NURTURE COURSE

TRIGONOMETRIC EQUATION

CONTENTS

TRIGONOMETRIC EQUATIONS & INEQUATIONS

IIT-JEE Syllabus :

General solution of trigonometric equations.

TRIGONOMETRIC EQUATION

1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) Principal solution :- The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

(b) General solution :- Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

(c) Particular solution :- The solution of the trigonometric equation lying in the given interval.

3. GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED) :

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$, $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha$, $n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, $n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$ $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n$, $n \in I$

If n is an odd integer, then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}, \cos\frac{n\pi}{2} = 0,$

$$\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$$

$$\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$$

Illustration 1 : Find the set of values of x for which $\frac{\tan 3x - \tan 2x}{1+\tan 3x.\tan 2x}=1$.

Solution : We have, $\frac{\tan 3x - \tan 2x}{1+\tan 3x.\tan 2x} = 1 \quad \Rightarrow \quad \tan(3x - 2x) = 1 \Rightarrow \tan x = 1$

$\Rightarrow \quad \tan x = \tan\frac{\pi}{4} \Rightarrow x = n\pi + \frac{\pi}{4}, n \in I \qquad$ {using $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$)

But for this value of x, tan 2x is not defined.

Hence the solution set for x is ϕ. **Ans.**

Do yourself-1 :

(i) Find general solutions of the following equations :

(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\left(\frac{3\theta}{2}\right) = 0$ (c) $\tan\left(\frac{3\theta}{4}\right) = 0$

(d) $\cos^2 2\theta = 1$ (e) $\sqrt{3}\sec 2\theta = 2$ (f) $\text{cosec}\left(\frac{\theta}{2}\right) = -1$

4. IMPORTANT POINTS TO BE REMEMBERED WHILE SOLVING TRIGONOMETRIC EQUATIONS :

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions. Reject extra solutions if they do not satisfy the given equation.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(i) Check that denominator is not zero at any stage while solving equations.

(ii) If tan θ or sec θ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(iii) If cot θ or cosec θ is involved in the equation, θ should not be multiple of π or 0.

5. DIFFERENT STRATEGIES FOR SOLVING TRIGONOMETRIC EQUATIONS :

(a) Solving trigonometric equations by factorisation.

e.g. $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\therefore \quad (2\sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$\therefore \quad (1 + \cos x)(2\sin x - \cos x - 1 + \cos x) = 0$

$\therefore \quad (1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow \quad \cos x = -1$ or $\sin x = \frac{1}{2}$

$\Rightarrow \quad \cos x = -1 = \cos\pi \quad \Rightarrow \quad x = 2n\pi + \pi = (2n + 1)\pi, n \in I$

or $\quad \sin x = \frac{1}{2} = \sin\frac{\pi}{6} \quad \Rightarrow \quad x = k\pi + (-1)^k\frac{\pi}{6}, k \in I$

Illustration 2 : If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is -

(A) $2n\pi\pm\frac{\pi}{3}$ (B) $2n\pi\pm\frac{\pi}{6}$ (C) $n\pi\pm\frac{\pi}{3}$ (D) none of these

Solution : Since, $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P.

$\Rightarrow \quad \cos^2\theta = \frac{1}{6}\sin\theta \,.\, \tan\theta \quad \Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\therefore \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3} \quad \Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in I.$ **Ans. (A)**

(b) Solving of trigonometric equation by reducing it to a quadratic equation.

e.g. $6 - 10\cos x = 3\sin^2 x$

$\therefore \quad 6 - 10\cos x = 3 - 3\cos^2 x \quad \Rightarrow \quad 3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow \quad (3\cos x - 1)(\cos x - 3) = 0 \quad \Rightarrow \quad \cos x = \frac{1}{3}$ or $\cos x = 3$

Since $\cos x = 3$ is not possible as $-1 \le \cos x \le 1$

$\therefore \quad \cos x = \frac{1}{3} = \cos\left(\cos^{-1}\frac{1}{3}\right) \quad \Rightarrow \quad x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right), n \in I$

Illustration 3 : Solve $\sin^2\theta - \cos\theta = \frac{1}{4}$ for θ and write the values of θ in the interval $0 \le \theta \le 2\pi$.

Solution : The given equation can be written as

$1 - \cos^2\theta - \cos\theta = \frac{1}{4} \quad \Rightarrow \quad \cos^2\theta + \cos\theta - 3/4 = 0$

$\Rightarrow \quad 4\cos^2\theta + 4\cos\theta - 3 = 0 \quad \Rightarrow \quad (2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}, -\frac{3}{2}$

Since, $\cos\theta = -3/2$ is not possible as $-1 \le \cos\theta \le 1$

$\therefore \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \cos\frac{\pi}{3} \quad \Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

For the given interval, n = 0 and n = 1.

$\Rightarrow \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **Ans.**

Illustration 4 : Find the number of solutions of tanx + secx = 2cosx in $[0, 2\pi]$.

Solution : Here, tanx + secx = 2cosx $\Rightarrow$ $\sin x + 1 = 2\cos^2 x$

$\Rightarrow$ $2\sin^2 x + \sin x - 1 = 0$ $\Rightarrow$ $\sin x = \frac{1}{2}, -1$

But $\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$ for which tanx + secx = 2 cosx is not defined.

Thus $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ number of solutions of tanx + secx = 2cos x is 2. **Ans.**

Illustration 5 : Solve the equation $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$

Solution : To solve this equation we use the fundamental formula of trigonometric identities,

$\sin^2 x + \cos^2 x = 1$

writing the equation in the form,

$5\sin^2 x - 7\sin x \,.\, \cos x + 16\cos^2 x = 4(\sin^2 x + \cos^2 x)$

$\Rightarrow$ $\sin^2 x - 7\sin x\cos x + 12\cos^2 x = 0$

dividing by $\cos^2 x$ on both side we get,

$\tan^2 x - 7\tan x + 12 = 0$

Now it can be factorized as :

$(\tan x - 3)(\tan x - 4) = 0$

$\Rightarrow$ $\tan x = 3, 4$

i.e., $\tan x = \tan(\tan^{-1}3)$ or $\tan x = \tan(\tan^{-1}4)$

$\Rightarrow$ $x = n\pi + \tan^{-1}3$ or $x = n\pi + \tan^{-1}4$, $n \in I$. **Ans.**

Illustration 6 : If $x \neq \frac{n\pi}{2}$, $n \in I$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$, then find the general solutions of x.

