Quadratic Equations & Inequations

INTRODUCTION TO POLYNOMIALS — Section - 1

1.1 Real Polynomial :

Let $a_0, a_1, a_2, ..., a_n$ be real numbers and x is a real variable. Then,

$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

is called a real polynomial of real variable x with real coefficient.

For Example **:**

$f(x) = 2x^2 + 3x + 1$, $f(x) = x + 3$, $f(x) = 5x^4 + 3x^2 - 4x - 1$ are some examples of real polynomials.

Note : *How to identify a polynomial ?*

Polynomial in x should be an expression in terms of various powers of x where every power should be a positive integer.

For Example **:** $f(x) = x + \frac{1}{x} + 2$, $f(x) = x^2 + x^{1/2} + 3$,

$f(x) = x^{-2} - x^{-1} + 1$ are not polynomials.

1.2 Degree of a Polynomial

The degree of a real polynomial is the highest power of x in the polynomial.

$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0x^0$

is a polynomial of degree n as highest power of x is n where n is a positive integer and $a_n \neq 0$.

For Example **:**

$f(x) = x^3 - 2x^2 + x - 1$ is a polynomial of degree 3.

$f(x) = x^5 + x^2 - 1$ is a polynomial of degree 5.

➤ **Linear Polynomial :** Polynomial of degree one is known as linear polynomial.

For Example **:**

$f(x) = 2x + 3$, $f(x) = -x + 5$ are linear polynomials.

Note : $f(x) = ax + b$ *is a general degree one polynomial known as linear polynomial* $(a \neq 0)$.

- **Quadratic Polynomial :** Polynomial of degree two is known as quadratic polynomial.

For Example :

$f(x) = x^2 + x + 1$, $f(x) = -x^2 + 2x - 1$ are quadratic polynomials.

Note : *$f(x) = ax^2 + bx + c$, $a \neq 0$ is a general degree two polynomial known as quadratic polynomial.*

1.3 Polynomial Equation :

If $y = f(x)$ is a real polynomial of degree n, then $f(x) = 0$ is the corresponding real polynomial equation of degree n.

For Example : If $f(x) = x^2 - 2x - 8$ is a quadratic polynomial, then $x^2 - 2x - 8 = 0$ is the corresponding quadratic equation.

1.4 Roots of an Equation :

Roots of an equation in x are those values of x which satisfy the equation

OR

If $f(\alpha) = 0$, then $x = \alpha$ is the root of the equation $f(x) = 0$.

For Example :

$x = 2$, $x = 3$ are roots of $x^2 - 5x + 6 = 0$ because when we replace $x = 2$ or $x = 3$ in the equation,

We get : $0 = 0$. This implies $x = 2$ and $x = 3$ satisfy equation. Hence $x = 2$ and $x = 3$ are roots of the equation.

Note :
- *Real roots of an equation $f(x) = 0$ are the x-co-ordinates of the points where graph of $y = f(x)$ intersects X-axis.*
- *An equation of degree n has n roots.* **(*not necessarily all real*).**

QUADRATIC EQUATION & INEQUATION — Section - 2

2.1 Introduction :

The standard form of the quadratic equation is :

$ax^2 + bx + c = 0$ where a, b, c are real numbers and $a \neq 0$.

2.2 Roots of a Quadratic Equation

Roots of a quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$, $a, b, c \in R$) are given by :

$$\alpha,\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Sum of the roots $= \alpha + \beta = -\dfrac{b}{a}$
- Product of roots $= \alpha\beta = \dfrac{c}{a}$
- Factorized form of $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.
- If S be the sum and P be the product of roots, then quadratic equation is : $x^2 - Sx + P = 0$.

