13. RELATIONS AND FUNCTIONS

1. INTRODUCTION TO SETS

A collection of well-defined objects that are distinct are known as sets.

Well-defined object clearly defines if the object belongs to a given collection or not.

Well-defined collections

(a) Odd natural numbers less than 10, i.e., 1, 3, 5, 7, 9.

(b) Rivers of India.

(c) Vowels in the English alphabet a, e, i, o, u.

Not well-defined collections

(a) Collection of bright students in class XI of a Nucleus Academy.

(b) Collection of renowned mathematicians of the world.

(c) Collection of beautiful girls of the world.

(d) Collection of fat people.

CONCEPTS

- The terms objects, elements and member of a set are synonymous.
- Sets are generally denoted by capital letters A, B, C, ..., X, Y, Z.
- The elements of a set are represented by small letters a, b, c, ..., x, y, z.
- If a is an element of a set A, then we can say that a belongs to A. The Greek symbol $\in$ represents 'belongs to'. Thus we write $a \in A$.

  If b is not an element of a set A, then we write $b \notin A$.

Uday Kiran G (JEE 2012, AIR 102)

2. SOME IMPORTANT SYMBOLS AND THEIR MEANINGS

Symbol	Meaning
$\Rightarrow$	Implies
$\in$	Belongs to
$A \subset B$	A is subset of B
$\Leftrightarrow$	Implies and is implied by
$\notin$	Does not belong to
| or : or S.t.	Such that
$\forall$	For all
$\exists$	There exists
i*ff*	If and only if
&	And

Symbol	Meaning
a/b	a is divisor of b
N	Set of natural numbers
W	Set of whole numbers
I or Z	Set of integers
Q	Set of rational numbers
Q^c	Set of irrational numbers
R	Set of real numbers
C	Set of complex numbers

3. REPRESENTATION OF SETS

The following two methods are used to represent sets:

(a) Roster or Tabular form

(b) Set builder form

3.1 Roster or Tabular Form

In the roster form, all the elements of a set are enclosed within braces { } and each element is separated by a comma.

Few examples are listed as follows:

(a) The set of all even positive number less than 7 is {2, 4, 6}.

(b) The set of vowels in the English alphabets is {a, e, i, o, u}.

(c) The set of odd natural numbers is represented as {1, 3, 5, ...}. The three dots (ellipses) denote that the list is endless.

Note: (i) In the roster form, the elements of a set are not repeated, i.e. all the elements are taken as distinct, e.g. "SCHOOL" $\Rightarrow$ {S, C, H, O, L}

(ii) The order in which the element of a set is written is immaterial.

E.g. The set {1, 2, 3} and {2, 1, 3} are same.

3.2 Set Builder Form

In the set builder form, all the elements of a set possess a common property. This common property does not match with any element outside this set.

E.g. (i) all the elements 'a, e, i, o, u' possess a common property, i.e. each alphabet is a vowel which none other letters possessing this property.

This can be represented in set builder form as follows:

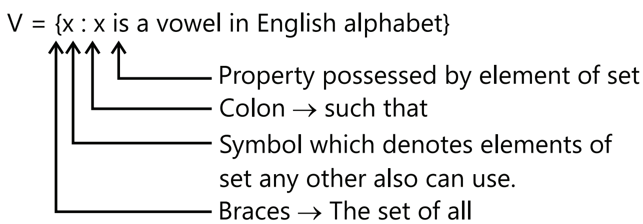


Figure 13.1

i.e. The set of all x such that x is a vowel in English alphabet.

(ii) A = {x: x is a natural number and $3 < x < 10$}

The set of all x such that x is a natural and $3 < x < 10$. Hence the numbers 4, 5, 6, 7, 8 and 9 are the elements of set A.

4. SUBSET

Set A is a subset of B if B has all the elements of A, denoted by $A \subset B$ (read as A is subset of B).

4.1 Number of Subset

If a set X contains n elements $\{x_1, x_2, ..., x_n\}$, then total number of subsets of X = 2^n.

Proof: Number of subsets of set X is equal to the number of selections of elements taking any number of them at a time out of the total n elements and it is equal to 2^n.

$\because {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$.

4.2 Types of Subset

Set A is said to be a proper subset of set B if all elements of subset A is present in set B and at least one element in set B is not an element of subset A, i.e. $A \subset B$ and $A \neq B$.

The set A itself is an improper subset of A.

E.g., If X = $\{x_1, x_2, ..., x_n\}$, then total number of proper sets = $2^n - 1$ (excluding itself). The statement $A \subset B$ can be written as $B \supset A$, then B is called the superset of A.

5. TYPE OF SETS

5.1 Null or Empty Set

Any set is called empty or null set if no elements are present in that set. It is denoted by { }.

Few examples are as follows:

(a) Set of odd numbers divisible by 2 is null set, as we know that odd numbers are not divisible by 2 and hence the resultant set is a null set.

(b) Set of even prime numbers is not a null set because 2 is a prime number divisible by 2.

(c) {y: y is a point common to any two parallel lines} is null set because two parallel lines do not intersect.

5.2 Finite and Infinite Set

Any set is said to be finite set, if finite number of elements are present in it.

Few examples are listed as follows:

(a) { } [null set is a finite set]

(b) {a, e, i, o, u}

(c) {Jan, Feb,... Dec} etc.

Any set is said to be infinite set if the number of elements are not finite.

Few examples are as follows:

(a) S = {men living presently in different parts of the world} is non-countable. Therefore it is an infinite set.

(b) S = {x: $x \in R$} is infinite set.

(c) S = {x: $2 < x > 3$, $x \in R$} is infinite set.

5.3 Equal and Equivalent Sets

Any two sets A and B are said to equal set if all elements of set A are in B and vice versa.

E.g. {a, e, i, o, u} = A and B = {e, i, a, u, o} then A = B.

Any two sets A and B are said to be equivalent sets if their number of elements in both the sets are same.

E.g. A = {1, 2, 3} and B = {a, b, c} are equivalent set, i.e. both the sets have three elements ($A \approx B$).

CONCEPTS

1. Number of elements in any set is said to be cardinality/cardinal number of that set.

E.g. 1. A = {1, 2, 3, 4}, then cardinality of set A is 4.

2. B = {a, e, i, o, u}, then cardinality of set B is 5.

2. Any set does not change if one or more elements of the set are repeated.

E.g. 1. A = {June, Nov., April, Sept}

2. B = {June, Nov., June, Sept., April, Sept.} are equal.

Shivam Agarwal (JEE 2009, AIR 27)

5.4 Singleton Set

A set with single element is called a singleton set, i.e. n(X) = 1,

E.g. {x: $x \in N$, $1 < x < 3$}, { { } }, i.e. set of null set, {π} is a set containing alphabet ϕ.

5.5 Universal Set

It is a set which includes all the sets under consideration, i.e. it is a super set of each of the given set. Thus, a set that contains all sets in a given context is called the universal set. It is denoted by U.

E.g. If A = {1, 2, 3}, B = {2, 4, 5, 6} and C = {1, 3, 5, 7}, then U = {1, 2, 3, 4, 5, 6 7} is a universal set which contains all elements of sets A, B and C.

5.6 Disjoint Set

Two sets are said to be disjoint if they have no elements in common, i.e. if A and B are two sets, then $A \cap B = \phi$.

If $A \cap B \neq \phi$, then A and B are said to be intersecting or overlapping sets.

E.g. (i) If A = {1, 2, 3}, B= {4, 5, 6} and C = {4, 7, 9}, then A and B are disjoint sets, and B and C are intersecting sets.
(ii) Set of even natural numbers and odd natural numbers are disjoint sets.

5.7 Complementary Set

Complementary set of A is a set containing all those elements of universal except the element in set A. It is denoted by $\overline{A}$, A^C or A′. So A^C = {x: x ∈ U and x ∉ A}, e.g. If set A = {1, 2, 3, 4, 5} and universal set U = {1, 2, 3, 4, ..., 50}, then $\overline{A}$ = {6, 7, ..., 50}

Note: All disjoint sets are not complementary sets and vice versa.

5.8 Power Set

The collection of all subsets of set A is called the power set of A and is denoted by P(A) i.e., P(A) = {x: x is a subset of A}. If X = {x_1 x_2 x_3 ..... x_n} then n(P(x)) = 2^n; n(P(P(x))) = 2^{2^n} .

Illustration 1: Which of the two sets are equal?

(i) A = {4, 8, 12, 18}, B = {8, 4, 12, 16}

(ii) A = {2, 4, 6, 8, 10} B = {x : x is positive even integer and x ≤ 10}

(iii) A = {x: x is a multiple of 10} B = {x: x is a multiple of 5 and x ≥ 10} **(JEE MAIN)**

Sol: Refer to the definition of different types of sets in the above section.

(i) The elements 4, 8, 12 belong to both sets A and B. But 16 ∈ B and 18 do not belong to both A and B. So $A \neq B$.

(ii) All elements present in set A is present in set B, and vice versa. Therefore A = B.

(iii) Given

A = {10, 20, 30, 40, ...} and B = {10, 15, 20, 25, 30, ...}

Since 15, 25, 35 are not multiples of 10, B does not belong to A.

$\therefore A \neq B$.

Illustration 2: From the following sets, select equal sets:

A = {2, 4, 8, 12}, B = {1, 2, 3, 4}, C = {4, 8, 12, 14}, D = {3, 1, 4, 2}, E = {–1, 1}

F = {0, a}, G = {1, –1}, H = {0, 1} **(JEE MAIN)**

Sol: Similar to the previous question.

Since numbers 2, 4, 8, 12 belongs to set A, but 8, 12 ∉ B, 2 ∉ C, 8, 12, ∉ D, 2, 4, 8, 12, ∉ E, F, G and H.

Therefore, $A \neq B$, $A \neq C$, $A \neq D$, $A \neq E$, $A \neq F$, $A \neq G$, $A \neq H$

Elements 1, 2, 3, 4 belong to set B but 1, 2, 3 $\notin$ C; 2, 3, 4 $\notin$ F, G and H. Only D has all elements of B.

Therefore, $B \neq C$; $B \neq E$; $B \neq F$; $B \neq G$; $B \neq H$ and $B = D$.

Elements 4, 8, 12, 14 $\notin$ C; elements 8, 12, 14 $\notin$ D, E, F, G and H.

Therefore, $C \neq D$; $C \neq E$; $C \neq F$; $C \neq G$; $C \neq H$.

Repeating this procedure completely, we find only one equal set $E \neq F$, $D \neq E$, $D \neq F$, $D \neq G$, $D \neq H$, $E \neq H$ $G \neq H$, $F \neq G$, $F \neq H$.

Finally, we have two equal sets from given sets, i.e. B = D and E = G.

6. VENN DIAGRAMS

The diagrams drawn to represent sets and their relationships are called Venn diagrams or Euler–Venn diagrams. Here we represents the universal U as set of all points within rectangle and the subset A of the set U is represented by a circle inside the rectangle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then we represent A and B by two intersecting circles.

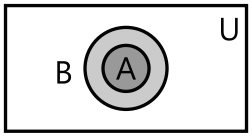


Figure 13.2

E.g., If A is subset of B then it is represent diagrammatically in Fig. 13.2. Let U be the universal set, A is a subset of set B. Then the Venn diagram is represented as follows:

If A is set, then the complement of A is represent as follows (Refer Fig. 13.3):

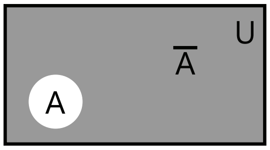


Figure 13.3

7. SET OPERATIONS

7.1 Union of Sets

If A and B are two sets, then union (∪) of A and B is the set of all elements belonging to set A and set B. It is also defined as $A \cup B = \{x : x \in A \text{ or } x \in B\}$. It is represented by shaded area in following figures.

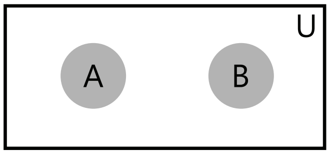


Figure 13.4

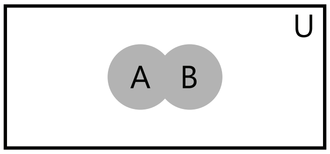


Figure 13.5

7.2 Intersection of Sets

If A and B are two sets, then intersection (∩) of A and B is the set of elements which belongs to both A and B in common, i.e. $A \cap B = \{x: x \in A \text{ and } x \in B\}$ represented with shaded area in Venn diagram (see Fig. 13.6)

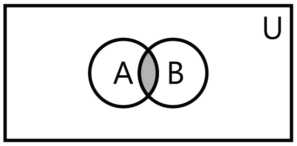


Figure 13.6

7.3 Difference of Two Sets

If A and B are two sets, then the difference of A and B is the set of elements which belongs to A and not B.

Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

$A - B \neq B - A$.

It is represented through the Venn diagrams in Fig. 13.7.

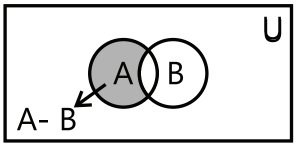


Figure 13.7

7.4 Symmetric Difference of Two Sets

Set of those elements which are obtained by taking the union of the difference of A and B: (A-B) and the difference of B and A: (B-A) is known as the symmetric difference of two sets A and B denoted by ($A \Delta B$).

Thus $A \Delta B = (A - B) \cup (B - A)$

Venn diagram is represented in Fig. 13.8.

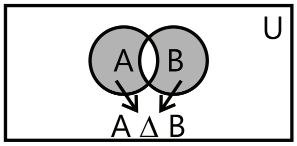


Figure 13.8

8. NUMBER OF ELEMENTS IN DIFFERENT SETS

If A, B and C are finite sets and U be the finite universal set, then

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(b) $n(A \cup B) = n(A) + n(B)$ (if A and B are disjoint sets)

(c) $n(A - B) = n(A) - n(A \cap B)$

(d) $n(A \Delta B) = n[(A - B) \cup (B - A)] = n(A) + n(B) - 2n(A \cap B)$

(e) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(f) $n(A' \cup B') = n(A \cap B)' = n(\cup) - n(A \cap B)$

(g) $n(A' \cap B') = n(A \cup B)' = n(\cup) - n(A \cup B)$

9. ALGEBRAIC OPERATIONS ON SETS

9.1 Idempotent Operation

For any set A, we have (i) $A \cup A = A$ and (ii) $A \cap A = A$

9.2 Identity Operation

For any set A, we have

(a) $A \cup \phi = A$

(b) $A \cap A = A$, i.e. ϕ and U are identity elements for union and intersection, respectively.

9.3 Commutative Operation

For any set A and B, we have

(a) $A \cup B = B \cup A$ and (ii) $A \cap B = B \cap A$

i.e. union and intersection are commutative.

9.4 Associative Operation

If A, B and C are any three sets, then

(a) $(A \cup B) \cup C = A \cup (B \cup C)$

(b) $(A \cap B) \cap C = A \cap (B \cap C)$

i.e. union and intersection are associative.

9.5 Distributive Operations

If A, B and C are any three sets, then

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union, respectively.

10. DE MORGAN'S PRINCIPLE

If A and B are any two sets, then

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

Proof: (a) Let x be an arbitrary element of $(A \cup B)'$. Then $x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow$ $x \notin A$ and $x \notin B$ $\Rightarrow$ $x \in A' \cap B'$

Again let y be an arbitrary element of $A' \cap B'$. Then $y \in A' \cap B'$

$\Rightarrow$ $y \in A'$ and $y \in B'$ $\Rightarrow$ $y \notin A$ and $y \notin B$

$\Rightarrow$ $y \notin (A \cup B)$ $\Rightarrow$ $y \in (A \cup B)'$

$\therefore$ $A' \cap B' \subseteq (A \cup B)'$

Hence $(A \cup B)' = A' \cap B'$

Similarly (b) can be proved.

RELATIONS

1. ORDERED PAIR

A pair of elements written in a particular order is called an ordered pair. Let A and B be two sets. If $a \in A$, $b \in B$, then elements (a, b) denotes an ordered pair, with first component a and second component b. Here, the order in which the elements a and b appear is important. The ordered pair (1, 2) and (2, 1) are different, because they represent different points in the co-ordinate plane.

Equality of ordered pairs: Ordered pair (a_1, b_1) is equal to (a_2, b_2) *iff* $a_1 = a_2$ and $b_1 = b_2$.

2. CARTESIAN PRODUCT OF SETS

2.1 Cartesian Product of Two Sets

The Cartesian product of two sets A and B is the set of all those ordered pair whose first co-ordinate is an element of set A and the second co-ordinate is an element of set B.

It is denoted by $A \times B$ and read as 'A cross B' or 'product set of A and B'.

i.e. $A \times B = \{(a, b): a \in A \text{ and } b \in B\}$

E.g. Let $A = \{1, 2, 3\}$

$B = \{3, 5\}$

Then $A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$ and

$B \times A = \{(3, 1), (3, 2), (3, 3), (5, 1), (5, 2), (5, 3)\}$

Hence $A \times B \neq B \times A$

Thus "Cartesian product of sets is not commutative".

E.g. $A = \{1, 2\}$, $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

Properties of Cartesian product

(a) $A \times B \neq B \times A$ (non-commutative)

(b) $n(A \times B) = n(A)\, n(B)$ and $n(P(A \times B)) = 2n(A)n(B)$

(c) $A = \phi$ and $B = \phi \Leftrightarrow A \times B = \phi$

(d) If A and B are two non-empty sets with n elements in common, then $(A \times B)$ and $(B \times A)$ have n^2 element in common.

(e) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(f) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(g) $A \times (B - C) = (A \times B) - (A \times C)$

Illustration 3: If $A = \{2, 4\}$ and $B = \{3, 4, 5\}$, then find $(A \cap B) \times (A \cup B)$ **(JEE MAIN)**

Sol: Use Cartesian Product of two Sets.

$A \cap B = \{4\}$ and $A \cup B = \{2, 3, 4, 5\}$

$\therefore \quad (A \cap B) \times (B \cup A) = \{(4, 2), (4, 3), (4, 4), (4, 5)\}$

2.2 Cartesian Product of More than Two Sets

The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \ldots \times A_n$ and is defined as $A_1 \times A_2 \times \ldots \times A_n = \{ (x_1\, x_2, \ldots, x_n): x_{i\in}\, A$, where $i = 1, 2, \ldots, n\}$

Note:

(i) Elements of $A \times B$ are called 2-tuples.

(ii) Elements of $A \times B \times C$ are called 3-tuples.

(iii) Elements of $A_1 \times A_2 \times \ldots \times A_n$ are also called n-tuples.

E.g. P = {1, 2}, from the set P × P × P.

P × P × P = {(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)}

2.3 Number of Elements in the Cartesian Product

If A and B are two finite sets then

n(A × B) = n(A) · n(B)

Thus, if A and B have m elements and n elements, respectively, then A × B has mn elements.

Proof: Let A = $\{x_1, x_2, x_3, ..., x_n\}$

and B = $\{y_1, y_2, y_3, ..., y_n\}$

Then A × B = $\{(x_1, y_1), (x_2, y_2) ... (x_1, y_n),$

$(x_2, y_1), (x_2, y_2) ... (x_2, y_n),$

⋮

$(x_m, y_1), (x_m, y_2) ... (x_m, y_n)\}$

Clearly each row has n ordered pairs and there are m such rows. Therefore A × B has mn elements.

Similarly, n(A × B × C) = n(A) · n(B) · n(C)

Illustration 4: If n(A) = 7, n(B) = 8 and n(A ∩ B) = 4, then match the following columns: **(JEE MAIN)**

(i)	n(A ∪B)	(a) 56
(ii)	n(A × B)	(b) 16
(iii)	n((A × B) × A)	(c) 392
(iv)	n((A × B) ∩ (B × A))	(d) 96
(v)	n((A × B) ∪ (B × A))	(e) 11

Sol: Use the formula studied in the above section.

(i) n (A ∪ B) = n (A) + n (B) – n (A ∩ B) = 7 + 8 – 4 = 11

(ii) n(A × B) = n(A) n(B) = 7 * 8 = 56 = n(B × A)

(iii) n((A × B) × A) = n(B × A) · n(A) = 56 × 7 = 392

(iv) n((A × B) ∩ (B × A)) = $(n(A \cap B))^2 = 4^2$ = 16

(v) n((A × B) ∪ (B × A)) = n(A × B) + n(B × A) – n(A × B) ∩ (B × A) = 56 + 56 – 16 = 96

2.4 Representation of Cartesian Product

Let A = {1, 2, 3} and B = {a, b}

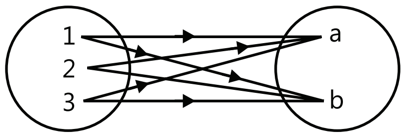


Figure 13.9

Each element of set A to each element of set B is represented by lines from A to B in Fig. 13.9.

3. RELATION

A relation R from one set to another say from set X to set Y ($R : X \rightarrow Y$) is a correspondence between set X to set Y through which some or more elements of X are associated with some or more elements of Y. Therefore a relation R, from a non-empty set X to another non-empty set Y, is a subset of $X \times Y$, i.e. $R: X \rightarrow Y$ is a subset of $A \times B$.

Every non-zero subset of $A \times B$ is defined as a relation from set A to set B. Therefore, if R is a relation from $A \rightarrow B$ then $R = \{(a, b) \mid (a, b) \in A \times B \text{ and } a\,R\,b\}$.

If A and B are two non-empty sets and $R: A \rightarrow B$ be a relation such that $R: \{(a, b) \mid (a, b) \in R, \text{ and } a \in A \text{ and } b \in B\}$, then

(a) 'b' is an image of 'a' under R.

(b) 'a' is a pre-image of 'b' under R.

For example consider sets X and Y of all male and female members of a royal family of the kingdom Ayodhya. X = {Dashrath, Ram, Bharat, Laxman, Shatrughan} and Y = {Koshaliya, Kaikai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti}. A relation R_H is defined as "husband of" from set X to set Y.

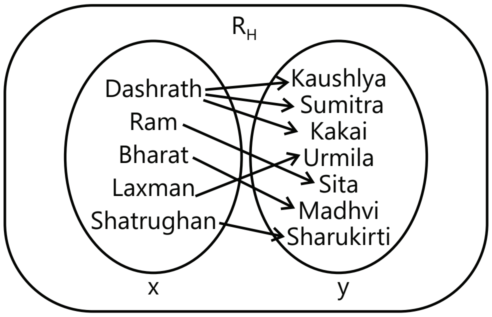


Figure 13.10

R_H = {(Dashrath, Koshaliya), (ram, sita), (Bharat, Mandavi), (Laxman, Urmila), (Shatrughan, Shrutkirti), (Dashrath, Kakai), (Dashrath, Sumitra)}

(a) If a is related to b, it is symbolically written as a R b.

(b) It is not necessary for each and every element of set A to have an image in set B, and set B to have a pre-image in set A.

(c) Elements of set A having image in B are not necessarily unique.

(d) Number of relations between A and B is the number of subsets of $A \times B$.

Number of relations = no. of ways of selecting a non-zero subset of $A \times B$.

$= {}^{mn}C_1 + {}^{mn}C_2 + \ldots + {}^{mn}C_{mn} = 2^{mn} - 1.$

3.1 Domain

Domain of a relation is a collection of elements of the first set participating in the correspondence, i.e. it is set of all pre-images under the relation. For example, from the above example, domain of R_H = {Dashrath, Ram, Bharat, Laxman, Shatrughan}.

3.2 Codomain

All elements of any set constitute co-domain, irrespective of whether they are related with any element of correspondence set or not, e.g. Y = {Koshaliya, Kakai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti} is co-domain of R_H.

3.3 Range

Range of relation is a set of those elements of set Y participating in correspondence, i.e. set of all images. Range of R_H = {Koshaliya, Kakai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti}.

Illustration 5: A = {1, 2, 3, 4, 5} and B = {2, 4, 5}. Relation between A and B is defined as a R b $\Rightarrow$ a and b are relatively prime or co-prime (i.e., HCF is 1). Find domain and range of R. **(JEE MAIN)**

Sol: Write the elements of the Relation and then write the domain and range.

R = {(1, 2), (1, 4), (1, 5), (2, 5), (3, 2), (3, 4), (3, 5), (4, 5), (5, 2), (5, 4)}

Domain of R {1, 2, 3, 4, 5}

Range of R {2, 4, 5}

Illustration 6: A = {Jaipur, Patna, Kanpur, Lucknow} and B = {Rajasthan, Uttar Pradesh, Bihar}

a R b $\Rightarrow$ a is capital of b, a $\in$ A and b $\in$ B. Find R. **(JEE MAIN)**

Sol: Use the concept / definition studied above.

R = {(Jaipur, Rajasthan), (Patna, Bihar), (Lucknow, Uttar Pradesh)}

Illustration 7: If A = {1, 3, 5, 7}, B = {2, 4, 6, 8}

Relation is a R b $\Rightarrow$ a > b, a $\in$ A, b $\in$ B. Find domain and range of R. **(JEE MAIN)**

Sol: Similar to the Illustration 3.

R = {(3, 2), (5, 2), (5, 4), (7, 2) (7, 4), (7, 6)}

Domain = {3, 5, 7}

Range = {2, 4, 6}

4. REPRESENTATION OF A RELATION

4.1 Roster Form

In this form we represent set of all ordered pairs (a, b) such that (a, b) $\in$ R where a $\in$ A, b $\in$ B.

4.2 Set Builder Form

Relation is denoted by the rule which co-relates the two set. This is similar to set builder form in sets.

4.3 Arrow-Diagram (Mapping)

It is a pictorial notation of any relation.

E.g. Let A = {–2, –1, 4} and B = {1, 4, 9}

A relation from A to B, i.e. a R b, is defined as a < b.

1. Roster form

R = {(0-2, 1), (–2, 4), (–2, 9), (–1, 1), (–1, 4), (–1, 9), (4, 9)}

2. Set builder notation

R = {(a, b): a $\in$A and b $\in$B, a is less than b}

3. Arrow-diagram (mapping)

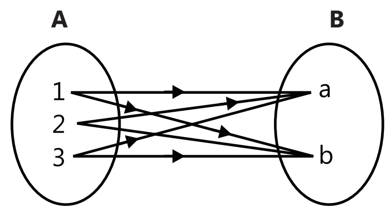


Figure 13.11

5. TYPES OF RELATION

5.1 Reflexive Relation

R: X $\rightarrow$ Y is said to be reflexive iff x R x $\forall$ x $\in$ X, i.e. every element in set X must be a related to itself. Therefore $\forall$ x $\in$ X; (x, x) $\in$ R, then relation R is called as reflexive relation.

5.2 Identity Relation

Consider a set X. Then the relation I = {(x, x): x $\in$ X} on X is called the identity relation on X i.e. a relation I on X is identity relation if every element of X related to itself only. For example, y = x.

Note: All identity relations are reflexive but all reflexive relations are not identity.

5.3 Symmetric Relation

R: X $\rightarrow$ Y is said to be symmetric iff (x, y) $\in$ R $\Rightarrow$ (y, x) $\in$ R.

For example, perpendicularity of lines in a plane is symmetric relation.

5.4 Transitive Relation

R: X $\rightarrow$ Y is transitive iff (x, y) $\in$ R and (y, z) $\in$ R $\Rightarrow$ (x, z) $\in$ R,

i.e. x R y and y R z $\Rightarrow$ x R z.

For example, the relation "being sister of" among the numbers of a family is always transitive.

Note:

(i) Every null relation is a symmetric and transitive relation.

(ii) Every singleton relation is a transitive relation.

(iii) Universal and identity relation are reflexive, symmetric as well as transitive.

5.5 Antisymmetric Relation

Let A be any set. A relation R on set A is said to be an antisymmetric relation iff (a, b) $\in$ R and (b, a) $\in$R $\Rightarrow$ a = b for all a, b $\in$A. For example, relations "being subset of"; "is greater than or equal to" and "identity relation on any set A" are anti-symmetric relations.

5.6 Equivalence Relation

A relation R from a set X to set Y ($R \rightarrow X \rightarrow Y$) is said to be an equivalence relation iff it is reflexive, symmetric as well as transitive. The equivalence relation is denoted by, e.g. relation "is equal to" equality similarity and congruence of triangle, parallelism of lines are equivalence relation.

5.7 Inverse Relation

If relation R is defined from A to B then inverse relation would be defined from B to A, i.e.

$R: A \rightarrow B \Rightarrow a\ R\ b$, where $a \in A$, $b \in B$.

$R^{-1}: B \rightarrow A \Rightarrow b\ R\ a$, where $a \in A$, $b \in B$.

Domain of R = Range of R^{-1}

and range of R = Domain of R^{-1}.

$\therefore \quad R^{-1} = \{b, a\} \mid (a, b) \in R\}$

For example, a relation R is defined on the set of 1st ten natural numbers.

$\therefore \quad N = \{1, 2, 3, ..., 10\}$ and $a, b \in N$.

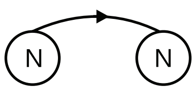


Figure 13.12

$a\ R\ b \Rightarrow a + 2b = 10$

$R = \{(2, 4), (4, 3), (6, 2)\ (8, 1)\}$

$R^{-1} = \{(4, 2), (3, 4), (2, 6)\ (1, 8)\}$

For example, a relation defined on the set of lines.

1. $aR\ b \Rightarrow a \parallel b$

It is a symmetric relation because if line 'a' is || to 'b', the line 'b' is || to 'a'.

where $(a, b) \in L$ {L is a set of || lines}

2. L_1RL_2 $L_{1\perp} L_2$ It is a symmetric relation

$L_{1,} L_{2 \in} L$ {L is a set of lines}

3. a R b $\Rightarrow$ 'a' is brother of 'b' is not a symmetric relation as 'b' may be sister of 'a'.

4. a R b $\Rightarrow$ 'a' is cousin of 'b'. This is a symmetric relation. If R is symmetric.

5. $R = R^{-1}$.

6. Range of R = Domain of R.

FUNCTIONS

1. INTRODUCTION

The concept of function is of fundamental importance in almost all branches of Mathematics. It plays a major role to solve real world problems in mathematics. As a matter of fact, functions are some special type of relations.

General definition:

Definition 1: Consider two sets A and B and let there exist a rule or manner or correspondence 'f' which associates to each element of A to a unique element in B. Then f is called a function or mapping from A to B. It is denoted by symbol f and represented by f: A →B (read as 'f' is a function from A to B or 'f maps A to B').

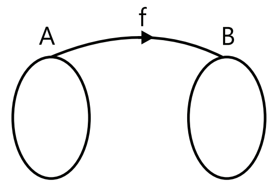


Figure 13. 13

If element $a \in A$ is associated with an element $b \in B$, then b is called the 'f image of a' or 'image of a under f' or 'the value of function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as

b = f(a) or f: a → b or f: (a, b)

Function as a set of ordered pairs:

A function f: A → B can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to A and second element is the corresponding element of B.

As such a function f: A → B can be considered as a set of ordered pairs (a, f(a)), where $a \in A$ and $f(a) \in B$, which is the f image of a. Hence, f is a subset of A × B.

Definition 2: A relation R from a set A to a set B is called a function if

(i) each element of A is associated with some element of B.

(ii) each element of A has unique image in B.

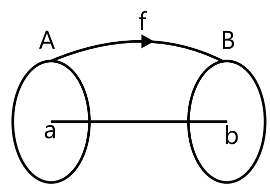


Figure 13. 14

Thus a function 'f' from set A to set B is a subset of A × B in which element a belonging to A appears in only one ordered pair belonging to f. Hence, a function f is a relation from A to B satisfying the following properties:

(i) $f \subset A \times B$ (ii) $\forall a \in A \Rightarrow (a, (f(a)) \in f$ and (iii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.

CONCEPTS

Every function is a relation, but every relation may not necessarily be a function.

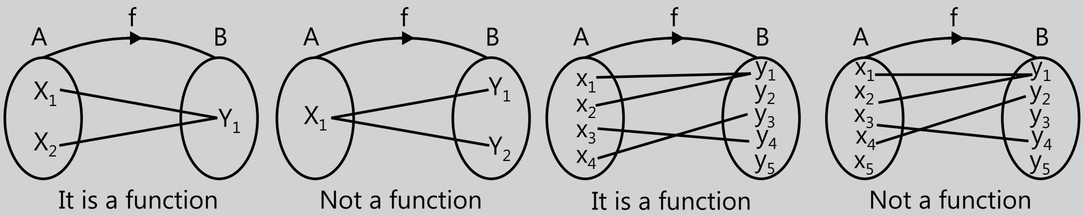


Figure 13.15

Easy way to differentiate function from relation is as follows :

1. Consider every element in set A is a guest and every element in set B is hosting a function at the same time and invited every element from A.

2. None of the elements can be at two functions simultaneously.

3. So if an element is attending two functions at the same time, then it is just a relation and if an element is attending only one function then it is said to be a function.

Chen Reddy Sundeep Reddy (JEE 2012, AIR 63)

2. RELATION VS FUNCTION

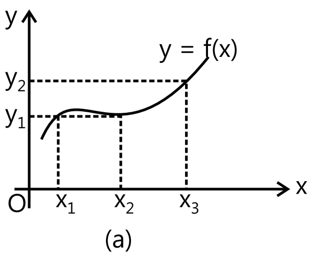


Figure 13.16

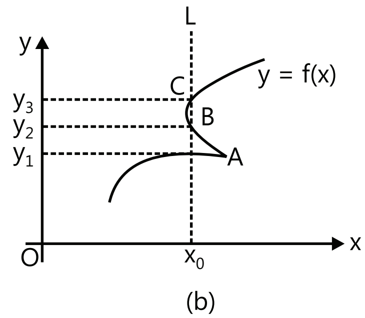


Figure 13.17

These figures show the graph of two arbitrary curves. In figure 16, any line drawn parallel to y-axis would meet the curve at only one point. That means each element of X would have only one image. Thus figure 16 (a) represents the graph of a function.

In figure 17, certain line parallel to y-axis (e.g., line L) would meet the curve in more than one point (A, B and C). Thus element x_0 of X would have three distinct images. Thus, this curve does not represent a function.

Hence, if y = f(x) represents a function, then lines drawn parallel to y-axis through different point corresponding to points of set X should meet the curve at one point.

Equation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a relation, which is a combination of two functions

$y=b\sqrt{1-\frac{x^2}{a^2}}$ and $y=-b\sqrt{1-\frac{x^2}{a^2}}$. Similarly, the equation of the parabola $y^2 = x$ is a combination of two functions as shown in Fig. 13.18.

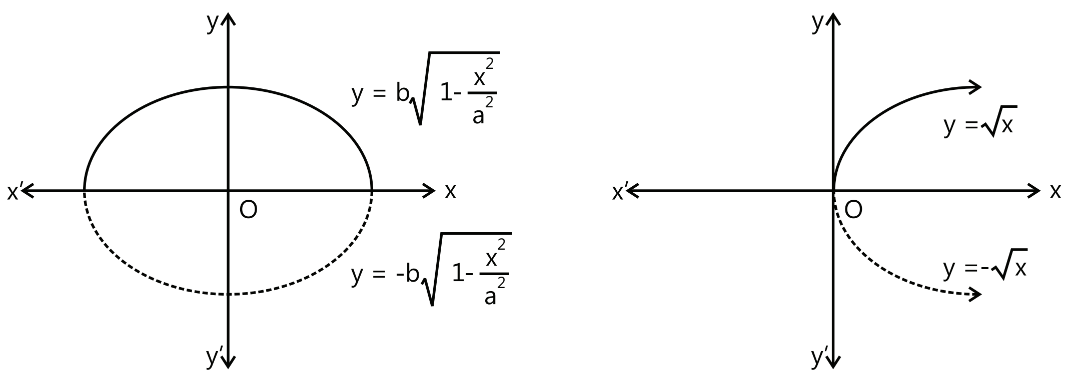


Figure 13.18

3. DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Let f: A → B, then the set A is known as the domain of f and the set B is known as co-domain of f.

The set of all f images of elements of A is known as the range of f. Thus:

Domain of f = {a |a ∈ A, (a, f(a)) ∈ f} Range of f = {f(a) ∀ a ∈ A and f(a) ∈ B}

It should be noted that range is a subset of co-domain. If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined.

Note: If domain of f(x) is D_1 and domain of g(x) is D_2 then domain of f(x) + g(x) = $D_1 \cap D_2$.

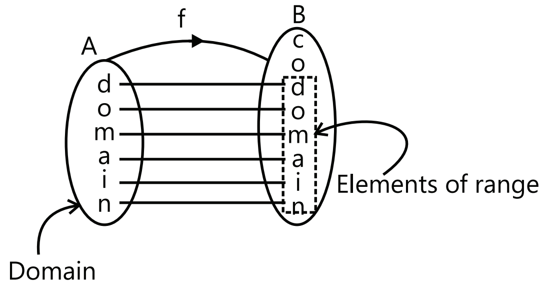


Figure 13.19

4. COMMON FUNCTIONS

4.1 Polynomial Function

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... a_{n-1}x + a_n$ where n is a non-negative integer and a_0, a_1, a_2, ..., a_n are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

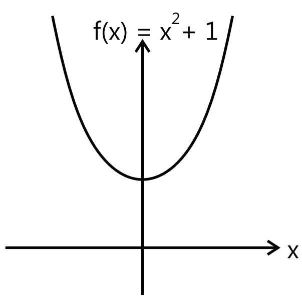


Figure 13.20

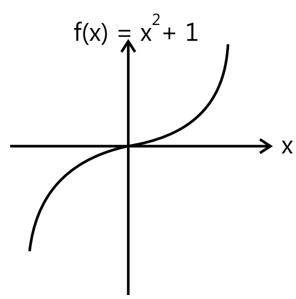


Figure 13.21

CONCEPTS

(a) A polynomial of degree one with no constant term is called an odd linear function i.e., $f(x) = a_x$, $a \neq 0$

(b) There are two polynomial functions, satisfying the relation ; f(x) f(1/x) = f(x) + f(1/x)

They are :

(i) $f(x) = x^n + 1$ and (ii) $f(x) = 1 - x^n$, where n is positive integer.

(c) Domain of a polynomial function is R

(d) Range for odd degree polynomial is R whereas for even degree polynomial range is a subset of R.

(i) If f(x) + f(y) = f(xy) then f(x) = k log x

(ii) If f(x) · f(y) = f(x + y) then $f(x) = a^{kx}$

(iii) If f(x) + f(y) = f(x+ y) then f(x) = kx

Rohit Kumar (JEE 2012, AIR 79)

4.2 Algebraic Function

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x) = y^n + P_1(x)\, y^{n-1} + ... + P_{n-1}(x)\, y + P_n(x) = 0$ where n is a positive integer and $P_0(x)$,

$P_1(x)$ .... are polynomials in x e.g., $x^3 + y^3 - 3xy = 0$ or

$y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

CONCEPTS

All polynomial functions are Algebraic but not the converse. A function that is not algebraic is called Transcendental function.