Solution : As $x \neq \frac{n\pi}{2}$ $\Rightarrow$ $\cos x \neq 0, 1, -1$

So, $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$ $\Rightarrow$ $\sin^2 x - 3\sin x + 2 = 0$

$\therefore$ $(\sin x - 2)(\sin x - 1) = 0 \Rightarrow$ $\sin x = 1, 2$

where sinx = 2 is not possible and sinx = 1 which is also not possible as $x \neq \frac{n\pi}{2}$

$\therefore$ no general solution is possible. **Ans.**

Illustration 7 : Solve the equation $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \,.\, \cos x$.

Solution : $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \,.\, \cos x$ $\Rightarrow$ $(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = \frac{7}{2}\sin x \,.\, \cos x$

$\Rightarrow$ $1 - \frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x)$ $\Rightarrow 2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow$ $(2\sin 2x - 1)(\sin 2x + 4) = 0$ $\Rightarrow$ $\sin 2x = \frac{1}{2}$ or $\sin 2x = -4$ (which is not possible)

$\Rightarrow \quad 2x = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

i.e., $\quad x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}, n \in I$ **Ans.**

Do yourself-2 :

(i) Solve the following equations :

(a) $3\sin x + 2\cos^2 x = 0$ (b) $\sec^2 2\alpha = 1 - \tan 2\alpha$

(c) $7\cos^2\theta + 3\sin^2\theta = 4$ (d) $4\cos\theta - 3\sec\theta = \tan\theta$

(ii) Solve the equation : $2\sin^2\theta + \sin^2 2\theta = 2$ for $\theta \in (-\pi, \pi)$.

(c) Solving trigonometric equations by introducing an auxilliary argument.

Consider, $a \sin\theta + b\cos\theta = c$ .... (i)

$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$

equation (i) has a solution only if $|c| \le \sqrt{a^2+b^2}$

let $\quad \frac{a}{\sqrt{a^2+b^2}} = \cos\phi, \quad \frac{b}{\sqrt{a^2+b^2}} = \sin\phi \quad \& \quad \phi = \tan^{-1}\frac{b}{a}$

by introducing this auxillary argument ϕ, equation (i) reduces to

$\sin(\theta + \phi) = \frac{c}{\sqrt{a^2+b^2}}$ Now this equation can be solved easily.

Illustration 8 : Find the number of distinct solutions of $\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$.

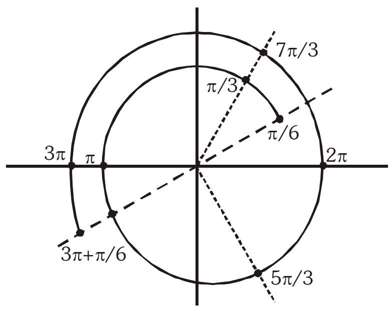
Solution : Here, $\sec x + \tan x = \sqrt{3} \Rightarrow 1 + \sin x = \sqrt{3}\cos x$

or $\sqrt{3}\cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2+b^2}$ i.e. $\sqrt{4} = 2$, we get

$\Rightarrow \quad \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$

$\Rightarrow \quad \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x = \frac{1}{2} \Rightarrow \cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$



As $\quad 0 \le x \le 3\pi$

$\frac{\pi}{6} \le x + \frac{\pi}{6} \le 3\pi + \frac{\pi}{6}$

$\Rightarrow \quad x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$

But at $x = \frac{3\pi}{2}$, tanx and secx is not defined.

$\therefore$ Total number of solutions are 2. **Ans.**

Illustration 9 : Prove that the equation kcosx – 3sinx = k + 1 possess a solution iff $k \in (-\infty, 4]$.

Solution : Here, k cosx – 3sinx = k + 1, could be re-written as :

$$\frac{k}{\sqrt{k^2+9}}\cos x - \frac{3}{\sqrt{k^2+9}}\sin x = \frac{k+1}{\sqrt{k^2+9}}$$

or $\cos(x+\phi) = \frac{k+1}{\sqrt{k^2+9}}$, where $\tan\phi = \frac{3}{k}$

which possess a solution only if $-1 \le \frac{k+1}{\sqrt{k^2+9}} \le 1$

i.e., $\left|\frac{k+1}{\sqrt{k^2+9}}\right| \le 1$

i.e., $(k+1)^2 \le k^2+9$

i.e., $k^2 + 2k + 1 \le k^2 + 9$

or $k \le 4$

$\Rightarrow$ The interval of k for which the equation (kcosx – 3sinx = k + 1) has a solution is $(-\infty, 4]$. **Ans.**

Do yourself-3 :

(i) Solve the following equations :

(a) $\sin x + \sqrt{2} = \cos x$.

(b) $\operatorname{cosec}\theta = 1 + \cot\theta$.

(d) Solving trigonometric equations by transforming sum of trigonometric functions into product.

e.g. cos 3x + sin 2x – sin 4x = 0

cos 3x – 2 sin x cos 3x = 0

$\Rightarrow$ (cos3x) (1 – 2sinx) = 0

$\Rightarrow$ cos3x = 0 or $\sin x = \frac{1}{2}$

$\Rightarrow$ $\cos 3x = 0 = \cos\frac{\pi}{2}$ or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow$ $3x = 2n\pi \pm \frac{\pi}{2}$ or $x = m\pi + (-1)^m \frac{\pi}{6}$

$\Rightarrow$ $x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$ or $x = m\pi + (-1)^m \frac{\pi}{6}$; $(n, m \in I)$

Illustration 10 : Solve : $\cos\theta + \cos3\theta + \cos5\theta + \cos7\theta = 0$

Solution : We have $\cos\theta + \cos7\theta + \cos3\theta + \cos5\theta = 0$

$\Rightarrow\ 2\cos4\theta\cos3\theta + 2\cos4\theta\cos\theta = 0 \Rightarrow \cos4\theta(\cos3\theta + \cos\theta) = 0$

