8)

**Ans.** **(17)**

**Sol.** Differentiate both side w.r.t. 'x'

$f'(x) + \dfrac{f(x)}{x} = \dfrac{1}{2\sqrt{x+1}}$

$\dfrac{dy}{dx} + \dfrac{1}{x}\ y = \dfrac{1}{2\sqrt{x+1}}$

$\text{If} = e^{\int\frac{1}{x}dx} = x$

So solution is $y.x = \displaystyle\int \frac{x}{2\sqrt{x+1}}\,dx$

$xy = \left|\dfrac{(t^2-1)}{2t^2}2t\,dt = \dfrac{t^3}{3} - t + 6\right.$

$yx = \dfrac{1}{3}(x+1)^{3/2} - (x+1)^{1/2} + c$ {Given f(3) = 2}

$x = 3, y = 2 \Rightarrow 6 = \dfrac{8}{3} - 2 + c \Rightarrow c = 6 - \dfrac{2}{3} = \dfrac{16}{3}$

so $12f(6) = 12\left(\dfrac{1}{8}\left(\dfrac{1}{3}\times 27 - 3 + \dfrac{16}{3}\right)\right) = \dfrac{12}{8}\left[6+\dfrac{16}{3}\right] = \dfrac{12}{8}\ .\ \dfrac{34}{3} = 17$

**12.** y = f(x) is parabola with focus $\left(-\frac{1}{2}, 0\right)$ & directrix $y = -\frac{1}{2}$. Given that

$\tan^{-1}\sqrt{f(x)} + \sin^{-1}\sqrt{f(x)+1} = \frac{\pi}{2}$, then find number of solution for x.

**Ans.** **(2)**

**Sol.** $\left(x+\frac{1}{2}\right)^2 + y^2 = \left|\frac{y+\frac{1}{2}}{1}\right|^2 \quad \Rightarrow \quad y = x^2 + x$

$\therefore \tan^{-1}\sqrt{x^2+x} + \sin^{-1}\left(\sqrt{x^2+x+1}\right) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right) + \sin^{-1}\left(\sqrt{x^2+x+1}\right) = \frac{\pi}{2}$

$\Rightarrow \sqrt{x^2+x+1} = \frac{1}{\sqrt{x^2+x+1}}$

$x^2 + x + 1 = 1$

$x(x + 1) = 0$

$x = 0$ or $x = -1 \Rightarrow 2$ solutions

**13.** $x + y + z = 7$

$\alpha x + \beta y + 7z = 8$

$3x + 2y + z = 2023$

which of the following is incorrect ?

(1) If $\alpha = \beta = 7$ then equation has no solution

(2) If $\alpha - 2\beta + 7 = 0$ and $(\alpha, \beta) \neq (7, 7)$ then equation has unique solution

(3) If $(\alpha, \beta) = (-1, 4)$, then the equation has unique solution

(4) There exists $(\alpha, \beta)$ lying on line $x + 2y = 21$ for which system has no solution

**Ans.** **(2)**

**Sol.** $\Delta = \alpha - 2\beta + 7$

(A) $\alpha = \beta = 7$

equation (1) and (2) are $x + y + z = 7$

and $x + y + z = \frac{8}{7}$

$\Rightarrow$ no solution

(B) $(\alpha, \beta) \neq (7, 7) \Rightarrow D \neq 0 \Rightarrow$ unique solution

(C) $(\alpha, \beta) = (-1, 4) \Rightarrow D \neq 0 \Rightarrow$ unique solution

**14.** A bag contains 6 balls. If two balls chosen from bag & found to be black, find the probability that bag contains atleast 5 black balls.

(1) $\frac{5}{7}$ (2) $\frac{3}{7}$ (3) $\frac{6}{7}$ (4) $\frac{4}{7}$

**Ans.** **(1)**

**Sol.** 4 white & 2 black, 3 white & 3 black, 2 white & 4 black, 1 white & 5 black, 0 white & 6 black,

probability = $\frac{{}^5C_2 + {}^6C_2}{{}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 + {}^6C_2} = \frac{10+15}{1+3+6+10+15} = \frac{25}{35} = \frac{5}{7}$

**15.** Find the number of 5 digits numbers which are less than 28000 and are divisible by 3 and 11.

(1) 6070 (2) 7091 (3) 8070 (4) 7080

**Ans.** (B)

**Sol.** 5-digit number div. by 3 : {10002, 10005, ...... , 27999}

$\Rightarrow$ 6000 number

5- digit number div. by 11 : {10010, ....... , 27995}

$\Rightarrow$ 1636

5-digit numbers div. by 33 : {10032, ..... , 27984}

$\Rightarrow$ 545 numbers

$\therefore$ total number = 6000 + 1636 – 576 = 7091

**16.** For the ellipse $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$, minimum distance of any normal from centre is 1 then electricity of ellipse is