Anvit Tawar (JEE 2012, AIR 9)

4.3 Fractional/Rational Function

It is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) and h(x) are polynomial and $h(x) \neq 0$.

4.4 Exponential/Logarithmic Function

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function. The inverse of exponential function is called the logarithmic function.

i.e. $g(x) = \log_a x$.

CONCEPTS

(a) $f(x) = e^x$ domain is R and range is R^+.

(b) $f(x) = e^{1/x}$ domain is $R - \{0\}$ and range is $R^+ - \{1\}$.

(c) f(x) and g(x) are inverse of each other and their graphs are as shown.

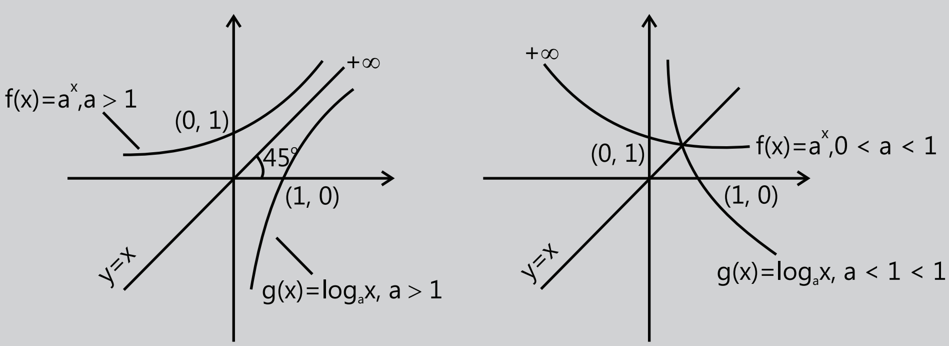


Figure 13.22

Shivam Agarwal (JEE 2009, AIR 27)

4.5 Absolute Value Function

A function $y = f(x) = |x|$ is called the absolute value function or modulus function. It is defined as $y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

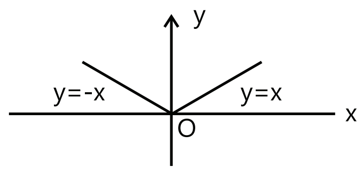


Figure 13.23

Note: (a) f(x) = | x |, domain is R and range is $R^+ \cup \{0\}$.

(b) f(x) = $\frac{1}{f|x|}$, domain is R – {0} and range is R^+.

4.6 Signum Function

A function y = f(x) = sgn(x) is defined as follows: y = f(x) = $\begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ If is also written as sgn x = | x |/ x or $\frac{x}{|x|}$;x ≠ 0; f(0) = 0

Note: Domain is $x \in R$ and its range = {–1, 0, 1}

sgn x = $\frac{d}{dx}|x|$ when $x \neq 0$

= 0 when x = 0

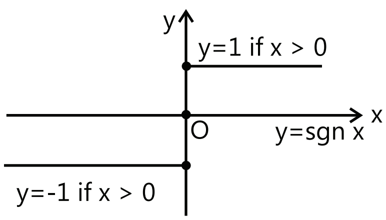


Figure 13.24

4.7 Greatest Integer Function or Step Function

The function y = f(x) = [x] is called the greatest integer function, where [x] denotes the greatest integer less than or equal to x.

Note:

(a) $-1 \le x < 0 \Rightarrow$ [x] = –1

$0 \le x < 1 \Rightarrow$ [x] = 0

$1 \le x < 2 \Rightarrow$ [x] = 1

$2 \le x < 3 \Rightarrow$ [x] = 2 etc.

(b) f(x) = [x], domain is R and range is I.

(c) f(x) = $\frac{1}{[x]}$ domain is R – [0, 1) and range is $\left\{\frac{1}{n} \middle| n \in I - \{0\}\right\}$

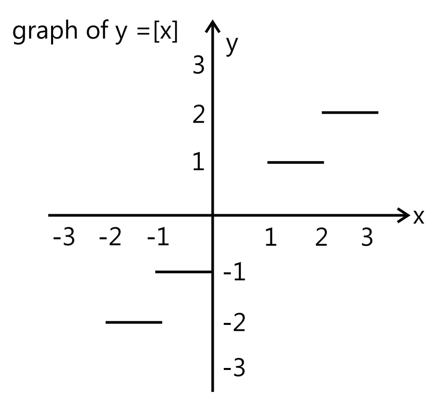


Figure 13.25

Properties of greatest integer function:

(a) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$.

(b) [x + m] = [x] + m if m is an integer.

(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$.

(d) [x] + [–x] = 0 if x is an integer= –1 otherwise.

4.8 Fractional Part Function

It is defined as g(x) = {x} = x – [x].

For example, the fractional part of the number 2.1 is 2.1 – 2 = 0.1 and the fractional part of –3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.

Note: (a) f(x) = {x}, domain is R and range is [0, 1)

(b) $f(x) = \frac{1}{\{x\}}$, domain is R – I, range is (1, ∞)

{x + n} = {x}, where n ∈ I

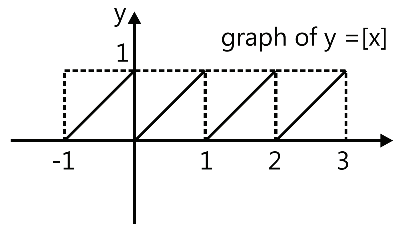


Figure 13.26

5. TRANSFORMATION OF CURVES

(a) Given graph of a function y = f(x) and we have to draw the graph of y = f(x – a) [means replacing x by x–a, a > 0] then shift the entire graph through a distance a units in positive direction of x-axis.

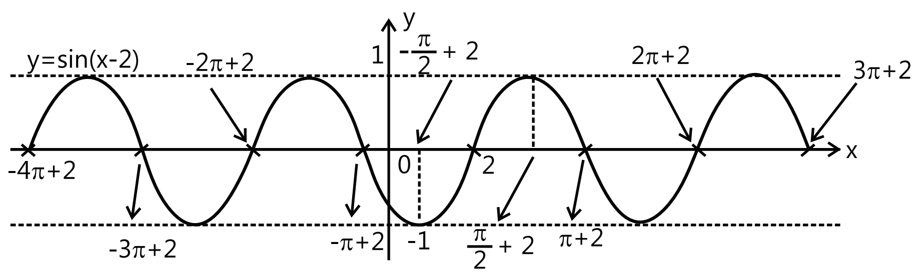


Figure 13.27

(b) Given graph of a function y = f(x), draw a graph of y = f(x + a) [means replacing x by x + a, a > 0]. Shift the entire graph through in negative direction of x-axis.

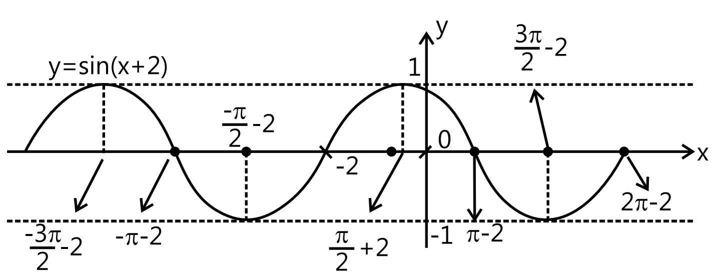


Figure 13.28

(c) Given graph of a function y = f(x), draw a graph of y = af(x) [means replacing y by y/a, a >0]. Then multiply all the values by a on y-axis.

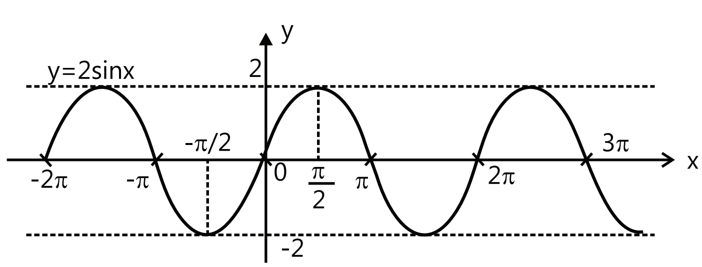


Figure 13.29

(d) Given graph of a function y = f(x), draw a graph of y = f(ax) [means replacing x by ax, a >0]. Then divide all the values by a on x-axis.

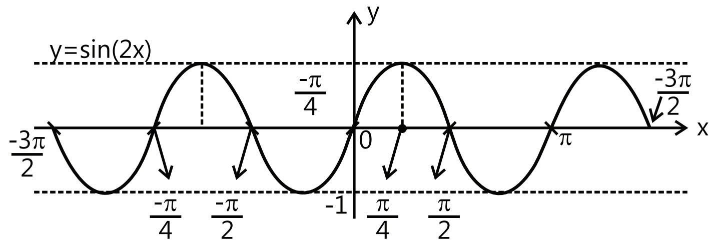


Figure 13.30

(e) Given graph of a function y = f(x), draw the graph of y = f(x) + a [means replacing y by y – a, a >0]. Then shift the entire graph in positive direction of y-axis.

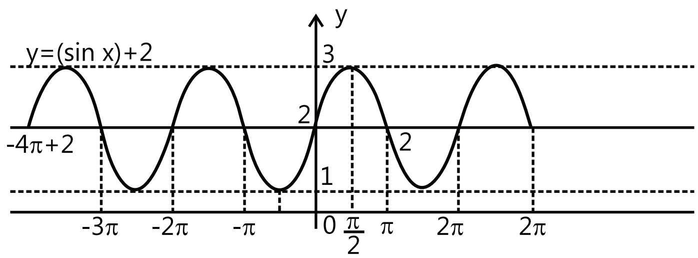


Figure 13.31

(f) Given graph of a function y = f(x), draw the graph of y = f(x) –a [means replacing y by y + a, a > 0]. Then shift the entire graph in negative direction of y-axis.

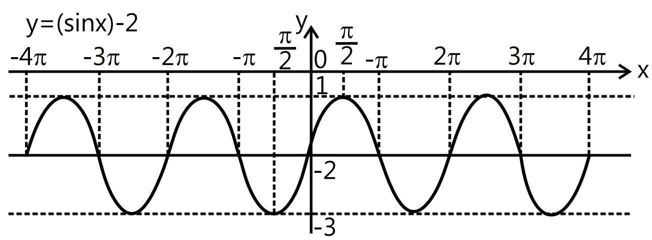


Figure 13.32

(g) Given graph of a function y = f(x), draw the graph of y = f(–x) [means replacing x by –x]. Then take the reflection of the entire curve in y-axis.

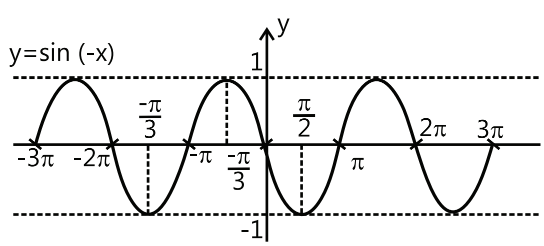


Figure 13.33

(h) Given graph of a function y = f(x), draw the graph of y = –f(x) [means replacing y by –y]. Then take the reflection of the entire curve in x-axis.

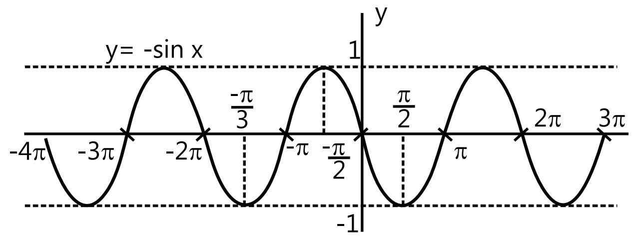


Figure 13.34

(i) Given graph of a function y = f(x), draw the graph of y = f(|x|) [means replacing x by |x|]

(a) Remove the portion of the curve, on left-hand side of y-axis.

(b) Take the reflection of right-hand size on the left-hand side.

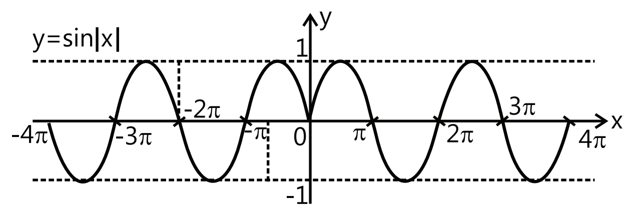


Figure 13.35

(j) Given graph of a function y = f(x), draw the graph of y = | f(x) |.The projection of the curve lying below x-axis will go above the axis.

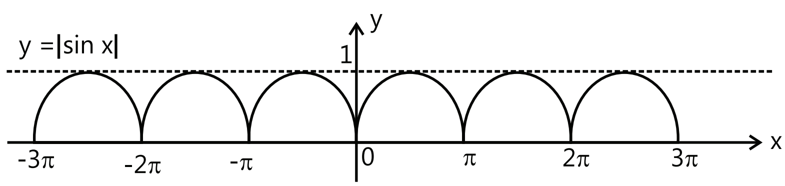


Figure 13.36

(k) Given graph of a function y = f(x), draw the graph of |y| = f(x) [means replacing y by |y|]. Then remove a portion of the curve below x-axis and then take the reflection of the upper part on the lower part.

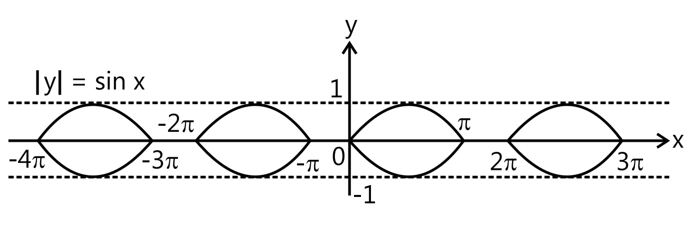


Figure 13.37

6. TIPS FOR PLOTTING THE GRAPH OF A RATIONAL FUNCTION

(a) Examine whether denominator has a root or not. If no root, then graph is continuous and f is non-monotonic.

For example, $f(x) = \dfrac{x}{x^2 - 5x + 9} \Rightarrow$ for $D_rD < 0$, D_r will never be zero.

f(x) is discontinuous, only when dominator has roots and hence non-monotonic.

For example, $f(x) = \frac{x^2+2x-3}{x^2+2x-8} = \frac{(x+3)(x-1)}{(x+4)(x-2)}$

D = 0 at x = –4, 2.

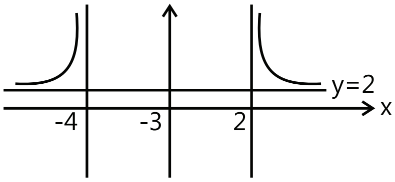


Figure 13.38

(b) If numerator and denominator has a common factor (say x = a) it would mean removable discontinuity at x = a, E.g. $f(x) = \frac{(x-2)(x-1)}{(x+3)(x-2)}$. Such a function will always be monotonic, i.e. either increasing or decreasing and removable discontinuity at x = 2.

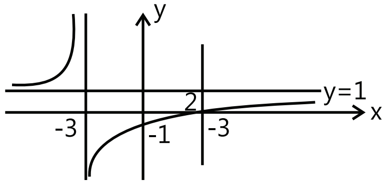


Figure 13.39

(c) Compute point where the curve cuts both x-axis and y–axis by putting y = 0 and x = 0, respectively, and mark points accordingly.

f(x) = x – 1

x = 0, y = –1

y = 0, x = 1

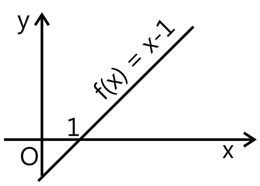


Figure 13.40

(d) Compute $\frac{dy}{dx}$ and find the intervals where f(x) increases or decreases and also where it has horizontal tangent.

$y = x^2 - 3x + 2$

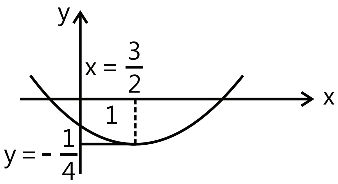


Figure 13.41

(e) In regions where curves is monotonic, compute y if $x \to \infty$ or $x \to -\infty$ to find whether y is asymptotic or not.

$f(x) = \frac{x-1}{x-3}$

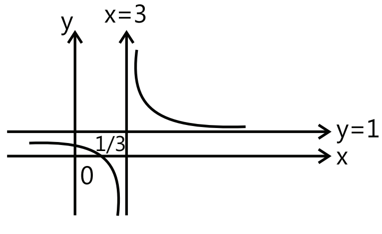


Figure 13.42

(f) If denominator vanishes, say at x = a and (x – a) is not a common factor between numerator and denominator, then examine $\lim_{x\to a^-}$ and $\lim_{x\to a^+}$ to find whether f approaches ∞ or –∞. Plot the graphs of the following function.

Illustration 8: Draw the graph of functions $f(x) = \dfrac{x}{\ell nx}$

Sol: Calculate the domain of the given function. Then use the derivative of the given function to trace the given curve.

Domain of f(x) is $x \in R\ (0, 1) \cup (1, \infty)$, $f'(x) = \dfrac{1.\ell nx - x\cdot(1/x)}{(\ell nx)^2}$

$f'(x) = 0$ at $x = e$

also as x approaches zero f(x) approaches zero from negative side and x approaches ∞ f(x) approaches +∞.

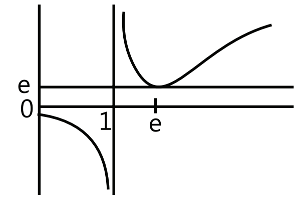


Figure 13.43

From the graph we can observe the range of f(x) is $(-\infty, 0) \cup [e, \infty)$.

7. TO FIND DOMAIN AND RANGE

To calculate domain or range of a function, the following points are considered.

7.1 Domain

(a) Expression under even root (i.e. square root, fourth root, etc.) ≥ 0 and denominator ≠ 0.

(b) If domain of y = f(x) and y = g(x) are D_1 and D_2 respectively, then the domain of f(x) + g(x) or f(x) · g(x) is $D_1 \cap D_2$.

(c) Domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

(d) Domain of $\sqrt{f(x)} = D_1 \cap \{x: f(x) \geq 0\}$

(e) Expression inside logarithm should be positive, i.e. for $\log_a E$ to exist, E should positive.

7.2 Range

(a) For the real valued function, real values of y for x ∈ domain of f are the range of the function.

Therefore, find domain of f and then impose restriction upon y, for the values of x in domain.

(b) If f is a continuous real valued function, then the range of function = [minimum f, maximum f].

Method of finding range:

(a) Range of the function in restricted domain

For the range of y = f(x) in the interval [a, b], retain the portion of the curve y = f(x) below the lines x = a and x = b. Then the required range is the projection of y-axis.

(b) Range of composite function

To find the range of f(g(x)), first find the range of g(x), say A. Then find the range of f(x) in domain A.

(c) If f = (A sin x + B cos x) + C then range of function is $\left[-\sqrt{A^2+B^2}+C,\ \sqrt{A^2+B^2}+C\right]$.

(d) Range of periodic function can be found only for the interval whose length is a period of a function.

(e) Similarly for odd functions, if range on right-hand side on x-axis is (α, β). Then range on left-hand side of x-axis will be (–α, –β), to get the final range, union of both these.

(f) Change of variable

y = f(g(x)); to get the range of f(g(x)) substitute g(x) at t and then find the range of f(t) the domain of f(t).

CONCEPTS

Whenever we substitute the variable t for g(x), care should be taken that the corresponding condition on t should be written immediately. Further analysis of the function will be according to the condition.

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 9: Find domain and range of f(x) = $\cot^{-1}\log_{\frac{4}{5}}(5x^2 - 8x + 4)$. **(JEE MAIN)**

Sol: Use the definition of Domain and Range.

Consider the quadratic expression P(x) = $5x^2 - 8x + 4$.

For the above quadratic, D < 0 $\Rightarrow$ the expression always positive.

$\therefore$ e expression always $\in$ R.

$P_{maximum} = \infty$, $P_{minimum} = 4/5$.

$\therefore$ Range of $\log_{\frac{4}{5}}(5x^2 - 8x + 4)$ is $(-\infty, 1]$

Now, draw the graph of $\cot^{-1}(t)$ for t $(-\infty, 1]$

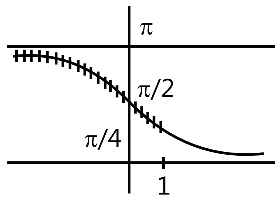


Figure 13.44

From the graph we can observe that range of f(x) is $\left[\frac{\pi}{4}, \pi\right)$

Illustration 10: Draw a graph of f(x) = $\frac{x^2 - x + 1}{x^2 + x + 1}$ and also evaluate its domain and range. **(JEE MAIN)**

Sol: Find derivative of the given function and use the techniques of curve tracing.

For drawing the graph of f(x) first find out those point where f′(x) = 0

$f'(x) = \frac{2(x^2 - 1)}{(x^2 + x + 1)^2}$

f′(x) = 0 at x = –1, 1

$f(1) = \frac{1}{3}$; f(–1) = 3

Also when x approach ±∞, f approaches 1.

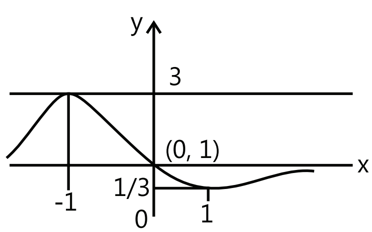


Figure 13.45

From the graph, domain is x $\in$ R and range $\left[\frac{1}{3}, 3\right]$

Note: Graph of f(x) = $\frac{ax + b}{cx + d}$ is always monotonic.

Illustration 11: If f(x) = $\sin^{-1} x^2 + \left[\ln\sqrt{x - [x]}\right] + \cot\left(\frac{1}{1 + \sqrt{2}x^2}\right)$. Then find its domain and range. **(JEE ADVANCED)**

Sol: Follow the steps discussed above.

Domain of function is {–1, 1} – {0}, because x – [x] = 0 for integral value of x; hence, middle term will not be defined.

Also [{f}] = 0, whenever f is meaningful.

Therefore value of f(x) = $\sin^{-1}x^2 + \tan^{-1}(1 + \sqrt{2}\,x^2)$ $\left(\cot^{-1} x = \tan^{-1}\frac{1}{x} \text{ when } x > 0\right)$

Function is continuous and is even.

Least value of the function will occur when $x \to 0$ and is $\frac{\pi}{4}$.

Maximum value = $\lim_{x \to \pm 1} f(x) = \sin^{-1} 1 + \tan^{-1}(1 + \sqrt{2}) = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}$

Therefore, range of f(x) is $\left(\frac{\pi}{4}, \frac{7\pi}{8}\right)$.

8. DOMAIN AND RANGE OF COMMON FUNCTION

A. ALGEBRAIC FUNCTION

Function	Domain	Range
(i) x^n, $(n \in N)$	R = set of real numbers	R, if n is odd $R^+ \cup \{0\}$, if n is even
(ii) $\frac{1}{x^n}$, $(n \in N)$	R – {0}	R – {0}, if n is odd R^+, if n is even
(iii) $x^{1/n}$ $(n \in N)$	R, if n is odd $R^+ \cup \{0\}$, if n is even	R, if n is odd $R^+ \cup \{0\}$, if n is even
(iv) $\frac{1}{x^{1/n}}$, $(n \in N)$	R – {0}, if n is odd R^+, if n is even	R – {0}, if n is odd R^+, if n is even

B. TRIGONOMETRIC FUNCTION

Function	Domain	Range
(i) sin x	R	[–1, 1]
(ii) cos x	R	[–1, 1]
(iii) tan x	$R - (2k + 1)\frac{\pi}{2}$, $k \in I$	R
(iv) sec x	$R - (2k + 1)\frac{\pi}{2}$, $k \in I$	$(-\infty, -1] \cup [1, \infty)$
(v) cosec x	$R - k\pi$, , $k \in I$	$(-\infty, -1] \cup [1, \infty)$
(vi) cot x	$R - k\pi$, , $k \in I$	R

C. INVERSE TRIGONOMETRIC FUNCTION

Function	Domain	Range
(i) $\sin^{-1} x$	[–1, 1]	$\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$
(ii) $\cos^{-1} x$	[–1, 1]	$[0, \pi]$
(iii) $\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv) $\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left[\frac{\pi}{2}\right]$
(v) $\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(vi) $\cot^{-1} x$	R	$(0, \pi)$

D. EXPONENTIAL FUNCTION

Function	Domain	Range
(i) e^x	R	R^+
(ii) $e^{1/x}$	R – {0}	$R^+ - \{1\}$
(iii) a^x, a > 0	R	R^+
(iv) $a^{1/x}$, a > 0	R – {0}	$R^+ - \{1\}$

E. LOGARITHIMIC FUNCTION

Function	Domain	Range
(i) $\log_a x$, $(a > 0)(a \neq 1)$	R^+	R
(ii) $\log_x a = \frac{1}{\log_a x}$ $(a > 0)\ (a \neq 1)$	$R^+ - \{1\}$	R – {0}

F. INTEGRAL PART FUNCTION

Function	Domain	Range
(i) [x]	R	I
(ii) $\frac{1}{[x]}$	R – [0, 1)	$\left\{\frac{1}{n}, n \in I - \{0\}\right\}$

G. FRACTIONAL FUNCTION

Function	Domain	Range
(i) {x}	R	[0, 1)
(ii) $\frac{1}{\{x\}}$	R – I	$(1, \infty)$

H. MODULUS FUNCTION

Function	Domain	Range
(i) $\|x\|$	R	$R^+ \cup \{0\}$
(ii) $\frac{1}{\|x\|}$	R – {0}	R^+

I. SIGNUM FUNCTION

Function	Domain	Range
$sgn(x) = \frac{\|x\|}{x}$	R	{–1, 0, 1}

J. CONSTANT FUNCTION

Function	Domain	Range
f(x) = c	R	{c}

9. EQUAL OR IDENTICAL FUNCTION

Two functions *f* and g are said to be equal if the following conditions are satisfied:

(i) The domain of *f* is equal to the domain of g

(ii) The range of f is equal to the range of g and

(iii) f(x) = g(x), for every x belonging to their common domain,

E.g. $f(x) = \frac{1}{x}$ and $g(x)\ \frac{x}{x^2}$ are identical functions.

Few examples of equal functions are listed as follows:

(i) $f(x) = \ln x^2;\ g(x) = 2\ln x$ (N.I.)

(ii) $f(x) = \sin^{-1}(3x - 4x^3);\ g(x) = 3\sin^{-1} x$ (N.I.)

(iii) $f(x) = \sec^{-1}x + \operatorname{cosec}^{-1}x;\ g(x) = \frac{\pi}{2}$ (N.I.)

(iv) $f(x) = \cot^2x.\cos^2x : g(x) = \cot^2x - \cos^2x$ (I)

(v) $f(x) = \operatorname{Sgn}(x^2 + 1) ;\ g(x) = \sin^2x + \cos^2x$ (I)

(vi) $f(x) = \tan^2x.\sin^2x;\ g(x) = \tan^2x - \sin^2x$ (I)

(vii) $f(x) = \sec^2x - \tan^2x; : g(x) = 1$ (N.I.)

(viii) $f(x) = \tan(\cot^{-1}x);\ g(x) = \cot(\tan^{-1} x)$ (I)

(ix) $f(x) = \sqrt{x^2 - 1}\ ;\ g(x) = \sqrt{x-1}\sqrt{x+1}$ (N.I.)

(x) $f(x) = \tan x.\ \cot x\ ;\ g(x) = \sin x\ .\ \operatorname{cosec} x$ (N.I.)

(xi) $f(x) = e^{\ell n\, e^x}\ ;\ g(x) = e^x$ (I)

(xii) $f(x) = \sqrt{\frac{1 - \cos 2x}{2}}\ ;\ g(x) = \sin x$ (N.I.)

(xiii) $f(x) = \sqrt{x^2}$; $g(x) = \left(\sqrt{x}\right)^2$ (N.I.)

(xiv) $f(x) = \log(x + 2) + \log(x - 3)$; $g(x) = (x^2 - x - 6)$ (N.I.)

(xv) $f(x) = \frac{1}{|x|}$; $g(x) = \sqrt{x^{-2}}$ (I)

(xvi) $f(x) = x\,|x|$; $g(x) = x^2 \text{ sgn } x$ (I)

(xvii) $f(x) = \underset{n\to\infty}{\text{Limit}} \frac{x^{2n}-1}{x^{2n}+1}$; $g(x) = \text{sgn}(|x| - 1)$ (I)

(xviii) $f(x) = \sin(\sin^{-1} x)$; $g(x) = \cos(\cos^{-1} x)$ (I)

(xix) $f(x) = \frac{1}{1+\frac{1}{x}}$; $g(x) = \frac{x}{1+x}$ (N.I.)

(xx) $f(x) = [\{x\}]$; $g(x) = \{[x]\}$ (I) (note that f(x) and g(x) are constant functions)

(xxi) $f(x) = e^{\ell n \cot^{-1} x}$; $g(x) = \cot^{-1} x$ (I)

(xxii) $f(x) = e^{\ell n \sec^{-1} x}$; $g(x) = \sec^{-1} x$ (N.I.) Identical if $x \in (-\infty, -1] \cup (1, \infty)$ (xxiii)

(xxiii) f(x) = (fog)(x); G(x) = (gof)(x) where f(x) = ex; g(x) = ln x (N.I.)

10. CLASSIFICATIONS OF FUNCTIONS

10.1 One–One Function

One–one function (Injective mapping)

A function $f : A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ and $f(x_1) \in A$.

$f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_{1\neq}\, x_2 \Leftrightarrow f(x_1)\ {}^1 f(x_2)$

Example: $f_1 \to R \to R$, $f(x) = x^3 + 1$; $f(x) = e^{-x}$; $f(x) = \log x$

Diagrammatically an injective mapping is shown as follows:

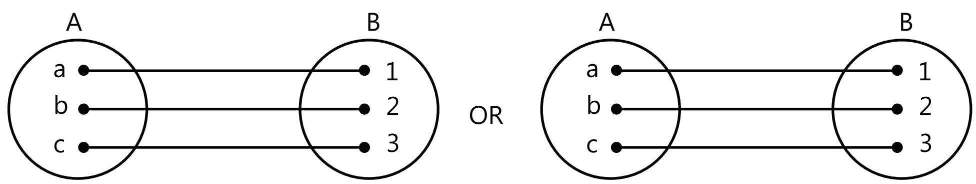


Figure 13.46

Note:

(i) If there is increase or decrease of continuous function in whole domain, then f(x) is one–one.

(ii) If any line parallel to x-axis cuts the graph of the function only at one point, then the function is one–one.

10.2 Many–One Function (Not Injective)

A function f: $A \to B$ is said to be a many–one function if two or more elements of A have the same f image in B. Thus for f: $A \to B$ is many–one if for $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$ but $x_{1\neq}\, x_2$

For example, $f_{1\to}$ $R \to R$, $f(x) = [\ x\ ]$; $f(x) = |\ x\ |$; $f(x) = ax^2 + bx = c$; $f(x) = \sin x$.

Diagrammatically a many–one mapping is shown as follows:

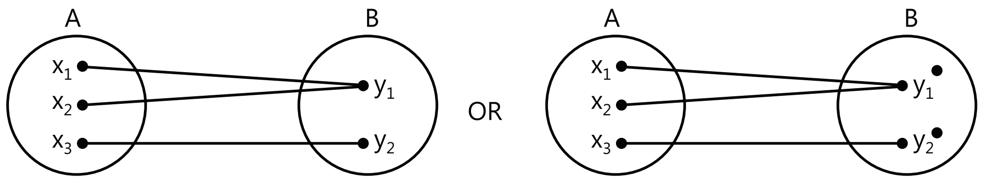


Figure 13.47

Note:

(i) If any continuous function has at least one local maximum or local minimum, then f(x) is many–one. In other words, if a line parallel to x-axis cuts the graph of the function at least at two points, then f is many–one.

(ii) If a function is one–one, it cannot be many–one and vice versa.

One–one + Many–one = Total number of mapping.

10.3 Onto Functions

If the function $f : A \to B$ is such that each element in B (co-domain) is the f image of at least one element in A, then we say that f is a function of A 'onto' B. Thus $f: A \to B$ is surjective $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

Diagrammatically surjective mapping is shown as follows:

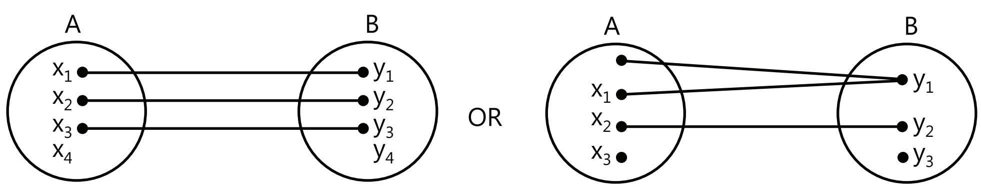


Figure 13.48

Note: if range is equal to co-domain, then f(x) is onto.

10.4 Into Functions

If $f: A \to B$ is such that there exists at least one element in co-domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function is shown as follows:

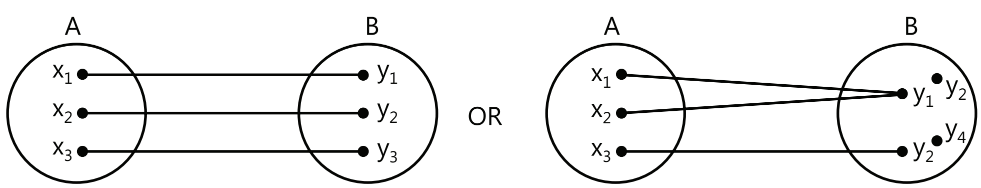


Figure 13.49

Note: If a function is onto, it cannot be into and vice versa. A polynomial of degree even and odd defined from R $\to$ R will always be into and onto, respectively.

Thus a function can be one of these four types:

(i) one–one onto (injective & surjective) $(I \cap S)$

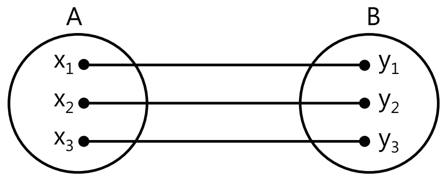


Figure 13.50

(ii) one–one onto (injective but not surjective) $(I \cap \bar{S})$

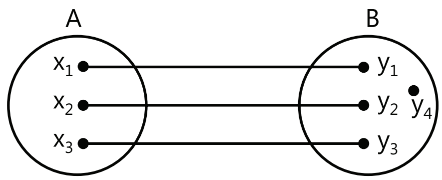


Figure 13.51

(iii) Many–one onto (surjective but not injective) $(S \cap \bar{I})$

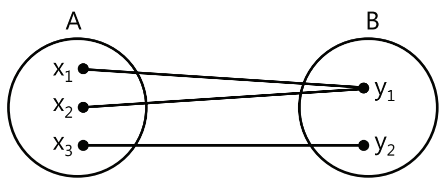


Figure 13.52

(iv) Many–one onto (neither surjective nor injective) $(\bar{I} \cap \bar{S})$

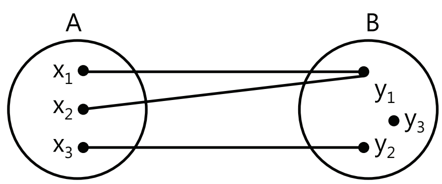


Figure 13.53

Illustration 12: Let A be a finite set. If *f*: A → A is an onto function, then show that f is one–one. **(JEE MAIN)**

Sol: Use the definition of one-one and onto function.

Let A = $\{a_1, a_2, ..., a_n\}$. To prove that f is a one–one function, we will have to show that $f(a_1), f(a_2), ..., f(a_n)$ are distinct elements of A. We have,

Range of f = $\{f(a_1), \cdot f(a_2), ..., f(a_n)\}$

Since f : A → A is a onto function. Therefore,

Range of f = A.

$\Rightarrow f = \{f(a_1), f(a_2), ..., f(a_n)\} = A$

But, A is a finite set consisting of n elements.

Therefore, $f(a_1), f(a_2), f(a_3), ..., f(a_n)$ are distinct element of A.

Hence, f : A → A is one–one.

Illustration 13: Let C and R denote the set of all complex and all real numbers respectively. Then show that *f*: C → R given by f(z) = | z |, for all z ∈ C is neither one–one nor onto. **(JEE MAIN)**

Sol: Using two complex conjugate numbers, we can prove that the given function is not one-one. For the second part, use the fact that modulus of a number cannot be negative.

Injectivity: We find that z_1 = 1 – i and z_2 = 1 + i are two distinct complex numbers such that | z_1 | = | z_2 |, i.e. $z_{1'}z_2$ but $f(z_1) = f(z_2)$.

It is clear that different elements may have the same image. So, f is not an injection.

Surjectivity: f is not a surjection, because negative real numbers in R do not have their pre-image in C. In other words, for every negative real number there is no complex number z ∈ C such that f(z) = | z | = a. So, *f* is not a surjection.

Illustration 14: For function *f*: A →A, fof = f. Prove that f is one–one if and only if f is onto. **(JEE MAIN)**

Sol: Starting with the relation given in the question fof = f and use the definition of one-one and onto function.

Suppose f is one–one.

Then, (fof) (x) = f(x)

f(f(x)) = f(x)

f(x) = x (f is one–one)

Thus, f(x) = x, ∀ x ∈ A.

for each x A, there is an x ∈A such that f(x) = x

∴ f is onto.

Now suppose f is onto.

Then, for each y ∈A, there is an x ∈A such that f(x) = y.

Let x_1, x_2 ∈ A and let $f(x_1) = f(x_2)$.

Then there exist y_1, y_2 ∈ A such that $x_1 = f(y_1) \cdot x_2 = f(y_2)$

$f(f(y_1)) = f(f(y_2))$

$(fof)(y_1) = (fof)(y_2)$

$f(y_1) = f(y_2)$ (fof = f)

$x_1 = x_2$

∴ f is one–one.

Illustration 15: Show that the function *f*: R → R given by f(x) = x^3 + x is a bijection. **(JEE ADVANCED)**

Sol: Consider two elements x and y in the domain and prove that f(x) = f(y) implies x = y. Use the definition of onto to prove that the function f is a bijection.

Injectivity: Let x, y ∈ R such that

f(x) = f(y)

⇒ $x^3 + x = y^3 + y$

⇒ $x^3 - y^3 + (x - y) = 0$

⇒ $(x - y)(x^2 + xy + y^2 + 1) = 0$

⇒ x – y = 0 [Q$x^2 + xy + y^{2\geq} 0$ for all x, y ∈R. $x^2 + xy + y^2 + 1 \geq 1$ for all x,y ∈ R]

⇒ x = y

Thus, f(x) = f(y)

$\Rightarrow$ x = y for all x, y $\in$ R

So, f is an injective map.