$\Rightarrow\ \cos4\theta(2\cos2\theta\cos\theta) = 0$

$\Rightarrow$ Either $\cos\theta = 0 \Rightarrow \theta = (2n_1 + 1)\pi/2,\ n_1 \in I$

or $\cos2\theta = 0 \Rightarrow \theta = (2n_2 + 1)\dfrac{\pi}{4},\ n_2 \in I$

or $\cos4\theta = 0 \Rightarrow \theta = (2n_3 + 1)\dfrac{\pi}{8},\ n_3 \in I$ **Ans.**

(e) Solving trigonometric equations by transforming a product into sum.

e.g. $\sin5x.\cos3x = \sin6x.\cos2x$

$\sin8x + \sin2x = \sin8x + \sin4x$

$\therefore\ 2\sin2x.\cos2x - \sin2x = 0$

$\Rightarrow\ \sin2x(2\cos2x - 1) = 0$

$\Rightarrow\ \sin2x = 0$ or $\cos2x = \dfrac{1}{2}$

$\Rightarrow\ \sin2x = 0 = \sin0$ or $\cos2x = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

$\Rightarrow\ 2x = n\pi + (-1)^n \times 0,\ n \in I$ or $2x = 2m\pi \pm \dfrac{\pi}{3},\ m \in I$

$\Rightarrow\ x = \dfrac{n\pi}{2},\ n \in I$ or $x = m\pi \pm \dfrac{\pi}{6},\ m \in I$

Illustration 11 : Solve : $\cos\theta\cos2\theta\cos3\theta = \dfrac{1}{4}$; where $0 \le \theta \le \pi$.

Solution : $\dfrac{1}{2}(2\cos\theta\cos3\theta)\cos2\theta = \dfrac{1}{4} \Rightarrow (\cos2\theta + \cos4\theta)\cos2\theta = \dfrac{1}{2}$

$\Rightarrow\ \dfrac{1}{2}[2\cos^2 2\theta + 2\cos4\theta\cos2\theta] = \dfrac{1}{2} \Rightarrow 1 + \cos4\theta + 2\cos4\theta\cos2\theta = 1$

$\therefore\ \cos4\theta(1 + 2\cos2\theta) = 0$

$\cos4\theta = 0$ or $(1 + 2\cos2\theta) = 0$

Now from the first equation : $2\cos4\theta = 0 = \cos(\pi/2)$

$\therefore\ 4\theta = \left(n + \dfrac{1}{2}\right)\pi \Rightarrow \theta = (2n+1)\dfrac{\pi}{8},\ n \in I$

for $n = 0,\ \theta = \dfrac{\pi}{8};\ n = 1,\ \theta = \dfrac{3\pi}{8};\ n = 2,\ \theta = \dfrac{5\pi}{8};\ n = 3,\ \theta = \dfrac{7\pi}{8}$ $(\because\ 0 \le \theta \le \pi)$

and from the second equation :

$$\cos 2\theta = -\frac{1}{2} = -\cos(\pi/3) = \cos(\pi-\pi/3) = \cos(2\pi/3)$$

$\therefore \quad 2\theta = 2k\pi \pm 2\pi/3 \quad \therefore \quad \theta = k\pi \pm \pi/3,\ k \in I$

again for $k = 0,\ \theta = \frac{\pi}{3};\ k = 1,\ \theta = \frac{2\pi}{3} \quad (\because\ 0 \le \theta \le \pi)$

$\therefore \quad \theta = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ **Ans.**

Do yourself-4 :

(i) Solve $4\sin\theta \sin 2\theta \sin 4\theta = \sin 3\theta$.

(ii) Solve for x : $\sin x + \sin 3x + \sin 5x = 0$.

(f) Solving equations by a change of variable :

(i) Equations of the form $P(\sin x \pm \cos x, \sin x.\cos x) = 0$, where $P(y,z)$ is a polynomial, can be solved by the substitution :

$\cos x \pm \sin x = t \quad \Rightarrow \quad 1 \pm 2\sin x.\cos x = t^2$.

Illustration 12 : Solve : $\sin x + \cos x = 1 + \sin x.\cos x$.

Solution : put $\sin x + \cos x = t$

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$

$\Rightarrow 2\sin x \cos x = t^2 - 1 \quad (\because \sin^2 x + \cos^2 x = 1)$

$\Rightarrow \sin x.\cos x = \left(\frac{t^2-1}{2}\right)$

Substituting above result in given equation, we get :

$t = 1 + \frac{t^2-1}{2}$

$\Rightarrow 2t = t^2 + 1 \Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow \quad t = 1$

$\Rightarrow \sin x + \cos x = 1$

Dividing both sides by $\sqrt{1^2+1^2}$ i.e. $\sqrt{2}$, we get

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}} \Rightarrow \quad \cos x \cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad \Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow x = 2n\pi$ or $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2},\ n \in I$

(ii) Equations of the form of $a\sin x + b\cos x + d = 0$, where a, b & d are real numbers can be solved by changing sin x & cos x into their corresponding tangent of half the angle.

Illustration 13 : Solve : 3 cos x + 4 sin x = 5

Solution : $\Rightarrow 3\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)+4\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right)=5$

$$\Rightarrow \frac{3-3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{8\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=5$$

$$\Rightarrow 3-3\tan^2\frac{x}{2}+8\tan\frac{x}{2}=5+5\tan^2\frac{x}{2} \quad \Rightarrow \quad 8\tan^2\frac{x}{2}-8\tan\frac{x}{2}+2=0$$

$$\Rightarrow 4\tan^2\frac{x}{2}-4\tan\frac{x}{2}+1=0 \quad \Rightarrow \quad \left(2\tan\frac{x}{2}-1\right)^2=0$$

$$\Rightarrow 2\tan\frac{x}{2}-1=0 \Rightarrow \tan\frac{x}{2}=\frac{1}{2}=\tan\left(\tan^{-1}\frac{1}{2}\right)$$

$$\Rightarrow \frac{x}{2}=n\pi+\tan^{-1}\left(\frac{1}{2}\right),\ n\in I \quad \Rightarrow \quad x=2n\pi+2\tan^{-1}\frac{1}{2},\ n\in I$$

(g) Solving trigonometric equations with the use of the boundness of the functions involved.