***Illustration the Concept* :**

(a) *If α and β are the roots of equation $ax^2 + bx + c = 0$, find the value of following expressions.*

(i) $\alpha^2 + \beta^2$ **(ii)** $\alpha^3 + \beta^3$ **(iii)** $\alpha^4 + \beta^4$ **(iv)** $(\alpha - \beta)^2$ **(v)** $\alpha^4 - \beta^4$

SOLUTION :

In such type of problems, try to represent the given expression in terms of a + b (sum of roots) and ab (product of roots). In the given problem :

$$\alpha + \beta = \frac{-b}{a}, \ \alpha\beta = \frac{c}{a}$$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(-\dfrac{b}{a}\right)^2 - \dfrac{2c}{a} = \dfrac{b^2 - 2ac}{a^2}$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = \left(-\dfrac{b}{a}\right)^3 - 3\left(\dfrac{c}{a}\right)\left(-\dfrac{b}{a}\right) = \dfrac{-b^3 + 3abc}{a^3}$

(iii) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

$$= \left(\frac{b^2 - 2ac}{a^2}\right)^2 - 2\frac{c^2}{a^2} = \frac{(b^2 - 2ac)^2 - 2c^2a^2}{a^4}$$

(iv) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \dfrac{b^2}{a^2} - \dfrac{4c}{a} = \dfrac{b^2 - 4ac}{a^2}$

(v) $\alpha^4 - \beta^4 = (\alpha^2 + \beta^2)(\alpha + \beta)(\alpha - \beta)$

$$= \left(\frac{b^2 - 2ac}{a^2}\right)\left(-\frac{b}{a}\right)\left(\pm\sqrt{\frac{b^2 - 4ac}{a^2}}\right) \qquad \left[\text{Using (i) and } \left(\alpha - \beta = \pm\frac{\sqrt{D}}{a}\right)\right]$$

$$= \pm\frac{b}{a^4}(b^2 - 2ac)\sqrt{b^2 - 4ac}$$

(b) *If α and β are the roots of equation $ax^2 + bx + c = 0$, form an equation whose roots are :*

(i) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$ **(ii)** $\frac{1}{\alpha + \beta}, \frac{1}{\alpha} + \frac{1}{\beta}$

SOLUTION :

We know that to form an equation whose roots are known we have to find sum and product of the roots.

(i) Sum (S) $= \left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right) = (\alpha + \beta) + \frac{(\alpha + \beta)}{\alpha\beta} = \frac{-b(a + c)}{ac}$

Product (P) $= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right) = \alpha\beta + \frac{1}{\alpha\beta} + 2 = \frac{(c + a)^2}{ac}$

Product (P) $= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right) = \alpha\beta + \frac{1}{\alpha\beta} + 2 = \frac{(c + a)^2}{ac}$

The equation is : $x^2 - Sx + P = 0$

$$\Rightarrow \quad x^2 - \left(\frac{-b(a + c)}{ac}\right)x + \frac{(c + a)^2}{ac} = 0$$

$\Rightarrow$ $acx^2 + b(c + a)x + (c + a)^2 = 0$ is the required equation.

(ii) Sum (S) $= \left(\frac{1}{\alpha + \beta}\right) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \left(\frac{1}{\alpha + \beta}\right) + \frac{(\alpha + \beta)}{\alpha\beta} = -\frac{(ac + b^2)}{bc}$

Product (P) $= \left(\frac{1}{\alpha + \beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \left(\frac{1}{\alpha\beta}\right) = \frac{a}{c}$

The equation is : $x^2 - Sx + P = 0$

$$\Rightarrow \quad x^2 - \left(-\frac{(ac + b^2)}{bc}\right)x + \frac{a}{c} = 0$$

$\Rightarrow$ $bcx^2 + (ac + b^2)x + ab = 0$ is the required equation.

2.3 Nature of Roots of a Quadratic Equation

Nature of roots of a quadratic equation $ax^2 + bx + c = 0$ means whether the roots are real or complex. By analyzing the expression ***b^2 - 4ac (called as discriminant, D)***, one can get an idea about the nature of the roots as follows :

1.(a) **If $D < 0$** ($b^2 - 4ac < 0$), then the roots of the quadratic equation are non – real *i.e.* complex roots.

(b) **If $D = 0$** ($b^2 - 4ac = 0$), then the roots are real and equal.

Equal root $= -\dfrac{b}{2a}$

(c) **If $D > 0$** ($b^2 - 4ac > 0$), then the roots are real and unequal.

2. If D *i.e.* ($b^2 - 4ac$) is a perfect square and a, b and c are rational, *then the roots are rational*.

3. If D *i.e.* ($b^2 - 4ac$) is not a perfect square and a, b and c are rational, then roots are of the form $m+\sqrt{n}$ and $m-\sqrt{n}$.