Surjectivity: Let y be an arbitrary element of R then

$f(x) = y \Rightarrow x^3 + x = y$

$\Rightarrow x^3 + x - y = 0$

We know that an odd degree equation has at least one real root. Therefore, for every real value of y, the equation $x^3 + x - y = 0$ has a real root α, such that

$a^3 + \alpha - y = 0$

$a^3 + \alpha = y$

$f(\alpha) = y$

Thus, for every y $\in$ R, there exist $\alpha \in$ R such that $f(\alpha) = y$.

So, *f* is a surjective map.

Hence, $f : R \rightarrow R$ is a bijection.

Illustration 16: Let f: N $\cup$ {0} $\rightarrow$ N $\cup$ {0} be defined by f(n) = $\begin{cases} n+1, & \text{if n is even} \\ n-1, & \text{if n is odd} \end{cases}$. Show that f is a bijective function. **(JEE MAIN)**

Sol: Divide the solution in three cases.

Case I – When both the numbers are even, Case II – When both the numbers are odd & Case III – When one is even and other is odd.

Let f(n) = f(m).

Case I : both n, m are even.

Then, n + 1 = m + 1. So, n = m.

Case II : Both n, m are odd.

Then, n – 1 = m – 1, which implies m = n.

Case III : n is even and m is odd. Then, f(n) is odd and f(m) is even. So, f(n) $\neq$ f(m)

In any case, f(n) = f(m) implies n = m. Thus, f is one–one.

Now f(2n) = 2n + 1, f(2n + 1) = 2n for all n $\in$ N. So, *f* is onto. Hence *f* is a bijective function.

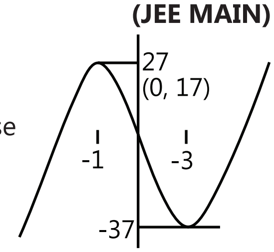


Figure 13.54

Illustration 17: Draw the graph of the function under the following condition and also check whether the function is one–one and onto or not, $f : R \rightarrow R$, $f(x) = 2x^3 - 6x^2 - 18x + 17$ **(JEE MAIN)**

Sol: Find the derivative of given function and understand the nature of the curve. Also find the values of f(x) at some particular values of x to trace the given curve.

Domain of the function is x $\in$ R

Here, $f'(x) = 6x^2 - 12x - 18 = 6(x^2 - 2x - 3) = 6(x + 1)(x - 3)$

$f'(x) = 0$ at x = –1, 3

$f(\infty) = \infty$

$f(-\infty) = -\infty$

f(0) = 17, f(–1) = 27, f(3) = –37

From the graph we can say that f is many one onto function.

11. COMPOSITE FUNCTIONS

Let *f*: A $\rightarrow$ B and g: B $\rightarrow$ C be two functions. Then the function gof: A $\rightarrow$ C defined by (gof) (x) = g (f(x)) $\forall$ x $\in$A is called the composite of the two functions f and g. Diagrammatically

it is shown as follows:

$$\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$$

Thus the image of every x $\in$A under the function gof is g-image of the f-image of x.

Note that gof is defined only if x $\in$ A, f(x) is an element of the domain of g so that we can take its g-image. Hence, for the product gof, the range of f must be a subset of the domain of g. In general, gof not equal to fog.

12. PROPERTIES OF COMPOSITE FUNCTIONS

(i) The composite of functions is not commutative, i.e. gof $\neq$ fog.

(ii) The composite of functions is associative, i.e. if f, g, h are functions such that fo(goh) and (fog)oh are defined, then fo(goh) = (fog)oh.

Associativity: f: N $\rightarrow$ I_0. f(x) = 2x

g: $I_0 \rightarrow$ Q, g(x) = $\frac{1}{x}$; h: Q $\rightarrow$ R h(x) = $e^{\frac{1}{x}} \Rightarrow$ (hog)of = ho(gof) = e^{2x}

(iii) The composite of two bijections is a bijection, i.e. if f and g are two bijections such that gof is defined, then gof is also a bijection.

Proof: Let *f* : A $\rightarrow$ B and g: B $\rightarrow$ C be two bijections. Then gof exists such that

gof: A $\rightarrow$ C

We have to prove that gof is one–one and onto.

One–one: Let a_1, $a_2 \in$ A such that (gof)(a_1) = (gof)(a_2), then

(gof)(a_1) = (gof)(a_2) $\Rightarrow$ g[f(a_1)] = g[f(a_2)]

$\Rightarrow$ f(a_1) = f(a_2) [$\because$ g is one–one]

$\Rightarrow a_1 = a_2$ [$\because$ f is one–one]

$\therefore$ gof is also one–one function.

Onto: Let c $\in$ C, then

c $\in$ C $\Rightarrow \exists$ b $\in$ B s.t. g(b) = c [$\because$ g is onto]

and b $\in$ B $\Rightarrow \exists$ a$\in$A s.t. f(a) = b [$\because$ f is into]

Therefore, we see that

c $\in$ C $\Rightarrow \exists$ a $\in$ A s.t. gof(a) = g[f(a)] = g(b) = c

i.e. every element of C is the gof image of some element of A. As such gof is onto function. Hence gof being one–one and onto is a bijection.

Illustration 18: Let f: R $\rightarrow$ R, g: R $\rightarrow$ R be defined by f(x) = $x^2 + 3x + 1$, g(x) = 2x – 3. Find fog and gof. **(JEE MAIN)**

Sol: Use the concept of composite functions.

We have (fog)(x) = f(g(x)) = f(2x – 3)= $(2x - 3)^2 + 3(2x - 3) + 1 = 4x^2 - 6x + 1$

and (gof)(x) = g(f(x)) = g($x^2 + 3x + 1$)= $2(x^2 + 3x + 1) - 3 = 2x^2 + 6x - 1$

Hence fog $\neq$ gof.

Illustration 19: If (x) = $\frac{1}{x^2}$ and g(x) = 0 are two real functions, show that fog is not defined. **(JEE MAIN)**

Sol: Find the domain of fog(x).

We have,

Domain (f) = R – {0}, Range (f) = R – {0}

Domain (g) = R and Range (g) = {0}

Clearly, Range (g) $\not\subset$ Domain (f)

$\therefore$ Domain (fog) = {x: x $\in$ Domain (g) and g(x) $\in$ Domain (f)}

$\Rightarrow$ Domain (fog) = {x: x $\in$ R and g(x) $\in$ Domain (f)}

$\Rightarrow$ Doman (fog) = f

[$\because$ g(x) = 0 $\notin$ Domain (f) for any x $\in$ R]

Hence, fog is not defined.

Illustration 20: If f(x) = $\frac{1}{2x+1}$, $x \neq -\frac{1}{2}$, then show that, f(f(x)) = $\frac{2x+1}{2x+3}$, provided that $x \neq -\frac{1}{2}, -\frac{3}{2}$. **(JEE MAIN)**

Sol: Check the domain of f(x) and use the concept of composite functions.

We have, f(x) = $\frac{1}{2x+1}$

Clearly, domain (f) = $R - \left\{-\frac{1}{2}\right\}$

Let, y = $\frac{1}{2x+1} \Rightarrow 2x + 1 = \frac{1}{y} \Rightarrow x = \frac{1-y}{2y}$

Since x is a real number distinct from $-\frac{1}{2}$, y can take any non-zero real value.

So, Range (f) = R – {0}

We observe that range

(f) = R – {0} $\not\subset$ domain (f) = $R - \left\{\frac{1}{2}\right\}$

$\therefore$ Domain (fof) = {x: x $\in$ domain (f) and f(x) $\in$ Domain (f)}

$\Rightarrow$ Domain (fof) = $\left\{x : x \in R - \left\{-\frac{1}{2}\right\} \text{ and } f(x) \in R - \left\{-\frac{1}{2}\right\}\right\}$

$\Rightarrow$ Domain (fof) = $\left\{x : x \neq -\frac{1}{2} \text{ and } f(x) \neq -\frac{1}{2}\right\}$

$\Rightarrow$ Domain (fof) = $\left\{x : x \neq -\frac{1}{2} \text{ and } \frac{1}{2x+1} \neq -\frac{1}{2}\right\}$

$\Rightarrow$ Domain (fof) = $\left\{x : x \neq -\frac{1}{2} \text{ and } x \neq -\frac{3}{2}\right\}$

$R - \left\{-\frac{1}{2}, -\frac{3}{2}\right\}$

Also, fof(x) = f(f(x)) = $f\left(\frac{1}{2x+1}\right) = \frac{1}{2\left(1/(2x+1)\right)+1} = \frac{2x+1}{2x+3}$

Thus, fof : $R - \left\{-\frac{1}{2}, -\frac{3}{2}\right\} \rightarrow R$ is defined by fof(x) = $\frac{2x+1}{2x+3}$

Hence, $f(f(x)) = \frac{2x+1}{2x+3}$ for all $x \in R$. $\quad x \neq -\frac{1}{2}, -\frac{3}{2}$.

Illustration 21: If $f(x) = \log_{100x}\left(\frac{2\log_{10} x+2}{-x}\right)$ and g(x) = {x}. If the function (fog)(x) exists then find the range of g(x). **(JEE ADVANCED)**

Sol: Find the domain of f(x) and use the given information that fog(x) exists.

To define f(x), the following condition must hold good:

(i) $100x > 0$ and $100x \neq 1 \Rightarrow x \neq \frac{1}{100}$

(ii) $x > 0$ and $\log_{10}x + 1 < 0 \Rightarrow 0 < x < \frac{1}{10}$ and $x \neq \frac{1}{100}$

$\therefore$ Domain of f(x) is $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{10}\right)$.

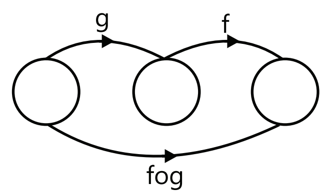


Figure 13.55

Here, g(x) = {x}, range of g(x) is [0, 1).

But, (fog) (x) exists $\Rightarrow$ range of g(x) $\subset$ domain of f(x).

$\therefore$ Range of g(x) is $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{10}\right)$.

Illustration 22: Consider functions f and g such that composite gof is defined and is one–one. Should f and g necessarily be one–one? **(JEE MAIN)**

Sol: Take an example to prove it.

Consider f: {1, 2, 3, 4} $\to$ {1, 2, 3, 4, 5, 6} defined as f(x) = x, $\forall$ x = 1, 2, 3, 4 and

g: {1, 2, 3, 4, 5, 6} ®{1, 2, 3, 4, 5, 6} as g(x), for x = 1, 2, 3, 4 and g(5) = g(6) = 5.

Then, gof(x) = x, $\forall$ x = 1, 2, 3, 4, which shows that gof is one–one. But g is clearly not one–one.

14. INVERSE OF A FUNCTION

Let f: A $\to$ B be a one–one and onto function, then there exists a unique function.

g: B $\to$ A such that f(x) = y $\Leftrightarrow$ g(y) = x, $\forall x \in A$ and $y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1}: B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

14.1 Properties of Inverse Functions

(i) The inverse of a bijection is unique.

Proof: Let f : A $\to$ B be a bijection, and let g: B $\to$ A and h: B $\to$ A be two inverse functions of f. Also let $a_1, a_2 \in A$ and $b \in B$, such that $g(b) = a_1$ and $h(b) = a_2$. Then

$g(b) = a_1 \Rightarrow f(a_1) = b$

$h(b) = a_2 \Rightarrow f(a_2) = b$

Since f is one–one, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \Rightarrow g(b) = h(b)$, $b \in B$.

(ii) If f: A $\to$ B is a bijection and g: B $\to$ A is the inverse of f, then fog = I_B and gof = I_A, where I_A and I_B are identity functions on the sets A and B, respectively.

Note that the graphs of f and g are the mirror images of each other in the line y = x. As shown in the figure given below a point (x′, y′) corresponding to $y = x^2 (x \geq 0)$ changes to (y′ x′) corresponding to $y = \sqrt{x}$, the changed form of $x = \sqrt{y}$.

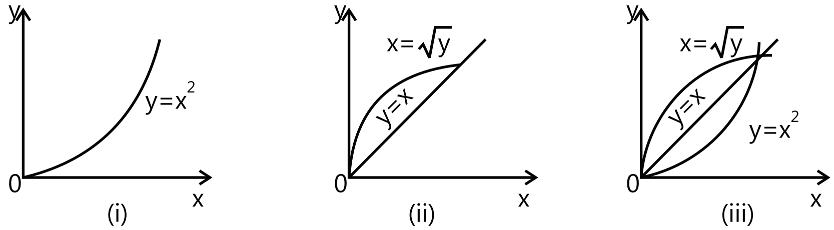


Figure 13.56

(iii) The inverse of a bijection is also a bijection.

Proof: Let f : A → B be a bijection and g : B → A be its inverse. We have to show that g is one–one and onto.

One-one: Let $g(b_1) = a_2$ and $g(b_2) = a_2$; $a_1, a_2 \in A$ and $b_1, b_2 \in B$

Then $g(b_1) = g(b_2) \Rightarrow a_1 = a_2$

$\Rightarrow$ $f(a_1) = f(a_2)$ [$\because$ f is bijection]

$\Rightarrow$ $b_1 = b_2$ [$\because g(b_1) = a_1 \Rightarrow b_1 = f(a_1); g(b_2) = a_2 \Rightarrow b_2 = f(a_2)$]

Which proves that g is one–one.

Onto: Again, if $a \in A$, then

$a \in A \Rightarrow \exists\ b \in B$ s.t. f(a) = b(by definition of f)

$\Rightarrow \exists\ b \in B$ s.t. a = g(b) [$\because f(a) = b \Rightarrow a = g(b)$]

Which proves that g is onto. Hence g is also a bijection.

(iv) If f and g are two bijections f : A → B, g : B → C then inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof: since f : A → B and g : B → C are two bijections.

$\therefore$ gof: A → C is also a bijection.

[By theorem the composite of two bijection is a bijection.]

As such gof has an inverse function $(gof)^{-1} : C \to A$. We have to show that

$(gof)^{-1} = f^{-1}og^{-1}$

Now let $a \in A$, $b \in B$, $c \in C$ such that f(a) = b and g(b) = c.

So (gof) (a) = g[f(a)] = g(b) = c

Now f(a) = b $\Rightarrow$ $a = f^{-1}(b)$ ....(i)

g(b) = c $\Rightarrow$ $b = g^{-1}(c)$ .... (ii)

(gof)(a) = c $\Rightarrow$ $a = [gof)^{-1}(c)$ .... (iii)

Also $(f^{-1}og^{-1})(c) = f^{-1}[g^{-1}(c)]$ [by definition]

$= f^{-1}(b)$ [by (ii)]

= a [by (i)]

$= (gof)^{-1}(c)$ [by (iii)]

$\therefore$ $(gof)^{-1} = f^{-1}og^{-1}$, which proves the theorem.

CONCEPTS

In the line y = x, the graphs of f and g are the mirror images of each other. As shown in the following figure, a point (x′, y′) corresponding to y = ln x(x > 0) changes to (y′, x′) corresponding to $y = e^x$, the changed form of $x = e^y$.

The inverse of a bijection is also a bijection.

If f and g are two bijections f: A → B, g: B → C, then inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

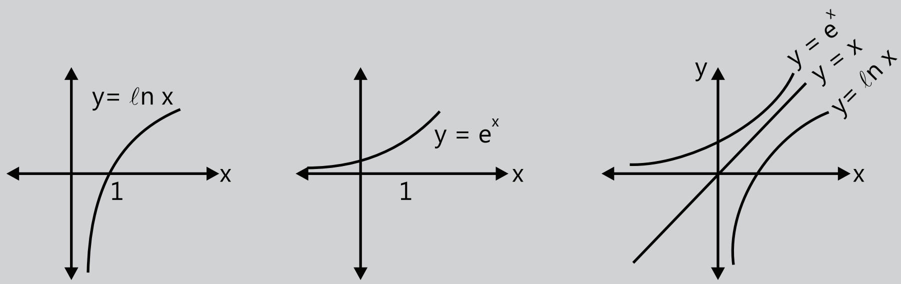


Figure 13.57

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 23: Find inverse of the function $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$ **(JEE MAIN)**

Sol: Put f(x) = y and solve for x.

Graph of f(x)

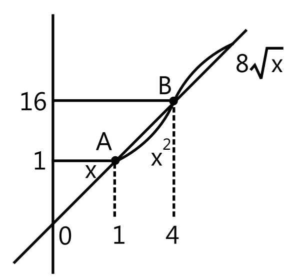


Figure 13.58

Using the above graph $f^{-1}(y) = x = \begin{cases} y & \text{if} \quad y < 1 \\ \sqrt{y} & \text{if} \quad 1 \le y \le 16 \\ \dfrac{y^2}{164} & \text{if} \quad y > 16 \end{cases}$

or $\quad f^{-1}(x) = \begin{cases} x & \text{if} \quad x < 1 \\ \sqrt{x} & \text{if} \quad 1 \le x \le 16 \\ \dfrac{x^2}{16} & \text{if} \quad x > 16 \end{cases}$.

Illustration 24: Let *f*: N → R be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that f : N → S, where S is the range of f, is invertible. Find the inverse of f. **(JEE MAIN)**

Sol: Put $4x^2 + 12x + 15 = y$ and solve for x. Then put $f^{-1}(y)$ in place of x.

Let y be an arbitrary element of range f. Then $y = 4x^2 + 12x + 15$, for some x in N, which implies that $y = (2x + 3)^2 + 6$. This gives $x = \dfrac{(\sqrt{y-6} - 3)}{2}$, as y ≥ 6.

Let us define g: S → N by $g(y) = \dfrac{(\sqrt{y-6} - 3)}{2}$.

Now, $\quad gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = g((2x + 3)^2 + 6)$

$= \frac{\sqrt{((2x+3)^2+6-6)}-3}{2} = \frac{(2x+3-3)}{2} = x$

and $fog(y) = f\left(\frac{\sqrt{y-6}-3}{2}\right) = \left(\frac{2\sqrt{y-6}-3}{2}+3\right) + 6$

$= (\sqrt{y-6}-3+3)^2 + 6 = (\sqrt{y-6})^2 + 6 = y - 6 + 6 = y$

Hence, gof = I_N and fog = I_S. This implies that f is invertible with $f^{-1} = g$.

Illustration 25: For the function f: R – {4} → R – {–2} : $f(x) = \frac{2x-5}{4-x}$. Find

(a) zero's of f(x)

(b) range of f(x)

(c) intervals of monotonocity

(d) $f^{-1}(x)$

(e) local maxima and minima if any

(f) interval when f(x) is concave upward and concave downward

(g) asymptotes

(h) $\int_1^2 f(x)\,dx$

(i) nature of function whether one–one or onto

(j) graph **(JEE MAIN)**

Sol:

(a) 5/2

(b) $(-\infty, -2) \cup (-2, \infty)$

(c) ↑ in its domain i.e., $(-\infty, 4) \cup (4, \infty)$

(d) $f^{-1}(x) = \frac{4x+5}{x+2}$

(e) no,

(f) $(-\infty, 4)$ upwards and $(4,\infty)$ downwards

(g) (g) y = –2

(h) $-\left(2+3\ln\frac{2}{3}\right)$

(i) both one-one and onto

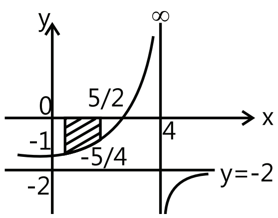
(j) 

Figure 13.59

Illustration 26: f: R → R, f(x) = 3x + 2. Find the inverse of f, if it exists. **(JEE MAIN)**

Sol: Check whether the given function is one-one onto. Put 3x + 2 = y and solve for x. Then put $f^{-1}(y)$ in place of x.

$\forall\, x_1, x_2 \in A,\ f(x_1) = f(x_2) \Rightarrow 3x_1 + 2 = 3x_2 + 2 \Rightarrow x_1 = x_2$

∴ f: R → R is one–one.

Now let y ∈ R. Then $y = 3x + 2 \Rightarrow x = \frac{y-2}{3}$

Hence, for every y ∈ R, there is a corresponding $x = \frac{y-2}{3} \in R$ such that y = f(x). Hence, the range of f is R and so f is onto, and the inverse of f exists.

Again, $y = f(x) \Leftrightarrow x = f^{-1}(y)$

$\therefore f^{-1} : R \to R, f^{-1}(y) = \frac{y-2}{3}$ or equivalently, $f^{-1}(x) = \frac{x-2}{3}$.

15. DIFFERENT TYPES OF FUNCTIONS

15.1 Homogeneous Functions

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x and y. Symbolically if,

$f(tx, ty) = t^n. f(x,y)$, then f(x, y) is homogeneous function of degree n.

Examples of Homogeneous function:

$f(x, y) = \frac{x - y\cos x}{y\sin x + x}$ is not a homogeneous function and

$f(x, y) = \frac{x}{y}\ln\frac{y}{x} + \frac{y}{x}\ln\frac{x}{y}$; $\sqrt{x^2 - y^2} + x$; $x + y\cos\frac{y}{x}$ are homogeneous function of degree one.

15.2 Bounded Function

A function is said to be bounded if $| f(x) | \le M$, where M is a finite quantity.

15.3 Implicit and Explicit Function

A function defined by an equation not solved for the dependent variable is called an implicit function. For example, the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of only x, then it is called explicit function.

Examples on implicit and explicit function (x, y) = 0

1. $x\sqrt{1+y} + y\sqrt{1+x} = 0$; explicit

$y = -\frac{x}{1+x}$ or y = x (rejected)

2. $y^2 = x$ represents two separated branches

3. $x^3 + y^3 - 3xy = 0$ folium of desecrates

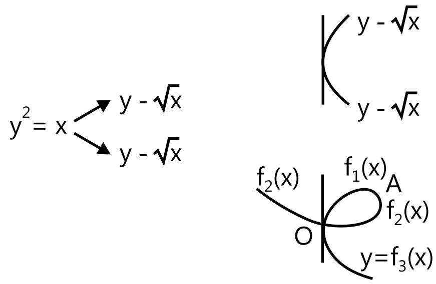


Figure 13.60

15.4 Odd and Even Function

A function f(x) defined on the symmetric interval (–a, a) then,

If $f(-x) = f(x)$ $\forall$ x in the domain of 'f' then f is said to be an even function.

E.g., $f(x) = \cos x$; $g(x) = x^2 + 3$;

If $f(-x) = -f(x)$ $\forall$ x in the domain of 'f' then f is said to be an odd function.

E.g., $f(x) = \sin x$; $g(x) = x^3 + x$.

Examples on odd and even functions:

Odd	**Even**	**Neither odd nor even**
1. $\ell n\left(x + \sqrt{1+x^2}\right)$	1. $x\frac{2^x + 1}{2^x - 1}$	1. $2x^3 - x + 1k$

2. $\ell n \frac{1-x}{1+x}$

3. $x \sin^2 x - x^3$

4. $\sqrt{1+x+x^2} - \sqrt{1+x+x^2}$

5. $\frac{1+2^{Kx}}{1-2^{Kx}}$

2. $\sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$

3. constant

4. $x^2 - |x|$

5. $\frac{(1+2^x)^2}{2^x}$

2. sin x + cos x

CONCEPTS

(a) f(x) – f(–x) = 0 ⇒ f(x) is even and f(x) + f(–x) = 0 ⇒ f(x) is odd.

(b) A function may neither be odd nor even.

(c) Inverse of an even function is not defined and an even function cannot be strictly monotonic.

(d) Every even function is symmetric about the y-axis and every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even and an odd function.

E.g.,

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}} \qquad 2^x = \underbrace{\frac{2^x + 2^{-x}}{2}}_{\text{EVEN}} + \underbrace{\frac{2^x - 2^{-x}}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line and is even and odd at the same time is f(x) = 0.

(g) If f and g are either both even or odd, then the function f · g will be even. But if any one of them is odd then f · g will be odd.

f(x)	g(x)	f(x)+g(x)	f(x) – g(x)	f(x) . g(x)	f(x) / g(x)	(gof) / (x)	(fog) (x)
odd	odd	Odd	odd	even	even	odd	odd
even	even	Even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Shrikant Nagori (JEE 2009, AIR 30)

15.5 Periodic Functions

A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that f(x + T) = f(x), for all values of x within the domain of x.

E.g., Both the functions sin x and cos x are periodic over 2π, and tan x is periodic over π.

Examples on periodic function

(i) $f(x) = \cos\frac{2x}{3} - \sin\frac{4x}{5}$ (15π)

(ii) $f(x) = \cos(\sin x)$ (π)

(iii) $f(x) = \sin(\cos x)$ (2π)

(iv) $f(x) = \sin^4 x + \cos^4 x$ $\left(\frac{\pi}{2}\right)$

(v) $f(x) = x - [x] = \{x\}$ (one)

CONCEPTS

- $f(T) = f(0) = f(-T)$, where 'T' is the period.
- Inverse of a periodic function does not exist.
- Every constant function is always periodic, with no fundamental period.
- If f(x) has a period T and g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. For example, $f(x) = |\sin x| + |\cos x|$
- If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.
- If f(x) has a period T then f(ax + b) has a period T/a (a >0).

Proof : Let $f(x + T) = f(x)$ and $f[a(x + T') + b] = f(ax + b)$

$f(ax + b + aT') = f(ax + b)$

$f(y + aT') = f(y) = f(y + T) \Rightarrow T = aT' \Rightarrow T' = \frac{T}{a}$

Vaihbav Gupta (JEE 2009, AIR 54)

15.6 Special Functions

If x, y are independent variables, then

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$

(b) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n$, $n \in R$

(c) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$

(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant

Illustration 27: Which of the following function(s) is(are) bounded on the intervals as indicated

(A) $f(x) = 2^{\frac{1}{x-1}}$ on (0, 1)
(B) $g(x) = x\cos\frac{1}{x}$ on $(-\infty, \infty)$
(C) $h(x) = xe^{-x}$ on $(0, \infty)$
(D) $I(x) = \arctan 2^x$ on $(-\infty, \infty)$

Sol: Check for the continuity of the given functions. If the function is continuous then to find the value of f(x) at the boundary points.

(A) $\underset{x\to 0^+}{\text{Limit}} f(x) = \underset{h\to 0}{\text{Limit}}\, 2^{\frac{1}{h-1}} = \frac{1}{2}$; $\underset{x\to I^-}{\text{Limit}} f(x) = \underset{h\to 0}{\text{Limit}}\, 2^{\frac{1}{-h}} = 0$

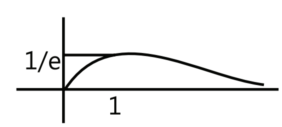


Figure 13.61

$\Rightarrow$ $f(x) \in \left(0, \frac{1}{2}\right) \Rightarrow$ bounded

(C) $\underset{h\to 0}{\text{Limit}}\ xe^{-x} = \underset{h\to 0}{\text{Limit}}\ he^{-h} = 0; \underset{x\to\infty}{\text{Limit}}\ x e^{-x} = \underset{x\to\infty}{\text{Limit}} \frac{x}{e^{x}} = 0$

$\Rightarrow$ also $y = \frac{x}{e^x} \Rightarrow y' = \frac{e^x - xe^x}{e^{2x}}\ e^{x(1-x)} \Rightarrow h(x) = \left(0, \frac{1}{e}\right]$

FORMULAE SHEET

Table: Domain and range of some standard functions-

Functions	**Domain**	**Range**
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	(K)
Reciprocal function $\frac{1}{x}$	R_0	R_0
$X^2, \lvert x \rvert$ (modulus function)	R	$R^+ \cup \{x\}$
$x^3, x\lvert x \rvert$	R	R
Signum function $\frac{\lvert x \rvert}{x}$	R	{-1,0,1}
$X + \lvert x \rvert$	R	$R^+ \cup \{x\}$
$x - \lvert x \rvert$	R	$R^- \cup \{x\}$
[x] (greatest integer function)	R	1
x-{x}	R	[0,1]
$\sqrt{x}$	(0, ∞)	[0,∞]
a^x(exponential function)	R	R^+
Log x(logarithmic function)	R^+	R

Inverse Trigo Functions	Domain	Range
$\sin^{-1}x$	(-1,1]	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	[-1,1]	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\sec^{-1}x$	R-(-1,1)	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\operatorname{cosec}^{-1}x$	R-(-1,1)	$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Inverse function: f^{-1} exists iff f is both one–one and onto.

$f^{-1}:B\to A$, $f^{-1}(b)=a \Rightarrow f(a)=b$

Even and odd function: A function is said to be

(a) Even function if f(x)=f(x) and

(b) Odd function if f(–x)= –f(x)

Properties of even & odd function:

(a) The graph of an even function is always symmetric about y-axis.

(b) The graph of an odd function is always symmetric about origin.

(c) Product of two even or odd function is an even function.

(d) Sum & difference of two even (odd) function is an even (odd) function.

(e) Product of an even or odd function is an odd function.

(f) Sum of even and odd function is neither even nor odd function.

(g) Zero function, i.e. f(x) = 0, is the only function which is both even and odd.

(h) If f(x) is an odd (even) function, then $f'(x)$ is even (odd) function provided f(x) is differentiable on R.

(i)A given function can be expressed as sum of even and odd function.

i.e. $f(x)=\frac{1}{2}\left[f(x)+f(-x)\right]+\frac{1}{2}\left[f(x)-f(-x)\right]$ =even function + odd function.

Increasing function: A function f(x) is an increasing function in the domain, D if the value of the function does not decrease by increasing the value of x.

Decreasing function: A function f(x) is a decreasing function in the domain, D if the value of function does not increase by increasing the value of x.

Periodic function: Function f(x) will be periodic if a +ve real number T exists such that

$f(x+T)=f(x), \forall x \in$ Domain.

There may be infinitely many such real number T which satisfies the above equality. Such a least +ve number T is called period of f(x).

(i) If a function f(x) has period T, then period of f(xn+a)=T/n and period of (x/n+a)=nT.

(ii) If the period of f(x) is T_1 & g(x) has T_2 then the period of f(x) ± g(x) will be L.C.M. of T_1& T_2 provided it satisfies definition of periodic function.

(iii) If period of f(x) & f(x) are same T, then the period of af(x)+bg(x) will also be T.

Function	Period
sin x, cos x	2π
sec x, cosec x	
tan x, cot x	π
sin (x/3)	6π
tan 4x	$\pi/4$
cos $2\pi x$	1
$\lvert\cos x\rvert$	π
$\sin^4 x+\cos^4 x$	$\pi/2$
$2\cos\left(\frac{x-\pi}{3}\right)$	6π
$\sin 3x + \cos^3 x$	$2\pi/3$
$\text{Sin}^3 x + \cos^4 x$	2π
$\frac{\sin x}{\sin 5x}$	2π
$\tan^2 x - \cot^2 x$	π
x-[x]	1
[x]	1

Solved Examples

JEE Main/Boards

Example 1: If a set A = {a, b, c}, then find the number of subsets of the set A and also mention the set of all the subsets of A.

Sol: Use the formula for the no. of subsets and list it down.

Since n(A) = 3.

$\therefore$ Number of subsets of A is $2^3 = 8$ and set of all those subsets is P(A) named as power set.

P(A): {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, {a, b, c}}

Example 2: Show that n {P[P(ϕ)]} = 4

Sol: We have P(ϕ) = {ϕ}

$\therefore$ P(P(ϕ)) = {ϕ, {f}}

$\Rightarrow$ P[P(P)] = {ϕ, { ϕ }, {{ ϕ }}, {ϕ, { ϕ }}

Hence n {P[P(ϕ)]} = 4.

Example 3: If A = {x: x = 2n + 1, n $\in$ Z} and B = {x: x = 2n, n $\in$ Z}, then find A ∪ B.

Sol: Write the two sets and then take union.

A $\cup$ B = {x: x is an odd integer} $\cup$ {x: x is an even integer} = {x: x is an integer} = Z.

Example 4: If A = {x: x = 3n, n $\in$ Z} andB = {x: x = 4n, n $\in$ Z}, then find A ∩ B.

Sol: Clearly, set A is a multiple of 3 and the other set is a multiple of 4. Hence, the intersection of the two would be set having multiples of 12.

We have

x $\in$ A ∩ B $\Leftrightarrow$ x = 3n, n $\in$ Z and x = 4n, n $\in$ Z

$\Leftrightarrow$ x is a multiple of 3 and x is a multiple of 4.

$\Rightarrow$ x is a multiple of 3 and 4 both.

$\Leftrightarrow$ x is a multiple of 12 $\Leftrightarrow$ x = 12n, n $\in$ Z.

Hence, A ∩ B = {x: x = 12n, n $\in$ Z}.

Example 5: If A = {2, 3, 4, 5, 6, 7} andB = {3, 5, 7, 9, 11, 13}, then find A – B andB – A.

Sol: Use algebra of Sets.

A – B = {2, 4, 6} and B – A = {9, 11, 13}

Example 6: Let A be the set of all students of a boys school Show that the relation R in A given by R = {(a, b) : a is sister of b} is the empty relation and R′ = {(a, b) : the difference between heights of a and b is less than 3 meters} is the universal relation.

Sol: Show that the relation R = ϕ and the relation R′ is true for any two.

Since the school is boys school, no student of the school can be sister of any student of the school. Hence, R = ϕ, showing that R is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 metres. This shows that R′ = A × A is the universal relation.

Example 7: Let T be the set of all triangles in a plane with R a relation in T given by R = {(T_1, T_2)} : T_1 is congruent to T_2. Show that R is an equivalence relation.

Sol: Prove that R satisfies the conditions for reflexive, symmetric and transitive relation.

R is reflexive, since every triangle is congruent to itself. Further (T, T_1) $\in$ R_2.

$\Rightarrow$ T_1 is congruent to T_2 $\Leftrightarrow$ T_2 is congruent to T_1. (T_2, T_1) $\in$ R, Hence R is symmetric.

Moreover, (T_1, T_2), (T_2, T_3) $\in$ R.

$\Rightarrow$ T_1 is congruent to T_2 and T_2 is congruent to T.

$\Rightarrow$ T_1 is congruent to T_3 $\Rightarrow$ (T_1, T_3) $\in$ R. Therefore, R is an equivalence relation.

Example 8: Let L be the set of all lines in a plane and R be the relation in L defined as R = {(L_1, L_2): L_1 is perpendicular to L_2}. Show that R is symmetric but neither reflexive nor transitive.

Sol: Use the definition of Symmetric relation.

R is not reflexive as the line L_1 cannot be perpendicular to itself, i.e. (L_1, L_1) $\notin$ R. R is symmetric as(L_1, L_2) $\in$ R.

$\Rightarrow$ L_1 is perpendicular to L_2.

$\Rightarrow$ L_2 is perpendicular to L_1.

$\Rightarrow$ (L_2, L_1) $\in$ R.

R is not transitive. Indeed, if L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 can never be perpendicular to L_3. In fact, L_1 is parallel to L_3, i.e. $(L_1, L_3) \in R$ $(L_2, L_3) \in R$ but $(L_1, L_3) \in R$.

Example 9: Show that the relation R in the set {1, 2, 3} given by R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} is reflexive but neither symmetric nor transitive.

Sol: R is reflexive, since (1, 1), (2, 2) and (3, 3) lie in R. Also R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$. Similarly, R is not transitive, as $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

Example 10: Show that the relation R in the set Z of integers given by R = {(a, b): 2 divides a – b} is an equivalence relation.

Sol: Similar to example 7.

R is reflexive, as 2 divides (a – a) for all $a \in Z$. Further, if $(a, b) \in R$, then 2 divides a – b].

Therefore, 2 divides b – a. Hence, $(b, a) \in R$, which shows that R is symmetric. Similarly if $(a, b) \in R$ and $(b, c) \in R$, then a – b and b – c are divisible by 2.

Now, a – c = (a – b) + (b – c) is even. So, (a – c) is divisible by 2. This shows that R is transitive. Thus, R is an equivalence relation in Z.

Example 11: Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Let R_1 be a relation in X given by R_1 = {(x, y): x – y is divisible by 3} and R_2 be another relation on X given by R_2. = {(x, y): $\{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}$}. Show that $R_1 = R_2$.

Sol: Prove that R_1 is a subset of R_2 and vice versa.

Note that the characteristic of sets {1, 4, 7}, {2, 5, 8} and {3, 6, 9} is that difference between any two elements of these sets is a multiple of 3. Therefore $(x, y) \in R_1$.

$\Rightarrow$ x – y is multiple of 3 $\Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$. Hence $R_{1\subset} R_2$.

Similarly, $\{x, y\} \in R_2 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}$

$\Rightarrow$ x – y is divisible by 3 $\Rightarrow \{x, y\} \in R_1$. This shows that $R_2 \in R_1$. Hence, $R_1 = R_2$.

Example 12: Let f: X $\rightarrow$ Y be a function. Define a relation R in X given by R = {(a, b): f(a) = f(b)}. Examine if R is an equivalence relation.

Sol: For every $a \in X$, $(a, a) \in R$, since f(a) = f(a), showing that R is reflexive.

Similarly, $(a, b) \in R \quad \Rightarrow f(a) = f(b)$

$\Rightarrow f(b) = f(a) \quad \Rightarrow (b, a) \in R$

Therefore, R is symmetric.

Further, $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow$ f(a) = f(b) and f(b) = f(c)

$\Rightarrow$ f(a) = f(c)

$(a, c) \in R$, which implies that R is transitive.

Hence, R is an equivalence relation.

Example 13: Let R be a relation from Q into Q defined by R = {(a, b): $a, b \in Q$ and $a - b \in Z$}. Show that,

(i) $(a, a) \in R$ for all $a \in Q$.

(ii) $(a, b) \in R$ implies $(b, a) \in R$.

(iii) $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$.

Sol: Do it by yourself.

(i) Since $a - a = 0 \in Z$, it follows that $(a, a) \in R$, for all $a \in Q$

(ii) $(a, b) \in R$ implies $a - b \in Z$. So, $b - a \in Z$. Therefore $(b, a) \in R$.

(ii) $(a, b) \in R$, $(b, c) \in R$ implies $a - b \in Z$,

$b - c \in Z$. So $a - c = (a - b) + (b - c) \in Z$. Therefore, $(a, c) \in R$.

Functions

Example 14: Show that f : N $\rightarrow$ N, given by

$$f(x)= \begin{cases} x+1, & \text{if x is odd} \\ x-1, & \text{if x is even} \end{cases}$$ Is both one–one and onto.

Sol: Use the definition, and consider the case when the two are either even or odd.

Suppose $f(x_1) = f(x_2)$. Note that if x_1 is odd and x_2 is even, then we will have $x_1 + 1 = x_2 - 1$, i.e., $x_2 - x_1 = 2$ which is impossible. Similarly the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument. Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd. Then

$f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1$

$\Rightarrow x_1 = x_2$.

Also any odd number 2r + 1 in the co–domain N is the image of 2r + 2 in the domain N and any even number 2r in the co–domain N is the image of 2r – 1 in the domain N. Thus f is onto.