Illustration 14 : Solve the equation $(\sin x+\cos x)^{1+\sin 2x}=2$, when $0\le x\le\pi$.

Solution : We know, $-\sqrt{a^2+b^2}\le a\sin\theta+b\cos\theta\le\sqrt{a^2+b^2}$ and $-1\le\sin\theta\le 1$.

$\therefore$ (sinx + cosx) admits the maximum value as $\sqrt{2}$

and (1 + sin 2x) admits the maximum value as 2.

Also $\left(\sqrt{2}\right)^2=2$.

$\therefore$ the equation could hold only when, $\sin x+\cos x=\sqrt{2}$ and $1+\sin 2x=2$

Now, $\sin x+\cos x=\sqrt{2} \quad \Rightarrow \quad \cos\left(x-\frac{\pi}{4}\right)=1$

$\Rightarrow \quad x=2n\pi+\pi/4,\ n\in I$ ...... (i)

and $1+\sin 2x=2 \quad \Rightarrow \quad \sin 2x=1=\sin\frac{\pi}{2}$

$\Rightarrow \quad 2x=m\pi+(-1)^m\frac{\pi}{2},\ m\in I \quad \Rightarrow \quad x=\frac{m\pi}{2}+(-1)^m\frac{\pi}{4}$ ...... (ii)

The value of x in $[0,\pi]$ satisfying equations (i) and (ii) is $x=\frac{\pi}{4}$ (when n = 0 & m = 0)

Ans.

Note : $\sin x+\cos x=-\sqrt{2}$ and $1+\sin 2x=2$ also satisfies but as $x\ge 0$, this solution is not in domain.

Illustration 15 : Solve for x and y : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+1/2}\le 1$

Solution : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+1/2}\le 1$ .... (i)

$$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\le 1$$

Minimum value of $2^{\frac{1}{\cos^2 x}}=2$

Minimum value of $\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{2}$

$\Rightarrow$ Minimum value of $2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+\frac{1}{2}}$ is 1

$\Rightarrow$ (i) is possible when $2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=1$

$\Rightarrow$ $\cos^2x = 1$ and $y = 1/2$ $\Rightarrow \cos x = \pm 1 \Rightarrow x = n\pi$, where $n \in I$.

Hence $x = n\pi$, $n \in I$ and $y = 1/2$. **Ans.**

Illustration 16 : The number of solution(s) of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$, $0 \le x \le \pi/2$, is/are -

(A) 0 (B) 1 (C) infinite (D) none of these

Solution : Let $y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$ $\Rightarrow$ $y = (1+\cos x)\sin^2 x$ and $y = x^2+\frac{1}{x^2}$

when $y = (1+\cos x)\sin^2 x$ = (a number < 2)(a number ≤ 1) $\Rightarrow$ $y < 2$ .... (i)

and when $y = x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2\ge 2$ $\Rightarrow$ $y \ge 2$ .... (ii)

No value of y can be obtained satisfying (i) and (ii), simultaneously

$\Rightarrow$ No real solution of the equation exists. **Ans. (A)**

Note: If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different values of θ, then solution does not exist.

Do yourself-5 :

(i) If $x^2 - 4x + 5 - \sin y = 0$, $y \in [0, 2\pi)$, then -

(A) $x = 1, y = 0$ (B) $x = 1, y = \pi/2$ (C) $x = 2, y = 0$ (D) $x = 2, y = \pi/2$

(ii) If $\sin x + \cos x = \sqrt{y+\frac{1}{y}}$, $y > 0$, $x \in [0, \pi]$, then find the least positive value of x satisfying the given condition.

6. TRIGONOMETRIC INEQUALITIES :

There is no general rule to solve trigonometric inequations and the same rules of algebra are valid provided the domain and range of trigonometric functions should be kept in mind.

Illustration 17 : Find the solution set of inequality sin x > 1/2.

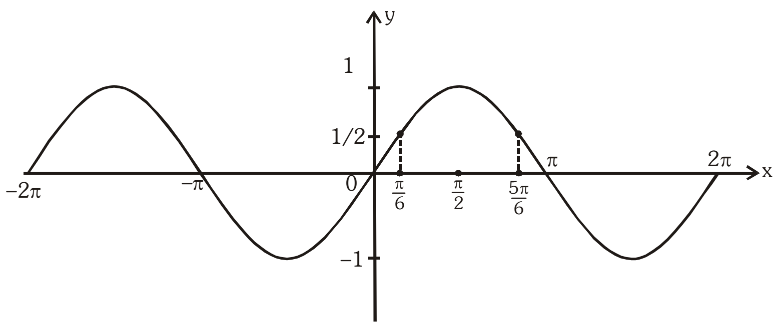
Solution : When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$.

From the graph of y = sin x, it is obvious that between 0 and 2π,

$\sin x > \frac{1}{2}$ for $\pi/6 < x < 5\pi/6$

Hence, sin x > 1/2

$\Rightarrow \quad 2n\pi + \pi/6 < x < 2n\pi + 5\pi/6,\ n \in I$

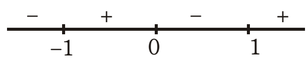


Thus, the required solution set is $\bigcup_{n\in I}\left(2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right)$ **Ans.**

Illustration 18 : Find the values of α lying between 0 and π for which the inequality : $\tan\alpha > \tan^3\alpha$ is valid.

Solution : We have : $\tan\alpha - \tan^3\alpha > 0 \Rightarrow \tan\alpha\,(1 - \tan^2\alpha) > 0$

$\Rightarrow \quad (\tan\alpha)(\tan\alpha + 1)(\tan\alpha - 1) < 0$



So $\quad \tan\alpha < -1,\ 0 < \tan\alpha < 1$

$\therefore \quad$ Given inequality holds for $\alpha \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ **Ans.**

Do yourself - 6 :

(i) Find the solution set of the inequality : $\cos x \geq -1/2$.

(ii) Find the values of x in the interval $[0, 2\pi]$ for which $4\sin^2 x - 8\sin x + 3 \leq 0$.

