Parabola **Chapter 28**

PARABOLA

A parabola is the locus of a point which moves in a plane in such a way that its distance from a fixed point is always equal to its distance from a fixed straight line.

This fixed point is called *focus* and the fixed straight line is called directrix.

Let S be the focus, ZZ' be the directrix and P be any point on the parabola. Then by definition,

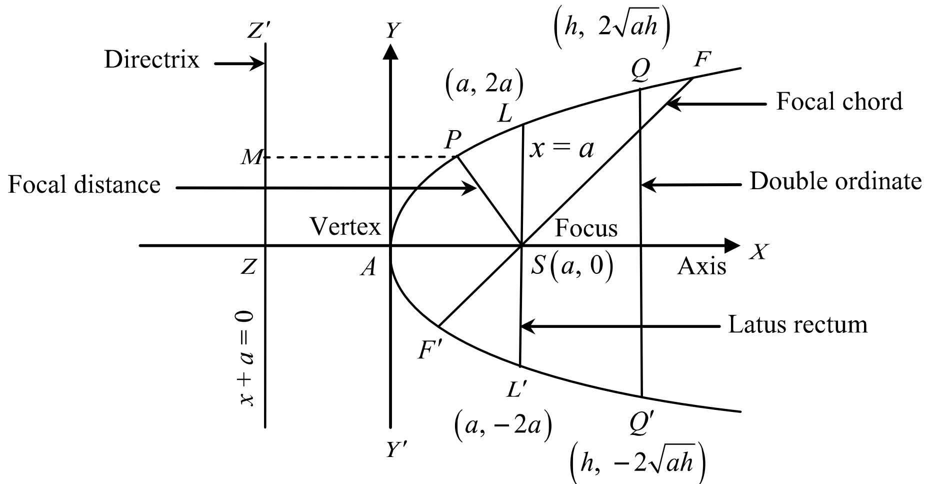
PS = PM

where PM is the length of the perpendicular from P on the directrix ZM.

STANDARD EQUATION OF THE PARABOLA

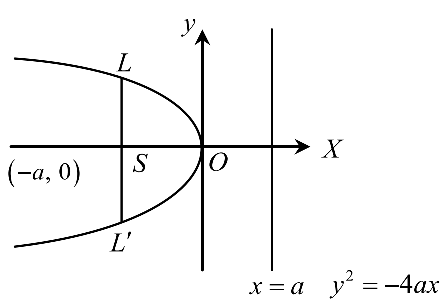
Let S be the focus, zz' be the directrix of the parabola and (x, y) be any point on parabola, then standard form of the parabola is $y^2 = 4ax$.

Some terms related to parabola

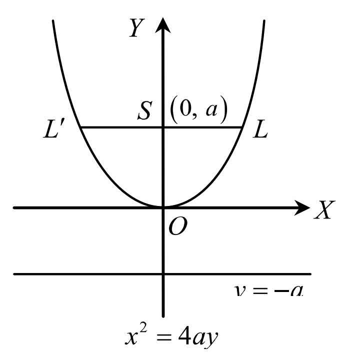


Some other standard forms of parabola are

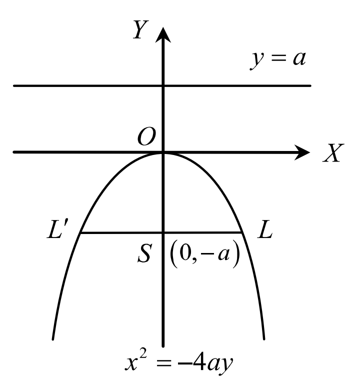
(i) Parabola opening to left i.e. $y^2 = -4ax$



(ii) Parabola opening upwards i.e. $x^2 = 4ay$,

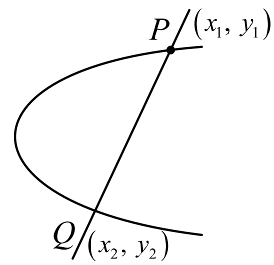


(iii) Parabola opening downwards i.e., $x^2 = -4ay$



Distance of a point $P(x_1, y_1)$ from focus of the parabola $y^2 = 4ax$ is $= a + x_1$