Example 15: Which of the following functions are even/odd ?

(i) $f(x) = \sin x + \cos x$

(ii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

Sol: A function is even if f(-x) = f(x) and odd if f(-x) = -f(x).

(i) f(x) sin x + cos x

$f(-x) = -\sin x + \cos x$

$f(-x)$ $^1f(x)$ $\forall\ x \in R$;

hence, f is not an even function.

$f(-x) \neq -f(x)\ \forall\ x \in R$; hence f is not an odd function.

$\therefore$ f is neither even nor odd.

(ii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

$f(-x) = \sqrt{1-x+x^2} - \sqrt{1+x+x^2}$

$= -f(x)$

$\therefore$ f is an odd function.

Example 16: Let $f(x) = \begin{cases} 2+x: & x \geq 0 \\ 2-x: & x < 0 \end{cases}$.

Find f of(x)

Sol: Use the concept of composite functions.

$f: R \to R$

$fof(x) = f(f(x))$

Let $x \geq 0$; $fof(x) = f(2 + x) = 2 + (2 + x)$ as $2 + x \geq 0$

$\therefore$ $fof(x) = 4 + x$ when $x \geq 0$

Let $x < 0$; $fof(x) = f(2 - x) = 2 + (2 - x)$

(as $2 - x \geq 0$) $= 4 - x$

$\therefore$ $fof(x) = \begin{cases} 4+x \text{ if } & x \geq 0 \\ 4-x \text{ if } & x < 0 \end{cases}$

Example 17: Let,

$f(x) = x + 1, x \leq 1 = 2x + 1, 1 < x \leq 2$

$g(x) = x^2$, –1 £x < 2 = x + 2, $2 \leq x \leq 3$

Find fog and gof.

Sol: Similar to the previous example.

$f\{g(x)\} = g(x) + 1, g(x) \leq 1$

$= 2g(x) + 1, 1 < g(x) \leq 2$

$\Rightarrow f\{g(x)\} = x^2 + 1, -1 \leq x \leq 1$

$= 2x^2 + 1, 1 < x \leq \overline{2}$; $g\{f(x)\} = \{f(x)\}^2$,

– 1 £f(x) < 2= f(x) + 2,   2 £f(x) £3

gof(x) $= (x + 1)^2, -2 \leq x < 1$

$= (x + 1)^2 - 2 \leq x \leq 1$

Example 18: If $f(x) = (2 + (x - 3)^3)^{1/3}$, find f^{-1}.

Sol: Take $(2 + (x - 3)^3)^{1/3} = y$ and solve for x. Then put $f^{-1}(y)$ in place of x.

$y = [(2 + (x - 3)^3]^{1/3}$

$y^3 = 2 + (x - 3)^3$; $(x - 3)^3 = y^3 - 2$

$x - 3 = (y^3 - 2)^{1/3}$; $x = 3 + (y^3 - 2)^{1/3}$

$g(y) = x = 3 + (y^3 - 2)^{1/3}$ is the inverse function.

Example 19: Let $f(x) = x^2 + x$ be defined on the interval [0, 2]. Find the odd and even extensions of f(x) in the interval [–2, 2].

Sol: The definition is given for $0 \leq x \leq 2$, so in order to find the even and odd extension, define the function for $-2 \leq x < 0$.

Odd extension

$$f(x) = \begin{cases} f(x), & 0 \leq x \leq 2 \\ -f(-x) & -2 \leq x < 0 \end{cases} = \begin{cases} x^2 + x, & 0 \leq x \leq 2 \\ -x^2 + x, & -2 \leq x < 0 \end{cases}$$

Even extension

$$f(x) = \begin{cases} f(x), & 0 \leq x \leq 2 \\ f(-x), & -2 \leq x < 0 \end{cases} = \begin{cases} x^2 + x, & 0 \leq x \leq 2 \\ x^2 - x, & -2 \leq x < 0 \end{cases}$$

Example 20: The value of $n \in I$ for which the function $f(x) = \dfrac{\sin nx}{\sin(x/n)}$ has 4p as its period is

(A) 2   (B) 3   (C) 5   (D) 4

Sol: For n = 2,

We have $\dfrac{\sin 2x}{\sin(x/2)} = 4(\cos x/2) \cos x$.

The period of cos x is 2p and that of cos(x/2) is 4p.

Hence the period of $\dfrac{\sin 2x}{\sin(x/2)}$ is 4p.

Also, the period of $\dfrac{\sin 3x}{\sin(x/3)}$, $\dfrac{\sin 5x}{\sin(x/5)}$ and $\dfrac{\sin 4x}{\sin(x/4)}$ cannot be 4p.

Example 21: Find the range of the following functionf(x) $= \dfrac{3}{2-x^2}$

Sol: First find the domain of the given function and then proceed to find the values f(x) can take..

Let $y = \frac{3}{2-x^2} = f(x)$ ... (i)

The function y is not defined for $x = \pm\sqrt{2}$

From (i), $x^2 = \frac{2y-3}{y}$

since for real x, $x^2 \geq 0$, We have $\frac{2y-3}{y} \geq 0$

$\therefore$ $y \geq 3/2$ or $y < 0$ (Note that $y \neq 0$)

Hence the range of the function is

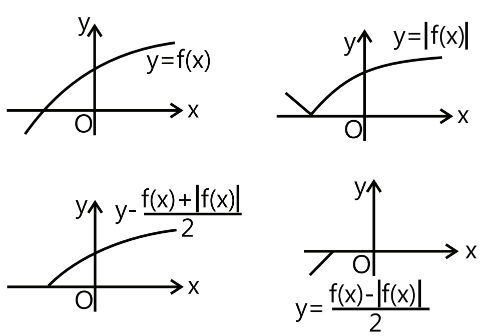
$[-\infty, 0] \cup [3/2, \infty)$

Example 22: Knowing the graph of y = f(x) draw

$y = \frac{f(x)+|f(x)|}{2}$ and $y = \frac{f(x)-|f(x)|}{2}$

Sol: Use the basics of curve tracing.

Let graph

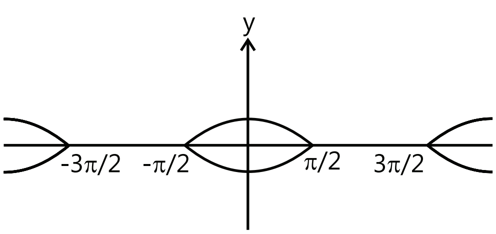


Example 23: Draw the following graphs :

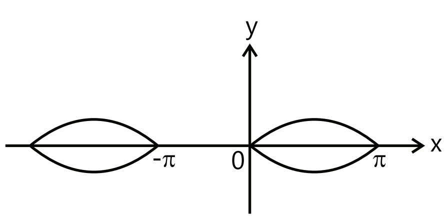
(i) | y | = cos x (ii) | y | = sin x

Sol: Similar to the previous example.

(i) | y | = cos x



(ii) | y | = sin x



JEE Advanced/Boards

Example 1: Let A = {1, 2, 3}. Then show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive, but not symmetric, is four.

Sol: Try all the possibilities and prove that the number of such relations can be four.

The smallest relation R_1 containing (1, 2) and (2, 3) which is reflexive and transitive, but not symmetric, is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}. Now, if we add the pair (2, 1) to R_1 to get R_2, then the relation R_2 will be reflexive and transitive, but not symmetric. Similarly, we can obtain R_3 and R_4 by adding (3, 2) and (3, 1), respectively, to R_1 to get the desired relations. However, we cannot add any two pairs out of (2, 1), (3, 2) and (3, 1) to R_1 at a time, as by doing so, we will be forced to add the remaining third pair in order to maintain transitivity and in the process, the relation will become symmetric also which is not required. Thus, the total number of desired relations is four.

Example 2: Show that the number of equivalence relation in the set {1, 2, 3} containing (1, 2) and (2, 1) is two.

Sol: Similar to the previous one.

The smallest equivalence relation R_1 containing (1, 2) and (2, 1) is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}. Now we are left with only four pairs namely (2, 3), (3, 2), (1, 3) and (3, 1). If we add any one, say (2, 3) to R_1, then for symmetry we must add (3, 2) also and now for transitivity we are forced to add (1, 3) and (3, 1). Thus, the only equivalence relation bigger than R_1 is the universal relation. This shows that the total number of equivalence relations containing (1, 2) and (2, 1) is two.

Example 3: Let R be a relation on the set A of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if xv = yu. Show that R is an equivalence relation.

Sol: Prove that the relation is reflexive, symmetric and transitive.

Clearly, (x, y) R (x, y), $\forall \in$ (x, y) $\forall \in$ A, since xy = yx. This shows that R is reflexive.

Further, (x, y) R (u, v) $\Rightarrow$ xv = yu $\Rightarrow$ uy = vx and hence (u, v) R (x, y).

This shows that R is symmetric.

Similarly, (x, y) R (u, v) and (u, v) R (a, b)

$\Rightarrow$ xv = yu and ub = va

$\Rightarrow \quad xv\frac{a}{u} = yu\frac{a}{u} \Rightarrow xv\frac{b}{v} = yu\frac{a}{u}$

$\Rightarrow \quad$ xb = ya and hence (x, y) R (a, b). Thus, R is transitive.

Thus, R is an equivalence relation.

Example 4: If R_1 and R_2 are equivalence relation in set A, show that $R_1 \cap R_2$ is also an equivalence relation.

Sol: Refer to the type of relation.

Since R_1 and R_2 are equivalence relations, $(a, a) \in R_1$ and $(a, a) \in R_2 \ \forall \ a \in A$. This implies that $(a, a) \in R_1 \cap R_2$, $\forall$ a, showing $R_1 \cap R_2$, is reflexive. Further, $(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2 \Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2 \Rightarrow (b, a) \in R_1 \cap R_2$, hence $R_1 \cap R_2$ is symmetric.

Similarly, $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$

$\Rightarrow \quad (a, c) \in R_1$ and $(a, c) \in R_2$

$\Rightarrow \quad (a, c) \in R_1 \cap R_2$. This shows that $R_1 \cap R_2$ is transitive. Thus, $R_1 \cap R_2$ is an equivalence.

Example 5: If A and B be two sets containing 3 and 6 elements, respectively, what can be the minimum number of elements in $A \cup B$? Also find the maximum number of elements in $A \cup B$.

Sol: Vary the number of elements in the intersection of the two sets and find the maximum and the minimum.

We have,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This shows that $n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum, respectively.

Case 1: When $n(A \cap B)$ is minimum, i.e. $n(A \cap B) = 0$. This is possible only when $A \cap B = \phi$. In this case,

$n(A \cup B) = n(A) + n(B) - 0$

$= n(A) + n(B) = 3 + 6 = 9$

So, maximum number of elements in $A \cup B$ is 9.

Case 2: When $n(A \cap B)$ is maximum.

This is possible only when $A \subseteq B$. In this case, $n(A \cap B) = 3$

$\therefore \ n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= (3 + 6 - 3) = 6$

So, minimum number of elements in $A \cup B$ is 6.

Example 6: If f, g, h are functions from R to R such that $f(x) = x^2 - 1$, $g(x) = \sqrt{x^2+1}$,

$h(x) = 0$, if $x \le 0 = x$, if $x \ge 0$,

then find the composite function ho(fog) and determine whether the function fog is invertible and the function h is identity function.

Sol: Use the concept of composite functions and check if the function is on-one onto.

Here $f(x) = x^2 - 1$ for all x.

and $g(x) = \sqrt{x^2+1}$ for all x

$\therefore f\{g(x)\} = \{g(x)\}^2 - 1 = x^2 + 1 - 1 = x^2$ for all x

$\therefore h\{f(g(x))\} = h(x)^2 = x^2$

because $x^2 \ge 0$ [from definition of h(x)]

Now, $\quad f\{g(x)\} = x^2$ for all x.

As $\quad x^2 \ge 0$, (fog) (x) cannot be negative.

So fog is not invertible.

Again, $h(x) = x$ for $x \ge 0$.

But, by definition $h(x) \ne x$ for $x < 0$.

Hence, h is not the identity function.

Example 7: Let $f : R \to R$ be given by

$f(x) = (x+1)^2 - 1$, $x \ge -1$. Show that f is invertible. Also, find the set $S = \{x: f(x) = f^{-1}(x)\}$.

Sol: Check if the function is one-one and onto.

In order to show that f(x) is invertible, it is sufficient to show that f(x) is a bijection.

f is an injection. For any x, y $\in$ R satisfying $x \ge -1, y \ge -1$.

We have f(x) = f(y)

$\Rightarrow \quad (x + 1)^2 - 1 = (y + 1)^2 - 1$

$\Rightarrow \quad x^2 + 2x = y^2 + 2y$

$\Rightarrow \quad x^2 - y^2 = -2(x - y)$

$\Rightarrow \quad (x - y)(x + y) = -2(x - y)$

$\Rightarrow \quad (x - y)\,[x + y + 2] = 0$

$\Rightarrow \quad x - y$ or $x + y + 2 = 0$

$\Rightarrow \quad x = y$ or $x = y = -1$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all

$x \ge -1, y \ge -1$

So, f(x) is an injection.

f is a surjection: For all $y \ge -1$ there exists.

$x = -1 + \sqrt{y+1} \ge -1$ such that f(x) = y

So, f(x) is a surjection.

Hence, f is bijection. Consequently, it is invertible.

$f(x) = f^{-1}(x) \quad \Rightarrow f(x) = x$

$(x + 1)^2 - 1 = x \Rightarrow x = 0, -1$

Example 8: Show that the function

f: R → R defined by $f(x) = 3x^3 + 5$ for all $x \in R$ is a bijection.

Sol: Similar to the previous example.

Injectivity: Let x, y be any two elements of R (domain). Then, f(x) = f(y)

$\Rightarrow 3x^3 + 5 = 3y^3 + 5;\ x^3 = y^3 \Rightarrow x = y$

Thus, f(x) = f(y)

$\Rightarrow x = y$ for all $x, y \in R$. so, f is an injective map. Surjectivity: Let y be an arbitrary element of R(co-domain).

Then, f(x) = y

$\Rightarrow 3x^3 + 5 = y \Rightarrow x^3 = \frac{y-5}{3};\ x = \left(\frac{y-5}{3}\right)^{1/3}$

Thus we find that for all $y \in R$ (co-domain) there exists $x = \left(\frac{y-5}{3}\right)^{1/3} \in R$ (domain) such that f(x) =

$$f\left(\left(\frac{y-5}{3}\right)^{1/3}\right) = 3\left[\left(\frac{y-5}{3}\right)^{1/3}\right]^3 + 5$$

$= y - 5 + 5 = y$

This shows that every element in the co-domain has its pre-image in the domain. So, f is a surjection. Hence, f is a bijection.

Example 9: Find the range of the following functions:

$$f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$$

Sol: Put f(x) = y and simplify. Then use the range of the trigonometric function to find the answer.

$$\Rightarrow f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$$

$$= \log_2\left(\sin\left(\pi - \frac{\pi}{4}\right) + 3\right) = y(\text{let})$$

$$\because 2^y = \sin\left(\pi - \frac{\pi}{4}\right) + 3$$

$$= 2^y - 3 = \sin\left(\pi - \frac{\pi}{4}\right)$$

But $-1 \le \sin\left(\pi - \frac{\pi}{4}\right) \le 1$

$\therefore -1 \le 2^y - 3 \le 1$

$\Rightarrow 2 \le 2^y \le 4 \quad \Rightarrow 2 \le 2^y \le 2^2$

Hence $y \in [1, 2]$

Hence range of f(x) is [1, 2]

Example 10: Find the period of the following functions

(i) f(x) = tan 2x

(ii) $f(x) = \sin^4 x + \cos^4 x$

(iii) f(x) = x – [x] + | cos px |

+ | cos 2 px | + ... + | cos npx |

Sol: Proceed according to the section 15.5.

(i) f(x) = tan 2x has period π /2as tan x has period π.

(ii) $f(x) = \sin^4 x + \cos^4 x$

$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$

$= 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{4}(1 - \cos 4x) = \frac{3}{4} - \frac{1}{4}\cos 4x$

Since cos x has period 2π,

$\therefore$ cos 4x has period $\frac{2\pi}{4} = \frac{\pi}{2}$.

$\therefore$ $f(x) = \frac{3}{4} + \frac{1}{4}\cos 4x$ has period $\frac{\pi}{2}$.

(iii) x – [x] has period 1

| cos x | has period p

| cos πx | has period $\frac{\pi}{\pi} = 1$

| cos 2πx | has period $\frac{\pi}{2\pi} = \frac{1}{2}$

| cos nπx | has period $\frac{\pi}{n\pi} = \frac{1}{n}$

l.c.m. $\left\{1, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}\right\} = 1$

$\therefore$ f(x) has period 1.

Example 11: Find the domain of the following functions

(i) $f(x) = \sqrt{x-1} + \sqrt{6-x}$

(ii) $f(x) = \log_{1/2}(x^2 + 4x + 3)$

(iii) $f(x) = \dfrac{1}{\sqrt{|x|-x}}$

Sol: Use the fact that the quantity inside the square root should be positive. For (ii), use the condition for logarithm to be defined.

(i) $\sqrt{x-1}$ is defined only if $x - 1 \geq 0$

$\sqrt{6-x}$ is defined only if $6 - x \geq 0$

$\therefore$ f(x) is defined.

$\forall\, x \in \{x : x - 1 \geq 0\} \cap \{x : 6 - x \geq 0\}$

Domain of f is = [1, 6]

(ii) $f(x) = \log_{1/2}(x^2 + 4x + 3)$

f is defined if $x^2 + 4x + 3 > 0$

i.e. if $(x + 1)(x + 3) > 0$

i.e. if $x > -1$ or $x < -3$

Domain of f is $(-\infty, -3) \cup (-1, \infty)$

(iii) f is defined if $|x| - x > 0$

If $x \geq 0$ then $|x| = x$

If $x < 0$ then $|x| = -x > 0$ and $|x| > x$.

$\therefore$ Domain of f = {x: x < 0}

Example 12: A function

f: $\left[\dfrac{3}{2}, \infty\right) \to \left[\dfrac{7}{4}, \infty\right)$ defined as

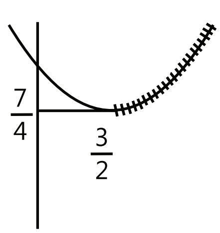
$f(x) = x^2 - 3x + 4$. Solve the equation

f(x) = f–1(x).

Sol: It is equivalent to solving for f(x) = x.

Domain of f(x) is $\left[\dfrac{3}{2}, \infty\right)$ and its co-domain gives as $\left[\dfrac{7}{4}, \infty\right)$.

We can plot the graph of f(x) for the above domain as



For solving $f(x) = f^{-1}(x)$, we can solve $f(x) = x$ or $f^{-1}(x) = x$

Any one of the above equation can be solved depending on the fact that easiest equation is given priority.

Here, $x^2 - 3x + 4 = x \Rightarrow (x - 2)^2 = 0$

x = 2 is the solution for the equation.

Example 13: Draw the graph of following function and also find their domain and range.

(i) $f(x) = \dfrac{(x+2)(x-1)}{x(x+1)}$

(ii) $f(x) = \dfrac{x^2-5x+4}{x^2+2x-3}$

Sol: Use the method of curve tracing.

(i) $f(x) = \dfrac{(x+2)(x-1)}{x(x+1)}$, domain of f(x) is

$x \in R - \{-2, 1\}$

$f'(x) = \dfrac{2(2x+1)}{[x(x+1)]^2}$

$f'(x) = 0$ at $x = -1/2$

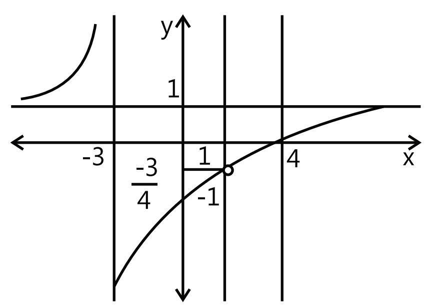
Here, $f(-\infty) = 1^- = f(+\infty)$

$f(-1^-) = -\infty = f(0^-) f(-1^+) = +\infty = f(0^+)$

Range is $(-\infty, 1) \cup (9, \infty)$

(ii) $f(x) = \dfrac{x^2-5x+4}{x^2+2x-3} = \dfrac{(x-4)(x-1)}{(x+3)(x-1)}$

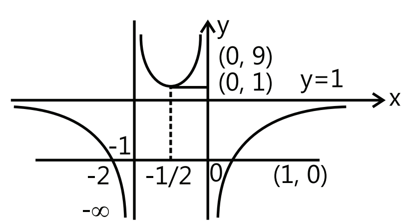
Domain of f(x) is R – {1, –3}



$f'(x) > 0 \Rightarrow f(-\infty) = 1 = f(+\infty)$

$f(-3^-) = +¥ \quad \Rightarrow f(-3^+) = -\infty$

f(4) = 0 Range is $R - \left\{-\dfrac{3}{4}, 1\right\}$



JEE Main/Boards

Exercise 1

Sets and Relations

Q.1 If R be a relation and N defined by $x + 2y = 8$ then find the domain of R.

Q.2 R is a relation from {11, 12, 13} to {8, 10, 12} defined by $y = x - 3$ then find R^{-1}.

Q.3 Let A be the set of all students of a boy's school. Let relation R in set A is given by R = {(a, b) ∈ A × A is a sister of b}. Can we say that R is an empty relation? Give reason.

Q.4 Let A = {1, 2} and B = {1, 3} and R be a relation from set A to set B defined as R = {(1, 1), (1, 3), (2, 1), (2, 3)}. Is R a universal relation? Explain.

Q.5 Let R be a relation in the set of natural numbers N, defined by R = {(a, b) ∈ N × N : a < b}. Is relation R reflexive? Explain.

Q.6 Let A be any non-empty set and P(A) be the power set of A. A relation R defined on P(A) by $X R Y \Leftrightarrow X \cap Y = X$, Y ∈ P(A). Examine whether R is symmetric.

Q.7 Let A = {a, b, c} and R is a relation in A given by R = {(a, a), (a, b), (a, c), (b, a), (c, c)}. Is R symmetric? Explain.

Q.8 Given a relation R = {(yellow, black), (cat, dog), (red, green)}. White R^{-1}.

Q.9 Let A = {1, 3, 5}, B = {9, 11} and let R = {(a, b)∈ A × B: a – b is odd}. Write the relation r.

Q.10 Let A = {a, b, c} and relation R in the set A be given by R = {(a, c), (c, a)}. Is relation R symmetric? Explain.

Functions

Q.1 If $f(x) = \dfrac{2\tan x}{1+\tan^2 x}$, then find $f\left(\dfrac{\pi}{4}\right)$.

Q.2 If $f(x) = \dfrac{|x|}{x}$, $x \neq 0$, prove that

$|f(\alpha) - f(-\alpha)| = 2\alpha, a \neq 0$.

Q.3 If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, prove that

$f\left(\dfrac{2x}{1+x^2}\right) = 2f(x)$.

Q.4 Find the domain of the function $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$

Q.5 Find the domain of definition of the function $y = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

Q. 6 Find the range of the function $y = \dfrac{x}{1+x^2}$.

Q.7 Find the domain and range of the function $\left\{\left(x, \dfrac{1}{1-x^2} : x \in R, x \neq \pm 1\right)\right\}$.

Q.8 Find the domain and range of the function $y = \dfrac{1}{2-\sin 3x}$.

Q.9 If f: R → R is defined by $f(x) = x^3 + 1$ and g : R → R is defined by $g(x) = x + 1$, then find f + g, f – g, f · g, $\dfrac{f}{g}$ and α f(a ∈ R).

Q.10 Let f: R → R is defined by f(x) = x and g : R → R is defined by g(x) = | x |. Find

(i) f + g (ii) f – g

(iii) f · g (iv) a f, a ∈ R

(v) f/g

Q.11 Let f be the exponential function and g be the logarithmic function defined by $f(x) = e^x$ and $g(x) = \log_e x$. Find

(i) (f + g) (1) (ii) (f – g) (1) (iii) (f · g) (1)

Q.12 If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where [x] denotes the integral part of x, write the value of $f\left(\dfrac{\pi}{2}\right)$.

Q.13 ({x}, represents fractional part function)

(i) Domain of the function

$f(x) = \ln(1 - \{x\}) + \sqrt{\sin x + \frac{1}{2}} + \sqrt{4 - x^2}$ is ______.

(ii) Range of the function cos(2 sin x) is ______.

(iii) Period of the function

$f(x) = \sin\left(\frac{\pi x}{3}\right) + \{x\} + \tan^2(\pi x)$ is ______.

Q.14 Let A = {1, 2, 3}, B = {3, 6, 9, 10}. Which of the following relations are functions from A to B? Also find their range if they are function.

f = {(1, 9), (2, 3), (3, 10)}

g = {(1, 6), (2, 10), (3, 9), (1, 3)}

h = {(2, 6), (3, 9)}

$u = \{(x, y): y = 3x, x \in A\}$

Q.15 Let A = {a, b, c, d}. Examine which of the following relation is a function on A?

(i) f = {(a, a), (b, c), (c, d), (d, c)}

(ii) g = {(a, c), (b, d), (b, c)}

(iii) h = {(b, c), (d, a), (a, a)}

Q.16 (i) Let f = {(1, 1), (2, 3), (0, –1), (–1, –3)} be a function from Z to Z defined by f(x) = ax + b for some integers a, b determine a and b.

(ii) Let f = {(1, 1), (2, 3), (0, –1), (–1, –3)} be a linear function from Z to Z, find f(x).

Q.17 Function f is given by f = {(4, 2), (9, 1), (6, 1), (10, 3)}. Find the domain and range of f.

Q.18 If A = {(–3, –2, –1, 0, 1, 2, 3)} and $f(x) = x^2 - 1$ defines $f: A \to R$. Then find range of f.

Q.19 Find the domain and range of the following functions.

(i) f(x) = x (ii) f(x) = 2 – 3x

(iii) $f(x) = x^2 - 1$ (iv) $f(x) = x^2 + 2$ (v) $f(x) = \sqrt{x-1}$

Q.20 Find the domain of the function $f(x) = \frac{x^2 + 3x + 5}{x^2 - 5x + 4}$

Q.21 Find the domain of the definition and range of the function defined by the rules:

(i) $f(x) = x^2$ (ii) g(x) = | x |

(iii) $h(x) = \frac{1}{3 - x^2}$ (iv) $u(x) = \sqrt{4 - x^2}$

Q.22 Consider the following rules:

(i) $f: R \to R : f(x) = \log_e x$

(ii) $g: R \to R : g(x) = \sqrt{x}$

(iii) $h: A \to R : h(x) = \frac{1}{x^2 - 4}$, where A = R – {–2, 2}

Which of them are functions? Also find their range, if they are function.

Q.23 Let $f: R - \{2\} \to R$ be defined by $f(x) = \frac{x^2 - 4}{x - 2}$ and $g: R \to R$ be defined by g(x) = x + 2. Find whether f = g or not.

Q.24 Let f = {(1, 1), (2, 3), (0, –1), (–1, –3)} be a linear function from Z into Z and g(x) = x. Find f + g.

Q.25 Find f + g, f – g, f · g, f/g and a f ($a \in R$) if

(i) $f(x) = \frac{1}{x+4}$, $x \neq -4$ and $g(x) = (x + 4)^3$

(ii) f(x) = cos x, $g(x) = e^x$.

Q.26 If f(x) = x, g(x) = | x |, find (f+g)(–2), (f – g) (2), (f·g)(2), $\left(\frac{f}{g}\right)$(–2), and 5f(2).

Q.27 Define the function $f: R \to R$ by $y = f(x) = x^2$. Complete the table given below:

x	–4	–3	–2	–1	0	1	2	3	4
$y=f(x)=x^2$									

Q.28 Define the real valued function on

$f: R - \{0\} \to R$ as f(x) = 1/x

Complete the figure given below :

x	–2	–1.5	–1	–0.5	0	1	1	2	2
$y=f(x)=\frac{1}{x_1}$									

Find the domain and range of f.

Q.29 If $f(x + 3) = x^2 - 1$, write the expression for f(x).

Exercise 2

Sets and Relations

Single Correct Choice Type

Q.1 Let A = {1, 2, 3, 4}, and let R = {(2, 2),(3, 3),(4, 4), (1, 2)} be a relation on A. Then R is

(A) Reflexive (B) Symmetric

(C) Transitive (D) None of these

Q.2 The void relation on a set A is

(A) Reflexive

(B) Symmetric and transitive

(C) Reflexive and symmetric

(D) Reflexive and transitive

Q.3 For real number x and y, we write x R y ⇔ x – y + is an irrational number. Then the relation R is

(A) Reflexive (B) Symmetric

(C) Transitive (D) None of these

Q.4 Let R be a relation in N defined by

$R = \{(1 + x, 1 + x^2) : x \leq 5, x \in N\}$.

Which of the following is false

(A) R = {(2, 2), (3, 5), (4, 10), (5, 17), (6, 25)}

(B) Domain of R = {2, 3, 4, 5, 6}

(C) Range of R = {2, 5, 10, 17, 26}

(D) None of these

Q.5 The relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)} on the set A = {1, 2, 3} is

(A) Reflexive but not symmetric

(B) Reflexive but not transitive

(C) Symmetric and transitive

(D) Neither symmetric nor transitive

Q.6 Let A = {2, 3, 4, 5} and let R = {(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)} be a relation in A. Then R is

(A) Reflexive and transitive

(B) Reflexive and symmetric

(C) Reflexive and anti-symmetric

(D) None of these

Q.7 If A = {2, 3} and B = {1, 2}, then A × B is equal to

(A) {(2, 1), (2, 2), (3, 1), (3, 2)}

(B) {(1, 2), (1, 3), (2, 2), (2, 3)}

(C) {(2, 1), (3, 2)}

(D) {(1, 2), (2, 3)}

Q.8 If A = {1, 2, 3} and B = {1, 4, 6, 9} and R is a relation from A to B defined by 'x is greater than y'. The range of R is

(A) {1, 4, 6, 9} (B) {4, 6, 9}

(C) {1} (D) None of these

Q.9 Let R = {(1, 3), (4, 2), (2, 4),(2, 3), (3, 1)} be a relation on the set A = {1, 2, 3, 4}. The relation R is

(A) Transitive (B) Not symmetric

(C) Reflexive (D) A function

Q.10 If A, B and C are these sets such that A ∩B = A ∩ C and A ∪ B = A ∪ C, then

(A) A = B (B) A = C

(C) B = C (D) A ∩ B = f

Q.11 The relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)} on the set A = {1, 2, 3} is

(A) Reflexive but not symmetric

(B) Reflexive but not transitive

(C) Symmetric and transitive

(D) Neither symmetric nor transitive

Q.12 Let A be the set of all children in the world and R be a relation in A defined by x R y if x and y have same sex. Then R is

(A) Not reflexive (B) Not symmetric

(C) Not transitive (D) An equivalence relation

Q.13 Let A = {2, 3, 4, 5} and R = {(2 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)} be a relation on A. Then R is

(A) Reflexive and transitive

(B) Reflexive and symmetric

(C) An equivalence relation

(D) None of these

Q.14 Let L be the set of all straight lines in the xy-plane. Two lines l_1 and l_2 are said to be related by the relation R if l_1 is parallel to l_2. Then the relation R is

(A) Reflexive (B) Symmetric

(C) Transitive (D) Equivalence

Q.15 Given the relation R = {(2, 3), (3, 4)} on the set {2, 3, 4}. The number of minimum number of ordered pair to be added to R so that R is reflexive and symmetric

(A) 4 (B) 5 (C) 7 (D) 6

Q.16 The minimum number of elements that must be added to the relation R = {(1, 2),(2, 3)} on the set {1, 2, 3}, so that it is equivalence is

(A) 4 (B) 7 (C) 6 (D) 5

Functions

Single Correct Choice Type

Q.1 If f(x+ay, x–ay) = ay then f(x, y) is equal to:

(A) $\frac{x^2 - y^2}{4}$ (B) $\frac{x^2 + y^2}{4}$

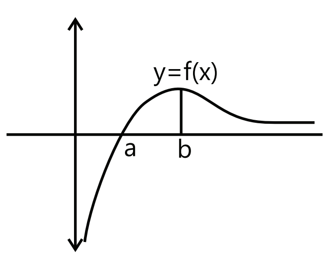
(C) 4xy (D) None of these

Q.2 The set of values of 'a' for which f: R → R f(x) = ax + cos x is bijective is

(A) [–1, 1] (B) R – {–1, 1}

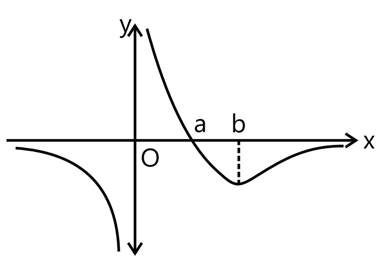
(C) R – (–1, 1) (D) R – {0}

Q.3 The graph of function f(x) is as shown, adjacently

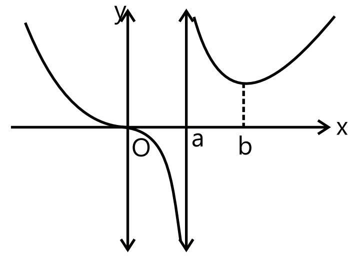


Then the graph of $\frac{1}{f(|x|)}$ is

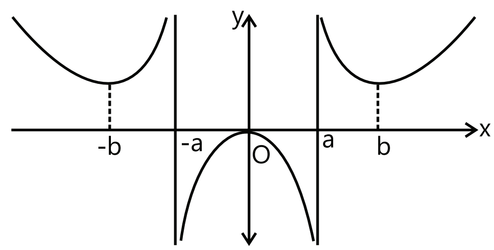
(A)



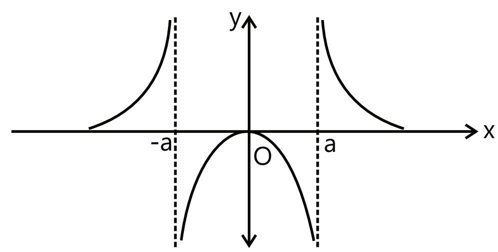
(B)



(C)



(D)



Q.4 Period of the function f(x) = $\frac{1}{2}\left(\frac{|\sin x|}{\cos x} + \frac{\sin x}{|\cos x|}\right)$ is

(A)π /2 (B) p

(C) 2 p (D) 4p

Q.5 Let f : R → R be a function defined by

f(x) = $\frac{2x^2 - x + 5}{7x^2 + 2x + 10}$ then f is

(A) One–one but not onto

(B) Onto but not one–one

(C) Onto as well as one–one

(D) Neither onto nor one–one

Q.6 If f(x) = $\cos\left[\frac{1}{2}\pi^2\right]x + \sin\left[\frac{1}{2}\pi^2\right]x$, [x] denoting the greatest integer function, then

(A) f(0) = 0 (B) $f\left(\frac{\pi}{3}\right) = \frac{1}{4}$

(C) $f\left(\frac{\pi}{2}\right) = 1$ (D) f(π) = 0

Q.7 Let f(x)= ln x and g(x) = $\frac{x^4 - x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1}$.

The domain of the composite function fog(x) is

(A) (–∞, ∞) (B) (0, ∞)

(C) (0, ∞) (D) (1, ∞)

Previous Years' Questions

Q.1 Let f(x) = | x – 1 |. Then, ***(1983)***

(A) $f(x^2) = [f(x)]^2$

(B) f(x + y) = f(x) + f(y)

(C) f(| x |) = | f(x) |

(D) None of the above

Q.2 If f(x) = cos(log x), then $f(x).f(y) - \frac{1}{2}\frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value ***(1983)***

(A) –1 (B) $\frac{1}{2}$

(C) –2 (D) None of these

Q.3 The domain of definition of the function ***(1983)***

$y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$ is

(A) (–3,–2) excluding –2.5

(B) [0, 1] excluding 0.5

(C) (–2, 1) excluding 0

(D) None of the above

Q.4 Which of the following functions is periodic? ***(1983)***

(A) f(x) = x – [x] where [x] denotes the greatest integer less than or equal to the real number x

(B) f(x) = sin $\frac{1}{x}$ for x 0, f(0) = 0

(C)f(x) = x cos x

(D) None of the above

Q.5 For real x, the function $\frac{(x-a)(x-b)}{(x-c)}$ will assume all real values provided ***(1984)***

(A) a > b > c (B) a < b < c

(C) a > c < b (D) a ≤ c ≤ b

Q.6 If g{f(x)} = | sin x| and f{g(x)} = $(\sin \sqrt{x})^2$, then ***(1998)***

(A) $f(x) = \sin^2 x$, g(x) = $\sqrt{x}$

(B) f(x) = sinx, g(x) = | x |

(C) $f(x) = x^2$, g(x) = sin $\sqrt{x}$

(D) f and g cannot be determined

Q.7 If f(x) = 3x – 5, then $f^{-1}(x)$ ***(1998)***

(A) Is given by $\frac{1}{3x-5}$

(B) Is given by $\frac{x+5}{3}$

(C) Does not exist because f is not one–one

(D) Does not exist because f is not onto

Q.8 If the function f : [1, ∞) → [1, ∞) is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is ***(1999)***

(A) $\left(\frac{1}{2}\right)^{x(x-1)}$ (B) $\frac{1}{2}(1+\sqrt{1+4\log_2 x})$

(C) $\frac{1}{2}(1-\sqrt{1+4\log_2 x})$ (D) Not defined

Q.9 Let f(θ) = sin θ (sin θ + sin 3θ). Then f(θ) ***(2000)***

(A) ≥ 0 only when q≥ 0 (B) ≤ 0 for all real q

(C) ≥ 0 for all real q (D) ≤ 0 only when θ≤ 0

Q.10 The domain of definition of the function y(x) is given by the equation $2^x + 2^y = 2$, is ***(2000)***

(A) 0 < x ≤ 1 (B) 0 ≤ x ≤ 1

(C) –∞< x ≤ 0 (D) –∞< x < 1

Q.11 Let f:N→Y be a function defined as f (x) = 4x + 3, where $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$. Show that f is invertible and its inverse is ***(2008)***

(A) $g(y) = \frac{3y+4}{3}$ (B) $g(y) = 4 + \frac{y+3}{4}$

(C) $g(y) = \frac{y+3}{4}$ (D) $g(y) = \frac{y-3}{4}$

Q.12 For real x, let $f(x) = x^3 + 5x + 1$, then ***(2009)***

(A) f is one-one but not onto R

(B) f is onto R but not one-one

(C) f is one-one and onto R

(D) f is neither one-one nor onto R

Statement-I (assertion) and statement-II (reason).