Miscellaneous Illustration :

Illustration 19 : Solve the following equation : $\tan^2\theta + \sec^2\theta + 3 = 2(\sqrt{2}\sec\theta + \tan\theta)$

Solution : We have $\tan^2\theta + \sec^2\theta + 3 = 2\sqrt{2}\sec\theta + 2\tan\theta$

$\Rightarrow \quad \tan^2\theta - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 3 = 0$

$\Rightarrow \quad \tan^2\theta + 1 - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 2 = 0$

$\Rightarrow \quad (\tan\theta - 1)^2 + (\sec\theta - \sqrt{2})^2 = 0 \quad \Rightarrow \quad \tan\theta = 1$ and $\sec\theta = \sqrt{2}$

As the periodicity of $\tan\theta$ and $\sec\theta$ are not same, we get

$\theta = 2n\pi + \frac{\pi}{4},\ n \in I$

Ans.

Illustration 20 : Find the solution set of equation $5^{(1 + \log_5 \cos x)} = 5/2$.

Solution : Taking log to base 5 on both sides in given equation :

$(1 + \log_5 \cos x).\log_5 5 = \log_5(5/2) \quad \Rightarrow \log_5 5 + \log_5 \cos x = \log_5 5 - \log_5 2$

$\Rightarrow \log_5 \cos x = -\log_5 2 \Rightarrow \cos x = 1/2 \quad \Rightarrow x = 2n\pi \pm \pi/3,\ n \in I$

Ans.

Illustration 21 : If the set of all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x + \sqrt{2}| < \sqrt{6}$ is $\left(\frac{a\pi}{24}, \frac{b\pi}{24}\right)$ then find the value of $\left|\frac{a-b}{3}\right|$.

Solution : $|4\sin x + \sqrt{2}| < \sqrt{6}$

$\Rightarrow \quad -\sqrt{6} < 4\sin x + \sqrt{2} < \sqrt{6} \quad \Rightarrow \quad -\sqrt{6} - \sqrt{2} < 4\sin x < \sqrt{6} - \sqrt{2}$

$\Rightarrow \quad \frac{-(\sqrt{6}+\sqrt{2})}{4} < \sin x < \frac{\sqrt{6}-\sqrt{2}}{4} \quad \Rightarrow \quad -\frac{5\pi}{12} < x < \frac{\pi}{12}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Comparing with $\frac{a\pi}{24} < x < \frac{b\pi}{24}$, we get, $a = -10, b = 2$

$\therefore \left|\frac{a-b}{3}\right| = \left|\frac{-10-2}{3}\right| = 4$

Ans.

Illustration 22 : The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2x - 7\sin x + 2 = 0$ is - **[JEE 98]**

(A) 0 (B) 5 (C) 6 (D) 10

Solution : $3\sin^2x - 7\sin x + 2 = 0$

$\Rightarrow$ $(3\sin x - 1)(\sin x - 2) = 0$

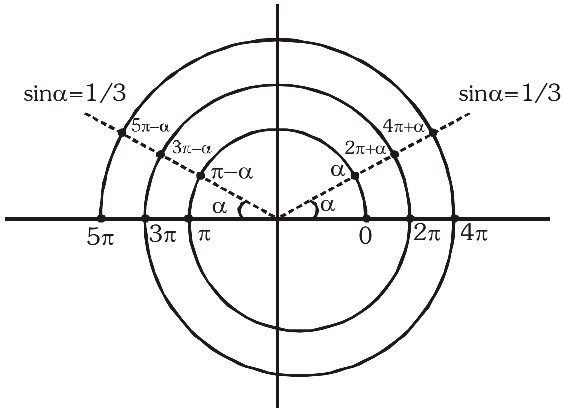
$\because$ $\sin x \neq 2$

$\Rightarrow$ $\sin x = \dfrac{1}{3} = \sin\alpha$ (say)

where α is the least positive value of x

such that $\sin\alpha = \dfrac{1}{3}$.

Clearly $0 < \alpha < \dfrac{\pi}{2}$. We get the solution,



$x = \alpha,\ \pi - \alpha,\ 2\pi + \alpha,\ 3\pi - \alpha,\ 4\pi + \alpha$ and $5\pi - \alpha$.

Hence total six values in $[0, 5\pi]$ **Ans. (C)**

ANSWERS FOR DO YOURSELF

1 : **(i)** **(a)** $\theta = n\pi + (-1)^n\dfrac{\pi}{6},\ n \in I$ **(b)** $\theta = (2n+1)\dfrac{\pi}{3},\ n \in I$ **(c)** $\theta = \dfrac{4n\pi}{3},\ n \in I$

(d) $\theta = \dfrac{n\pi}{2},\ n \in I$ **(e)** $\theta = n\pi \pm \dfrac{\pi}{12},\ n \in I$

(f) $\theta = 2n\pi + (-1)^{n+1}\pi,\ n \in I$

2 : **(i)** **(a)** $x = n\pi + (-1)^{n+1}\dfrac{\pi}{6},\ n \in I$ **(b)** $\alpha = \dfrac{n\pi}{2}$ or $\alpha = \dfrac{k\pi}{2} + \dfrac{3\pi}{8},\ n, k \in I$

(c) $\theta = n\pi \pm \dfrac{\pi}{3},\ n \in I$

(d) $\theta = n\pi + (-1)^n\alpha$, where $\alpha = \sin^{-1}\left(\dfrac{\sqrt{17}-1}{8}\right)$ or $\sin^{-1}\left(\dfrac{-1-\sqrt{17}}{8}\right),\ n \in I$

(ii) $\theta = \left\{-\dfrac{\pi}{4}, -\dfrac{3\pi}{4}, -\dfrac{\pi}{2}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{\pi}{2}\right\}$

3 : **(i)** **(a)** $x = 2n\pi - \dfrac{\pi}{4},\ n \in I$ **(b)** $2m\pi + \dfrac{\pi}{2},\ m \in I$

4 : **(i)** $\theta = n\pi$ or $\theta = \dfrac{m\pi}{3} \pm \dfrac{\pi}{9};\ n, m \in I$ **(ii)** $x = \dfrac{n\pi}{3},\ n \in I$ and $k\pi \pm \dfrac{\pi}{3},\ k \in I$