4. If $a = 1$, $b, c \in I$ and the roots are rational numbers, then the roots must be integer.

5. If a quadratic equation in x has more than two roots, then it is an identity in x (*i.e.* true for all real values of x) and $a = b = c = 0$.

Illustration the Concept :

Comment upon the nature of roots of the following equations :

(i) $x^2+(a+b)x-c^2=0$

(ii) $(a+b+c)x^2-2(a+b)x+(a+b-c)=0$

(iii) $(b-c)x^2+(c-a)x+(a-b)=0$

(iv) $x^2+2(3a+5)x+2\left(9a^2+25\right)=0$

(v) $(y-a)(y-b)+(y-b)(y-c)+(y-c)(y-a)=0$

SOLUTION :

To comment upon the nature of roots of quadratic equation we have to find 'D' (Discriminant)

(i) Find discriminant (D).

$$D=(a+b)^2-4(1)\left(-c^2\right)=(a+b)^2+4c^2$$

$\Rightarrow$ $D \geq 0$, hence the roots are real

(ii) $D=4(a+b)^2-4(a+b+c)(a+b-c)$

$$=4\left[(a+b)^2-(a+b)^2+c^2\right]=4c^2=(2c)^2$$

$\Rightarrow$ $D \geq 0$ and also a perfect square, hence the roots are rational .

(iii) $D=(c-a)^2-4(b-c)(a-b)$

$$=c^2+a^2-2ac-4ab+4b^2+4ac-4bc$$

$$=c^2+a^2+(2b)^2-4ab-4bc+2ac=(c+a-2b)^2$$

$\Rightarrow$ $D\geq 0$ and also a perfect square, hence the roots are rational .

(iv) $D=4(3a+5)^2-8\left(9a^2+25\right)=-4(3a-5)^2$

$\Rightarrow$ $D\leq 0$, so the roots are non real if $a\neq 5/3$ and real and equal if $a=5/3$

(v) Simplifying the given equation

$$3y^2-2(a+b+c)y+(ab+bc+ca)=0$$

Now $D=4(a+b+c)^2-12(ab+bc+ca)$

$$=4\left(a^2+b^2+c^2-ab-bc-ca\right)$$

$$=2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

Using : $(a^2+b^2+c^2-ab-bc-ca)=\frac{1}{2}\left[(a-b)^2+(b-c)^2(c-a)^2\right]$

$\Rightarrow$ $D\geq 0$, so the root are real

Note : If $D=0$, then $(a-b)^2+(b-c)^2+(c-a)^2=0$

$\Rightarrow$ $a=b=c$ $\Rightarrow$ if $a=b=c$, then the roots are equal

2.4 Condition for Common Root(s) :

Consider two quadratic equations :

$ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$

(a) For two common roots :

In such a case, two equations should be identical. For that, the ratio of coefficients of x^2, x and x^0 must be same,

i.e. $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

(b) For one common root :

Let α be the common root of two equations. So α should satisfy the two equations.

$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$ and $a'\alpha^2 + b\alpha + c' = 0$

Solving the two equations by using **Cramer's Rule** (or cross multiplication method) :

$$\Rightarrow \quad \frac{\alpha^2}{bc' - b'c} = \frac{-\alpha}{ac' - a'c} = \frac{1}{ab' - a'b}$$

$$\Rightarrow \quad \alpha = \frac{a'c - ac'}{ab' - a'b}, \ \alpha^2 = \frac{bc' - b'c}{ab' - a'b}$$

$$\Rightarrow \quad (bc' - b'c)(ab' - a'b) = (a'c - ac')^2$$

This is the condition for one root of two quadratic equations to be common.