Each of these questions also have four alternative choices, only one of which is the correct answer. You have to select the correct choice

Q.13 Let $f(x) = (x+1)^2 - 1, x \geq -1$

Statement-I : The set

$\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$

Statement-2 : f is a bijection. ***(2009)***

(A) Statement-I is true, statement-II is true; statement-II is a correct explanation for statement-I

(B) Statement-I is true, statement-II is true; statement-II is nota correct explanation for statement-I

(C) Statement-I is true, statement-II is false

(D) Statement-I is false, statement-II is true

Q.14 Let $f(x) = x|x|$ and $g(x) = \sin x$ ***(2009)***

Statement-I : gof is differentiable at x = 0 and its derivative is continuous at that point. Statement-II : gof is twice differentiable at x = 0.

(A) Statement-I is true, statement-II is true; statement-II is a correct explanation for statement-I

(B) Statement-I is true, statement-II is true; statement-II is nota correct explanation for statement-I

(C) Statement-I is true, statement-II is false

(D) Statement-I is false, statement-II is true

Q.15 Consider the following relations:

$R = \left\{ \begin{matrix} (x, y) \mid x, \text{ are real numbers and } x \\ = wy \text{ for some rational number } w \end{matrix} \right\}$

$S = \left\{ \left(\frac{m}{n}, \frac{p}{q} \right) \middle| \begin{matrix} m, n, p \text{ and } q \text{ are integers such that } n, \\ q \neq 0 \text{ and } qm = pn \end{matrix} \right\}$

Then ***(2010)***

(A) Neither R nor S is an equivalence relation

(B) S is an equivalence relation but R is not an equivalence relation

(C) R and S both are equivalence relations

(D) R is an equivalence relation but S is not an equivalence relation

Q.16 The domain of the function

$f(x) = \frac{1}{\sqrt{|x| - x}}$ is ***(2011)***

(A) $(0, \infty)$ (B) $(-\infty, 0)$

(C) $(-\infty, \infty) - \{0\}$ (D) $(-\infty, \infty)$

Q.17 If $f : R \to R$ is a function defined by

$f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi,$

where [x] denotes the greatest integer function, then f is ***(2012)***

(A) Continuous for every real x

(B) Discontinuous only at x = 0

(C) Discontinuous only at non-zero integral values of x

(D) Continuous only at x = 0

Q.18 Consider the function

$f(x) = |x-2| + |x-5|, x \in R.$ ***(2012)***

Statement-I: $f'(4) = 0$

Statement-II: f is continuous in [2, 5], differentiable in (2, 5) and f(2) = f(5).

(A) Statement-I is false, statement-II is true

(B) Statement-I is true, statement-II is true; statement-II is a correct explanation for statement-I

(C) Statement-I is true, statement-II is true; statement-II is not a correct explanation for statement-I

(D) Statement-I is true, statement-II is false

Q.19 If $a \in R$ and the equation

$-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$

(where [x] denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval ***(2014)***

(A) (-2, -1) (B) $(-\infty, -2) \cup (2, \infty)$

(C) $(-1, 0) \cup (0, 1)$ (D) (1, 2)

Q.20 Let $f_k(x)=\frac{1}{k}\left(\sin^k x+\cos^k x\right)$ where $x \in R$ and $k \geq 1$. Then $f_4(x)-f_6(x)$ equals. ***(2014)***

(A) $\frac{1}{4}$ (B) $\frac{1}{12}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

Q.21 If $f(x)+2f\left(\frac{1}{x}\right)=3x, x \neq 0$, and $S=\{x \in R: f(x)=f(-x)\}$; then S: ***(2016)***

(A) Contains exactly one element

(B) Contains exactly two elements.

(C) Contains more than two elements.

(D) Is an empty set.

JEE Advanced/Boards

Exercise 1

Sets and Relations

Q.1 Is set a collection of objects or a collection of well-defined objects which are distinct and distinguishable?

Q.2 Is the set { x : x ∈ N, x is prime and 3 < x < 5} is void or non-void?

Q.3 A = {a, e, i, o, u} and B = {i, o} then $A \subset B$ or $B \subset A$?

Q.4 A set is defined as A = {x : x is irrational and 0.1 < x < 0.101} then comment on whether A is null set or A is finite set or A is infinite set.

Q.5 Which of these: f, {}, {2, 3} and {ϕ} is a singleton Set?

Q.6 Two points A and B in a plane are related if OA = OB, where O is a fixed point. Then comment whether this relation is reflexive, symmetric, transitive or equivalence?

Q.7 If A = {2, 3} and B = {-2, 3}, then what is the value of A ∪ B?

Q.8 Given the sets A = {1, 2, 3}, B = {3, 4}, C = {4, 5, 6}, then what is the value of A ∪(B ∩ C)?

Q.9 If N_a = {an : n ∈ N}, then what is the value of $N_6 \cap N_8$?

Q.10 Is it true that both I = { x : x ∈ R and $x^2 + x + 1 = 0$}, II = { x : x ∈ R and $x^2 - x + 1 = 0$} are empty sets?

Q.11 Two finite sets have m and n elements respectively. The total number of subsets of first set is 56 more than the total number of subsets of the second set. Calculate the values of m and n

Q.12 If A = {x | x/2 ∈ Z, 0 ≤ x ≤ 10}. B = { x | x is one digit prime} C= {x | x/3 ∈ N, x ≤ 12}, Then what is the value of A ∩ (B ∪ C) ?

Q.13 If n(A) = 10, n(B) = 15 and n(A ∪ B) = x, then what is the range of x?

Q.14 Among 1000 families of a city, 40% read newspaper A, 20% read newspaper B, 10% read newspaper C, 5% read both A and B, 3% read both B and C, 4% read A and C and 2% read all three newspapers. What is the number of families which read only newspaper A?

Q.15 If for three disjoint sets A, B, C; n(A) = 10, n(B) = 6 and n(C) = 5, then what is the value of n(A ∪ B ∪ C) ?

Q.16 If A and B are disjoint, then what is the value of n(A ∪ B)?

Q.17 If X and Y are two sets, then what is the value of $X \cap (Y \cup X)^c$?

Q.18 Let n(U) = 700, n(A) = 200, n(B) = 300 and n(A ∩ B) = 100, then what is the value of $n(A^c \cap B^c)$?

Q.19 What is the value of set $(A \cap B^c)^c \cup (B \cap C)$?

Q.20 Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in A ∪ B?

Q.21 In a class of 100 students, 55 students have passed in Mathematics and 67 students have passed in physics, no student fails. Calculate the number of student who have passed in Physics only?

Q.22 Let X = {1, 2, 3, 4, 5, 6} be an universal set. Sets A, B, C in the universal set X be defined by A = {1, 2, 3}, B = {2, 4,5} and C = {3, 4, 5, 6}, then what is the value of (A – B) ∪ (B – A), (A – B) – C and A ∩ C′?

Q.23 If A, B and C are any three sets, then is A × (B ∪ C) equal to (A × B) ∪ (A × C) or (A × B) ∩(A × C)?

Q.24 If A, B and C are any three sets, then is A × (B ∩ C) equal to (A × B) ∩ (A ×C) or (A ∩ B) × (A ∩C)?

Q.25 Let A = {a, b, c, d}, B = {b, c, d, e}. Then what is the value of [(A × B) ∩ (B × A)]?

Q.26 In the set A= {1, 2, 3, 4, 5}, a relation R is defined by R = {(x, y) | x, y ∈ A and x < y}. Then is R reflexive or transitive or symmetric?

Q.27 Let R be a relation on the set N of natural numbers defined by nRm ⇔ is a factor of m (i.e., n | m). Then is R symmetric?

Q.28 If R is a relation from a finite set A having m elements to a finite set B having n elements, then what will be the number of relations from A to B ?

Q.29 Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\alpha \; R \; \beta \Leftrightarrow \alpha \perp \beta$, $\alpha, \beta \in L$. Then is R symmetric?

Functions

Q.1 Find the domain of the definitions of the following functions: (Read the symbol [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $y = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$

(ii) $y = \sqrt{x^2-3x+2} + \sqrt{\dfrac{1}{3+2x-x^2}}$

(iii) $y = \sqrt{x} + \sqrt[3]{\dfrac{1}{x-2}} - \log_{10}(2x-3)$

(iv) $f(x) = \sqrt{\dfrac{1-5^x}{7^{-x}-7}}$

(v) $y = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

(vi) $f(x) = \log_{100x}\left(\dfrac{2\log_{10}x+1}{-x}\right)$

(vii) $f(x) = y = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

(viii) $y = \sqrt{\log_{10}\left(\dfrac{5x-x^2}{4}\right)}$

(ix) $f(x) = \sqrt{x^2-|x|} + \dfrac{1}{\sqrt{9-x^2}}$

(x) $f(x) = \sqrt{(x^2-3x-10)\,\ell n^2(x-3)}$

(xi) $f(x) = \sqrt{(\sin x+\cos x)^2-1}$

(xii) $f(x) = \sqrt{\dfrac{\cos x-(1/2)}{6+35x-6x^2}}$

(xiii) $f(x) = \sqrt{\log_{1/3}(\log_4([x]^2-5))}$

(xiv) $f(x) = \dfrac{1}{[x]} + \log_{(2|x|-5)}(x^2-3x+10) + \dfrac{1}{\sqrt{1-|x|}}$

(xv) $f(x) = \log_7\log_5\log_3\log_2(2x^3+5x^2-14x)$

(xvi) $f(x) = \sqrt{\cos 2x} + \sqrt{16-x^2}$

(xvii) $f(x) = \ln(\sqrt{x^2-5x-24}-x-2)$

(xvii) $y = \log 10\,(1-\log_{10}(x^2-5x+16))$

(xix) $f(x) = \log 4\left(2-\sqrt[4]{x}-\dfrac{2\sqrt{x}+1}{\sqrt{x}+2}\right)$

Q.2 Find the domain and range of the following functions. (Read the symbols [*] and {*} as greatest integers and fractional part function respectively)

(i) $y = \log_{\sqrt{5}}\left(\sqrt{2}(\sin x-\cos x)+3\right)$

(ii) $y = \dfrac{2x}{1+x^2}$

(iii) $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$

(iv) $f(x) = \dfrac{x}{1 + |x|}$

(v) $y = \sqrt{2 - x} + \sqrt{1 + x}$

(vi) $f(x) = \log_{(\text{cosec } x - 1)}(2 - [\sin x] - [\sin x]^2)$

(vii) $f(x) = \dfrac{x + 1}{x - 2}$

Q.3 Classify the following functions f(x) defined in $R \to R$ as injective, surjective, both or none.

(a) $f(x) = \dfrac{x^2 + x + 1}{x^2 + 2x + 3}$

(b) $f(x) = (x^2 + 5x + 9)(x^2 + 5x + 1)$

Q.4 Let $f(x) = \dfrac{1}{1 - x}$. Let $f_2(x)$ denote f[f(x)] and $f_3(x)$ denotes f[f{f(x)}]. Find $f_{3n}(x)$ where n is a natural number. Also state the domain of this composite function.

Q.5 The function f(x) is defined as follows : on each of the intervals $n \le x < n + 1$, where n is a positive integer, f(x) varies linearly, and $f(n) = -1$, $f\left(n + \dfrac{1}{2}\right) = 0$. Draw the graph of the function.

Q.6 (a) For what values of x is the inequality $|f(x) + \varphi(x)| < |f(x)| + |\phi(x)|$ true if, $f(x) = x - 3$, and $\varphi(x) = 4 - x$.

(b) For what values of x is the inequality $|f(x) - \varphi(x)| > |f(x)| - |\phi(x)|$ true if, $f(x) = x$, and $\varphi(x) = x - 2$.

Q.7 Find whether the following functions are even or odd or none:

(a) $f(x) = \log(x + \sqrt{1 + x^2})$

(b) $f(x) = \dfrac{(a^x + 1)}{a^x - 1}$

(c) $f(x) = x^4 - 2x^2$

(d) $f(x) = x^2 - |x|$

(e) $f(x) = x \sin^2 x - x^3$

(f) f(x) = K, where K is constant

(g) $f(x) = \ln\left(\dfrac{1 - x}{1 + x}\right)$

(h) $f(x) = \dfrac{(1 + 2^x)^2}{2^x}$

(i) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$

(j) $f(x) = [(x + 1)^2]^{1/3} + [(x - 1)^2]^{1/3}$

Q.8 Find the period for each of the following functions:

(a) $f(x) = \sin^4 x + \cos^4 x$

(b) $f(x) = |\sin x| + |\cos x|$

(c) $f(x) = \cos\dfrac{3}{5}x - \sin\dfrac{2}{7}x$

Q.9 Write explicitly, function of y defined by the following equations and also find the domains of definition of the given implicit functions:

(a) $10^x + 10^y = 10$ (b) $x + |y| = 2y$

Q.10 Find out for what integral values of n the number 3π is a period of the functions: $f(x) = \cos nx \cdot \sin(5/n)x$.

Q.11 Compute the inverse of the functions:

(a) $f(x) = \ln(x + \sqrt{x^2 + 1})$ (b) $f(x) = 2^{\frac{x}{x-1}}$

(c) $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

Q.12 Show if $f(x) = \sqrt[n]{a - x^n}$, $x > 0$, $n \ge 2$, $n \in N$, then (fof) (x) = x. Find also the inverse of f(x).

Q.13 $f : \left[-\dfrac{1}{2}, \infty\right) \to \left[-\dfrac{9}{4}, \infty\right)$,

Defined as $f(x) = x^2 + x - 2$. Find $f^{-1}(x)$ and solve the equation $f(x) = f^{-1}(x)$.

Q.14 f(x) is defined for x <0 as

$$f(x) = \begin{cases} x^2 + 1 & x < -1 \\ -x^3 & -1 \le x < 0 \end{cases}$$

Define f(x) for $x \ge 0$, if f is

(a) Odd (b) Even

Q.15 If f(x) = max $\left(x, \frac{1}{x}\right)$ for x > 0 where max(a, b) denotes the greater of the two real numbers a and b. Define the function $g(x) = f(x)\, f\left(\frac{1}{x}\right)$ and plot its graph.

Q.16 Show that the function $f(x) = \frac{x^2 + 2x + c}{x^2 + 4x + 3c}$ attains any real value if $0 < c \le 1$.

Q.17 (a) Find the domain and range of the function

$f(x) = \sqrt{\log_2(x^2 - 2x + 2)}$

(b) $f : R - \{2\} \to R - \{2\}$; $f(x) = \frac{2x+3}{x-2}$, find whether f(x) is bijective or not.

Q.18 Prove that function $f(x) = 1 + 2^{\sqrt{\frac{x(x-2)+1}{4}}}$ is many one.

Exercise 2

Sets and Relations

Single Correct Choice Type

Q.1 Let R be a relation on the set N of natural numbers defined by nRm⇔ n is a factor of m (i.e. n | m). Then R is

(A) Reflexive and symmetric

(B) Transitive and symmetric

(C) Equivalence

(D) Reflexive, transitive but not symmetric

Q.2 Let R be a relation defined in the set of real numbers by a R b ⇔ 1 + ab> 0. Then R is

(A) Equivalence relation

(B) Transitive

(C) Symmetric

(D) Anti–symmetric

Q.3 Which one of the following relations on R is equivalence relation

(A) $x\, R_1\, y \Leftrightarrow |x| = |y|$

(B) $x\, R_2\, y \Leftrightarrow x \ge y$

(C) $x\, R_3\, y \Leftrightarrow x \mid y$

(D) $x\, R_4\, y \Leftrightarrow x < y$

Q.4 The relation R defined in A = {1, 2, 3} by aRb if $|a^2 - b^2| \le 5$. Which of the following is false

(A) R = {(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)}

(B) $R^{-1} = R$

(C) Domain of R = {1, 2, 3}

(D) Range of R = {5}

Q.5 Let a relation R in the set N of natural numbers be defined as (x, y) ∈ R if and only if $x^2 - 4xy + 3y^2 = 0$ for all x, y ∈N. The relation R is

(A) Reflexive

(B) Symmetric

(C) Transitive

(D) An equivalence relation

Q.6 Let R {(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)} be a relation on the set A = {3, 6, 9, 12}. The relation is

(A) An equivalence relation

(B) Reflexive and symmetric only

(C) Reflexive and transitive only

(D) Reflexive only

Q.7 Let R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)} be a relation on the set A = {1, 2, 3, 4}. The relation R is

(A) Reflexive (B) Transitive

(C) Not symmetric (D) A function

Q.8 Let N denote the set of all natural numbers and R be the relation on N × N defined by (a, b) R (c, d) if ad(b + c) = bc(a + d), then R is

(A) Symmetric only

(B) Reflexive only

(C) Transitive only

(D) An equivalence relation

Q.9 Let W denote the words in the English dictionary. Define the relation R by R = {(x, y) ∈ W × W | the words x and y have at least one letter in common}. Then R is

(A) Reflexive, symmetric and not transitive

(B) Reflexive, symmetric and transitive

(C) Reflexive, not symmetric and transitive

(D) Not reflexive, symmetric and transitive

Q.10 Let R be the real line. Consider the following subsets of the plane R × R :

S = {(x, y) : y = x + 1 and 0 < x < 2}

T = {(x, y) : x – y is an integer}

Which one of the following is true?

(A) Both Sand T are equivalence relations on R

(B) S is an equivalence relation on R but T is not

(C) T is an equivalence relation on R but S is not

(D) Neither S nor T is an equivalence relation on R

Multiple Correct Choice Type

Q.11 Let X = {1, 2, 3, 4, 5} and Y = {1, 3, 5, 7, 9}. Which of the following is/are relations from X to Y

(A) $R_1 = \{(x, y) \mid y = 2 + x, x \in X, y \in Y\}$

(B) $R_2 = \{(1, 1),(2, 1), (3, 3), (4, 3), (5, 5)\}$

(C) $R_3 = \{(1, 1),(1, 3), (3, 5), (3, 7), (5, 7)\}$

(D) $R_4 = \{(1, 3),(2, 5), (2, 4), (7, 9)\}$

Functions

Q.1 Domain of the function,

$f(x) = \sin^{-1}\sqrt{x - x^2} + \sec^{-1}\left(\frac{1}{x}\right) + \ln x$ is

(A) [0, 1) (B) (0, 1] (C) (0, 1) (D) None of these

Q.2 In the square ABCD with side AB = 2, two points M and N are on the adjacent sides of the square such that MN is parallel to the diagonal BD. If x is the distance of MN from the vertex A and f(x) = Area (ΔAMN) then range of f(x) is

(A) $(0, \sqrt{2}]$ (B) (0, 2] (C) $(0, 2\sqrt{2}]$ (D) $(0, 2\sqrt{3}]$

Q.3 If 'f' and 'g' are bijective functions and gof is defined the, gofis :

(A) Injective (B) Surjective

(C) Bijective (D) Into only

Q.4 If y = 5[x] + 1 = 6[x – 1] – 10, where [.] denotes the greatest integer function, then [x + 2y] is equal to

(A) 76 (B) 61 (C) 107 (D) 189

Q.5 $f : R \to R$, $f(x) = ax^3 + e^x$ is one–one onto, then 'a' belongs to the interval

(A) $(-\infty, 0)$ (B) $(-\infty, 0]$ (C) $[0, \infty)$ (D) $(0, \infty)$

Q.6 The value of x in [–2π, 2π], for which the graph of the function $y = \sqrt{\frac{1+\sin x}{1-\sin x}} - \sec x$ and $y = -\sqrt{\frac{1-\sin x}{1+\sin x}} + \sec x$, coincide are

(A) $\left[-2\pi, -\frac{3\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$

(B) $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, -\frac{3\pi}{2}\right)$

(C) [–2 π, 2π]

(D) $[-2\pi, 2\pi] - \left\{\pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}\right\}$

Q.7 The period of the function

$f(x) = \sin\left(\cos\frac{x}{2}\right) + \cos(\sin x)$ equal

(A) $\frac{\pi}{2}$ (B) 2π (C) p (D) 4p

Q.8 Let $f : R \to R$ $f(x) = \frac{x}{1+|x|}$. Then f(x) is

(A) Injective but not surjective

(B) Surjective but not injective

(C) Injective as well as surjective

(D) Neither injective nor surjective

Multiple Correct Choice Type

Q.9 Let $f : I \to R$ (where I is the set of positive integers) be a function defined by, $f(x) = \sqrt{x}$, then f is

(A) One–one (B) Many one

(C) Onto (D) Into

Q.10 The function $f(x) = \sqrt{\log_{x^2} x}$ is defined for x belonging to

(A) $(-\infty, 0)$ (B) (0, 1) (C) $(1, \infty)$ (D) $(0, \infty)$

Q.11 If $f(x) = \frac{x\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1} - 1}$, then

(A) f(x) = –x if x < 2

(B) 2f(1.5) + f(3) is non negative integer

(C) f(x) = x if x > 2

(D) None

Q.12 Which of the following function(s) is(are) bounded on the intervals as indicated

(A) $f(x) = 2^{\frac{1}{x-1}}$ on (0, 1)

(B) $g(x) = x \cos\frac{1}{x}$ on $(-\infty, \infty)$

(C) $h(x) = xe^{-x}$ on $(0, \infty)$

(D) $\ell(x) = \arctan 2^x$ on $(-\infty, \infty)$

Q.13 Which of the following function(s) is/are periodic?

(A) f(x) – x – [x]

(B) g(x) = sin(1/x), $x \neq 0$ and g(0) = 0

(C) h(x) = x cos x

(D) w(x) = (sin x)

Q.14 On the interval [0, 1], f(x) is defined as,

$$f(x) = \begin{cases} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{cases}.$$

Then for all $x \in R$ the composite function f[f(x)] is

(A) A constant function

(B) An identity function

(C) An odd linear polynomial

(D) 1 + x

Q.15 Identify the pair(x) of functions which are identical.

(A) $y = \tan(\cos^{-1} x) : y = \frac{\sqrt{1-x^2}}{x}$

(B) $y = \tan(\cos^{-1}) : y = 1/x$

(C) $y = \sin(\arctan x) : y = \frac{x}{\sqrt{1+x^2}}$

(D) y = cos (arc tan x) : y = sin (arc cot x)

Previous Years' Questions

Q.1 If $y = f(x) = \frac{x+2}{x-1}$, then ***(1984)***

(A) x = f(y)

(B) f(1) = 3

(C) y increases with x for x < 1

(D) f is a rational function of x

Q.2 If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains ***(1986)***

(A) $\left(-\infty, -\frac{3}{2}\right)$ (B) $\left(-\frac{3}{2}, -\frac{1}{4}\right)$

(C) $\left(-\frac{1}{4}, \frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, 3\right)$

Q.3 Let g(x) be a function defined on [–1, 1]. If the area of the equilateral triangle with two of its vertices at (0, 0) and [x, g(x)] is $\sqrt{3}/4$, then the function g(x) is ***(1989)***

(A) $g(x) = \pm\sqrt{1-x^2}$ (B) $g(x) = \sqrt{1-x^2}$

(C) $g(x) = -\sqrt{1-x^2}$ (D) $g(x) = \sqrt{1+x^2}$

Q.4 If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where [x] stands for the greatest integer function, then ***(1991)***

(A) $f\left(\frac{\pi}{2}\right) = -1$ (B) $f(\pi) = 1$

(C) $f(-\pi) = 0$ (D) $f\left(\frac{\pi}{4}\right) = 1$

Q.5 Let $f : (0, 1) \to R$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that 0 < b < 1. Then, ***(2014)***

(A) f is not invertible on (0, 1)

(B) $f \neq f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(C) $f = f^{-1}$ on (0, 1)and $f'(b) = \frac{1}{f'(0)}$

(D) f^{-1} is differentiable on (0, 1)

Match the Columns

Q.6 Match the condition/expression in column I with statement in column II

Let the functions defined in column I have domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and range $(-\infty, \infty)$ ***(1992)***

Column I	Column II
(A) 1 + 2x	(p) Onto but not one–one
(B) tan x	(q) One–one but not onto
	(r) One–one and onto
	(s) Neither one–one nor onto

Q.7 Let f(x) = $\frac{x^2-6x+5}{x^2-5x+6}$ ***(2007)***

Column I	Column II
(A) If $-1 < x < 1$, then f(x) satisfies	(p) $0 < f(x) < 1$
(B) If $1 < x < 2$, then f(x) satisfies	(q) $f(x) < 0$
(C) If $3 < x < 5$, then f(x) satisfies	(r) $f(x) > 0$
(D) If $x > 5$, then f(x) satisfies	(s) $f(x) < 1$

Q.8 Match the statements/expressions in column I with the values given in column II. ***(2009)***

Column I	Column II
(A) The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$	(p) 1
(B) Value(s) of k for which the planes kx + 4y + z = 0, 4x + ky + 2z = 0 and 2x + 2y + z = 0 intersect in a straight line	(q) 2
(C) Value(s) of k for which $\|x-1\| + \|x-2\| + \|x+1\| + \|x+2\| = 4k$ has integer solution(s)	(r) 3
(D) If $y' = y + 1$ and $y(0) = 1$ then value(s) of y(ln2)	(s) 4
	(t) 5

Q.9 Match the statements/expressions in column I with the values given in column II. **(2009)**

Column I	Column II
(A) Root(s) of the expression $2\sin^2\theta + \sin^2 \theta - 2$	(p) $\frac{\pi}{6}$
(B) Points of discontinuity of the function $f(x) = \left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$, where [y] denotes the largest integer less than or equal to y	(q) $\frac{\pi}{4}$
(C) Volume of the parallelopiped with its edges represented by the vectors $\hat{i}+\hat{j}, \hat{i}+2\hat{j}$ and $\hat{i}+\hat{j}+\pi\hat{k}$	(r) $\frac{\pi}{3}$
(D) Angle between vectors $\vec{a}$ and $\vec{b}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a}+\vec{b}+\sqrt{3}\,\vec{c} = \vec{0}$	(s) $\frac{\pi}{2}$
	(t) π

Q.10 If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is ***(2009)***

Q.11 Let f be a real-valued function defined on the interval (−1, 1) such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$, for all $x \in (-1, 1)$ and f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to ***(2010)***

(A) 1 (B) 1/3 (C) 1/2 (D) 1/e

Q.12 For any real number, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [-10, 10] by

$$f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd,} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$ is ***(2010)***

Q.13 Let $\vec{a} = -\hat{i} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r}.\vec{a} = 0$, then the value of $\vec{r}.\vec{b}$ is ***(2011)***

Q.14 Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in$ R and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in$ R. Let (f o g)(x) denote f(g(x)) and (g o f)(x) denote g(f(x)). Then which of the following is (are) true? ***(2015)***

(A) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(B) Range of f o g is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(C) $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\pi}{6}$

(D) There is an $x \in$ R such that (g o f)(x) = 1

Q.15 Match the statements given in column I with the intervals/union of intervals given in column II ***(2011)***

	Column I		Column II
(A)	The set $\left\{Re\left(\frac{2iz}{1-z^2}\right): z \text{ is a complex number}, \lvert z\rvert = 1, z \neq 1\right\}$ is	(p)	$(-\infty, -1) \cup (1, \infty)$
(B)	The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q)	$(-\infty, 0) \cup (0, \infty)$
(C)	If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta): 0 \le \theta < \frac{\pi}{2}\right\}$ is	(r)	$[2, \infty)$
(D)	If $f(x) = x^{3/2}(3x-10), x \ge 0$, then f (x) is increasing in	(s)	$(-\infty, -1] \cup [1, \infty)$
		(t)	$(-\infty, 0] \cup [2, \infty)$

Q.16 The function $f: [0,3] \to [1,29]$, defined by $f(x) = 2x^3 - 15x^2 + 36 + 1$, is **(2012)**

(A) One–one and onto (B) Onto but not one–one

(C) One–one but not onto (D) Neither one–one nor onto

Q.17 Consider the statements:

P: There exists some $x \in IR$ such that $f(x) + 2x = 2(1+x^2)$

Q: There exists some $x \in IR$ such that $2f(x) + 1 = 2x(1+x)$

Then ***(2012)***

(A) Both P and Q ate true (B) P is true and Q is true

(C) P is false and Q is true (D) Both P and Q are false

Q.18 Let f : (-1 , 1) $\to$ IR be such that

$\to f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Then the value(s) of $f\left(\frac{1}{3}\right)$ is (are) ***(2012)***

Q.19 If the function $e^{-1}f(x)$ assumes its minimum in the interval [0, 1] at $x = \frac{1}{4}$, which of the following is true ? ***(2013)***

(A) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ (B) $f'(x) > f(x), 0 < x < \frac{1}{4}$

(C) $f'(x) < f(x), 0 < x < \frac{1}{4}$ (D) $f'(x) < f(x), \frac{3}{4} < x < 1$

Q.20 Let $f_1: R \to R, f_2: [0,\infty) \to R, f_3: R \to R$ and $f_4: R \to [0,\infty)$ be defined by ***(2014)***

$f_1(x) = \begin{cases} |x| \text{ if } x < 0 \\ e^x \text{ if } x \ge 0 \end{cases}$; $f_2(x) = x^2$; $f_3(x) = \begin{cases} \sin x \text{ if } x < 0 \\ x \quad \text{if } x \ge 0 \end{cases}$

and

$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0 \\ f_2(f_1(f_1(x))) - 1 & \text{if } x \ge 0 \end{cases}$

List I	List II
(1) f_4 is	(p) Onto but not one-one
(2) f_3 is	(q) Neither continuous nor one-one
(3) f_2 o f_1 is	(r) Differentiable but not one-one
(4) f_2 is	(s) Continuous and one-one

Codes:

	1	2	3	4
(A)	r	p	s	q
(B)	p	r	s	q
(C)	r	p	q	s
(D)	p	r	q	s

Q.21 let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ be given by $f(x)-\left(\log(\sec x+\tan x)\right)^3$. Then ***(2014)***

(A) f (x) is an odd function

(B) f (x) is a one-one function

(C) f (x) is an onto function

(D) f (x) is an even function

Q.22 Let f : R → R be a continuous odd function, which vanishes exactly at one point and $f(1)=\frac{1}{2}$. Suppose that $F(x)=\int_{-1}^{x} f(t)dt$ for all $x\in[-1,2]$ and $G(x)=\int_{-1}^{x} t\left|f(f(t))\right|dt$ for all $x\in[-1,2]$. If $\lim_{x\to1}\frac{F(x)}{G(x)}=\frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is ***(2015)***

Questions

JEE Main/Boards

Exercise 1

Sets and Relations

Q.4 Q.11 Q.14 Q.16

Functions

Q.5 Q.13 Q.19 Q.22

Q.26 Q.28 Q.31

Exercise 2

Sets and Relations

Q.2 Q.4 Q.8

Functions

Q.3

Previous Years' Questions

Q.4 Q.10

JEE Advanced/Boards

Exercise 1

Sets and Relations

Q.6 Q.11 Q.14 Q.21

Q.28 Q.30

Functions

Q.9 Q.11 Q.15 Q.19

Exercise 2

Sets and Relations

Q.1 Q.5 Q.9

Functions

Q.6 Q.8 Q.11 Q.15

Previous Years' Questions

Q.5 Q.7

Answer Key

JEE Main/Boards

Exercise 1

Sets and Relations

Q.1 {2, 4, 6}

Q.2 {(8, 11), (10, 13)}

Q.3 Empty relation

Q.4 yes

Q.5 No

Q.6 yes

Q.7 No

Q.8 R^{-1} = {(black, yellow), (dog, cat), (green, red)}

Q.9 Empty relation

Q.10 Yes

Functions

Q.1 1 **Q.2** $2\alpha, a \neq 0$ **Q.3** 2f(x)

Q.4 f = R− {−2, −6}

Q.5 $[-2, 0) \cup (0, 1)$

Q.6 $\left[-\frac{1}{2}, \frac{1}{2}\right]$

Q.7 $x \in R - \{-1, 1\}$; $y \in (-\infty, 0) \cup [1, \infty)$

Q.8 $\left[\frac{1}{3}, 1\right]$

Q.9 $\alpha x^2 + \alpha$

Q.10 (i) $\begin{cases} 0, & x \le 0 \\ 2x, & x \ge 0 \end{cases}$ (ii) $\begin{cases} 2x, & x \le 0 \\ 0, & x \ge 0 \end{cases}$

(iii) $\begin{cases} -x^2, & x \le 0 \\ x^2, & x \ge 0 \end{cases}$ (iv) ax

(v) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{|x|} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

Q.11 (i) e (ii) e (iii) 0

Q.12 −1

Q13 (i) $x \in \left[\frac{-\pi}{6}, 2\right]$ (ii) [cos 2, 1] (iii) 6

Q.14 f, u are function g, h are not function. Range f = {3, 9, 10}, Range u = (3, 6, 9)

Q.15 Only f is a function from A to A

Q.16 (i) a = 2, b = −1 (ii) f(x) = 2x − 1

Q.17 Domain = {4, 6, 9, 10}. Range = {1, 2, 3}

Q.18 Range f = {−1, 0, 3, 8}

Q.19 (i) Dom. f = R; Range of f = R

(ii) Dom. f = R; Range of f = R

(iii) Dom f = R; Range of f = $[1, \infty) = \{x : x \ge -1\}$

(iv) Dom f = R; Range of f = $[2, \infty) = \{x : x \ge 2\}$

(v) Dom. f = $[1, \infty)$; Range f = $[0, \infty) = \{x : x \ge 0\}$

Q.20 Domain of f = R – {1, 4}

Q.21 (i) Dom. f = R, Range f = $[0, \infty)$, (ii) Dom. g = R, Range g = $[0, \infty)$

(iii) Dom h = R – $\{-\sqrt{3}, \sqrt{3}\}$, Range h = $(-\infty, 0) \cup [1/3, \infty)$, (iv) Dom u = [–2, 2], Range u = [0, 2]

Q.22 f and g are not functions as they are not defined form negative values of x. h is function. Range h = $\left(-\infty, -\frac{1}{4}\right] \cup [0, \infty)$.

Q.23 $f \neq g$ as dom. $f \neq$ dom. g.

Q.24 f + g = {(1, 2), (2, 5), (0, –1), (–1, –4)}

Q.25 (i) $\frac{1+(x+4)^4}{x+4}$; $\frac{1-(x+4)^4}{x+4}$; $(x+4)^2$; $\frac{1}{(x+4)^4}$; $x \neq -4$; $\frac{\alpha}{x+4}$ (ii) $\cos x + e^x$; $\cos x - e^x$; $e^x \cos x$; $e^x \cos x$; a cos x

Q.26 0; 4; 4; –1; 10

Q.27

x	–4	–3	–2	–1	0	1	2	3	4
$y=f(x)=x^2$	16	9	4	1	0	1	4	9	16

Domain of f = $\{x : x \in R\}$ = R

Range of f = $\{x : x \geq 0, x \in R\} = [0, \infty)$

Graph of y = f(x) i.e., $y = x^2$ is as shown in the following figure.

Q.28 Domain f = R – {0}, Range f = R – {0}

Q.29 $f(x) = (x-3)^2 - 1$

Exercise 2

Sets and Relation

Q.1 C **Q.2** B **Q.3** B **Q.4** A **Q.5** A **Q.6** B

Q.7 A **Q.8** C **Q.9** B **Q.10** C **Q.11** A **Q.12** D

Q.13 B **Q.14** D **Q.15** B **Q.16** B

Functions

Q.1 A **Q.2** C **Q.3** C **Q.4** C **Q.5** D **Q.6** C **Q.7** A

Previous Years' Questions

Q.1 D **Q.2** D **Q.3** C **Q.4** A **Q.5** D **Q.6** A

Q.7 B **Q.8** B **Q.9** C **Q.10** D **Q.11** D **Q.12** C

Q.13 C **Q.14** C **Q.15** B **Q.16** B **Q.17** A **Q.18** B

Q.19 C **Q.20** B **Q.21** B

JEE Advanced/Boards

Exercise 1

Sets and relations

Q.8 {1, 2, 3, 4} **Q.11** 6, 3 **Q.14** 330 families **Q.16** n $(A \cup B) = n(A) + n(B)$

Q.25 9

Functions

Q.1 (i) defined no where (ii) $-1 < x \le 1$ and $2 \le x < 3$

(iii) $\frac{3}{2} < x < 2$ and $2 < x <$¥ (iv) $(-\infty, -1) \cup [0, \infty)$

(v) $-2 \le x < 0$ and $0 < x < 1$ (vi) $\left(0, \frac{1}{10}\right) \cup \left(\frac{1}{10}, \frac{1}{\sqrt{10}}\right)$

(vii) $(-2 \le x < 1)$-{0} (viii) $1 \le x \le 4$

(ix) $(-3, -1] \cup \{0\} \cup [1, 3)$ (x) $\{4\} \cup [5, \infty)$

(xi) $\left[n\pi, \left(n + \frac{1}{2}\right)\pi\right]$, $n \in I$

(xii) $\left(-\frac{1}{6}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 6\right) \cup \left[2K\pi + \frac{\pi}{3}, 2K\pi + \frac{5\pi}{3}\right]$, $k \in I - \{0\}$

(xiii) $[-3, -2) \cup [3, 4)$ (xiv) ϕ (xv) $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$

(xvi) $\left[-\frac{5\pi}{4}, \frac{-3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

(xvii) $(-\infty, -3]$ (xviii) $2 < x < 3$ (xix) $[0, 1)$

Q.2 (i) $D : x \in R$, $R : [0, 2]$

(ii) D = R; range $[-1, 1]$

(iii) D: $\{x \mid x \in R;\ x \ne -3;\ x \ne 2\}$, R: $\{f(x) \mid f(x) \in R,\ f(x) \ne 1/5:\ f(x) \ne 1\}$

(iv) D:R;R: $(-1, 1)$

(v) D: $-1 \le x \le 2$ R: $[\sqrt{3}, \sqrt{6}]$

(vi) $D : x \in (2n\pi, (2n+1)\pi) - \left\{2n\pi + \frac{\pi}{6}, 2n\pi + \frac{\pi}{2}, 2n\pi + \frac{5\pi}{6}, n \in I\right\}$ and $R : \log_a 2 : a \in (0, \infty) - \{1\} \Rightarrow$ Range is $(-\infty, \infty) - \{0\}$

(vii) $x \in R - \{2\}$, $f(x) \in R - \{1\}$

Q.3 (a) Neither surjective nor injective (b) Neither injective nor surjective

Q.4 $f_{3n}(x) = x$; Domain $= R - \{0, 1\}$

Q.5 2

Q.6 (a) $x < 3$ or $x > 4$ (b) $x < 2$

Q.7 (a) Odd (b) Odd (c) Even (d) Even (e) Odd (f) Even (g) Odd (h) Even

(i) Neither odd nor even (j) Even

Q.8 (a) $\pi/2$ (b) $\pi/2$ (c) 70p

Q.9 (a) $y = \log(10 - 10^x), -\infty < x < 1$ (b) $y = x/3$ when $-\infty < x < 0$ and $y = x$ when $0 \le x < +\infty$

Q.10 $\pm1, \pm3, \pm5, \pm15$

Q.11 (a) $\frac{e^x - e^{-x}}{2}$ (b) $\frac{\log_2 x}{\log_2 x - 1}$ (c) $\frac{1}{2}\log\frac{1+x}{1-x}$

Q.12 $f^{-1}(x) = (a - x^n)^{1/n}$

Q.13 $x = \pm\sqrt{2}$

Q.14 (a) $f(x) = \begin{cases} -(x^2+1) & x > 1 \\ x^3 & x \in (0,1] \end{cases}$ (b) $\begin{cases} (x^2+1) & x > 1 \\ -x^3 & x \in (0,1] \end{cases}$

Q.15 $g(0) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$

Q.17 (a) Domain $x \in R$, Range $[0, \infty)$ (b) yes

Exercise 2

Sets and Relations

Single Correct Choice Type

Q.1 D **Q.2** C **Q.3** A **Q.4** D **Q.5** A **Q.6** C **Q.7** C

Q.8 D **Q.9** A **Q.10** C

Multiple Correct Choice Type

Q.11 A, B, C

Functions

Single Correct Choice Type

Q.1 C **Q.2** B **Q.3** A **Q.4** D **Q.5** D **Q.6** A **Q.7** D

Q.8 A

Multiple Correct Choice Type

Q.9 A, D **Q.10** B, C **Q.11** B, C **Q.12** A, C, D **Q.13** A, D **Q.14** B, C **Q.15** A, C, D

Previous Years' Questions

Q.1 A, D **Q.2** A, D **Q.3** B, C **Q.4** A, C **Q.5** A **Q.6** A $\to$ q; B $\to$ r

Q.7 A $\to$ p; B $\to$ q; C $\to$ q; D $\to$ p

Q.8 A$\to$ p; B $\to$ q, s; C$\to$ q, r, s, t; D $\to$ r

Q.9 A$\to$ q, s; B$\to$ p, r, s, t; C$\to$ t; D$\to$ r

Q.11 B **Q.12** D **Q.14** A, B, C

Q.15 A $\to$ s; B $\to$ t; C$\to$r; D $\to$ r

Q.16 B **Q.17** C **Q.18** A, B

Solutions

JEE Main/Boards

Exercise 1

Sets and Relations

Sol 1: Relation R: x + 2y = 8 (defined in N)

x = 8 – 2y

x, y ∈ N

so for

$$\left.\begin{array}{ll} y=1 & x=8-2(1)=6 \\ y=2 & x=8-2(2)=4 \\ y=3 & x=8-2(3)=2 \end{array}\right\} \text{natural numbers}$$

y = 4 x = 8 – 2(4) = 0

so domain of R = {2, 4, 6}

Sol 2: Given relation R

R: {11, 12, 13} → {8, 10, 12}

y = x – 3

for R^{-1} x = y + 3

{8, 10, 12} → {11, 12, 13}

for y=8x =10 + 3 = 11 ✓
y=10x =10 + 3 = 13✓ } Element of {11, 12, 13}
y=12x =12 + 3 = 15

R^{-1} → {(8, 11), (10, 13)}

Sol 3: R = {(a, b) ∈ A × H: a is sister of b}

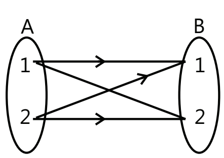
Domain and range both (a, b) are set {A (a, b) ∈ A × A}

And it is given that A is only boy's school a and b both are boys. So It is not possible that a is sister of b.