5 : **(i)** D **(ii)** $x = \dfrac{\pi}{4}$

6 : **(i)** $\bigcup_{n\in I}\left[2n\pi - \dfrac{2\pi}{3},\ 2n\pi + \dfrac{2\pi}{3}\right]$ **(ii)** $\left[\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right]$

EXERCISE # (O-1)

1. The number of solutions of the equation sin 2x – 2cosx + 4 sinx = 4 in the interval $[0, 5\pi]$ is -

(A) 6 (B) 4 (C) 3 (D) 5

2. Let $A = \{\theta : \sin(\theta) = \tan(\theta)\}$ and $B = \{\theta : \cos(\theta) = 1\}$ be two sets. Then -

(A) $A = B$ (B) $A \subset B$ and $B - A \neq \phi$

(C) $A \not\subset B$ (D) $B \not\subset A$

3. The complete solution set of the inequality $\tan^2 x - 2\sqrt{2}\tan x + 1 \le 0$ is-

(A) $n\pi + \frac{\pi}{8} \le x \le \frac{3\pi}{8} + n\pi,\ n \in I$ (B) $n\pi + \frac{\pi}{4} \le x \le \frac{3\pi}{4} + n\pi,\ n \in I$

(C) $n\pi + \frac{\pi}{16} \le x \le \frac{3\pi}{8} + n\pi,\ n \in I$ (D) $n\pi + \frac{\pi}{3} \le x \le \frac{2\pi}{3} + n\pi,\ n \in I$

4. The general solution of the equation $\tan^2\alpha + 2\sqrt{3}\tan\alpha = 1$ is given by -

(A) $\alpha = \frac{n\pi}{2}\ (n \in I)$ (B) $\alpha = (2n+1)\frac{\pi}{2}\ (n \in I)$

(C) $\alpha = (6n+1)\frac{\pi}{12}\ (n \in I)$ (D) $\alpha = \frac{n\pi}{12}\ (n \in I)$

5. If $2\tan^2\theta = \sec^2\theta$, then the general solution of θ -

(A) $n\pi + \frac{\pi}{4}\ (n \in I)$ (B) $n\pi - \frac{\pi}{4}\ (n \in I)$ (C) $n\pi \pm \frac{\pi}{4}\ (n \in I)$ (D) $2n\pi \pm \frac{\pi}{4}\ (n \in I)$

6. Number of principal solution(s) of the equation $4 \cdot 16^{\sin^2 x} = 2^{6\sin x}$ is

(A) 1 (B) 2 (C) 3 (D) 4

7. The general solution of equation $4\cos^2 x + 6\sin^2 x = 5$ is -

(A) $x = n\pi \pm \frac{\pi}{2}\ (n \in I)$ (B) $x = n\pi \pm \frac{\pi}{4}\ (n \in I)$ (C) $x = n\pi \pm \frac{3\pi}{2}\ (n \in I)$ (D) None of these

8. If $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \tan 4\theta \tan 7\theta$, then $\theta =$

(A) $\frac{n\pi}{4}$ (B) $\frac{n\pi}{7}$ (C) $\frac{n\pi}{12}$ (D) $n\pi$

where $n \in I$

9. If $\frac{1-\cos 2\theta}{1+\cos 2\theta} = 3$, then the general solution of θ is -

(A) $2n\pi \pm \pi/6$ (B) $n\pi \pm \pi/6$ (C) $2n\pi \pm \pi/3$ (D) $n\pi \pm \pi/3$

where $n \in I$

10. The number of solutions of the equation $2\cos\left(\frac{x}{2}\right) = 3^x + 3^{-x}$ is-

(A) 1 (B) 2 (C) 3 (D) None

11. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is-

(A) 0 (B) 1 (C) 2 (D) infinitely many

12. If $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$, then the greatest positive solution of $1 + \sin^4 x = \cos^2 3x$ is -

(A) π (B) 2π (C) $\frac{5\pi}{2}$ (D) none of these

13. The general value of θ satisfying $\sin^2\theta + \sin\theta = 2$ is-

(A) $n\pi\ (-1)^n\frac{\pi}{6}$ (B) $2n\pi + \frac{\pi}{4}$ (C) $n\pi + (-1)^n\frac{\pi}{2}$ (D) $n\pi + (-1)^n\frac{\pi}{3}$

14. The number of solutions of the equation $\tan^2 x - \sec^{10} x + 1 = 0$ in (0, 10) is -

(A) 3 (B) 6 (C) 10 (D) 11

15. The solution set of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in the interval $[0, 2\pi]$ is :

(A) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (B) $\left\{\frac{\pi}{3}, \pi\right\}$ (C) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ (D) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

16. The equation $\sin x \cos x = 2$ has :

(A) one solution (B) two solutions (C) infinite solutions (D) no solution

17. If $\tan^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 0$, then the general value of θ is :

(A) $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (B) $n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (C) $n\pi + \frac{\pi}{4}, n\pi - \frac{\pi}{3}$ (D) $n\pi - \frac{\pi}{4}, n\pi - \frac{\pi}{3}$

where $n \in I$

18. If $0 \le x \le 3\pi$, $0 \le y \le 3\pi$ and $\cos x . \sin y = 1$, then the possible number of values of the ordered pair (x, y) is-

(A) 6 (B) 12 (C) 8 (D) 15

19. If $\frac{\tan 2\theta + \tan\theta}{1 - \tan\theta\tan 2\theta} = 0$, then the general value of θ is -

(A) $n\pi$; $n \in I$ (B) $\frac{n\pi}{3}$; $n \in I$ (C) $\frac{n\pi}{4}$ (D) $\frac{n\pi}{6}$; $n \in I$

where $n \in I$

20. The most general values of x for which $\sin x + \cos x = \min_{a\in R} \{1, a^2 - 4a + 6\}$ is given by-

(A) $2n\pi$ (B) $2n\pi + \frac{\pi}{2}$ (C) $n\pi + (-1)^n.\frac{\pi}{4} - \frac{\pi}{4}$ (D) None of these

where $n \in I$

EXERCISE # (O-2)

1. Number of values of x satisfying the equation $\log_2(\sin x) + \log_{1/2}(-\cos x) = 0$ in the interval $(-\pi,\pi]$ is equal to-