Note : *To find the common root between the two equations, make the coefficient of α^2 common and then subtract the two equations.*

Illustration - 1 *The equation whose roots are squares of the sum and the diffrence of the roots of the equation* $2x^2 + 2(m+n)x + m^2 + n^2 = 0$ *is :*

(A) $x^2 - 4mnx + (m^2 - n^2)^2 = 0$ **(B)** $x^2 + 4mnx + (m^2 - n^2)^2 = 0$

(C) $x^2 - 4mnx - (m^2 - n^2)^2 = 0$ **(D)** $x^2 - 4mx - (m^2 - n^2)^2 = 0$

SOLUTION : (C)

Let α, β be the root of given equaiton.

$$\Rightarrow \quad \alpha + \beta = -(m+n) \text{ and } \alpha\beta = \frac{(m^2 + n^2)}{2}$$

Now we have to make an eqution whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$

Sum $(S) = (\alpha+\beta)^2 + (\alpha-\beta)^2 = 2(\alpha^2 + \beta^2) = 2\left[(\alpha+\beta)^2 - 2\alpha\beta\right] = 4mn$

Product $(P) = (\alpha+\beta)^2 . (\alpha-\beta)^2 = (\alpha+\beta)^2 . \left[(\alpha+\beta)^2 - 4\alpha\beta\right]$

$$P = (m+n)^2\left[(m+n)^2 - 2(m^2 + n^2)\right] = -(m^2 - n^2)^2$$

The equation is : $x^2 - Sx + P = 0$

$\Rightarrow$ The requaired equation is $x^2 - 4mnx - (m^2 - n^2)^2 = 0$

Illustration - 2 *The value of k, so that the equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ may have one root in common.*

(A) $-3, -1$ (B) $-3, \frac{-27}{4}$ (C) $-1, -2$ (D) $3, \frac{27}{4}$

SOLUTION : (B)

Let a be the common root of two equations.

Hence $2\alpha^2 + k\alpha - 5 = 0$ and $\alpha^2 - 3\alpha - 4 = 0$

Solving the two equations ;

$$\frac{\alpha^2}{-4k-15} = \frac{-\alpha}{-8+5} = \frac{1}{-6-k}$$ [Using : [2.4 (*b*)]

$\Rightarrow$ $(-3)^2 = (4k + 15)(6 + k)$ $\Rightarrow$ $4k^2 + 39k + 81 = 0$

$\Rightarrow$ $k = -3$ or $k = -27/4$

Illustration - 3 *If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a root in common, find the relation between a, b and c.*

(A) $a = 0$ or $a^3 + b^3 + c^3 = 3abc$ (B) $a = 0$ or $a^3 + b^3 + c^3 = -3abc$

(C) $a = 0$ or $a^3 - b^3 - c^3 = 3abc$ (D) $a = 0$ or $a^3 + b^3 - c^3 = -3abc$

SOLUTION : (A)

Solve the two equations as done in last illustration :

$ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$

$$\frac{x^2}{ba-c^2} = \frac{-x}{a^2-bc} = \frac{1}{ac-b^2}$$ [Using : [2.4 (*b*)]

$\Rightarrow$ $(a^2 - bc)^2 = (ba - c^2)(ac - b^2)$

Simplify to get : $a(a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow$ $a = 0$ or $a^3 + b^3 + c^3 = 3abc$

This is the relation between *a*, *b* and *c*.

Illustration - 4 *If the equations $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$ have one root in common and second equatio has equal roots, prove that ac* is :

(A) b (B) $2(b+d)$ (C) $b-d$ (D) bd

SOLUTION : (B)

The equation $x^2 - cx + d = 0$ has equal roots.

$\Rightarrow$ $D = 0$ $\Rightarrow$ $D = c^2 - 4d = 0$ . . . . . . (i)

$\Rightarrow \quad x = \frac{c}{2}$ is the equal root of this equation. [As equal root of $ax^2 + bx + c = 0$ *are* $x = \frac{-b}{2a}$]

Now this should be the common root.

$\therefore \quad x = \frac{c}{2}$ will satisfy the first equation

$$\Rightarrow \quad \frac{c^2}{4} - a\left(\frac{c}{2}\right) + b = 0 \quad \Rightarrow \quad c^2 + 4b = 2ac \quad \Rightarrow \quad 4d + 4b = 2ac \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 2(d+b) = ac \qquad \text{Hence } ac = 2(b+d)$$

Illustration - 5 *If the ratio of roots of the equation $x^2 + px + q = 0$ be equal to the ratio of roots of the equation $x^2 + bx + c = 0$, then prove that $p^2c - b^2q =$*

(A) -1 **(B)** 1 **(C)** 0 **(D)** None of these

SOLUTION : (C)