Sol 4: Given A = {1, 2}, B = {1, 3}

R: A → B {(1, 1),(1, 3),(2, 1),(2, 3)}

Universal relation: Each element of A is related to every element of B.



So R is universal relation.

Sol 5: R = {(a, b) ∈ N × N: a < b}

Reflective relation → R: A → B is said to be reflective iff a R a ∀ a ∈ A (here A, B ⇒ N)

R = {(a, a) ∈ N × N: a < a}

we know a < a is universal false

so R is not a reflexive relation.

Sol 6: P(A) ⇒ Power set of A

X R Y ⇒ X ∩ Y = X , Y ∈ P(A)

X R Y ↔ – X ∩ Y = X, Y

YRX ↔ Y ∩ X

We know X ∩ Y = Y ∩ X

so XRY → YRX

so R is symmetric.

Sol 7: A = {a, b, c}

and R = {(a, a),(a, b), (a,c),(b,a),(c, c)}

R^{-1} = {(a, a), (a, b),(c, a),(b, a),(c, c)}

R ≠ R^{-1}

so R is not symmetric relation.

Sol 8: R = {(yellow, black), (cat, dog), (red, green)}

R → x → y

R–1 → y → x

So R^{-1} = {(black, yellow), (dog, cat), (green, red)}

Sol 9: A = {1, 3, 5}, B = {9, 11}

R = {(a, b) ∈ A × B: a – b is odd}

A – b will be odd when a, b both are not even and not odd.

In A, B all elements are odd

so (a – b) ∈ π

R is empty relation.

Sol 10: A = {a, b, c}

R = {(a, c), (c, a)}

R^{-1} = {c, a},(a, c)

$R \equiv R^{-1}$ relation R and R^{-1} both are same

so R is symmetric.

Functions

Sol 1: $f(x)=\dfrac{2\tan x}{1+\tan^2 x} \Rightarrow f\left(\dfrac{\pi}{4}\right)=\dfrac{2\tan \pi/4}{1+\tan^2(\pi/4)}$

We know that tan $(\pi/4)$ = 1

$$f\left(\frac{\pi}{4}\right)=\frac{2(1)}{1+(1)^2}=1$$

Sol 2: $f(x)=\dfrac{|x|}{x}$

If $x>0$ $f(x)=1$

If x < 0; $f(x)=\dfrac{-x}{x}=-1$

Case I $\alpha>0$

$f(\alpha)=1 \quad f(-\alpha)=-1$

$|1-(-1)|=2$

Hence proved

Sol 3: $f(x)=\log\left(\dfrac{1+x}{1-x}\right)$

$$= \log\left(\frac{1+x^2+2x}{1+x^2-2x}\right)=\log\left(\frac{1+x}{1-x}\right)^2$$

$$f\left(\frac{2x}{1+x^2}\right)=\log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)=2\log\left(\frac{1+x}{1-x}\right)=2f(x)$$

Sol 4: $f(x)=\dfrac{x^2+2x+1}{x^2+8x+12}$

$$f(x)=\frac{x^2+2x+1}{\left(x^2+8x+16\right)-16+12}=\frac{(x+1)^2}{(x+4)^2-4}$$

$(x+4)^2-4\neq 0$

$(x+4)\neq\pm 2$

$x\neq-2$ and $x\neq-6$

Domain $x\in R-\{-2,-6\}$

Sol 5: $y=\dfrac{1}{\log(1-x)}+\sqrt{x+2}$

Logarithm is not defined for $(1-x)\le 0,\ x\ge 1$

$\sqrt{x+2}$ is not defined for $x+2<0,\ x<-2$

$\log(1-x)\neq 0 \Rightarrow x\neq 0$

So domain will be $x\in(-2,0)\cup(0,1)$

Sol 6: Function $y=\dfrac{x}{1+x^2}$

$$y=\frac{1}{\left(\frac{1}{x}+x\right)}$$

$\left(x+\dfrac{1}{x}\right)$ is always greater than 2

From arithmetic mean > G.M

For $x>0$; $\dfrac{x+\frac{1}{x}}{2}>\sqrt{x\left(\frac{1}{x}\right)}$

For $x<0$; $x+\dfrac{1}{x}<-2,\ x+\dfrac{1}{x}>2$

So range will be $\dfrac{-1}{2}\le y\le\dfrac{1}{2}$

$$y\in\left[\frac{-1}{2},\frac{1}{2}\right]$$

Sol 7: $f(x)=\dfrac{1}{1-x^2}$ domain $x\in R-\{-1,1\}$

$x^2\ge 0 \Rightarrow$ So the range $y\in(-\infty,0)\cup[1,\infty)$

Sol 8: $y=\dfrac{1}{2-\sin 3x} \Rightarrow 2-\sin 3x\neq 0$

$\sin 3x\neq 2$

So the domain $x\in R$

$-1\le\sin 3x\le 1$

Range $y\in\left[\dfrac{1}{3},1\right]$

Sol 9: $f+g=x^3+1+x+1 = x^3+x+2 \quad R\to R$

$f-g=x^3+1-x-1 = x^3-x \quad R\to R$

$f.g=(x^3+1)=x^4+1+x+x^3$

$\dfrac{f}{g}=\dfrac{x^3+1}{x+1} \quad x\neq-1$

$\alpha f=ax^2+\alpha$

Sol 10: $f:R\to R \quad f(x)=x$

$g:R\to R \ g(x)=|x|$

if $x \ge 0$ $g(x) = x$

if $x \le 0$ $g(x) = -x$

$f+g = \begin{cases} 2x, & x > 0 \\ 0, & x \le 0 \end{cases}$; $f-g = \begin{cases} 0, & x \ge 0 \\ 2x, & x \le 0 \end{cases}$

$f \cdot g = \begin{cases} x^2 & x \ge 0 \\ -x^2 & x \le 0 \end{cases}$

$\frac{f}{g} = \begin{cases} 1 & x \ge 0 \\ -1 & x \le 0 \\ \text{not defined} & x \ne 0 \end{cases}$

$af = ax$

Sol 11: $f(x) = e^x$

$g(x) = \log_e x$

$(f+g)(1) = e^{(1)} + \log_e^{(1)} = e$

$(f-g)(1) = e^{(1)} - \log_e^{(1)} = e$

$f.g(1) = e^{(1)} \log_e^{(1)} = 0$

Sol 12: $\pi^2 = 9.8 \Rightarrow [\pi^2] = 9$ and $[-\pi^2] = -10$

$f(x) = \cos 9x + \cos(-10)x = \cos 9x + \cos 10x$

$f\left(\frac{\pi}{2}\right) = \cos 9\left(\frac{\pi}{2}\right) + \cos 10\frac{\pi}{2} = -1$

Sol 13: (i) $f(x) = \ell n(1-\{x\}) + \sqrt{\sin x + \frac{1}{2}} + \sqrt{4-x^2}$

$1 - \{x\} > 0$ (Always true)

AND $\sin x + \frac{1}{2} \ge 0$

$\sin x \ge -\frac{1}{2}$

$x \in \left[2n\pi - \frac{\pi}{6}, 2n\pi + \frac{7\pi}{6}\right]$

And $4 - x^2 \ge 0$

$x^2 \le 4$

$x \in [-2, 2]$

So domain is $x \in \left[\frac{-\pi}{6}, 2\right]$

(ii) Range cos(2sinx)

$-1 \le \sin \le 1$

$-2 \le 2\sin x \le 2$

$y = \cos(2\sin x) \in [\cos 2, 1]$

(iii) $f(x) = \sin\left(\frac{\pi x}{3}\right) + \{x\} + \tan^2 \pi x$

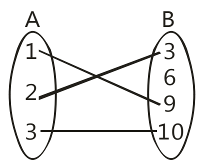
$\sin\frac{\pi x}{3}$ has time period = 6

{ x } has time period = 1

$\tan^2 \pi x$ has time period = 1

LCM (6,1,1) = 6

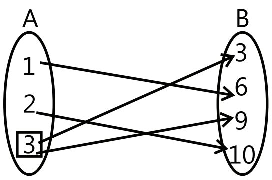
Sol 14: f



Function

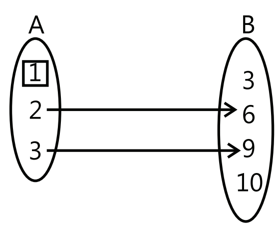
Range {3, 9, 10}

g



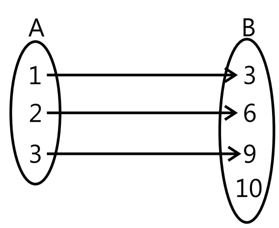
not a function as 3 has two values in Range.

h



not a function as 1 has no value

u



putting x =1 y = 3

x = 2y = 6

x = 3y = 3

So, it is a function and Range $\in \{3, 6, 9\}$

Sol 15: (i) Yes, it is because it cover all A and each corresponds to one value.

(ii) No, as it gives two value at $x = b$

(iii) No, as x=c has no image.

Sol 16: (i) f(x) = ax + b

at $x = 1,\ f(1) = 1$

$1 = a + b$ ... (i)

at $x = 2,\ f(2) = 3$

3 = 2a + b ... (ii)

Solving 1 & 2

$a = 2$

$b = -1$

(ii) Let $f(x) = a\times + b$

Some question as part (i)

So the answer will be $f(x) = 2x - 1$

Sol 17: Domain $= \{4, 9, 6, 10\}$

Range $= \{2, 1, 3\}$

Sol 18: $f(x) = x^2 - 1$

$x^2 \geq 0$

$f(0) = -1$

$f(1) = 0 = f(-1)$

$f(2) = 3 = f(-2)$

$f(3) = 8 = f(-3)$

So the range will be $\{-1, 0, 3, 8\}$

Sol 19: (i) Domain $x \in R$

Range $y \in R$

(ii) Domain $x \in R$

Range $y \in R$

(iii) Domain $x \in R$

$x^2 \geq 0$

So the range $y \in [-1, \infty)$

(iv) Domain $x \in R$

$x^2 \geq 0$

So the range $y \in [2, \infty)$

(v) Domain $x - 1 \geq 0$

$x \geq 1$

$x \in [1, \infty)$

Range $y \geq 0$ (Because value of under root is always non neg.) $y \in [0, \infty)$

Sol 20: $f(x) = \dfrac{x^2 + 3x + 5}{x^2 - 5x + 4} = \dfrac{x^2 + 3x + 5}{(x-4)(x-1)}$

$(x-4)(x-1) \neq 0$

$x \neq 4\ \ x \neq 1$

Domain $x \in R - \{1, 4\}$

Sol 21: (i) Domain =R

Range $y \in [0, \infty)$

(ii) Domain =R

Range $y \in [0, \infty)$

(iii) Domain $3 - x^2 \neq 0$

$x \neq \pm\sqrt{3}$

$x \in R - \{\sqrt{3}, -\sqrt{3}\}$

Range $x^2 \geq 0$

$3 - x^2 \leq +3$

$y \in (-\infty, 0) \cup \left[\dfrac{1}{3}, \infty\right)$

(iv)Domain $4 - x^2 \geq 0$

$4 \geq x^2$

$x^2 \leq 4$

$x \in [-2, 2]$

Range $u(2) = 0$

$u(0) = 2$

$y \in [0, 2]$

Sol 22: (i) $f : R \to R :\ f(x) = \log x$

Not a function

Because f(x) is not defined for negative values of x.

(ii) $g : R \to R :\ g(x) = \sqrt{x}$

Not a function

As f(x) is not defined for negative values.

(iii) $h : A \to R :\ h(x) \dfrac{1}{x^2 - 4}; A = R - \{-2, 2\}$

h(x) is defined for all values of x in A set.

So it is a function.

Sol 23: $f: R-\{2\} \to R$

$f(x)=\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{(x-2)}=(x+2)$ for $x \neq 2$

$g: R \to R \; g(x)=x+2$ at $x \in R$

$f \neq g$

As their domain are not same.

Sol 24: $f = a\times + b$ (Suppose)

$f(1)=1=a+b$ ...(i)

$f(2)=3=2a+b$ ...(ii)

$a=2 \quad b=-1$

$f(x)=2x-1$

$g(x)=\times$

$f+g=3x-1$

at $x=1 \; (f+g)(1)=2$

$(f+g)(2)=5$

$(f+g)(0)=-1$

$(f+g)(-1)=4$

So $f+g=[(1,2)(2,5)(0,-1)(-1,-4)\}$

Sol 25: (i) $f+g=\frac{1}{x+4}+(x+4)^3 \quad x \neq -4$

$f-g=\frac{1}{x+4}-(x+4)^3 \quad x \neq -4$

$f.g=(x+4)^2; \; x \neq -4$

$\frac{f}{g}=\frac{1}{(x+4)^4}; \; x \neq -4$

(ii) $f(x)=\cos x \;\; g(x)=e^x$

$f+g=\cos x+e^x$

$f-g=\cos x-e^x$

$f.g=e^x(\cos x)$

$f/g=\frac{\cos x}{e^x}$

$\alpha f=\alpha \cos x$

Sol 26: $f(x)=x \;\; g(x)=|x|$

If $x \geq 0 \;\; g(x)=x$

If $x<0 \;\; g(x)=-x$

$(f+g)(-2)=x-x=0$

$(f-g)(2)=2\,(2)=4$

$f.g\,(2)=(2)^2=4$

$\frac{f}{g}(-2)=-1$

$5f(2)=5(2)=10$

Sol 27:

x	–4	–3	–2	–1	0	1	2	3	4
$f(x) = x^2$	16	9	4	1	0	1	4	9	16

Sol 28:

x	–2	–15	–1	–0.5	0	0.5	1	1.5	2
f(x) = 1/x	–1/2	–2/3	–1	–2	X	2	1	2/3	1/2

Domain R–{0}

Range $(-\infty,0)\cup(0,\infty)$

Sol 29: $f(x+3)=x^2-1=(x+3)^2-9-6x=$

$(x+3)^2-6(x+3)+8-10$

Exercise 2

Sets and Relations

Single Correct Choice Type

Sol 1: (C) A = {1, 2, 3, 4}

R = {(2, 2), (3, 3), (4, 4), (1, 2)}

→ A = {1, 2, 3, 4}

and (1, 1) ∉ R so R is not reflexive

→ (1, 2) ∈ R but (2, 1) ∉ R

So R is not symmetric

→ (1,2) ∈ R, (2, 2) ∈ R

(1, 2) ∈ R

So R is transitive.

Sol 2: (B) Void relation: A → A

It is also called empty relation

A relation R is void relation, if no element of set A is related to any element of A.

(x, y) ⇒ x is not related to y

∴ (y, x) ⇒ y is not related to x

∴ Symmetric

(x, x) ⇒ x is any how will be related to itself

∴ So it can't be reflexive

(x, y)(y, z)

⇒ x is not related to y

y is not related to z

∴ This is transitive.

Sol 3: (B) x R y ↔ x − y is an irrational number if 0 = x – x is an irrational no. (say z)

so R is not reflexive

→ if x – y is a irrational number

than y – x is also an irrational number R is symmetric

→ (x, y) ∈ R and (y, z) ∈ R

say(z_1) and say(z_2)

$z_1 + z_2 = x - y + y - z = x - z$

Its not necessary that $z_1 + z_2$ will be an irrational no. R is not transitive.

Sol 4: (A) R = {(1 + x, 1 + x^2): x ≤ 5, x ∈ N}

(A)R = {(2, 2), (3, 5), (4, 10), (5, 17), (6, 25)}

$$(2, 2)\rightarrow \left.\begin{matrix} 1+x=2 \\ 1+x^2=2 \end{matrix}\right\} x = 1\ x \le 5$$

$$(3, 5)\rightarrow \left.\begin{matrix} 1+x=3 \\ 1+x^2=5 \end{matrix}\right\} x = 2,\ x \le 5$$

$$(4, 10)\rightarrow \left.\begin{matrix} 1+x=4 \\ 1+x^2=10 \end{matrix}\right\} x = 3,\ x \le 5$$

$$(5, 17)\rightarrow \left.\begin{matrix} 1+x=5 \\ 1+x^2=17 \end{matrix}\right\} x = 4,\ x \le 5$$

$$(6, 25)\rightarrow \left.\begin{matrix} 1+x=6 \\ 1+x^2=25 \end{matrix}\right\} \text{no solution}$$

Option A is false.

Sol 5: (A) R = {(1, 1), (2, 2), (3, 3),(1, 2), (2, 3), (1, 3)}

on the set A = {1, 2, 3}

→ (1, 1), (2, 2), (3, 3) ∈ R

R is reflexive

→ (1, 2) ∈ R but (2, 1) ∉ R

R is not symmetric

→ (1, 2) ∈ R and (2, 3) and also (1, 3) ∈ R

So R is transitive.

Sol 6: (B) R = {(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)}

in set A = {2, 3, 4, 5}

→ for set A = {2, 3, 4, 5}

(2, 2), (3, 3), (4, 4), (5, 5) ∈ R

R is reflexive

→ (2, 3) (3, 2) ∈ R

and (3, 5) (5, 3) ∈ R

(x, x) ∈ R

So R is symmetric

→ (3, 5) ∈ R, (5, 3) ∈ R and also (3, 3) ∈ R

[(a, b) ∈ R , (b, c) ∈ R for transitive (a, c) ∈ R]

so R is not transitive as (2, 3), (3, 5) ∈ R

but (2, 5) ∉ R

Sol 7: (A) A = {2, 3}B = {1, 2}

then A × B = {(2, 1) (2, 2), (3, 1)(3, 2)}

Sol 8: (C) A = {1, 2, 3}, B = {1, 4, 6, 9}

(x, y) ∈ R ∴ x > y

R: A → B

⇒ 2 > 1, 3 > 1 ⇒ R = {(2, 1), (3, 1)}

So range = {1}

Sol 9: (B) R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)}

on the set A = {1, 2, 3, 4}

→ for set A

(1, 1), (2, 2), (3, 3), (4, 4) ∉ R

R is not reflexive

→ (2, 3) ∈ R but (3, 2) ∉ R

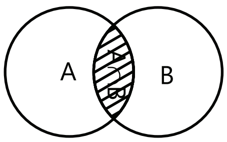
R is not symmetric

→ (1, 3) ∈ R, (3,1) ∈ R but

(1, 1) ∉ R so

R is not transitive.

Sol 10: (C)



Given $A \cap B = A \cap C$ ... (i)

and $A \cup B = A \cup C$ ... (ii)

we know that $A \cup B = A + B - A \cap B$

from (i) and (ii)

$A \cup C = A + B - A \cap C$ ... (iii)

but $A \cup C = A + C - A \cap C$ ... (iv)

From (iii) – (iv)

$O = B - C$

$B = C$

Sol 11: (A) R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}

Domain → {1, 2, 3}

→ there are pairs such that

(a, a) → (1, 1),(2, 2), (3,3)

So R is reflexive → $R^{-1} \neq R$

So R is not symmetric

→ transitive if (x, y) ∈ R, (y, z) ∈ R

⇒ (x, z) ∈ R here (1, 2) ∈ R, (2, 3) ∈ R and (1, 3) is also solution of R.

so R is transitive.

Sol 12: (D) A → set of all children in the world x R y if x and y have same sex.

So → x and x have same sex → Reflexive if x and y have same sex. So y and x also have same sex → symmetric

x and y have same sex, y and z have same sex so it is clear that x, y, z have same sex transitive

So R is an equivalence relation.

Sol 13: (B) A set A = {2, 3, 4, 5}

Given relation R={(2, 2), (3, 3), (4, 4), (5, 5), (2, 3),(3, 2), (3, 5) (5, 3)}

⇒ if (x, y) ∈ R and x = y

(x, y) ∈ R

Here x ∈ A

x = 2, 3, 4, 5 and (2, 2), (3, 3), (4, 4), (5, 5) are in solution of R

R is reflexive ⇒ (x, y) ∈ R

and (y, x) ∈ R

⇒ (2, 3) ∈ R ⇒ (3, 2) ∈ R

(5, 3) ∈ R ⇒ (3, 5) ∈ R

or say $R^{-1} = R$

so R is symmetric.

Sol 14: (D) L = get of all times in x-y plane

R = {l_1, l_2 = l_1 is parallel to l_2}

⇒ if (l_1, l_2) ∈ R

so (l_1, l_2) ∈ R (l_2 is also parallel to l_1)

symmetric

⇒ (l_1, l_2) ∈ R because a line is parallel to itself reflexive

⇒ (l_1, l_2) ∈ R, (l_2, l_3) ∈ R ⇒ (l_1, l_3) ∈ R

l_1, l_2 are parallel and (l_2 and l_3) are parallel, so l_1 and l_3 also are parallel R transitive

∴ R is equivalent

Sol 15: (B) Relation R = {(2, 3), (3, 4)}

on the set assume A = {2, 3, 4}

⇒ For reflexive → (x, x) ∈ R, x ∈ A

$\underbrace{(2, 2), (3, 3), (4, 4)}_{\text{3pairs}}$ should be pair of R

⇒ For symmetric → given (2, 3), (3, 4) ∈ R to be R → symmetric (3, 2) and (4, 3) should be pair of R

Total added pair is 5{(2, 2),(3, 3),(4, 4),(3, 2),(4,3)}

Sol 16: (B) R = {(1, 2), (2, 3)} on the set A (assume)

So A = {1, 2, 3}

→ Reflexive (x, x) ∈ R, x ∈ A

$\underbrace{(1, 1), (2, 2), (3, 3)}$ must be added

→ symmetric (x, y) ∈ R for (y, x) ∈ R

$\underbrace{(2, 1), (3, 2)}$ must be added

$\rightarrow$ Transitive (1, 2) $\in$ R and (2, 3) $\in$ R

so $\underbrace{(1, 3)}$ must be added for transitive relation

Now $\rightarrow$ R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3),(2, 1), (3, 2)}

Now for symmetric pair (1, 3), pair (3, 1) must be added

Total added pairs $\rightarrow$

7 {(1, 1)(2, 2)(3, 3)(2, 1)(3, 2)(1, 3) (3, 1)}

Functions

Single Correct Choice Type

Sol 1: (A) $f(x+ay, x-ay) = a\times y$

Put $x+ay=b$

$x-ay=c$

$2x=b+c$

$2ay=b-c$

$y=\dfrac{b-c}{2a}$

$f(b,c)=a\left(\dfrac{b+c}{2}\right)\left(\dfrac{b-c}{2a}\right)$

$f(b,c)=\dfrac{(b+c)(b-c)}{4}=\dfrac{b^2-c^2}{4}$

$f(x,y)\ =\dfrac{x^2-y^2}{4}$

Sol 2: (C) $f(x)=ax+\cos x$

It is bijective so it must be strictly increasing or decreasing

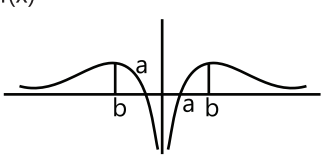
$f'(x)=a-\sin x$

$a\le -1$ or $a\ge 1$

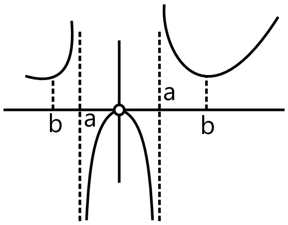
$x\in R-(-1\ ,1)$

Sol 3: (C) f(x)

f(x)



$\dfrac{1}{(f\,|\,x\,|)}$



Sol 4: (C) $f(x)=\dfrac{1}{2}\left(\dfrac{|\sin x|}{\cos x}+\dfrac{\sin x}{|\cos x|}\right)$

If $\sin x>0, \cos>0$

F(x)=tanx

If sinx>0, cosx<0

F(x) =0

If sinx<0, cosx<0

F(x) = –tanx

If sinx<0, cosx>0

F(x) = 0

} Time period = 2π

Sol 5: (D) $d(x)=\dfrac{2x^2-x+5}{7x^2+2x+10}$

Not onto as range is not R. the function

Does not goes to infinite at any x.

$$f(x)=\frac{\frac{2}{7}\left(7x^2-\frac{7}{2}x+\frac{25}{2}\right)}{7x^2+2x+10}$$

$$=\frac{\frac{2}{7}\left(7x^2+2x+10-\frac{11}{2}x+\frac{15}{2}\right)}{7x^2+2x+10}$$

$$=\frac{2}{7}-\frac{11x-15}{49x^2+14x+70}$$

Quadratic cannot be one-one function

So f(x) is not one-one function

Sol 6: (C) $f(x)=\cos\left[\dfrac{1}{2}\pi^2\right]x+\sin\left[\dfrac{1}{2}\pi^2\right]x$

$\left[\dfrac{\pi^2}{2}\right]=4$

$f(x)=\cos 4x+\sin 4x$

f(0)=1

$f\left(\dfrac{\pi}{3}\right)=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\ne 4$

$f\left(\dfrac{\pi}{2}\right)=1$

$f(\pi)=1$

Sol 7: (A) $f(x)=\ell nx$ & $g(x)=\dfrac{x^4-x^3+3x^2-2x+2}{2x^2-2x+1}$

$$fog(x)=\ell n\left(\frac{x^4-x^3+3x^2-2x+2}{2x^2-2x+1}\right)$$

$$=\ell n\left(\frac{x^4-x^3+3x^2-2x+2}{2x^2-2x+1}\right)$$

$2x^2-2x+1\neq 0$

$\Rightarrow 2\left(x-\frac{1}{2}\right)^2+\frac{1}{2}>0$

Let say $h(x)=x^4-x^3+3x^2-2x+2$

By hit and trial method, h(x) has no root. It is always ≥ 0 so domain $x\in(-\infty,\infty)$

Previous Years' Questions

Sol 1: (D) Given, $f(x) = |x-1|$

$\therefore$ $f(x^2) = |x^2-1|$ and $\{f(x)\}^2 = (x-1)^2$

$\Rightarrow f(x^2)\neq\{f(x)\}^2$, hence (a) is false

Also, $f(x+y) = |x+y-1|$ and $f(x) = |x-1|$, $f(y) = |y-1|$ Also,

$\Rightarrow f(x+y)\neq f(x) = f(y)$, hence (b) is false.

$f(|x|) = ||x|-1| = |f(x)| = ||x-1|| = |x-1|$

$\therefore$ $f(|x|)\neq|f(x)|$, hence c is false.

Sol 2: (D) Given, $f(x) = \cos(\log x)$

$\therefore$ $f(x)\,.\,f(y)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$

$=\cos(\log x).\cos(\log y)-\frac{1}{2}[\cos(\log x-\log y)+\cos(\log x+\log y)]$

$=\cos(\log x)\,.\,\cos(\log y)-\frac{1}{2}[(2\cos(\log x)\,.\,\cos(\log y)]$

$=\cos(\log x)\,.\,\cos(\log y)-\cos(\log x)\,.\,\cos(\log y)=0$

Sol 3: (C) For domain of y,

$1-x>0$, $1-x\neq 1$ and $x+2>0$

$\Rightarrow x<1$, $x\neq 0$ and $x>-2$

$\Rightarrow -2<x<1$ excluding $0\Rightarrow x\in(-2,1)-\{0\}$

Sol 4: (A) Clearly, $f(x) = x-[x] = \{x\}$ which has period 1 and $\sin\frac{1}{x}$, $x\cos x$ are non–periodic function.

Sol 5: (D) Let $y=\dfrac{x^2-(a+b)x+ab}{x-c}$

$\Rightarrow yx-cy = x^2-(a+b)x+ab$

$\Rightarrow x^2-(a+b+y)x+(ab+cy) = 0$

for real roots, $D\geq 0$

$\Rightarrow (a+b+y)^2-4(ab+cy)\geq 0$

$\Rightarrow (a+b)^2+y^2+2(a+b)y-4ab-4cy\geq 0$

$\Rightarrow y^2+2(a+b-2c)y+(a-b)^2\geq 0$

which is true for all real value of y.

$\Rightarrow D\leq 0$

$\Rightarrow 4(a+b-2c)^2-4(a-b)^2\leq 0$

$\Rightarrow (a+b-2c+a-b)(a+b-2c-a+b)\leq 0$

$\Rightarrow (2a-2c)(2b-2c)\leq 0$

$\Rightarrow (a-c)(b-c)\leq 0$

$\Rightarrow (c-a)(c-b)\leq 0$

$\Rightarrow$ c must lie between a and b
i.e. $a\leq c\leq b$ or $b\leq c\leq a$

Sol 6: (A) Let $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Now, $fog(x) = f[g(x)] = f(\sqrt{x}) = \sin^2\sqrt{x}$

and $gof(x) = g[f(x)] = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$

Again let $f(x) = \sin x$, $g(x) = |x|$

$fog(x) = f[g(x)] = f(|x|) = \sin|x|\neq(\sin\sqrt{x})^2$

When $f(x) = x^2$, $g(x) = \sin\sqrt{x}$

$fog(x) = f[g(x)] = f(\sin^{\sqrt{x-1}\sqrt{x+1}}) = (\sin\sqrt{x})^2$

and $(gof)(x) = g[f(x)] = g(x^2) = \sin\sqrt{x^2}$

$=\sin|x|\neq|\sin x|$

Therefore, (a) is the answer.

Sol 7: (B) Given, $f(x) = 3x-5$(given)

Let $y = f(x) = 3x-5\Rightarrow y+5 = 3x$

$\Rightarrow x=\dfrac{y+5}{3}$

$f^{-1}(y)=\dfrac{y+5}{3}\Rightarrow f^{-1}(x)=\dfrac{x+5}{3}$

Sol 8: (B) Let y $=2^{x(x-1)}$, where $y \ge 1$ as $x \ge 1$

Taking $\log_2$ on both sides, we get

$\log_2 y = \log_2 2^{x(x-1)}$

$\Rightarrow \log_2 y = x(x-1)$

$\Rightarrow x^2 - x - \log_2 y = 0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{1+4\log_2 y}}{2}$

For $y \ge 1$, $\log_2 y \ge 0 \Rightarrow 4\log_2 y \ge 0 \Rightarrow 1 + 4\log_2 y \ge 1$

$\Rightarrow \sqrt{1+4\log_2 y} \ge 1$

$\Rightarrow -\sqrt{1+4\log_2 y} \le -1$

$\Rightarrow 1 - \sqrt{1+4\log_2 y} \le 0$

But $x \ge 1$

So, $x = 1 - \sqrt{1+4\log_2 y}$ is not possible.

Therefore, we take $x = \dfrac{1}{2}(1+\sqrt{1+4\log_2 y})$

$\Rightarrow f^{-1}(y) = \dfrac{1}{2}(1+\sqrt{1+4\log_2 y})$

$\Rightarrow f^{-1}(x) = \dfrac{1}{2}(1+\sqrt{1+4\log_2 x})$

Sol 9: (C) It is given,

$f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$

$= (\sin\theta + 3\sin\theta - 4\sin^3\theta)\sin q$

$= (4\sin\theta - 4\sin^3\theta)\sin q$

$= \sin^2\theta(4 - 4\sin^2\theta)$

$= 4\sin^2\theta\cos^2\theta = (2\sin\theta\cos\theta)^2$

$= (\sin 2\theta)^2 \ge 0$

Which is true for all θ.

Sol 10: (D) Given that $2^x + 2^y = 2\ \forall x, y \in R$

But 2^x, $2y > 0$, $\forall\ x, y \in R$

Therefore, $2^x = 2 - 2^y < 2$

$\Rightarrow 0 < 2x < 2$

Taking log on both sides with base 2, we get

$\log_2 0 < \log_2 2^x < \log_2 2$

$\Rightarrow -\infty < x < 1$

Sol 11: (D) Function is increasing $x = \dfrac{y-3}{4} = g(y)$

Sol 12: (C) Given $f(x) = x^3 + 5x + 1$

Now $f'(x) = 3x^2 + > 0, \forall x \in R$

$\therefore f(x)$ is strictly increasing function

$\therefore$ It is one-one

Clearly, f(x) is a continuous function and also increasing on R,

$\underset{x\to-\infty}{\text{Lt}} f(x) = -\infty$ and $\underset{x\to\infty}{\text{Lt}} f(x) = \infty$

$\therefore f(x)$ takes every value between $-\infty$ and ∞.

Thus, f(x) is onto function.

Sol 13: (C) There is no information about co-domain therefore f(x) is not necessarily onto.

Sol 14: (C) $f(x) = x|x|$ and $g(x) = \sin x$

$$gof(x) = \sin(x|x|) = \begin{cases} -\sin x^2 & ,x<0 \\ \sin x^2 & ,x\ge 0 \end{cases}$$

$$\therefore (gof)'(x) = \begin{cases} -2x\cos x^2 & ,x<0 \\ 2x\cos x^2 & ,x\ge 0 \end{cases}$$

Clearly, $L(gof)'(0) = 0 = R(gof)'(0)$

$\therefore$ g of is differentiable at x = 0 and also its derivative is continuous at x = 0

$$\text{Now, } (gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2\sin x^2 & , x<0 \\ 2\cos x^2 - 4x^2\sin x^2 & , x \ge 0 \end{cases}$$

$\therefore L(gof)''(0) = -2$ and $R(gof)''(0) = 2$

$\therefore L(gof)''(0) \ne R(gof)''(0)$

$\therefore gof(x)$ is not twice differentiable at x = 0.

Sol 15: (B)

xRy need not implies yRx

$S : \dfrac{m}{s} S \dfrac{p}{q} \Leftrightarrow qm = pn$

$\dfrac{m}{n} S \dfrac{m}{n}$ reflexive

$\dfrac{m}{n} S \dfrac{p}{q} \Rightarrow \dfrac{p}{q} S \dfrac{m}{n}$ symmetric

$\frac{m}{n} S \frac{p}{q}, \frac{p}{q} S \frac{r}{s}$

$\Rightarrow qm = pn,\ ps = rq$

$\Rightarrow ms = rn$ transitive.

S is an equivalence relation.

Sol 16: (B)

$\frac{1}{\sqrt{|x| - x}} \Rightarrow |x| - x > 0 \Rightarrow |x| > x \Rightarrow x$ is negative

$x \in (-\infty, 0)$

Sol 17: (A)

$f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi = [x]\cos\left(x - \frac{1}{2}\right)\pi$

$= [x] \sin \pi x$ is continuous for every real x.

Sol 18: (B) $f(x) = 7 - 2x;\ x < 2$

$= 3;\quad 2 \le x \le 5$

$= 2x - 7;\ x > 5$

f(x) is constant function in [2, 5]

f is continuous in [2, 5] and differentiable in

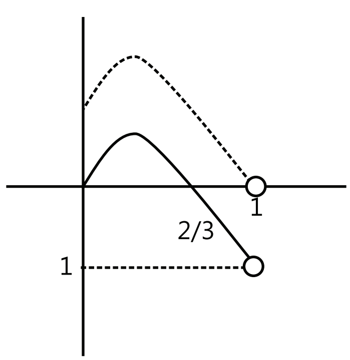
(2, 5) and f(2) = f(5)

by Rolle's theorem f' (4) = 0

$\therefore$ Statement-II and statement-I both are true and statement-II is correct explanation for statement-I.