(A) 0 (B) 1 (C) 2 (D) 3

2. Given $a^2 + 2a + \operatorname{cosec}^2\left(\frac{\pi}{2}(a+x)\right) = 0$ then, which of the following holds good?

(A) $a = 1$; $\frac{x}{2} \in I$ (B) $a = -1$; $\frac{x}{2} \in I$

(C) $a \in R$; $x \in \phi$ (D) a , x are finite but not possible to find

3. If the equation $\cot^4 x - 2\operatorname{cosec}^2 x + a^2 = 0$ has atleast one solution then, sum of all possible integral values of 'a' is equal to

(A) 4 (B) 3 (C) 2 (D) 0

4. The set of angles between 0 and 2π satisfying the equation $4\cos^2\theta - 2\sqrt{2}\cos\theta - 1 = 0$ is

(A) $\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$ (B) $\left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}\right\}$

(C) $\left\{\frac{5\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}\right\}$ (D) $\left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

5. In which one of the following intervals the inequality, $\sin x < \cos x < \tan x < \cot x$ can hold good?

(A) $\left(0, \frac{\pi}{4}\right)$ (B) $\left(\frac{3\pi}{4}, \pi\right)$ (C) $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ (D) $\left(\frac{7\pi}{4}, 2\pi\right)$

6. If the equation $\sin^4 x - (k+2)\sin^2 x - (k+3) = 0$ has a solution then k must lie in the interval :

(A) $(-4, -2)$ (B) $[-3, 2)$ (C) $(-4, -3)$ (D) $[-3, -2]$

7. The smallest positive angle satisfying the equation $1 + \cos 3x - 2\cos 2x = 0$, is equal to

(A) 15° (B) 22.5° (C) 30° (D) 45°

8. **Statement-1:** If $\sin\frac{3x}{2}\cos\frac{5y}{3} = k^8 - 4k^4 + 5$, where $x, y \in R$ then exactly four distinct real values of k are possible.

because

Statement-2: $\sin\frac{3x}{2}$ and $\cos\frac{5y}{3}$ both are less than or equal to one and greater than or equal to – 1.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

9. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}$, $0 < x \le \frac{\pi}{2}$ has

(A) one real solutions (B) more than one real solutions

(C) no real solution (D) none of the above

10. The number of solutions of the equation $\sin x = x^2 + x + 1$ is-

(A) 0 (B) 1 (C) 2 (D) None

11. Number of integral solution(s) of the inequality $2\sin^2x - 5\sin x + 2 > 0$ in $x \in [0,2\pi]$, is-

(A) 3 (B) 4 (C) 5 (D) 6

12. If $\tan\theta - \sqrt{2}\sec\theta = \sqrt{3}$, then the general solution of θ is -

(A) $n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}$ (B) $n\pi + (-1)^n\frac{\pi}{3} - \frac{\pi}{4}$ (C) $n\pi + (-1)^n\frac{\pi}{3} + \frac{\pi}{4}$ (D) $n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{3}$

where $n \in I$

EXERCISE # (S-1)

1. Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2}+\log_5(\sin x)} = 15^{\frac{1}{2}+\log_{15}\cos x}$

2. Find all the values of θ satisfying the equation; $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$.

3. Solve the equality: $2\sin 11x + \cos 3x + \sqrt{3}\sin 3x = 0$

4. Find all value of θ, between 0 & π, which satisfy the equation; $\cos\theta \cdot \cos 2\theta \cdot \cos 3\theta = 1/4$.

5. Solve for x, the equation $\sqrt{13 - 18\tan x} = 6\tan x - 3$, where $-2\pi < x < 2\pi$.

6. Determine the smallest positive value of x which satisfy the equation, $\sqrt{1+\sin 2x} - \sqrt{2}\cos 3x = 0$.

7. Find the number of principal solution of the equation, $\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x$.

8. Find the general solution of the trigonometric equation $3^{\left(\frac{1}{2}+\log_3(\cos x+\sin x)\right)} - 2^{\log_2(\cos x-\sin x)} = \sqrt{2}$.

9. Find all values of θ between 0° & 180° satisfying the equation ; $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$.

10. Find the general solution of the equation, $\sin \pi x + \cos \pi x = 0$. Also find the sum of all solutions in [0, 100].

11. Find the range of y such that the equation, $y + \cos x = \sin x$ has a real solution. For $y = 1$, find x such that $0 < x < 2\pi$.

12. Find the general values of θ for which the quadratic function $(\sin\theta)x^2 + (2\cos\theta)x + \frac{\cos\theta + \sin\theta}{2}$ is the square of a linear function.

13. Prove that the equations

(a) $\sin x \cdot \sin 2x \cdot \sin 3x = 1$ (b) $\sin x \cdot \cos 4x \cdot \sin 5x = -1/2$

have no solution.

14. Let $f(x) = \sin^6x + \cos^6x + k(\sin^4x + \cos^4x)$ for some real number k. Determine

(a) all real numbers k for which $f(x)$ is constant for all values of x.

(b) all real numbers k for which there exists a real number 'c' such that $f(c) = 0$.

(c) If $k = -0.7$, determine all solutions to the equation $f(x) = 0$.

15. If the set of values of x satisfying the inequality $\tan x \cdot \tan 3x < -1$ in the interval $\left(0, \frac{\pi}{2}\right)$ is (a,b), then the value of $\left(\frac{36(b-a)}{\pi}\right)$ is

EXERCISE (S-2)

1. Solve the equation : $\sin 5x = 16 \sin^5 x$.

2. Find all the solutions of $4\cos^2 x \sin x - 2\sin^2 x = 3 \sin x$.

3. Solve for x, $(-\pi \le x \le \pi)$ the equation; $2(\cos x + \cos 2x) + \sin 2x(1 + 2\cos x) = 2\sin x$.

4. Find the general solution of the following equation :
$2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$.

5. Find the values of x, between 0 & 2π, satisfying the equation $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2}$.

6. Solve: $\tan^2 2x + \cot^2 2x + 2\tan 2x + 2\cot 2x = 6$.

7. Solve the equation: $1 + 2\,\text{cosec}x = -\dfrac{\sec^2\frac{x}{2}}{2}$.

8. Solve: $\tan^2 x \,.\, \tan^2 3x \,.\, \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x$.

9. Find the set of values of x satisfying the equality
$\sin\left(x - \frac{\pi}{4}\right) - \cos\left(x + \frac{3\pi}{4}\right) = 1$ and the inequality $\dfrac{2\cos 7x}{\cos 3 + \sin 3} > 2^{\cos 2x}$.