EQUATION OF A CHORD



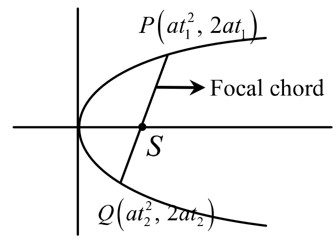
1. The equation of chord joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the parabola $y^2 = 4ax$ is

$$y(y_1 + y_2) = 4ax + y_1 y_2$$

2. The equation of chord joining $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$y(t_1 + t_2) = 2(x + at_1 t_2)$$

CONDITION FOR THE CHORD TO BE A FOCAL CHORD



The chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ passes through focus if $t_1 t_2 = -1$.

LENGTH OF FOCAL CHORD

The length of focal chord joining $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$PQ = a(t_2 - t_1)^2.$$

CONDITION OF TANGENCY AND POINT OF CONTACT

The line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ and the coordinates of the point of contact are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right).$$

EQUATION OF TANGENT IN DIFFERENT FORMS

1. **POINT FORM**

   The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is

$$yy_1 = 2a(x + x_1).$$

2. **PARAMETRIC FORM**

   The equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$ty = x + at^2.$$

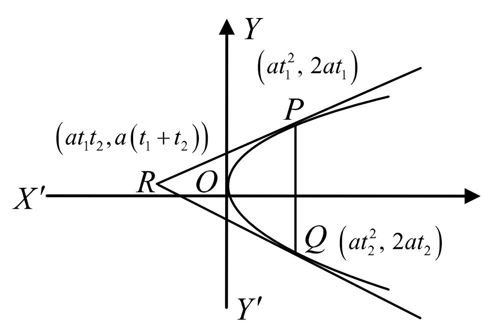
3. **SLOPE FORM**

   The equation of tangent to parabola $y^2 = 4ax$ in terms of slope 'm' is

$$y = mx + \frac{a}{m}.$$

   The coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

POINT OF INTESECTION OF TANGENTS



The point of intersection of tangents drawn at two different points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

$$R(at_1 t_2, a(t_1 + t_2)).$$

EQUATIONS OF NORMAL IN DIFFERENT FORMS

1. **POINT FORM**

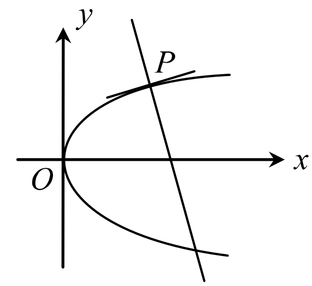
   The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is

   $$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

2. **PARAMETRIC FORM**

   The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

   $$y + tx = 2at + at^3.$$



3. **SLOPE FORM**

   The equation of normal to the parabola $y^2 = 4ax$ in terms of slope 'm' is

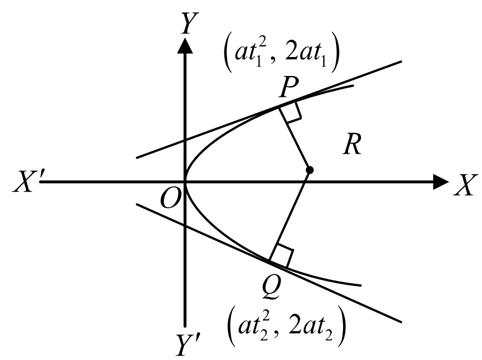
   $$y = mx - 2am - am^3.$$

   The coordinates of the foot of normal P are $(am^2, -2am)$.

CONDITION FOR NORMALITY

The line $y = mx + c$ is a normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$.

POINT OF INTERSECTION OF NORMALS



The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

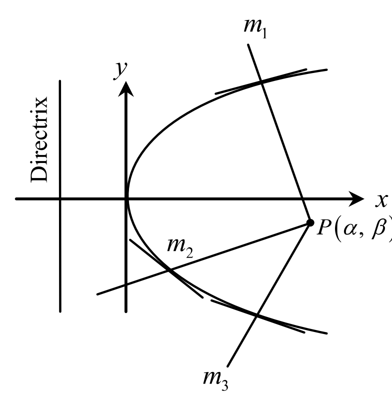
$$R \equiv \left[2a + a\left(t_1^2 + t_2^2 + t_1t_2\right), -at_1t_2(t_1 + t_2)\right]$$

Note :

(a) If the normal at the point 't_1' cuts the parabola again at the point 't_2' then $t_2 = -t_1 - \frac{2}{t_1}$

(b) If normals at t_1 and t_2 intersect on the parabola then $t_1t_2 = 2$.