Let α, β be the roots of $x^2 + px + q = 0$ so, $\alpha + \beta = -p, \alpha\beta = q$

and also let γ, δ $x^2 + bx + c = 0$ so, $\gamma + \delta = -b, \gamma\delta = c$

Now, $\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{(\alpha+\beta)^2}{(\alpha-\beta)^2} = \frac{(\gamma+\delta)^2}{(\gamma-\delta)^2}$ (Apply componendo-divideendo and take square on both sides)

$$\Rightarrow \quad \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2 - (\alpha-\beta)^2} = \frac{(\gamma+\delta)^2}{(\gamma+\delta)^2 - (\gamma-\delta)^2}$$

$$\Rightarrow \quad \frac{(\alpha+\beta)^2}{4\alpha\beta} = \frac{(\gamma+\delta)^2}{4\gamma\delta} \Rightarrow \quad \frac{p^2}{4q} = \frac{b^2}{4c} \qquad \Rightarrow \quad p^2c = b^2q.$$

Illustration - 6 *The condition that the equation $\frac{1}{x} + \frac{1}{x+b} = \frac{1}{m} + \frac{1}{m+b}$ has real roots that are equal in magnitude but opposite in sign is*

(A) $b^2 = m^2$ **(B)** $b^2 = 2m^2$ **(C)** $2b^2 = m^2$ **(D)** *None of these*

SOLUTION : (B)

Clearly $x = m$ is a root of the equation. Therefore, the other root must be $-m$. That is,

$$\frac{1}{-m} + \frac{1}{-m+b} = \frac{1}{m} + \frac{1}{m+b}$$

$$\Rightarrow \quad \frac{1}{b-m} - \frac{1}{b+m} = \frac{2}{m} \qquad \Rightarrow \quad \frac{b+m-b+m}{b^2-m^2} = \frac{2}{m}$$

$$\Rightarrow \quad 2m^2 = 2b^2 - 2m^2 \quad \text{or} \quad 2m^2 = b^2.$$

INEQUATION & INEQUALITIES — Section - 3

3.1 Inequalities

The following are some very useful points to remember :

- $a \le b \quad \Rightarrow$ Either $a < b$ or $a = b$
- $a < b$ and $b < c \quad \Rightarrow \quad a < c$
- $a < b \quad \Rightarrow a + c < b + c \ \forall \ c \in R$
- $a < b \quad \Rightarrow -a > -b$ *i.e.* inequality sign reverses if both sides are multiplied by a negative number.
- $a < b$ and $c < d \quad \Rightarrow \quad a + c < b + d \quad \Rightarrow \quad a - d < b - c$
- $a < b \quad \Rightarrow \quad ma < mb$ if $m > 0$ and $ma > mb$ if $m < 0$
- $0 < a < b \quad \Rightarrow \quad a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$
- $\left(a + \frac{1}{a}\right) \ge 2 \ \forall \ a > 0$ and equality holds for $a = 1$.
- $\left(a + \frac{1}{a}\right) \le -2 \ \forall \ a < 0$ and equality holds for $a = -1$.

3.2 Interval

An infinite continuous subset of R is called an interval.

3.3 Closed interval

The set of real number between a and b (where $a < b$) also including the end points a and b called a closed interval and is denote by $[a, b]$. Thus $[a,b] = \{x \in R : a \le x \le b\}$

3.4 Open interval

The set of real number between a and b (where $a < b$) also excluding the end points a and b is called on open interval and denoted by (a, b). Thus $(a, b) = \{x : a < x < b\}$.

The set of real number x such that $a < x \le b$ is called a semi - open or semi - closed interval and is denoted by $(a,b]$. Similarly we have $[a,b) = \{x : a \le x < b\}$.

The number $b - a$ is called length of the interval (a, b) or of $[a, b]$. The smallest and greatest elements in an open interval (a, b) do not exist.

3.5 Infinite intervals

The set of all real numbers greater than a certain real number, say 'a' is an-infinite interval and is denoted by (a, ∞). Thus $(a, \infty) = \{x : x > a\}$, and $[a, \infty) = \{x : x \geq a]$

Similarly, we define $(-\infty, a) = \{x : x < a\}$ and $[-\infty, a) = \{x : x \leq a]$

The infinite intervals have infinite length. In writing them, we use the symbol ∞ as notation only. The sides of $-\infty$ and ∞ in writing infinite intervals must be kept open because a real number is never equal to $-\infty$ or ∞.