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{\sqrt{3}}{4}$ (3) $\frac{2}{\sqrt{3}}$ (4) $\frac{2}{3}$

**Ans.** **(1)**

**Sol.** Minimum distance any normal from centre is a – b

so 2 – b = 1 $\Rightarrow$ b = 1

so ellipse is $\frac{x^2}{4} + \frac{y^2}{1} = 1$

ecc. = $\sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

**17.** Find $\alpha$ for the given data :

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_i$ | 2 | 4 | 5 | 6 | 7 | 9 |
| $f_i$ | 4 | 16 | $\alpha$ | 15 | 6 | 4 |

If variance is 3.01, find $\alpha$. **(Data may vary)**

(1) 2 (2) 3 (3) 5 (4) 4

**Ans.** **(3)**

**Sol.** As we know :

variance $= \frac{\sum f_i x_i^2}{\sum f_i} - (\overline{x})^2$

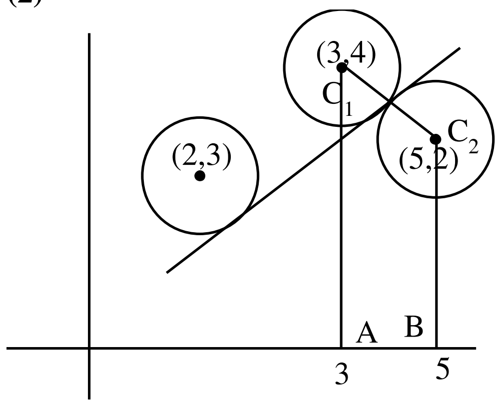
putting the given data in the formula we get $\alpha = 5$

**18.** A circle having centre (2, 3) & radius $\sqrt{2}$ rolls 4 unit upwards on tangent at (3, 2). Let it is $C_1$ circle. Now $C_2$ is a circle which is image of $C_1$ about tangent. Area of trapezium made by centre of $C_1$ & $C_2$ & their foot on x-axis -

(1) $10 + 8\sqrt{2}$ (2) $11 + 8\sqrt{2}$ (3) $11 + 6\sqrt{2}$ (4) $10 + 6\sqrt{2}$

**Ans.** **(2)**

**Sol.**




$(x-2)^2 + (y-3)^2 = 2 \Rightarrow x^2 + y^2 - 4x - 6y + 11 = 0$

Tangent at (3, 2) is $x \cdot 3 + y \cdot 2 - 2(x + 3) - 3(y + 2) + 11 = 0$

$\Rightarrow x - y - 1 = 0$

New centre of circle $\left(2 + 4\cos 45, 3 + 4\sin 45\right) = \left(2 + 2\sqrt{2}, 3 + 2\sqrt{2}\right)$

Image of $\left(2 + 2\sqrt{2}, 3 + 2\sqrt{2}\right)$ about tangent is

$$\frac{x - \left(2 + 2\sqrt{2}\right)}{1} = \frac{y - \left(3 + 2\sqrt{2}\right)}{-1} = \frac{-2\left(\left(2 + 2\sqrt{2}\right) - \left(3 + 2\sqrt{2}\right) - 1\right)}{2} = 2\ x = 4 + 2\sqrt{2}, \ y = 5 + 2\sqrt{2}$$

Area $= \frac{1}{2} - (4 + 2)2 = 6$

$\frac{1}{2}(2)\left(3 + 2\sqrt{2}\right)\left(1 + 2\sqrt{2}\right)$

$= 11 + 8\sqrt{2}$

**19.** Let R be a relation over the set N × N and it is defined by (a, b) R(c, d) ⇒ ad(b – c) = bc (a – d). Then relation R is

(1) Reflexive, symmetric and Transitive

(2) symmetric, Not Reflexive, and Not Transitive

(3) symmetric, Transitive, and Not Reflexive

(4) Transitive, not Reflexive and not symmetric

**Ans.** **(2)**

**Sol.** $((a, b), (a, b)) : ab(b - a) \neq ba(a - b)$

$\therefore$ Not Reflexive

Symmetric

$((a, b), (c, d))$ $ad(b - c) = bc(a - d)$

$((c, d), (a, b))$ $cb(d - a) = da(c - b)$

Hence symmetric

Transitive

$((2, 3), (3, 2)) \in R$ & $((3, 2), (5, 30)) \in R \Rightarrow ((2, 3), (5, 30)) \notin R$

Hence Not Transitive