10. Find the solution set of the equation, $\log_{\frac{-x^2-6x}{10}}(\sin 3x + \sin x) = \log_{\frac{-x^2-6x}{10}}(\sin 2x)$.

EXERCISE (JM)

1. Let A and B denote the statements

A : $\cos\alpha + \cos\beta + \cos\gamma = 0$

B : $\sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then - **[AIEEE 2009]**

(1) Both **A** and **B** are true (2) Both **A** and **B** are false

(3) **A** is true and **B** is false (4) **A** is false and **B** is true

2. The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are: **[AIEEE 2011]**

(1) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$ (2) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

(3) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$ (4) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

3. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation
$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is :- **[JEE(Main) 2016]**

(1) 9 (2) 3 (3) 5 (4) 7

4. If sum of all the solutions of the equation $8\cos x \cdot \left(\cos\left(\frac{\pi}{6} + x\right).\cos\left(\frac{\pi}{6} - x\right) - \frac{1}{2}\right) = 1$ in $[0, \pi]$ is $k\pi$, then k is equal to : **[JEE(Main) 2018]**

(1) $\frac{13}{9}$ (2) $\frac{8}{9}$ (3) $\frac{20}{9}$ (4) $\frac{2}{3}$

EXERCISE (JA)

1. The number of values of θ in the interval $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for n = 0, ±1,±2 and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$, is **[JEE 2010, 3]**

2. The positive integer value of n > 3 satisfying the equation

$$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$ is **[JEE 2011, 4]**

3. Let θ, φ ∈ [0,2π] be such that

$$2\cos\theta(1-\sin\varphi) = \sin^2\theta\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)\cos\varphi - 1,\quad \tan(2\pi-\theta) > 0 \text{ and } -1 < \sin\theta < -\frac{\sqrt{3}}{2}.$$

Then φ **cannot** satisfy- **[JEE 2012, 4M]**

(A) $0 < \varphi < \frac{\pi}{2}$ (B) $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$ (C) $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$ (D) $\frac{3\pi}{2} < \varphi < 2\pi$

4. For x ∈ (0, π), the equation sinx + 2sin2x – sin3x = 3 has

(A) infinitely many solutions (B) three solutions

(C) one solution (D) no solution **[JEE(Advanced)-2014, 3(–1)]**

5. The number of distinct solutions of equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval [0, 2π] is **[JEE 2015, 4M, –0M]**

6. Let $S = \left\{x \in (-\pi,\pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x + \text{cosec}x + 2(\tan x - \cot x) = 0$ in the set S is equal to -

[JEE(Advanced)-2016, 3(–1)]

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

7. Let a, b, c be three non-zero real numbers such that the equation

$$\sqrt{3}a\cos x + 2b\sin x = c,\quad x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

has two distinct real roots α and β with α + β = $\frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is ______

[JEE(Advanced)-2018, 3(0)]

ANSWERS

EXERCISE (O-1)

1. C **2.** C **3.** A **4.** C **5.** C **6.** C **7.** B **8.** C
9. D **10.** A **11.** A **12.** B **13.** C **14.** A **15.** C **16.** D
17. A **18.** A **19.** B **20.** C

EXERCISE (O-2)

1. B **2.** B **3.** D **4.** B **5.** A **6.** D **7.** C **8.** D
9. C **10.** A **11.** C **12.** D

EXERCISE (S-1)

1. $x = 2n\pi + \frac{\pi}{6},\ n \in I$ **2.** $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}\ \&\ \pi$ **3.** $x = \frac{n\pi}{7} - \frac{\pi}{84}$ or $x = \frac{n\pi}{4} + \frac{7\pi}{48},\ n \in I$

4. $\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ **5.** $\alpha - 2\pi\ ;\ \alpha - \pi, \alpha, \alpha + \pi$, where $\tan\alpha = \frac{2}{3}$ **6.** $x = \pi/16$

7. 10 solutions **8.** $x = 2n\pi + \frac{\pi}{12}$ **9.** $30°, 45°, 90°, 135°, 150°$

10. $x = n - \frac{1}{4},\ n \in I$; sum = 5025 **11.** $-\sqrt{2} \le y \le \sqrt{2}\ ;\ \frac{\pi}{2}, \pi$ **12.** $2n\pi + \frac{\pi}{4}$ or $(2n+1)\pi - \tan^{-1}2,\ n \in I$

14. (a) $-\frac{3}{2}$; (b) $k \in \left[-1, -\frac{1}{2}\right]$; (c) $x = \frac{n\pi}{2} \pm \frac{\pi}{6}$ **15.** 3

EXERCISE (S-2)

1. $x = n\pi$ **or** $x = n\pi \pm \frac{\pi}{6}$ **2.** $n\pi\ ;\ n\pi + (-1)^n \frac{\pi}{10}$ **or** $n\pi + (-1)^n\left(-\frac{3\pi}{10}\right)$ **3.** $\frac{\pm\pi}{3}, \frac{-\pi}{2}, \pm\pi$

4. $x = 2n\pi$ or $x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$ **or** $x = n\pi + (-1)^n \frac{\pi}{6}$ **5.** $\frac{\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{13\pi}{7}$

6. $x = \frac{n\pi}{4} + (-1)^n \frac{\pi}{8}$ **or** $\frac{n\pi}{4} + (-1)^{n+1}\frac{\pi}{24}$ **7.** $x = 2n\pi - \frac{\pi}{2}$

8. $\frac{(2n+1)\pi}{4}, k\pi$, where $n, k \in I$ **9.** $x = 2n\pi + \frac{3\pi}{4},\ n \in I$ **10.** $x = -\frac{5\pi}{3}$

EXERCISE (JM)

1. 1 **2.** 1 **3.** 4 **4.** 1

EXERCISE (JA)

1. 3 **2.** 7 **3.** A,C,D **4.** D **5.** 8 **6.** C **7.** **0.5**

Important Notes