Note : In any interval the smaller value is to be written first. For example suppose we want to write set of real numbers between 1 and 3 then we will denote it by (1, 3) and not by (3, 1).

QUADRATIC POLYNOMIAL — Section - 4

4.1 Introduction

The quadratic polynomial in x is $ax^2 + bx + c$; where a, b, c are real numbers and $a \neq 0$.

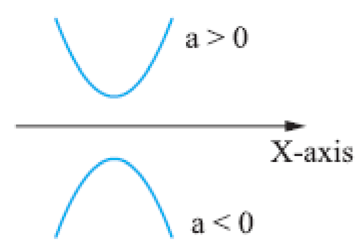
$ax^2 + bx + c$ is also known as the quadratic expression in x. Evidently $ax^2 + bx + c$ is a function in x. For different real values of x, we get different real values of $ax^2 + bx + c$.

So, in general quadratic expression is represented as : $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$

4.2 Graph of a Quadratic Polynomial

$$f(x) = ax^2 + bx + c \qquad (a \neq 0)$$

To draw the graph of $f(x)$, proceed according to following steps :



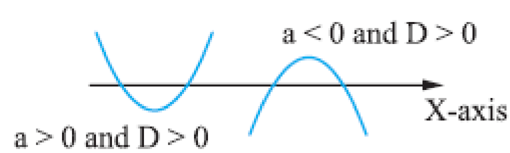
1. The shape of the curve $y = f(x)$ is **parabolic**.

2. For $a > 0$, the parabola opens upwards.

For $a < 0$, the parabola opens downwards.

3. Intersection with axes :

(i) with X-axis



➤ **For $D > 0$**

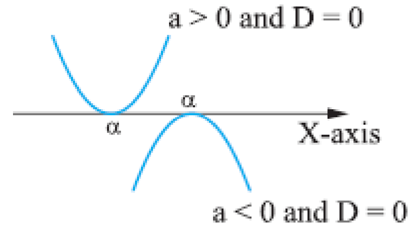
Parabola cuts X-axis in two points.

The points of intersection are $\alpha, \beta = \dfrac{-b \pm \sqrt{D}}{2a}$.

➤ **For $D = 0$**

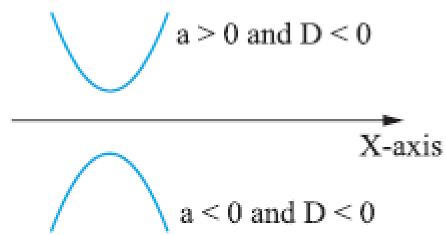
Parabola touches X-axis in one point.

The points of intersection is $\alpha = \frac{-b}{2a}$.



➤ **For $D < 0$**

Parabola does not cut X-axis at all *i.e.* no point of intersection with X-axis.



(ii) with Y-axis

The points of intersection with Y-axis is (0, c) {put $x = 0$ in the quadratic polynomial}

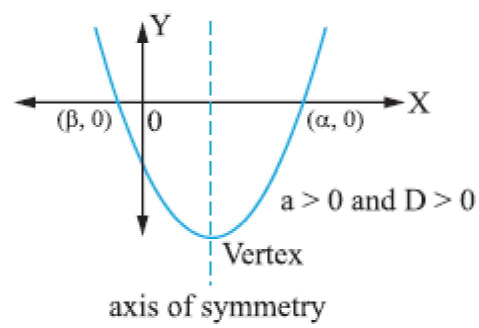
4. Maximum and Minimum value of $f(x)$:

V is called as vertex of parabola.

The coordinates of $V \equiv \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

The line passing through vertex and parallel to the Y-axis is called as axis of symmetry.

The parabolic graph of a quadratic polynomial is sym metrical about axis of symmetry.