Sol 19: (C) $-\{x\}^2 + 2\{x\} + a^2 = 0$

Now, $-3\{x\}^2 + 2\{x\}$



to have no integral roots $0 < a^2 < 1$

$\therefore a \in (-1, 0) \cup (0, 1)$

Sol 20: (B) $f_k(x) = \frac{1}{k}\left(\sin^k x + \cos^k x\right)$

$f_4(x) - f_6(x) = \frac{1}{4}\left(\sin^4 x + \cos^4 x\right) - \frac{1}{6}\left(\sin^6 x + \cos^6 x\right)$

$= \frac{1}{4}\left[1 - 2\sin^2 x \cos^2 x\right] - \frac{1}{6}\left[1 - 3\sin^2 x \cos^2 x\right]$

$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

Sol 21: (B) $f(x) + 2f\left(\frac{1}{x}\right) = 3x$

S : f(x) = f(–x)

$\because f(x) + 2f\left(\frac{1}{x}\right) = 3x$ ... (i)

$x \to \frac{1}{x} \quad f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$ ... (ii)

(1) – 2 × (2) $\quad -3f(x) = 3x - \frac{3}{x} \qquad f(x) = \frac{2}{x} - x$

Now, f (x) = f (-x)

$\therefore \frac{2}{x} - x = -\frac{2}{x} + x \qquad \frac{4}{x} = 2x$

$\frac{2}{x} = x \ \Rightarrow\ x = \pm\sqrt{2}$

Exactly two elements.

JEE Advanced/Boards

Exercise 1

Sets and Relations

Sol 1: Sets $\rightarrow$ a collection of well-defined objects which are distinct and distinguishable

Sol 2: Set {x: x $\in$ N, x is prime and 3 < x < 5} there is only one natural number 4 which follow 3 < x < 5, but 4 is not a prime no. therefore the set is void.

Sol 3: A = {a, e, i, Q, 4}, B {i,Q}

So B is subset of A

Sol 4: A = {x: x is irrational and 0. 1 < x < 0. 101 between 0.1 and 0. 101 there is infinite number which are irrational so A is infinite set}

Sol 5: Singleton set → also known as a unit set which is with exactly on element (i.e., {0}) {π} is singleton set

Sol 6: In a plane there are two points A and B such that OA = OB, O is a fixed point

→ if OA = OB (assume it is a relation R = {(A, B): OA = OB})

(B, A) ∈ R R is symmetric

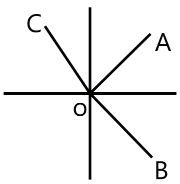
→ OA = OA (always true)

so (A, A) ∈ R R is reflexive

→ if (A, B) ∈ R⇒ OA = OB ...... (i)

and (B, C) ∈ R⇒ OB = OC ...... (ii)

from equation (i) and (ii)



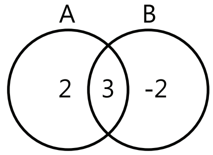
OA = OB = OC

⇒ OA = OC

(A, C) ∈ R ⇒ R is transitive

⇒ R is an equivalence relation

Sol 7:



A = {2, 3}; B = {– 2, 3}

So A ∪ B = {–2, 2, 3}

Sol 8: A = {1, 2, 3}, B = {3, 4}

C = {4, 3, 6}

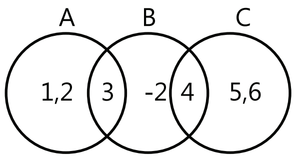
{4 is common is both set B and C}

B ∩ C = {4}

and A ∪ (B ∩ C)

= {1, 2, 3} ∪ ({4})

= {1, 2, 3, 4}



Sol 9: Na = {an: n ∈ N}

$N_6 \cap N_8$ → those element which are common in both sets N_6 and N_8

It means elements which are divisible by 6 and 8 both.

⇒ LCM (6, 8) = 24

so $N_{24} = N_6 \cap N_8$

which are divisible by 24 (6 and 8 both)

Sol 10: (A){x: x ∈ R and $x^2 + x + 1 = 0$}

$x^2 + x + 1 = 0 \Rightarrow (x^2 + 2x + 1) - x = 0$

$(x + 1)^2 - x = 0$

$(x + 1)^2 = x$

no solution

(A) is empty set

(B){x: x ∈ R and $x^2 - x + 1 = 0$}

$x^2 - x + 1 = 0$

$x = \frac{+1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$ $ax^2 + bx + c = 0$

$x = \frac{1 \pm \sqrt{-3}}{2}$ $x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

no solution

(B) is empty set

(C) {x: x ∈ R and $x^2 + 2x + 1 \le 0$}

$x^2 + 2x + 1$

$(x + 1)^2$

Which is zero at (x + 1) = 0

⇒ x = – 1

So (C) is not empty set

(D) {x: x ∈ R and $x^2 - 2x + 1 \ge 0$

$x^2 - 2x + 1 \ge 0$

$(x - 2)^2 \ge 0$

$(x - 2)^2$ is always greater than or equal to '0'

$(x - 2)^2 \ge 0$ always true for any x ∈ R

Sol 11: Total no. of subsets of M is 56 more than the total no. of subsets of N so we know no. of total subsets of any set A which have n element = 2^n

so $2^m - 2^n = 56$

m and n are integer

64 is the just bigger no. in 2^n terms after 56

so assume

$2^m = 64 = 2^6$

so $64 - 2^n = 56$

$2^n = 64 - 56 = 8$

$8 = 2^3$

n = 3 which is integer

(m, n) = (6, 3)

Sol 12: Given $A = \{x \mid x/2 \in z, 0 \le x \le 10\}$

$B = \{x \mid x \text{ is one digit prime}\}$

$C = \{x|x/3 \in N, x \le 12\}$

$(A) \Rightarrow \frac{x}{2} \in z, 0 \le x \le 10$

$x = 2z$, z is integer, $0 \le x \le 10$

x = 0, 2, 4, 6, 8, 10

(B) x is one digit prime no.

X = 2, 3, 5, 7

$(C) \frac{x}{3} \in N, x \le 12$

$x = 3N, x \le 12$

x = 3, 6, 9, 12

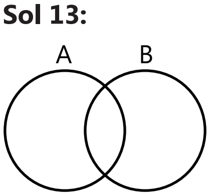
$B \cup C = \{3, 6, 9, 12\} \cup \{2, 3, 5, 7\}$

$= \{2, 3, 5, 6, 7, 9, 12\}$

$A \cap (B \cap C) = \{0, 2, 4, 6, 8, 10\}$

$\cap \{2, 3, 5, 6, 7, 9, 12\} = \{2, 6\}$

Sol 13:



n(A) = 10

n(B) = 15

Then $A \cup B$ will have

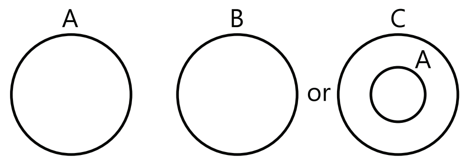
$\rightarrow A \cup B = A + B - A \cap B$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ ʻ

$= 10 + 15 - n(A \cap B)$

$= 25 - n(A \cap B)$

$n(A \cap B)$ can be zero to n(A)



$n(A \cap B) = 0$ $\qquad$ $n(A \cap B) = n(A)$

so $15 \le n(A \cup B) \le 25$

Sol 14: Total set = 1000 families of a city

40% read newspaper A

20% read newspaper B

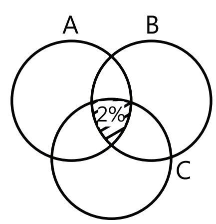
5% read newspaper both A and B$(A \cap B)$

10% read newspaper C

3% read newspaper both B and C$(B \cap C)$

4% read newspaper both A and C$(A \cap C)$

2% read newspaper both all $(A \cap B \cap C)$



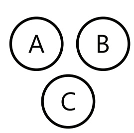
Families which read only news paper A

$= A - A \cap B - A \cap C + A \cap B \cap C$

40 – 5 – 4 + 2 = 33 %

33% of 1000 = 330 families

Sol 15:



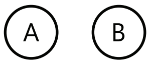
$\left.\begin{array}{l} n(A) = 10 \\ n(B) = 6 \\ n(C) = 5 \end{array}\right\}$ all are disjoint

So $n(A \cup B \cup C) = n(A) + n(B) + n(C)$

$= 10 + 6 + 5 = 21$

Other terms terns zero because there is no joints between any two sets

Sol 16: A and B are disjoint

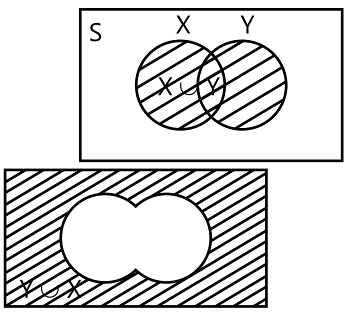


so $A \cap B = f$

$n(A \cap B) = 0$

$\therefore\ n(A \cup B) = n(A) + n(B)$

Sol 17:

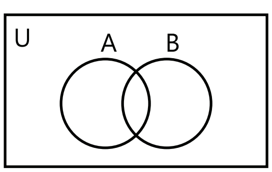


$X \cap (Y \cup X)^C$

so in $(Y \cup X)^C$, there is no elements of X

so $X \cap (Y \cup X)^C$ is π

Sol 18:



$n(U) = 700$

$n(A) = 200$

$n(B) = 300$, $n(A \cap B) = 100$

$A^c \cap B^c$

We know

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

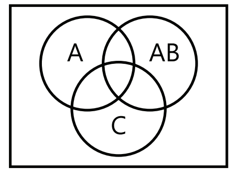
$\overline{(A \cup B)} = A^C \cap B^C$

So $n\overline{(A \cup B)} = n(A^C \cap B^C) = n(U) - n(A \cup B)$

$n(A \cup B) = 200 + 300 - 100 = 400$

$n(A^C \cap B^C) = 700 - 400 = 300$

Sol 19:



Let $(A \cap B) \cup (B \cap C)$

$(A \cap B^C)^C = (A^C \cup B)$

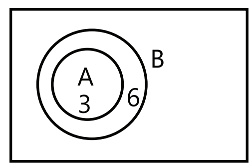
$\Rightarrow (A^C \cup B) \cup (B \cap C)$

We know that

$(B \cap C) \subset (B)$

so $(A^C \cup B) \cup (B \cap C) = A^C \cup B$

Sol 20:



Given $n(A) = 3$

$n(B) = 6$

we know

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

for minimize $n(A \cup B)$, $n(A \cap B)$ should be maximum which can be $n(A)$ (lower in both)

so $n(A \cup B) = 3 + 6 - 3 = 6$

Sol 21: Let $\rightarrow$ class of 100 students

55 students have passed in mathematics (M)

67 students have passed in physics (P)

no student fail

so $n(M \cap P) = -n(M \cup P) + n(M) + n(P)$

$= -100 + 55 + 67 = 22$

no of students which are passed in physics only

$= n(P) - n(M \cap P) = 67 - 22 = 45$

Sol 22: $X = \{1, 2, 3, 4, 5, 6\}$ universal set

$A = \{1, 2, 3\}$

$B = \{2, 4, 5\}$

$C = \{3, 4, 5, 6\}$

So $A - B = \{1, 2, 3\} - \{2, 4, 5\} = \{1, 3\}$

$B - A = \{2, 4, 5\} - \{1, 2, 3\} = \{4, 5\}$

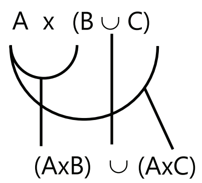
$(A - B) \cup (B - A) = \{1, 3) \cup \{4, 5\} = \{1, 3, 4, 5\}$

$(A - B) - C = \{1, 3\} - \{3, 4, 5, 6\} = \{1\}$

$C' = X - C = \{1, 2\}$

$A \cap C' = \{1, 2, 3\} \cap \{1, 2\} = \{1, 2\}$

Sol 23:



$A \times (B \cup C) \quad \Rightarrow (A \times B) \cup (A \times C)$

Sol 24: $A \times (A \cap C)$

$(A \times B) \cap (A \times C)$

Sol 25: $A = \{a, b, c, d\}$

$B = \{b, c, d, e\}$

for $\rightarrow (A \times B) \cap (B \times A)$

$A \times B = \{(a_i, b_i)\}$

= {(a, b)(a, c)(a, d)(a, e), (b, b)(b, c)(b, d) (b, e)(c, b)(c, c) (c, d)(c, e) (d, b)(d, c)(d, d) (d, e)}

B × A = {(b,a) (b,b) (b,c) (b,d) (c,a) (c,b) (c,c) (c,d) (d,a) (d,b) (d,c) (d,d) (e,a) (e,b) (e,c) (e,d)}

(A × B) ∩ (B × A) = {(b,b) (b,c) (b,d) (c,b) (c,c) (c,d) (d,b) (d,c) (d,d)}

total elements = 9

Sol 26: Given $A = \{1, 2, 3, 4, 5\}$

$R = \{x, y\}|x, y \in A$ and $x < y|$

$X < x$ always false

so $(x, x) \notin R$

R is not reflexive $\rightarrow$ if $(x, y) \in R \Rightarrow x < y$

so $y < x$ always false

$(y, x) \notin R$

R is not symmetric

$\Rightarrow$ if $(x, y) \in R \Rightarrow x < y$ .... (i)

and $(y, z) \in R \Rightarrow y < z$ .... (ii)

so from equation (i) and (ii)

$x < y < z$

so $x < z$

$\therefore (x, z) \in R$ R is transitive

Sol 27: $n R m \leftrightarrow n$ is a factor or of m (i. e. n/m)

Same as exercise –III question III

Sol 28: $n(A) = m$

$n(B) = n$ and $R: A \rightarrow B$

then total no. of relations form A to B is

$(2^m)^n = 2^{mn}$

Sol 29: $L \Rightarrow$ set of all straight lines in a plane

Relation $R \rightarrow \alpha R \beta \leftrightarrow \alpha\ \hat{}\beta, \alpha, \beta \in L$ any line never perpendicular to itself

so $(\alpha, \alpha) \notin R\ \alpha \perp \alpha$ (false)

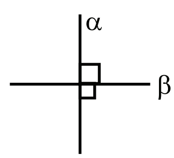
R is not reflexive

$\rightarrow$ if $(\alpha, \beta) \in R \Rightarrow \alpha \perp b$

so $\beta \perp a$

$\rightarrow (\beta, \alpha) \in R$

R is symmetric

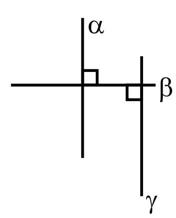


$\rightarrow$ if $(\alpha, \beta) \in R \Rightarrow \alpha \perp b$

and $(\beta, \gamma) \in R \Rightarrow \beta \perp g$

so $a || \gamma \Rightarrow (\alpha \perp \gamma) \rightarrow$ false

$\Rightarrow (\alpha, \gamma) \notin R$ R is not transitive.



Functions

Sol 1: (i) $y=\sqrt{\frac{x-2}{x+2}}+\sqrt{\frac{1-x}{1+x}}$

$x+2\neq 0\;\; x\neq -2$

$x+1\neq 0\;\; x\neq -1$

$\frac{x-2}{x+2}\geq 0$

Case I $x-2\geq 0$ & $x+2>0$

$x\geq 2$ & $x>-2\;\; x\in[2,\infty]$

Or Case II $x-2\leq 0$ & $X+2<0$

$x\leq 2$ & $x<-2$

$x\in(-\infty,-2)$

And $\frac{1-x}{1+x}\geq 0$

Case II $1-x\geq 0$ & $1+x>0$

$x\leq 1$ & $x>-1\Rightarrow x\in(-1,1]$

Or Case II $(1-x)\leq 0$ & $1+x<0$

$x\geq 1$ & $x>-1$

So the range domain will be $x\in(-1,1]\cap[2,\infty)$

$x\in\phi$

(ii) $y=\sqrt{x^2-3x+2}+\frac{1}{\sqrt{3+2x-x^2}}$

$y=\sqrt{(x-2)(x-1)}+\frac{1}{\sqrt{(3+x)(x+1)}}$

$(x-2)(x-1)\geq 0\Rightarrow x\in(-\infty,1]\cup[2,\infty)$

And $(3-x)(x+1)>0\Rightarrow x\in(-1,3)$

The domain $x\in(-1,1],\cup[2,3)$

(iii) $y=\sqrt{x}+\frac{1}{(x-2)^{1/3}}-\log_{10}(2x-3)$

$x\geq 0$

And $x-2\neq 0$

$x\neq 2$

And $2x-3>0$

$x>\frac{3}{2}$

So the domain $x\in\left(\frac{3}{2},\infty\right)-\{2\}$

(iv) $y=\sqrt{\frac{1-5^x}{7^{-x}-7}}$

$\frac{1-5^x}{7^{-x}-7}\geq 0$

5^x & 7^x are always greater than zero.

Case I $1-5^x\geq 0$ & $7^{-x}-7<0$

$5^x\leq 1$ & $7^{-x}>7$

$x\leq 0$ & $x<-1$

$x\in(-\infty,-1)$

Or Case II $1-5^x\leq 0$ & $7^{-x}-7<0$

$5^x\geq 1$ & $7^{-x}<7$

$x\geq 0$ & $x\geq -1$

$x\in[0,\infty)$

So the answer is $x\in(-\infty,-1)\cup[0,\infty)$

(v) $y=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$

$(1-x)>0$ & $1-x\neq 1$

$x<1\;\; x\neq 0$

And $x+2>0$

$x\geq -2\;\; x\in[-2,\infty)$

So the domain $x\in[-2,1)-\{0\}$

(vi) $f(x)=\log_{100x}\left(\frac{2\log_{10}x+1}{-x}\right)$

$100x\neq 0,1\;\; 100\,x>0$

$x\neq 0,\frac{1}{100}\;\; x>0$

And $\frac{2\log_{10}x+1}{-x}>0$

$\frac{2\log_{10}x+1}{x}<0$

$x>0$ So

$2\log_{10}x+1<0$

$\log_{10}x<-\frac{1}{2}\Rightarrow 0<x<\frac{1}{\sqrt{10}}$

So domain $x\in\left(0,\frac{1}{\sqrt{10}}\right)-\left(\frac{1}{10}\right)$

(vii) $f(x)=\frac{1}{\log(1-x)}+\sqrt{x+2}$

$x+2\geq 0\ \ \&x\geq -2$

And $(1-x)>0\ \ \&\ 1-x\neq 1$

$x<1\ \ \&\ \ x\neq 0$

$x\in(-\infty,1)-\{0\}$

So domain $[-2,1)-\{0\}$

(viii) $y=\sqrt{\log\left(\frac{5x-x^2}{4}\right)}$

$\log\left(\frac{5x-x^2}{4}\right)\geq 0$

$\frac{5x-x^2}{4}\geq 1$

$5x-x^2-4\geq 0$

$x^2-5x+4\leq 0$

$(x-4)(x-1)\leq 0$

$x\in[1,4]$

(ix) $f(x)=\sqrt{x^2=|x|}\ +\ \frac{1}{\sqrt{9-x^2}}$

$9-x^2>0$

$x^2<9$

$x\in(-3,3)$

And $x^2-|x|\geq 0$

$\sqrt{1+x+x^2}-\sqrt{1+x+x^2}\ \frac{1+2^{Kx}}{1-2^{Kx}}$

$x\in[1,\infty)$

$\text{if}\, x<0\ \ x(x+1)\geq 0$

$x\in(-\infty,-1]$

So the domain $x\in(-3,-1]\cup[1,3)$

(x) $\sqrt{(x^2-3x-10)\,\ell n^2(x-3)}$

$x-3>0\ \ \text{at}\, x=4$

$x>3\ \ \ f(x)=0$

And $x^2-3x-10\geq 0\ \ \ (x-5)(x+2)\geq 0$

$x\in(-\infty,-2]\cup[5,\infty)$

So domain $x\in(5,\infty)\cup\{4\}$

(xi) $f(x)=\sqrt{(\sin x)^2+\cos^2 x+2\sin x\cos x-1}$

$=\sqrt{2\sin x\cos x}=\sqrt{\sin 2x}$

$\sin 2x\geq 0$

$2x\in[2n\pi,(2n+1)\pi]$

$x\in[n\pi,\left(n+\frac{1}{2}\right)\pi]$

(xii) $f(x)=\sqrt{\frac{\cos x-1/2}{6+35x-6x^2}}$

$=\sqrt{\frac{\cos x-1/2}{(6-x)(6x+1)}}\ \ x\neq 6,\ \frac{-1}{6}$

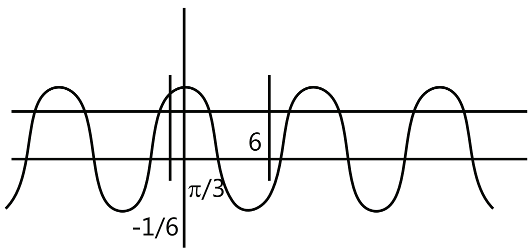
Case I: Let's say $\frac{-1}{6}<x<6$

So $\cos x-\frac{1}{2}>0$

Case II: $x<\frac{-1}{6}$

So $\cos x\frac{-1}{2}<0$

$\cos x<\frac{1}{2}$



Case III: $x>6$

$\cos x\leq\frac{1}{2}$

From group analysing solution

$\left(\frac{-1}{6},\frac{\pi}{3}\right]\cup\left[\frac{5\pi}{3},6\right)\cup$

$\left[2k\pi+\frac{\pi}{3},2k\pi+\frac{5\pi}{3}\right],k\in I-\{0\}$

(xiii) $f(x)=\sqrt{\log_{\frac{1}{3}}(\log_4[x]^2-5)}$

$\log_3(\log_4[x]^2-5)\geq 0$

$\log_4[x]^2-5)\leq 1$

$[x]^2 \le 9$

$[x] \in [-3,3]$

And $[x]^2 - 5 > 0$

$[x]^2 > 5 \Rightarrow$

And $[x]^2 - 5 \ne 1$ $[x]^2 - 5 > 1$

$[x]^2 \ne 6$ $[x]^2 > 6$ $[x] = 3, -3$

And $[x]^2 - 5 \ne 4$ $[x]^2 \ne 9$

$[x] \ne \pm 3$

So domain $x \in [-3,2) \cup [3,4)$

(xiv) $f(x) = \frac{1}{[x]} + \log_{(2\{x\}-5)}(x^2 - 3x + 10) + \frac{1}{\sqrt{1-|x|}}$

$[x] \ne 0 \quad x \in [0,1)$

$2\{x\} - 5 < 0$ So $\log_{2\{x\}-5}(x^2 - 3x + 10)$ is not defined. So $x \in \phi$

(xv) $f(x) = \log_7 \log_5 \log_3 \log_2 (2x^2 + 5x^2 - 14x)$

$\log_5 \log_3 \log_2 (2x^3 + 5x^2 - 14x) > 0$

$\log_3 \log_2 (2x^3 + 5x^2 - 14x) > 1$

$\log_2 (2x^3 + 5x^2 - 14x) > 3$

$2x^3 + 5x^2 - 14x > 8$

$2x^3 + 5x^2 - 14x - 8 > 0$

$(x-2)(2x^2 + 9x + 4) > 0$

$(x-2)(2x+4)(2x+1) > 0$

$x \in \left(-4, \frac{-1}{2}\right) \cup (2, \infty)$

(xvi) $f(x) = \sqrt{\cos 2x} + \sqrt{16 - x^2}$

$\cos 2x > 0$ and $16 - x^2 \ge 0$

$2x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$ & $x \in [-4, 4]$

$x \in \left[n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{4}\right]$ & $x \in [-4, 4]$

So domain $\in \left[-\frac{5\pi}{4}, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

(xvii) $f(x) = \ell n(\sqrt{x^2 - 5x - 24 - x - 2})$

$\sqrt{x^2 - 5x - 24} - x - 2 > 0$

And $x^2 - 5x - 24 > 0$

$x^2 - 3x - 8x - 24 > 0$

$(x+3)(x-8) > 0$

$x \in [-\infty, -3] > 0$

$x \in [-\infty, -3] \cup [8, \infty)$

Case I if $x + 2 > 0 \Rightarrow x > -2$

The $x^2 - 5x - 24 > (x+2)^2$

$-5x - 24 > 4 + 4x$

$9x + 28 < 0$

$x < \frac{-28}{9}$

So no answer

Case II if $x + 2 < 0$ then always true in interval

$\sqrt{x^2 - 5x - 24} > x + 2$

$x < -2$

So the domain $x \in (-\infty, -3]$

(xvii) $y = \log(1 - \log_{10}(x^2 - 5x + 16))$

$1 - \log_{10}(x^2 - 5x + 16) > 0$ AND $x^2 - 5x + 16 > 0$

$\log_{10}(x^2 - 5x + 16) < 1$ AND is the always positive as its minimum value of $x = \frac{-b}{2a} = \frac{5}{2}$ is positive.

$x^2 - 5x + 6 < 0$

$(x-3)(x-2) < 0$

$x \in (2,3)$

So the domain 2<x<3

(xix) $f(x) = \log_4 (2 - (x)^{1/4} - \frac{2\sqrt{x+1}}{\sqrt{x+2}}$

$2 - (x)^{1/4} - \frac{2\sqrt{x+1}}{\sqrt{x+2}} > 0$

$(x)^{1/4} - \frac{2\sqrt{x+1}}{\sqrt{x+2}} > 2$

$(x)^{3/4} + 2(x)^{1/4} + 2(x)^{2/4} + 1 < 2(x)^{2/4} + 4$

$x \ge 0$ so put $x = t^4$

$t^3 + 2t + 2t^2 + 1 < 2t^2 + 4$

$t^3 + 2t - 3 < 0$

$(t-1)(t^2+t+3)<0$

t^2+t+3

Is always greater then zero.

So $(t-1)<0$

$t<1$

$x^{1/4}<1 \Rightarrow 0\le x<1$

Sol 2: (i) $y=\log_{\sqrt{5}}\left(\sqrt{2}(\sin x-\cos x)+3\right)$

$\sqrt{2}(\sin x-\cos x)+3>0$

$\sqrt{2}\sqrt{2}(\sin x\cos 45^0-\cos x\sin 45^0)>-3$

$2\sin(x-45^0)>-3$

$\sin(x-45^0)>\frac{-3}{2}$ so the domain $x\in R$

Range $f(x)=\log_{\sqrt{5}}\left[2\sin\left(x-\frac{\pi}{4}\right)+3\right]$

$+1\le 2\sin\left(x-\frac{\pi}{4}\right)+3\le 5$

$\log_{\sqrt{5}}(1)\le\log_{\sqrt{5}}\left(2\sin\left(x-\frac{\pi}{4}\right)+3\right)\le\log_{\sqrt{5}}(5)$

$0\le f(x)\le 2$

(ii) $y=\frac{2x}{1+x^2}$

Domain $x\in R$

Range $y=\frac{2}{x+\frac{1}{x}}$

$x+\frac{1}{x}\ge 2$ for $x\ge 0$ from arithmetic mean > geometric mean

$x+\frac{1}{x}\le -2$ for $x\le 0$ from A.M >G.M

So [-1,1]

(iii) $f(x)=\frac{x^2-3x+2}{x^2+x-6}$

Domain $x^2+x-6\ne 0$

$x^2+3x-2x-6\ne 0$

$(x+3)\ (x-2)\ne 0$

$x\ne 2,-3$

Range $f(x)=\frac{(x-2)(x-1)}{(x-2)(x+3)}$

For $x\ne 2\, f(x)=1-\frac{4}{x+3}$

$x=2\, f(x)$ not defined

Range $f(x)\in R-\{\frac{1}{5}\}-\{1\}$

at $x=2\Rightarrow\left(\frac{1}{5}\right)$ at $x=\infty$ [1]

(iv) $f(x)=\frac{x}{1+|x|}$ domain $x\in R$

for $x>0$ $f(x)=\frac{x}{1+x}=1-\frac{1}{1+x}$

$0<f(x)<1$

for $x<0$ $f(x)=\frac{x}{1-x}=\frac{1}{\frac{1}{x}-1}$

$-1<f(x)<0$

So $f(x)\in(-1,1)$

(v) $y=\sqrt{2-x}+\sqrt{1+x}$

$z-x\ge 0$ and $1+x\ge 0$

$x\le 2$ and $x\ge -1$

Domain $x\in[-1,2]$

Range $\sqrt{2-x}$ decreases with increment in x $\sqrt{1+x}$ Increase with increment in x

Since both are linear function root, so we can say that it's maximum will be at middle point of boundary defined.

So $f(x)\le\sqrt{2-\left(\frac{1}{2}\right)}+\sqrt{1+\left(\frac{1}{2}\right)}$

$f(x)\le\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{2}}=\sqrt{6}$

$f(x)\ge\sqrt{3}$ (at boundary point)

$\sqrt{3}\le f(x)\le\sqrt{6}$

(vi) $f(x)=\log_{(\operatorname{cosec}x-1)}(2-[\sin x]-[\sin x]^2)$

$[\sin x]=0$ or -1

$2-[\sin x]-[\sin x]^2=2$

$\operatorname{cosec}x - 1 > 0$ & $\operatorname{cosec}x - 1 \neq 1$

$\sqrt{x-1} + \sqrt{6-x}$

$\frac{1}{\sqrt{|x|-x}}$

$x \neq 2n\pi + \frac{\pi}{6}, (2n+1) - \frac{\pi}{6}$

$x \in (2n\pi(2n+1)\pi) - \{2n\pi + \frac{\pi}{2}\}$

So domain

$\left[\frac{3}{2}, \infty\right) \to \left[\frac{7}{4}, \infty\right)$

Range $\log_a 2$;

$a \in (0,\infty) - \{1\} \Rightarrow$ Range $(-\infty,\infty) - \{0\}$

(vii) $f(x) = \frac{x+1}{x-2}$

Domain $x \in R - \{2\}$

Range $f(x) = 1 + \frac{3}{x-2}$

$f(x) \in (-\infty,\infty) - \{1\}$

Sol 3: Injective = one to one mapping

Surjective = onto function

(a) $f(x) = \frac{x^2+x+1}{x^2+2x+3} = 1 - \frac{(x+2)}{x^2+2x+3}$

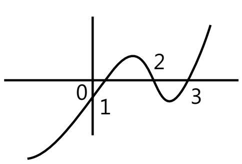
$= 1 - \frac{(x+2)}{(x+1)^2+1}$

It's range is not R. So not surjective not injective

$f(x) = x^3 - 6x^2 + 11x - 6$

Range $\in R$ onto function $\Rightarrow$ surjective

$f(x) = (x-1)(x^2-5x+6)$



$= (x-1)(x-3)(x-2)$

Not one-one as at x = 1, 2, 3 f(x) = 0

Not injective

(b) $f(x) = (x^2+5x+9)(x^2+5x+1)$

$= \left[\left(x+\frac{5}{2}\right)^2 + 9 - \frac{25}{4}\right]\left[\left(x+\frac{5}{2}\right)^2 + 1 - \frac{25}{4}\right]$

$= \left[\left(x+\frac{5}{2}\right)^2 - \frac{11}{4}\right]\left[\left(x+\frac{5}{2}\right)^2 - \frac{25}{4}\right]$

It has two roots so not injective nor Surjective (as the Ranges not R)

Sol 4: $f(x) = \frac{1}{1-x}$

$f_2(x) = f[f(x)] = f\left[\frac{1}{1-x}\right] = \frac{1}{1-\left[\frac{1}{1-x}\right]}$

$f_3(x) = f\left[f\{f(x)\}\right] = f\left[f\left\{\frac{1}{1-x}\right\}\right]$

$= \frac{1}{1-\frac{1}{1-\left\{\frac{1}{1-x}\right\}}} = \frac{1-\left\{\frac{1}{1-x}\right\}}{-\left\{\frac{1}{1-x}\right\}}$

Sol 5: $f(n) = -1$, $\quad f\left(n+\frac{1}{2}\right) = 0$

$f(n+x) = a(n+x)+b$ where $0 \le x < 1$

$x = 0$, $f(n) = -1 = ax + b$ ....(i)

$x = \frac{1}{2}$, $f\left(n+\frac{1}{2}\right) = 0 = a\left(n+\frac{1}{2}\right)+b$

$-\frac{a}{2} = ax + b$ ...(ii)

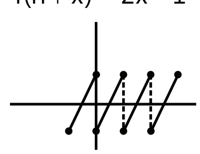
$-\frac{a}{2} = -1$

a = 2

$b = -1 - ax = -1 - 2x$

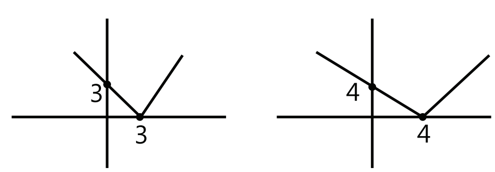
So $f(n+x) = 2(n+x) - 1 - 2x$

$f(n+x) = 2x - 1$ where $0 \le x < 1$, $n \in R^+$

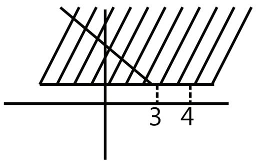


Sol 6: (a) $|(x-3)+(4-x)| < |x-3| + |4-x|$

$\Rightarrow 1 < |x+3| + |4-x|$



$|x+3|+|4-x|>1$



$x \in (-\infty, 3) \cup (4, \infty)$

(b) $|x-(x-2)| > |x|-|x-2|$

$2 > |x|-|x-2|$

$|x|-|x-2| < 2$

If $x > 2$ not true

If $0 \le x \le 2$ $2x-2 < 2$

$x \le 2$ $x \in [0,2]$

If $x < 0$ $-2 < 2$ always true

$(-\infty, 2)$

Sol 7: (a) $f(-x) = \log\left(-x+\sqrt{1+x^2}\right)$

$f(-x)+f(x) = \log\left(\sqrt{1+x^2}-x\right)\left(\sqrt{1+x^2}+x\right) = 0$

$f(x) = -f(x)$

So odd function

(b) $f(x) = \dfrac{a^x+1}{a^x-1}$

$f(-x) = \dfrac{\frac{1}{a^x}+1}{\frac{1}{a^x}-1} = \dfrac{1+a^x}{a^x-1} = -f(x)$

Odd function

(c) $f(x) = x^4-2x^2$

$f(-x) = x^4-2x^2 = f(x)$

Even

(d) $f(x) = x^2-|x|$

$f(-x) = x^2-|x|$

Even function

(e) $f(-x) = -x\sin^2 x + x^3 = -f(x)$

Odd function

(f) $f(x) = k = f(-x)$

Even

(g) $\ell n\left(\dfrac{1+x}{1-x}\right) = f(-x)$

$f(-x) = -\ell n\left(-\dfrac{1-x}{1+x}\right) = -f(x)$

Odd function

(h) $f(x) = \dfrac{(1+2^x)^2}{2^x}$

$f(-x) = \dfrac{(2^x+1)^2}{2^x} = f(x)$

Even

(i) $f(-x) = \dfrac{-x}{\frac{1}{e^x}-1} - \dfrac{x}{2} + 1 \ne f(x) \ne -f(x)$

So neither odd nor even

(j) $f(x) = [(x+1)^2]^{1/3} + [(x-1)^2]^{1/3}$

$f(-x) = \left[(1-x)^2\right]^{1/3} + \left[(-x-1)^2\right]^{1/3}$

$= \left[(x-1)^2\right]^{1/3} + \left[(1+x)^2\right]^{1/3} = f(x)$

Even function

Sol 8: (a) $f(x) = \sin^4 x + \cos^4 x$

$f(x) = (\sin^2 x)^2 + ((\cos^2)^3 x)^2 + 2\sin^2 x$

$\cos^2 x - 2\sin^2 \cos^2 x$

$f(x) = \left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x \cos^2 x$

$f(x) = 1 - \dfrac{(\sin 2x)^2}{2}$

$\sin 2x \Rightarrow$ Time period $\Rightarrow$ π

$(\sin 2x)^2 \Rightarrow$ Timeperiod $\Rightarrow \pi/2$

So f(x) has $\dfrac{\pi}{2}$ Time period

(b) $f(x) = |\sin x| + |\cos x|$

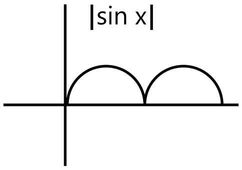
Case-I $0 < x < \dfrac{\pi}{2}$

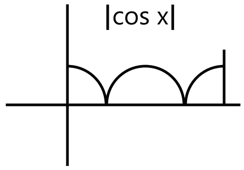
$f(x) = \sin x + \cos x$

Case-II $\dfrac{\pi}{2} < x < \pi$ $f(x) = \sin x - \cos x$

Case-III $\pi < x < \dfrac{3\pi}{2}$ $f(x) = -(\sin x + \cos x)$

Case-IV $\frac{3\pi}{2} < x < 2\pi \quad f(x) - \sin x + \cos x$





Since combining graphs, we can see $|\sin x| + |\cos x|$ has $\frac{\pi}{2}$ Time period.