CO-NORMAL POINTS



Any three points on a parabola normals at which pass through a common point are called co-normal points.

If three normals are drawn through a point $P(h, k)$, then their slopes are the roots of the cubic equation :

$$k = mh - 2am - am^3$$

Note :

(a) $m_1 + m_2 + m_3 = 0$ *i.e.,* the sum of the slopes of the normals at co-normals points is zero.

(b) The sum of the ordinates of the co-normal points (*i.e.*, $-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0$) is zero.

(c) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola [the vertices of the triangle formed by the co-normals points are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$ $\therefore$ y-coordinate of the centroid $= \frac{-2a(m_1 + m_2 + m_3)}{3} = \frac{-2a}{3} \times 0 = 0$. Hence, the centroid lies on the x-axis i.e. axis of the parabola].

(d) If three normals drawn to any parabola $y^2 = 4ax$ from a given point $(h, 0)$ be real, then $h > 2a$.

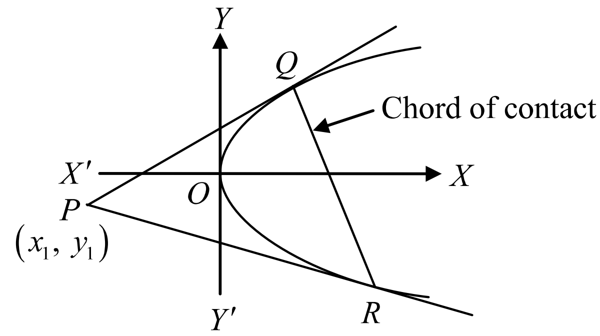
POSITION OF A POINT WITH RESPECT TO A PARABOLA

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =$ or < 0.

EQUATION OF THE CHORD OF CONTACT OF TANGENTS TO A PARABOLA

Let PQ and PR be tangents to the parabola $y^2 = 4ax$ drawn from any external point $P(x_1, y_1)$ then QR is called the chord of contact of the parabola $y^2 = 4ax$

The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

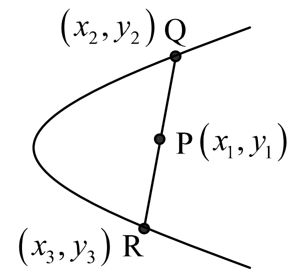


EQUATION OF THE CHORD OF THE PARABOLA WHICH IS BISECTED AT A GIVEN POINT

The equation of the chord of the parabola $y^2 = 4ax$ bisected at the point (x_1, y_1) is given by $T = S_1$ where

$T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$

i.e. $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

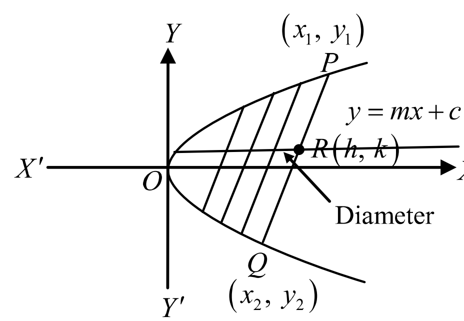


DIAMETERS OF A PARABOLA

The locus of the middle points of a system of parallel chords is called a diameter of the parabola. The diameter is a straight line parallel to the axis of the parabola.

The equation of the diameter bisecting chords of the parabola $y^2 = 4ax$ of slope m is $y = 2a/m$.

Tangents drawn at the ends of any of these chords meet on the diameter of these chords.