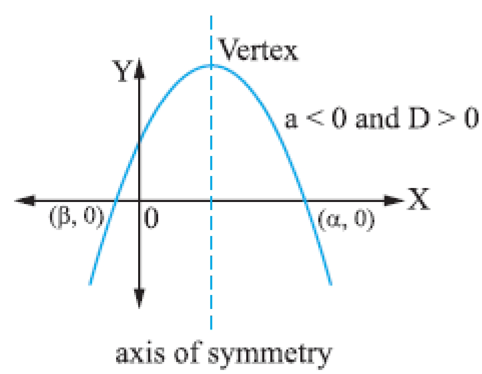


➤ $f(x)$ has minimum value at vertex if $a > 0$ and

$$f_{min} = -\frac{D}{4a} \text{ at } x = -\frac{b}{2a}.$$

➤ $f(x)$ has maximum value at vertex if $a < 0$ and

$$f_{max} = -\frac{D}{4a} \text{ at } x = -\frac{b}{2a}.$$

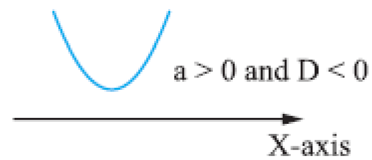


Note : *Graph of any quadratic polynomial can be plotted by following steps (1) to (4)*

4.3 Sign of a Quadratic Polynomial

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in R$ and $a \neq 0$.

1. $a > 0, D < 0$:



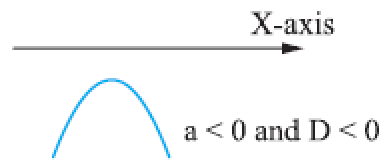
As $a > 0$, parabola opens upward.

As $D < 0$, parabola does not intersect X-axis.

So $f(x) > 0$ for all $x \in R$.

i.e., $f(x)$ is positive for all value of x.

2. $a < 0, D < 0$:



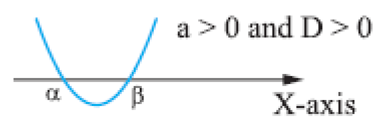
As $a < 0$, parabola opens downward.

As $D < 0$, parabola does not intersect X-axis.

So $f(x) < 0$ for all $x \in R$.

i.e. $f(x)$ is negative for all value of x.

3. $a > 0, D > 0$:



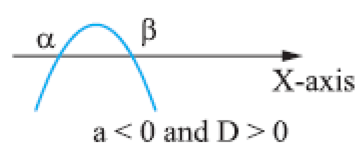
As $a > 0$, parabola opens upward.

As $D > 0$, parabola intersect X-axis in two points say α, β $(\alpha < \beta)$.

So $f(x) \geq 0$ for all $x \in (-\infty, \alpha] \cup [\beta, \infty)$ and $f(x) < 0$ for all $x \in (\alpha, \beta)$.

i.e. $f(x)$ is positive for some values of x and negative for other values for x.

4. $a < 0, D > 0$:



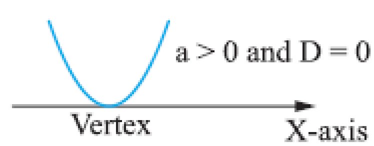
As $a < 0$, parabola opens downward.

As $D > 0$, parabola intersect X-axis in two points say α, β $(\alpha < \beta)$.

So $f(x) \leq 0$ for all $x \in (-\infty, \alpha] \cup [\beta, \infty)$ and $f(x) > 0$ for all $x \in (\alpha, \beta)$.

i.e. $f(x)$ is positive for some values of x and negative for other values of x.

5. $a > 0, D = 0$:



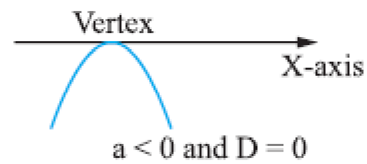
As $a > 0$, parabola opens upward.

As $D = 0$, parabola touches X-axis.

So $f(x) \geq 0$ for all $x \in R$.

i.e. $f(x)$ is positive for all values of x except at vertex where $f(x) = 0$.

6. ***a* < 0, *D* = 0 :**



As $a < 0$, parabola opens downward.

As $D = 0$, parabola touches X-axis.

So $f(x) \leq 0$ for all $x \in R$.

i.e. $f(x)$ is negative for all values of x except at vertex where $f(x) = 0$.

4.4 Quadratic Inequation :

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in R$ and $a \neq 0$. To solve the inequations of type :

$\{ f(x) \leq 0 \quad ; \quad f(x) < 0 \quad ; \quad f(x) \geq 0 \quad ; \quad f(x) > 0 \}$, we use the following precedure.