(c) $f(x) = \cos\frac{3}{5}x - \sin\frac{2x}{7}$

$\cos 3x$ has time period $\frac{2\pi}{3}$

$\cos\frac{3x}{5}$ has time period $\frac{10\pi}{3}$

$g(x) = \sin x$

has time period $\frac{2\pi}{2} \times (7)$

$$\text{L.C.M.}\left(\frac{10\pi}{3}, 7\right) = \frac{\text{L.C.M.}(10\pi, 7\pi)}{\text{H.C.F.}(3,1)} = 70\pi$$

Sol 9: (a) $10^x + 10^y = 10$

$y = \log(10 - 10^x)$

Domain $x \le 1$

(b) $(-\infty, \infty) - \{0\}$

If $y > 0 \ \ y = x$

If $y < 0 \ \ y = \frac{x}{3}$

Sol 10: $f(x) = \cos nx \cdot \sin\left(\frac{5}{n}x\right)$

$$= \left[2\cos nx \cdot \sin\frac{5}{n}x\right]\frac{1}{2}$$

$$= \left[\sin\left(\frac{5}{n} - x\right)x + \sin\left(\frac{5}{x} + n\right)x\right]\frac{1}{2}$$

$$\text{L.C.M.}\left[\frac{2x}{\frac{5}{n} - n}, \frac{2\pi}{\frac{5}{n} + n}\right] = 3\pi$$

$$\text{L.C.M.}\left(\frac{2n\pi}{5 - n^2}, \frac{2n\pi}{5 + n^2}\right) = 3\pi$$

$$\frac{2n\pi}{\text{H.C.F.}(5 - n^2, 5 + n^2)} = 3\pi$$

Sol 11: $f(x) = \ell n\left(x + \sqrt{x^2 + 1}\right)$

$y = \ell n\left(x + \sqrt{x^2 + 1}\right)$

$e^y - x = \sqrt{x^2 + 1}$

$e^{2y} + x^2 - 2e^y(x) = x^2 + 1$

$$x = -\frac{(1 - e^{2y})}{2e^y} = \frac{e^{2y} - 1}{2e^y}$$

$$\frac{1}{2}\left(e^y - \frac{1}{e^y}\right)$$

$$f^{-1}(x) = \frac{1}{2}\left(e^x - \frac{1}{e^x}\right)$$

(b) $f(x) = 2^{\frac{x}{x-1}}$

$$\log_2 y = \frac{x}{x - 1}$$

$$\log_2 y = 1 + \frac{1}{x - 1}$$

$$(x - 1) = \frac{1}{\log_2 y - 1}$$

$$x = 1 + \frac{1}{\log_2 y - 1}$$

$$x = \frac{\log_2 y}{\log_2 y - 1}$$

$$f^{-1}(x) = \frac{\log_2 y}{\log_2 x - 1}$$

(c) $y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$

$$y = \frac{10^{2x} - 1}{10^{2x} + 1}$$

$$\frac{y + 1}{y - 1} = \frac{2(10^{2x})}{-2}$$

$$\frac{y + 1}{y - 1} = 10^{2x} \Rightarrow x = \frac{\log_{10} - \left(\frac{y + 1}{y - 1}\right)}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{1}{2}\log\left(\frac{x + 1}{1 - x}\right)$$

Sol 12: $f(x) = (a - x^n)^{1/n}$

$$fof(x) = \left[a - \left[(a-x)^n\right]^n\right]^{1/n}$$

$$= \left[a - \left[(a-x^n)\right]^n\right]^{1/n}$$

to $f(x) = x$

If g is inverse of f then fog(x)=x from

Above we can say $f^{-1}(x) = f(x)$

Sol 13: $f(x) = x^2 + x - 2$

$$y = \left(x + \frac{1}{2}\right)^2 - \frac{9}{4}$$

$$x = -\frac{1}{2} + \sqrt{y - \frac{9}{4}}$$

$$f^{-1}(x) = y = -\frac{1}{2} + \sqrt{x - \frac{9}{4}}$$

$$f(x) = f^{-1}(x)$$

$$\Rightarrow fof(x) = x$$

$$(x^2 + x - 2)^2 + (x^2 + x - 2) - 2 = x$$

Solving this equation we get

$x = \pm\sqrt{2}$

Sol 14: $f(x) \begin{cases} x^2 + 1, & x < -1 \\ x^3 - 1, & x \le x < 0 \end{cases}$

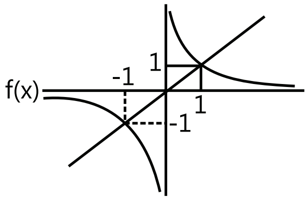
(a) If f(x) is odd

$$f(x)\begin{cases} x^2 + 1, & x < -1 \\ +x^3, & -1 \le x < 0 \\ +x^3, & 1 \ge x \ge 0 \\ -(x^2 + 1), & x > 1 \end{cases}$$

(b) Even function

$$f(x)\begin{cases} x^2 + 1, & x < -1 \\ +x^3, & -1 \le x < 0 \\ -x^3, & 1 \ge x \ge 0 \\ x^2 + 1, & x > 1 \end{cases}$$

Sol 15: $f(x) = \max\left(x, \frac{1}{x}\right)$

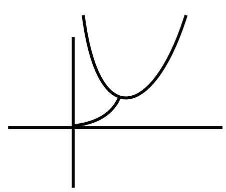


$$f\left(\frac{1}{x}\right) = f(x) = \max\left(\frac{1}{x}, x\right)$$

$$f(x)\begin{cases} \frac{1}{x}, & 0 < x < 1 \\ x, & x > 1 \\ \frac{1}{x}, & x < -1 \\ x, & 0 > x > -1 \end{cases}$$

$$f\left(\frac{1}{x}\right) = f(x)$$

$$g(x)\begin{cases} \frac{1}{x^2} & \text{if} \quad 0 < x \le 1 \\ x^2 & \text{if} \quad x > 1 \end{cases}$$



Sol 16: $f(x) = \dfrac{x^2 + 2x + c}{x^2 + 4x + 3c}$

$$f(x) = \frac{(x+1)^2 + c - 1}{(x+2)^2 + 3c - 4}$$

$(c - 1) \le 0$

$(36 - 4) \le 0$

So the value of f(x) is always F if $x \in R$

Sol 17: (a) $f(x) = \sqrt{\log_2(x^2 - 2x + 2)}$

$$\sqrt{\log_2(x-1)^2 + 1)}$$

$$\log_2\left[(x-1)^2 + 1\right] \ge 0$$

$$(x-1)^2 \ge 0$$

That is always true so domain is R

$(x-1)^2 + 1 \geq 1$

$\log_2[(x-1)^2 + 1] \geq 0$

Range $f(x) \geq 0$

(b) $f(x) = \frac{2x+3}{x-2}$

$= \frac{2(x-2)+7}{x-2} = 2 + \frac{7}{12}$

$f(x) \neq 7 \ f(x) \in R - \{2\}$

Bijective function

Sol 18: $-1 \leq x(x-2) < \infty$

$1 \leq \frac{x(x-2)+1}{4} < \infty$

$1 \leq 2^{\sqrt{\frac{x(x-2)+1}{4}}} < \infty$

$2 \leq f(x) < \infty$

At x=0 & x=2 value of f(x) is same.

So many one function.

Exercise 2

Sets and Relations

Single Correct Choice Type

Sol 1: (D) n R m $\leftrightarrow$ n is a factor of m

$\rightarrow$ every natural no. is a factor of itself

R is reflexive

$\rightarrow$ if n is factor of m, its not necessary that m is also factor of n

R is not symmetric

$\rightarrow$ if n is factor Q m

and m is factor Q ℓ

so n is also a factor of ℓ

(i.e., 3 R 6 and 6 R 18 and 3 R 18 is true)

Sol 2: (C) a R b $\leftrightarrow$ 1 + ab > 0

for a R b $\Rightarrow$ 1 + ab > 0

$\Rightarrow$ ab > – 1

$\rightarrow$ a R a $\Rightarrow$ 1 + a^2 > a

it is always true

so R is reflexive

$\rightarrow$ if a R b $\rightarrow$ 1 + ab > 0

1 + ab = 1 + ba

so 1 + ba also greater than zero

so (b, a) $\in$ R

R is symmetric

$\rightarrow$ if (a, b) $\in$ R and (b, c) $\in$ R

$\Rightarrow$ 1 + ab > 0,1 + bc > 0

Its not necessary to 1 + ac > 0

R is not transitive.

Sol 3: (A) (A)x R, y $\leftrightarrow$ |x| = |y|

|x| = |x| reflexive

|x| = |y| $\Rightarrow$ |y| = |x| symmetric

|x| = |y| and |y| = |z|

equivalence relation

so |x| = |y| = |z| $\Rightarrow$ |x| = |z| transitive

Sol 4: (D) Relation R defined in A = {1, 2, 3}

such that a R b $\Rightarrow$ $|a^2 - b^2| \leq 5$

(1, 2) $\Rightarrow$ $|(1^2 - 2^2)| = 3 \leq 5$

(1, 3) $\Rightarrow$ $|1^2 - 3^2| = 8 \leq 5$

(2, 3) $\Rightarrow$ $|2^2 - 3^2| = 5 \leq 5$

and (a, a) = $|a^2 - a^2| = 0 \leq 5$

and (a, b) = (b, a) $\Rightarrow$ $|a^2 - b^2| = |b^2 - a^2|$

so R = {(1, 2), (2, 1), (2, 3), (3,2), (1, 1), (2, 2), (3, 3)}

so R^{-1} = R

Domain = {1, 2, 3}

Range = {1, 2, 3} , so

Option D is false (Range = {5})

Sol 5: (A) (x, y) $\in$ R $\leftrightarrow$ $x^2 - 4xy + 3y^2 = 0$

for all x, y $\in$ N

$\rightarrow$ for (x, x) $\in$ R $\leftrightarrow$ $x^2 - 4x^2 + 3x^2 = 0$

= 0

R is reflexive

$\rightarrow$ if (x, y) $\in$ R $\Rightarrow$ $x^2 - 4xy + 3y^2 = 0$

then for (y, x) $\in$ R $\Rightarrow$ $y^2 - 4xy + 3x^2$ should be zero

but $3x^2 - 4xy + y^2$

$= x^2 - 4xy + 3y^2 + 2x^2 - 2y^2$

$= 0 + 2x^2 - 2y^2$

$\rightarrow (y, x) \in R \Rightarrow x^2 - y^2 = 0$

its not solution for all $(x, y) \in N$

so R is not symmetric

$\rightarrow (x, y) \in R\ x^2 - 4xy + 3y^2 = 0$

$(y, z) \in R \rightarrow y^2 - 4zy + 3z^2 = 0$

but we cannot find $x^2 - 4zx + 3z^2 = 0$ from above equations, so R is not transitive.

Sol 6: (C) R = {(3, 3), (6, 6), (9, 9),(12, 12), (6, 12), (3, 9), (3, 12), (3, 6)}

on the set A = {3, 6, 9, 12}

$\rightarrow$ for set A

(3, 3), (6, 6), (9, 9), (12, 12) $\in$ R

R is reflexive

$\rightarrow (6, 12) \in R$ but $(12, 6) \notin R$

So R is not symmetric

$\rightarrow (3, 6) \in R$ and $(6, 12) \in R$ and also $(3, 12) \in R$

R is transitive.

Sol 7: (C) R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)}

on the set A = {1, 2, 3, 4}

$\rightarrow$ for set A

(1, 1), (2, 2), (3, 3), (4, 4) $\notin$ A

R is not reflexive

$\rightarrow (2, 3) \in R$, $(3, 2) \notin R$

So R is not symmetric $(1,3) \in R$, $(3,1) \in R$

But $(1, 1) \notin R$

so R is not transitive.

Sol 8: (D) R: N × N

(a, b) R (c, d) if ad(b + c) = bc(a + d)

$\Rightarrow$ for (a, b)R(a, b) $\Rightarrow$ ab(b+a) = ba(a+b)

Which is true so R is reflexive

$\rightarrow$ for symmetric

(a, b) R (c, d) $\Rightarrow$ (c, d) R (a, b)

ad(b + c) = bc(a + d)

cb(d + a) = da(c + b) ... (ii)

ad(b + c) = bc(a + d)

$\Rightarrow$ cb(d + a) = da(c + b)

Which is equation (ii)

$\Rightarrow$ (a, b) R (c, d) Þ (c, d) R (a, b)

so R is symmetric relation

$\rightarrow$ assume (a, b) R (c, d) and (c, d) R (e, f)

ad(b + c) = bc(a + d) ... (i)

cf(d + e) = de(c + f) ... (ii)

for (a, b), (e, f)

$\Rightarrow$ af(b + e) = be(a + f)

If (e, f) = (c, d)

then (c, d) R (e, f) is always true

(R is reflexive)

so in equation (i) (c, d) $\rightarrow$ (e, f)

af (b + e) = be (a + f)

so ((a, b), (e, f)) $\in$ R

$\therefore$ R is transitive.

Sol 9: (A) W = all wards in the English dictionary

R = {(x, y) $\in$ w × w the words x and y have at least one letter in common}

$\rightarrow$ a word have all letter common to itself R is reflexive

$\rightarrow$ if x and y have one letter common soy and x is same condition

$(x, y) \in R \rightarrow (y, x) \in R$

R is symmetric

$\rightarrow$ if $(x, y) \in R$, $(y, z) \in R$

x and y have one letter common

y and z have one letter common

its not mean that it is necessary to x and z have one letter common

R is not transitive.

Sol 10: (C) R $\rightarrow$ real line

Given subset S = {(x, y): y = x + 1 and 0 < x < 2}

and T = {(x, y): x – y is an integer}

for S y = x + 1

but $x \neq x + 1$

s is not reflexive

so s is not equivalence

for T (x, y) $\in$ T $\Rightarrow$ x – y is an integer x – x = 0 is an integer

T is reflexive

$\rightarrow$ if (x, y) $\in$ T $\rightarrow$ x – y is an integer

so + (y – x) also an integer

so (y, x) $\in$ T

T is symmetric

$\rightarrow$ if (x, y) $\in$ T and (y, z) $\in$ T

$x - y = z_1$ ... (i);

$y - z = z_2$ ... (ii)

(assume z_1 and z_2 are integer)

(i) + (ii)

$x - y + y - z = z_1 + z_2$

$x - z = z_1 + z_2$

$\therefore$ sum of two integer is also an integer

so x – z is an integer

(x, z) $\in$ T

T is transitive

$\Rightarrow$ T is an equivalence relation

Multiple Correct Choice Type

Sol 11: (A, B, C) x = {1, 2, 3, 4, 5}, y= {1, 3, 5, 7, 9}

(A) R_1 = {(x, y) | y = 2 + x , x $\in$ X, y $\in$ Y }

R = {(1, 3) (3, 5)(5, 7)}

(B) R_2 = {(1, 1), (2, 1)(3, 3), (4, 3), (5, 5)}

is satisfied R: x $\rightarrow$ y

$$\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}\begin{pmatrix}1\\1\\3\\3\\5\end{pmatrix} \rightarrow \text{ a subset of y}$$

(C) R_3 = {(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)}

All elements are from x

$$\begin{pmatrix}1\\1\\3\\3\\5\end{pmatrix}\begin{pmatrix}1\\3\\5\\7\\7\end{pmatrix}$$

all element are from y

(D) R_4 = {(1, 3),(2, 5), (2, 4), (7, 2)}

in (2, 4), 4 is not belongs to y

So R is not relation from x to y.

Functions

Sol 1: (C)

$$f(x) = \sin^{-1}\sqrt{(x - x^2)} + \sec^{-1}\left(\frac{1}{x}\right) + \ell n|x - 1|$$

$|x - 1| \neq \Rightarrow x \neq 1$

AND $\sqrt{(x - x^2)} \leq 1$

$x - x^2 \leq 0$

$x(x - 1) \leq 0 \Rightarrow x \in [0,1]$

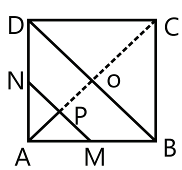
$\sqrt{x(x-1) \leq 1}$ always true in internal [0,1]

And $\frac{1}{x} \geq 1$ or $\frac{1}{x} \leq -1$ $x \neq 0$

$0 \leq x \leq 1$ or $1 \leq x \pi 0$

So domain $x \in (0,1)$

Sol 2: (B)



$AC = 2\sqrt{2}$

$BD = 2\sqrt{2}$

$AP = x$

$PC = 2\sqrt{2} - x \Rightarrow MN = 2x$

$(\Delta AMN)\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times (AP) \times (MN)$

$= \frac{1}{2} \times (2x)(x) = x^2$ (for $x < \sqrt{2}$)

$f(-1) = -1$ $f(-2) = -8$ ΔAMN is maximum

f(x) =maximum= $(\sqrt{2})^2 = 2$

So the range is (0,2)

Sol 3: (A) Bijective

gof

f $\to$ one-one function g $\to$ one-one

function so gof will be one-one function

f & g $\to$ onto function

gof(x) will be onto only of if domain of g = range of f

Sol 4: (D) $y = 5[x]+1 = 6[x-1]-10$

$5I+1 = 6I-6-10$

$I = 17$

$x \in [17,18)$

$y = 5(17)+1 = 86$

$[x+2y] = 189$

Sol 5: (D) $f(x) = ax^3 + e^x$

$f^{'}(x) = 2ax^2 + e^x$

For being a one-one $f^{'}(x) \ge 0$ or $f^{'}(x) \le 0$

f(x) is always greater for any x. if $a \ge 0$

$a \in (0,\infty)$ if $a \ne 0$ then f is not onto function.

Sol 6: (A)

$$\sqrt{\frac{1+\sin x}{1-\sin x}} - \sec x = -\sqrt{\frac{1-\sin x}{1+\sin x}} + \sec x$$

$$\sqrt{\frac{1+\sin x}{1-\sin x}} + \sqrt{\frac{1-\sin x}{1+\sin x}} = 2\sec x$$

$$\frac{2}{\sqrt{1-\sin^2 x}} = 2\sec x$$

$$\frac{2}{\sqrt{\cos^2 x}} = 2\sec x$$

$\sec x = |\sec x|$ if $\cos x \ne 0$

$$x \in \left[-2\pi, \frac{-3\pi}{2}\right) \cup \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$$

Sol 7: (D) $f(x) = \sin\left(\cos\frac{x}{2}\right) + \cos(\sin x)$

$\cos\frac{x}{2}$ has time period = 4π

sin x has time period= 2π

So the common time period = 4π

Sol 8: (A) $f(x) = \dfrac{x}{1+|x|}$

If x>0

$0 \le f(x) = 1 - \dfrac{1}{1+x} < 1$ one-one is the interval.

if x<0

$$f(x) = \frac{x}{1-x}$$

$$= -\left(\frac{-x}{1-x}\right) = -1\left(1 - \frac{-x}{1-x}\right)$$

$$0 \ge f(x) = \frac{1}{1-x} - 1 > -1$$

So f(x) is injective (one-one)

Multiple Correct Choice Type

Sol 9: (A, D) $f(x) = \sqrt{x}$ $\;f: I \to R$

F is not onto function.

But f is one-one function.

Sol 10: (B, C) $f(x) = \sqrt{\log_{x^2} x}$

$x \ne 0,\ x > 0\ x \ne 1$

$$\log_{(x)^2} = \frac{1}{2}\log_{x^x} = \frac{1}{2}$$

$x \in (0,\infty) - \{1\}$

x is defined for (0, 1) and (1, ∞)

Sol 11: (B, C) $y = \dfrac{x\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}-1}$

$x-1 \ge 0 \Rightarrow x \ge 1$

And $\sqrt{x-1} \ne 1 \;\Rightarrow x \ne 2$

And $x - 2\sqrt{x-1} \ge 0$

$x \ge 2\sqrt{x-1}$

Case I if $x \ge 0$ $x^2 \ge 4(x-1)$

$x^2 - 4x + 4 \ge 0$

$(x-2)^2 \ge 0$ which is always true.

Case II if $x \le 0$ then not true

So the domain is $x \in [1,\infty) - \{2\}$

$$2f(1.5)+f(3)=2\left[\frac{2/3\sqrt{\frac{3}{2}-2\sqrt{\frac{1}{2}}}}{\sqrt{\frac{1}{2}}-1}\right]+\frac{3\sqrt{3-2\sqrt{2}}}{\sqrt{2}-1}$$

$$=2\left[\frac{3\sqrt{3-2\sqrt{2}}}{2\sqrt{2}}\right]+\frac{(3\sqrt{3-2\sqrt{2})}}{\sqrt{2}-1}$$

= 0 = non negative

Putting $x-1=t^2$ for x>2

$$f(x)=\frac{(t^2+1)\sqrt{t^2+1-2t}}{t-1}=t^2+1=x$$

Putting for x<2, it is not always defined.

Sol 12: (A, C, D) (A) $f(x)=2^{\frac{1}{x-1}}$

Decreasing function

So at boundary conditions

$f(x)=2^{1/x-1}$

At $f(x)=2^{1/x-1}$ $f(x)=\frac{1}{2}$

At $x=1-h$ $f(x)=0$

Bounded

(B) $g(x)=x\cos\frac{1}{x}$

$\cos\frac{1}{x}$ will oscillate between [-1,1] for any is not bounded so g(x) is also not

(C) $h(x)^{-x}\geq 0$ in $(0,\infty)$

$h'(x)=e^{-x}-xe^{-x}$

If x > 1 then h (x) is decreasing

x < 1 then h (x) is increasing

at x = 0 ,h (x) = 0

at x = 1 , h (x) = $\frac{1}{e}$

at x = ∞ h (x) = 0 (so it is bounded)

(D) $\ell(x)=\tan^{-1}(2^x)$

2^x is strictly increasing and it is positive.

$\ell(x)$ is bounded as $=x-\infty,$ and $\ell(x)=0$

$=x^{-}\infty,\ \ell(x)=\pi/2$

Sol 13: (A, D) (A) $f(x)=x-[x]=\{x\}$

Periodic

(B) $g(x)=\sin\left(\frac{1}{x}\right),\ x\neq 0\ \&\ g(0)=0$

$\frac{1}{x}$ is not periodic so $\sin\frac{1}{x}$ is also not.

(C) $h(x)=x\cos x$

Not periodic C

(D) $w(x)=\sin x$

Periodic

Sol 14: (B, C) $f(x)=\begin{cases}x & \text{if } x\in Q\\ 1-x & \text{if } x\notin Q\end{cases}$

$x\in[0,1]$

$0\leq f(x)<1\quad [x\in[0,1]-\{1\}]$

At x=1, f(x) = 1

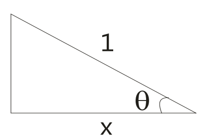
$[f(x)]=0$ for $x\in[0,1)$

at $n=1;\ [f(x)]=1$

$f([f(x)])=1$,for $x\in[0,1)$

Sol 15: (A, C, D) (A) $y=\tan(\cos^{-1}x)=\frac{\sqrt{1-x^2}}{x}$ (identical)

$\sqrt{1-x^2}$



Domain of both function, are not same at x=0

$y=\tan^{-1}(\cos x)$ is defined while the order is not.

(B) $y=\tan(\cos^{-1}x)$ (is not identical) $y=\frac{1}{x}$

(C) $y=\sin(\tan^{-1}(x))=\frac{x}{\sqrt{1+x^2}}$ (identical)

(D) $y=\cos(\tan^{-1}x)$; $y=\sin(\cot^{-1}x)$ (identical)

$$y=\frac{1}{\sqrt{1+x^2}}$$

Previous Years' Questions

Sol 1: (A, D) Given, $y = f(x) = \frac{x+2}{x-1}$

$\Rightarrow yx - y = x + 2 \Rightarrow x(y - 1) = y + 2$

$\Rightarrow x = \frac{y+2}{y-1} \Rightarrow x = f(y)$

Here, f(1) does not exist, so domain $\in R - \{1\}$.

$$\frac{dy}{dx} = \frac{(x-1)\cdot 1(x+2)\cdot 1}{(x-1)^2}$$

$$= -\frac{3}{(x-1)^2}$$

$\Rightarrow$ f(x) is decreasing for all $x \in R - \{1\}$

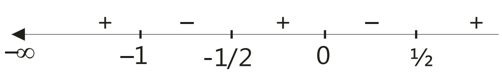
Also, f is rational function of x.

Hence, (a) and (d) are correct options.

Sol 2: (A, D) Since, $\frac{2x-1}{2x^3+3x^2+x} > 0$

$$\Rightarrow \frac{(2x-1)}{x(2x^2+3x+1)} > 0$$

$$\Rightarrow \frac{(2x-1)}{x(2x+1)(x+1)} > 0$$

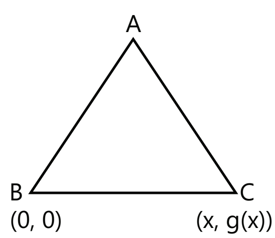


Hence, the solution set is,

$x \in (-\infty, -1) \cup (-1/2, 0) \cup (1/2, \infty)$

Sol 3: (B, C) Since, area of equilateral triangle = $\frac{\sqrt{3}}{4}(BC)^2$

$\Rightarrow \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}.[x^2 + g^2(x)] \Rightarrow g^2(x) = 1 - x^2$



$\Rightarrow g(x) = \sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

Hence, (b) and (c) are the correct options.

Sol 4: (A, C) Since, $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

$\Rightarrow f(x) = \cos(9)x + \cos(-10)x$,

(using $[\pi^2] = 9$ and $[-\pi^2] = -10$)

$\therefore f\left(\frac{\pi}{2}\right) = \cos\frac{9\pi}{2} + \cos 5\pi = -1$

$f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$

$f(-\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$

$f\left(\frac{\pi}{4}\right) = \cos\frac{9\pi}{4} + \cos\frac{10\pi}{4} = \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$

Hence, (a) and (c) are correct options.

Sol 5: (A) Here, $f(x) = \frac{b-x}{1-bx}$, where

$0 < b < 1, 0 < x < 1$

For function to be invertible it should be one–one onto.

$\therefore$ Check Range:

Let $f(x) = y \Rightarrow y = \frac{b-x}{1-bx}$

$\Rightarrow y - bxy = b - x$

$\Rightarrow x(1 - by) = b - y$

$\Rightarrow x = \frac{b-y}{1-by}$, where $0 < x < 1$

$\therefore 0 < \frac{b-y}{1-by} < 1$

$\frac{b-y}{1-by} > 0$ and $\frac{b-y}{1-by} < 1$

$\Rightarrow y < b$ or $y > \frac{1}{b}$ ... (i)

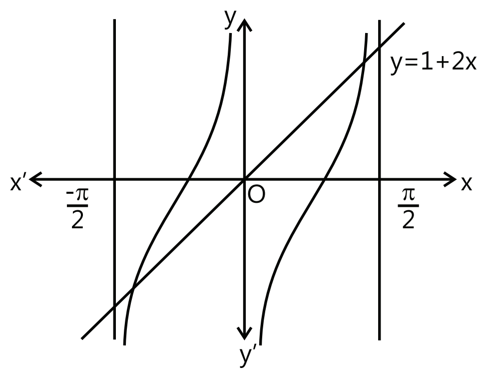
$\frac{(b-1)(y+1)}{1-by} < 0 \; -1 < y < \frac{1}{b}$ ... (ii)

From Eqs. (i) and (ii), we get $y \in \left(-1, \frac{1}{b}\right) \subset$ co–domain

Thus, f(x) is not invertible.

Sol 6: A $\rightarrow$ q; B $\rightarrow$ r

y = 1 + 2x is linear function therefore, it is one–one and its range is $(-\pi + 1, \pi + 1)$. Therefore, (1 + 2x) is one–one but not onto so (A) $\rightarrow$ (q) Again, see the figure.

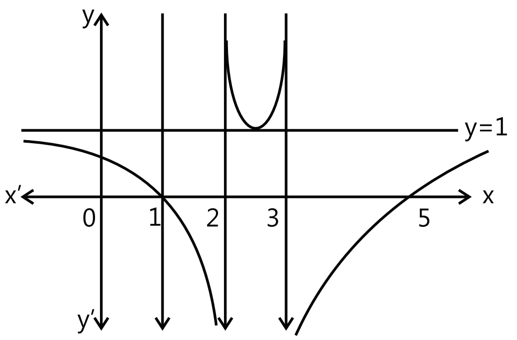


It is clear from the graph that y = tan x is one–one and onto, therefore (B) $\rightarrow$ (r)

Sol 7: A$\rightarrow$ p; B $\rightarrow$ q; C$\rightarrow$ q; D $\rightarrow$ p

Given, $f(x) = \frac{(x-1)(x-5)}{(x-2)(x-3)}$

The graph of f(x) is shown



(A) If $-1 < x < 1 \Rightarrow 0 < f(x) < 1$

(B) If $1 < x < 2 \Rightarrow f(x) < 0$

(C) If $3 < x < 5 \Rightarrow f(x) > 0$

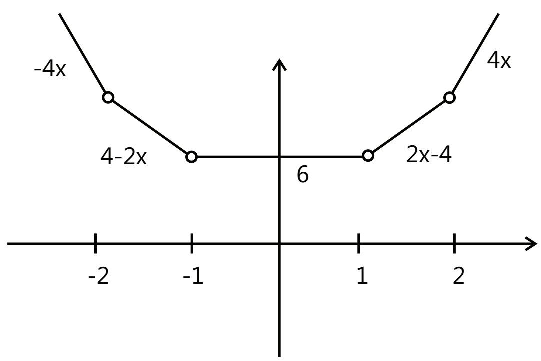
(D) If $x > 5 \Rightarrow f(x) < 1$

Sol 8: A$\rightarrow$ p; B $\rightarrow$ q, s; C$\rightarrow$ (q, r, s, t); D $\rightarrow$ r

(A) $f'(x) > 0, \forall x \in (0, \pi/2)$

f (0) < 0 and $f(\pi/2) > 0$

so one solution.



(B) Let (a, b, c) is direction ratio of the intersected line, then

ak + 4b + c = 0

4a + kb + 2c = 0

$$\frac{a}{8-k} = \frac{b}{4-2k} = \frac{c}{k^2-16}$$

We must have

$$2(8-k)+2(4-2k)+(k^2-16)=0$$

$\Rightarrow k = 2, 4$.

(C) Let f(x) = |x + 2| + |x + 1| + |x −1| + |x −2|

$\Rightarrow$ k can take value 2, 3, 4, 5.

(D) $\int \frac{dy}{y+1} = \int dx$

$\Rightarrow f(x) = 2e^x - 1 \;\Rightarrow f(\ln 2) = 3$

Sol 9: A$\rightarrow$ q, s; B$\rightarrow$ p, r, s, t; C$\rightarrow$ t; D$\rightarrow$ r

$$2\sin^2\theta + 4\sin^2\theta\cos^2\theta = 2$$

$$\sin^2\theta + 2\sin^2\theta(1-\sin^2\theta) = 1$$

$$3\sin^2\theta + 2\sin^2\theta - 1 = 0 \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{2}}, \pm 1$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{\pi}{2}.$$

(B) Let $y = \frac{3x}{\pi}$

$$\Rightarrow \frac{1}{2} \le y \le 3 \;\forall\; x \in \left[\frac{\pi}{6}, \pi\right]$$

Now f(y) = [2y] cos[y]

Critical points are $y = \frac{1}{2}, y = 1, y = \frac{3}{2}, y = 3$

$\Rightarrow$ points of discontinuity $\left\{\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \pi\right\}$.

(C) $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix} = \pi \Rightarrow$ volume of parallelepiped = π

(D) $|\vec{a}+\vec{b}| = \sqrt{3}$

$$\Rightarrow \sqrt{2+2\cos\alpha} = \sqrt{3}$$

$$\Rightarrow 2 + 2\cos\alpha = 3$$

$$\Rightarrow \alpha = \frac{\pi}{3}.$$

Sol 10: $f(0) = 1,\; f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

$$\Rightarrow f'(g(x))\, g'(x) = 1$$

Put $x = 0 \Rightarrow g'(1) = \frac{1}{f'(0)} = 2$.

Sol 11: (B) $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$ ... (i)

$f\left(f^{-1}(x)\right) = x$

$\Rightarrow f\left(f^{-1}(x)\right)'\left(f^{-1}(x)\right)' = 1 \Rightarrow \left(f^{-1}(2)\right)' = \dfrac{1}{f'\left(f^{-1}(2)\right)}$

$\Rightarrow f(0) = 2 \Rightarrow f^{-1}(2) = 0$

$\left(f^{-1}(2)\right)' = \dfrac{1}{f'(0)}$

$e^{-x}\left(f'(x) - f(x)\right) = \sqrt{x^4+1}$

Put x = 0

$\Rightarrow f'(0) - 2 = 1 \Rightarrow f'(0) = 3$

$\left(f^{-1}(2)\right)' = 1/3$

Sol 12: (D) $f(x) = \begin{cases} \{x\} & , 2n-1 \le x < 2n \\ 1-\{x\} & , 2n \le x < 2n+1 \end{cases}$

Clearly f(x) is a periodic function with period = 2

Hence f(x). cos π x is also periodic with period = 2

$\dfrac{\pi^2}{10}\int_{-10}^{10} f(x)\cos(\pi x)dx = \pi^2 \int_0^2 f(x)\cos(\pi x)dx$

$= \pi^2 \int_0^1 \left(\left(1-\{x\}\right) + \{x\}\right)\cos(\pi x)\,dx$

$= 2\pi^2 \int_0^1 (-x\cos \pi x)dx = -2\pi^2\left[\dfrac{\pi \sin \pi x}{\pi} + \dfrac{\cos \pi x}{\pi^2}\right]_0^1$

$= -2\pi^2\left(-\dfrac{2}{\pi^2}\right) = 4$

Sol 13: $\vec{r}.\times\vec{b} = \vec{c}\times\vec{b}$

taking cross with a

$\vec{a}\times\left(\vec{r}.\times\vec{b}\right) = \vec{a}\left(\vec{c}\times\vec{b}\right)$

$\left(\vec{a}.\vec{b}\right)\vec{r} - \left(\vec{a}.\vec{r}\right)\vec{b} = \vec{a}\times\left(\vec{c}\times\vec{b}\right)$

$\Rightarrow \vec{r} = -3\,\hat{i} + 6\,\hat{j} + 3\,\hat{k}$

$\vec{r}\,.\,\vec{b} = 3 + 6 = 9$

Sol 14: (A, B, C) Given $g(x) = \dfrac{\pi}{2}\sin x\;\forall x \in R$

$f(x) = \sin\left(\dfrac{1}{3}g\left(g(x)\right)\right)$

$\Rightarrow g\left(g(x)\right) \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\forall x \in R$

Also, $g\left(g\left(g(x)\right)\right) \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\forall x \in R$

Hence, f (x) and $f\left(g(x)\right) \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

$\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0}\dfrac{\sin\left(\frac{1}{3}g\left(g(x)\right)\right)}{\frac{1}{3}g\left(g(x)\right)}.\dfrac{\frac{1}{3}g\left(g(x)\right)}{g(x)}$

$\Rightarrow \lim\limits_{x\to 0}\dfrac{\pi}{6}.\dfrac{\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x} = \dfrac{\pi}{6}$

Range of $g\left(f(x)\right) \in \left[-\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right), \dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)\right]$

$\Rightarrow g\left(f(x)\right) \ne 1$.

Sol 15: A → s; B → t; C→r; D → r

(A) $z = \dfrac{2i(x+iy)}{1-(x+iy)^2} = \dfrac{2i(x+iy)}{1-\left(x^2-y^2+2ixy\right)}$

Using $1-x^2 = y^2$

$Z = \dfrac{2ix-2y}{2y^2-2ixy} = -\dfrac{1}{y}$.

$\because -1 \le y \le 1 \Rightarrow -\dfrac{1}{y} \le -1$ or $-\dfrac{1}{y} \ge 1$.

(B) For domain

$-1 \le \dfrac{8.3^{x-2}}{1-3^{2(x-1)}} \le 1$

$\Rightarrow -1 \le \dfrac{3^x - 3^{x-2}}{1-3^{2x-2}} - \le 1$

Case I: $\dfrac{3^x-3^{x-2}}{1-3^{2x-2}} - 1 \le 0$

$\Rightarrow \dfrac{\left(3^x-1\right)\left(3^{x-2}-1\right)}{\left(3^{2x-2}-1\right)} \ge 0$

$\Rightarrow x \in (-\infty, 0] \cup (1, \infty)$

Case – II: $\frac{3^x - 3^{x-2}}{1-3^{2x}-2}+1 \geq 0$

$\Rightarrow \frac{\left(3^{x-2}-1\right)\left(3^x+1\right)}{\left(3^x.3^{x-2}-1\right)} \geq 0$

$\Rightarrow x \in (-\infty, 1) \cup [2, \infty)$

So, $x \in (-\infty, 0) \cup [2, \infty)$.

(C) $R_1 \to R_1 + R_3$

$$f(\theta) = \begin{vmatrix} 0 & 0 & 2 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$$

$= 2\left(\tan^2\theta + 1\right) = 2\sec^2\theta$.

(D) $f'(x) = \frac{3}{2}(x)^{1/2}(3x-10) + (x)^{3/2} \times 3$

$= \frac{15}{2}(x)^{1/2}(x-2)$

Increasing, when $x \geq 2$.

Sol 16: (B) $f(x) = 2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36$

$= 6\left(x^2 - 5x + 6\right)$

$= 6(x-2)(x-3)$

f (x) is increasing in [0, 2] and decreasing in [2, 3]

f (x) is many one

f (0) = 1

f (2) = 29

f (3) = 28

Range is [1, 29]

Hence, f (x) is many-one-onto.

Sol 17: (C)

$f(x) + 2x = 1(1-x)^2 \sin^2 x + x^2 + 2x$

$\because f(x) + 2x = 2\left(1+x^2\right)$

$\Rightarrow (1-x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$

$(1-x)^2 \sin^2 x = x^2 - 2x + 1 + 1$

$= (1-x)^2 + 1$

$\Rightarrow (1-x)^2 \cos^2 x = -1$

Which can never be possible

P is not true

$\Rightarrow$ Let $H(x) = 2f(x) + 1 - 2x(1+x)$

H (0) = 2 f (0) + 1 – 0 = 1

H (1) = 2 f (1) + 1 – 4 = -3

$\Rightarrow$ So H (x) has a solution

So Q is true.

Sol 18: (A, B)

$\cos 4\theta = \frac{1}{3} \Rightarrow 2\cos^2 2\theta - 1 = \frac{1}{3}$

$\Rightarrow \cos^2 2\theta = \frac{2}{3} \Rightarrow \cos 2\theta = \pm\sqrt{\frac{2}{3}}$

Now $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta} = \frac{1+\cos 2\theta}{\cos 2\theta} = 1 + \frac{1}{\cos 2\theta}$

$\Rightarrow f\left(\frac{1}{3}\right) = 1 \pm \sqrt{\frac{3}{2}}$

Sol 19: (C) Let, $g(x) = e^{-x}f(x)$

As $g''(x) > 0$ so g′ (x) is increasing.

So, for $x < 1/4,\ g'(x) < g'(1/4) = 0$

$\Rightarrow \left(f'(x) - f(x)\right)e^{-x} < 0$

$\Rightarrow f'(x) < f(x)$ in $(0, 1/4)$

Sol 20: (D) $f_2(f_1) = \begin{cases} x^2 & , x < 0 \\ e^{2x} & x \geq 0 \end{cases}$

$f_4 R \to [0, \infty)$

$f_4(x) = \begin{cases} f_2(f_1(x)) & , x < 0 \\ f_2(f_1(x)) - 1 & , x \geq 0 \end{cases}$

$= \begin{cases} x^2 & , x < 0 \\ e^{2x} - 1 & , x \geq 0 \end{cases}$

Sol 21: (A, B, C)

$f(x) = \left(\log(\sec x + \tan x)\right)^3 \;\forall\; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f(-x) = -f(x)$, hence f (x) is odd function

Let g (x) = sec x + tan x $\forall\; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow g'(x) = \sec x(\sec x + \tan x) > 0 \;\forall\; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow g(x)$ is one-one function

Hence $\left(\log_e(g(x))\right)^3$ is one-one function.

And $g(x) \in (0, \infty) \;\forall\, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow \log(g(x)) \in R$. Hence f (x) is an onto function.

Sol 22: $G(1) = \int_{-1}^{1} t\left|f(f(t))\right| dt = 0$

f (- x) = - f (x)

Given $f(1) = \frac{1}{2}$

$$\lim_{x\to 1} \frac{F(x)}{G(x)} = \lim_{x\to 1} \frac{\frac{F(x)-F(1)}{x-1}}{\frac{G(x)-G(1)}{x-1}} = \frac{f(1)}{\left|f(f(1))\right|} = \frac{1}{14}$$

$$\Rightarrow \frac{1/2}{\left|f(1/2)\right|} = \frac{1}{14}$$

$$\Rightarrow f\left(\frac{1}{2}\right) = 7.$$