POLE AND POLAR

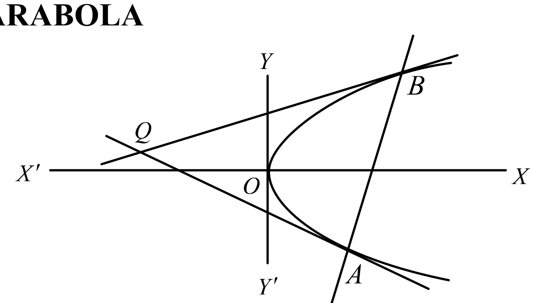
Let P be a given point. Let a line through P intersect the parabola at two points A and B. Let the tangent at A and B intersect at Q. The locus of point Q is a straight line called the polar of point P with respect to the parabola and the point P is called the pole of the polar.

EQUTION OF POLAR OF A POINT WITH RESPECT TO A PARABOLA

The polar of a point $P(x_1, y_1)$ with respect to the parabola

$$y^2 = 4ax \text{ is } T = 0$$

where $T \equiv yy_1 - 2a(x + x_1)$.



CONJUGATE POINTS

If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of Q will pass through P and such points are said to be **conjugate points**.

CONJUGATE LINES

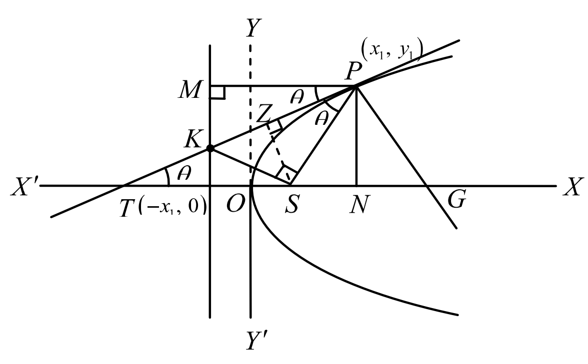
If the pole of line $ax+by+c=0$ lies on the another line $a_1x+b_1y+c_1=0$ then the pole of the second line will lie on the first and such lines are said to be **conjugate lines**.

Note :

(a) Polar of the focus is the directrix.
(b) Any tangent is the polar of its point of contact.
(c) The point of intersection of the polars of two points Q and R is the pole of QR.

SOME MORE IMPORTANT FACTS ABOUT PARABOLA

1. The *parametric equations* of the parabola or the coordinates of any point on it are $x=at^2,\ y=2at$.
2. The tangents at the extremities of any focal chord intersect at right angles on the directrix.
3. The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.
4. The area of the triangle formed by any three points on the parabola is twice the area of the triangle formed by the tangents at these points.
5. The circle described on any focal chord of a parabola as diameter touches the directrix.
6. If the normal at the point $(at_1^2, 2at_1)$ meets the parabola again at $(at_2^2, 2at_2)$ then $t_2=-t_1-2/t_1$.
7. Three normals can be drawn from a point (x_1, y_1) to the parabola. The points where these normals meet the parabola are called *feet of the normals* or *conormal points.* The sum of the slopes of these normals is zero and the sum of the ordinates of the feet of these normals is also zero.
8. If the normals 't_1' and 't_2' meet on the parabola then $t_1\ t_2=2$.
9. The pole of any focal chord of the parabola lies on its directrix.
10. A diameter of the parabola is parallel to its axis and the tangent at the point where it meets the parabola is parallel to the system of chords bisected by the diameter and tangent at the ends of any of the parallel chords of this diameter meet on the diameter.
11. The harmonic mean between the focal radii of any focal chord of a parabola is equal to semi-latus rectum.
12. If the tangent and normal at any point P on the parabola meet the axis of the parabola in T and G respectively, the



(i) $ST=SG=SP$, S being the focus
(ii) $\lfloor PSK=\pi/2$, where K is the point where the tangent at P meets the directrix.
(iii) The tangent at P is equally inclined to the axis of the parabola and the focal distance of P.
(iv) length of the subtangent is twice the abscissa at the point of the tangency for the parabola $y^2=4ax$
(v) length of the sub-normal is always of constant length and is equal to semi latus rectum of the parabola i.e. $2a$

13. If we draw a circle taking any focal radius as diameter will touch the tangent at the vertex.
14. The foot of $\perp$ from focus on any tangent to the parabola is the point where the tangent meets the tangent at vertex.

i.e. here Z lies on y-axis and $SZ=\sqrt{OS.SP}$