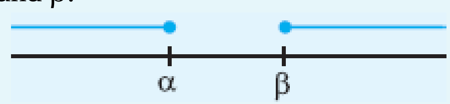
(a) $D > 0$

- Make the coefficient of x^2 positive
- Factorise the expression and represent the left hand side of inequality in the form $(x - \alpha)(x - \beta)$.
- If $(x - \alpha)(x - \beta) > 0$, then x lies outside α and β.

  $\Rightarrow \quad x \in (-\infty, \alpha) \cup (\beta, \infty)$

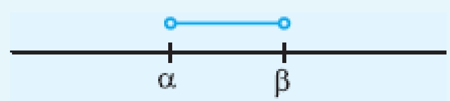
- If $(x - \alpha)(x - \beta) \geq 0$, then x lies on and outside α and β.

  $\Rightarrow \quad x \in (-\infty, \alpha] \cup [\beta, \infty)$



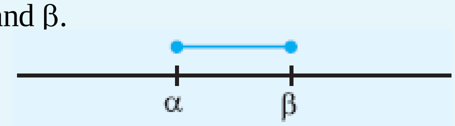
- If $(x - \alpha)(x - \beta) < 0$, then x lies inside α and β.

  $\Rightarrow \quad x \in (\alpha, \beta)$



- If $(x - \alpha)(x - \beta) \leq 0$, then x lies on and inside α and β.

  $\Rightarrow \quad x \in [\alpha, \beta]$



(b) $D < 0$ and $a > 0 : f(x) > 0$ for all $x \in R$.

(c) $D < 0$ and $a < 0 : f(x) < 0$ for all $x \in R$.

(d) $D = 0$ and $a > 0 : f(x) \geq 0$ for all $x \in R$.

(e) $D = 0$ and $a < 0 : f(x) \leq 0$ for all $x \in R$.

(f) $D \leq 0$, $a > 0 : f(x) \geq 0$ for all $x \in R$.

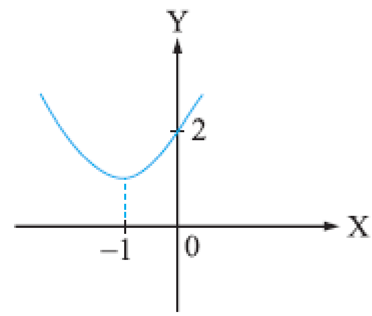
(g) $D \leq 0$, $a < 0 : f(x) \leq 0$ for all $x \in$ R

Illustration the Concept :

(a) *If* $f(x) = x^2 + 2x + 2$, *then solve the following inequalities :*

(i) $f(x) \geq 0$ **(ii)** $f(x) \leq 0$ **(iii)** $f(x) > 0$ **(iv)** $f(x) < 0$

$f(x) = x^2 + 2x + 2$



Let us find D

$\Rightarrow$ $D = b^2 - 4ac = (2)^2 - 4(1)(2) = -4 < 0 \Rightarrow D < 0$

$\Rightarrow$ roots of the corresponding equation ($f(x) = 0$) are non real.

$f(x)$ cannot be factorized into linear factor.

Also observe that a = coefficient of $x^2 = 1 > 0$

As $a > 0$ and $D < 0$, we get : $f(x) > 0 \ \forall \ x \in R$ **[Using result 3.5 (b)]**

(i) $f(x) \geq 0$ is true $\forall \ x \in R$ (ii) $f(x) \leq 0$ is true for no value of x *i.e.*, $x \in \{\ \}$.

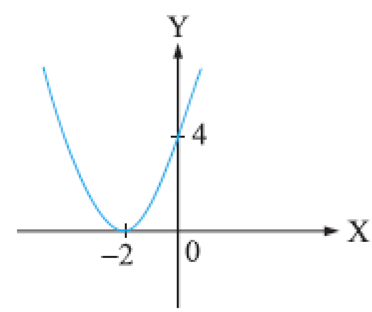
(iii) $f(x) > 0$ is true $\forall \ x \in R$. (iv) $f(x) < 0$ is true for no value of x *i.e.*, $x \in \{\ \}$.

[Using results given in section 3.5, (*b*) and (*c*)]

(b) *If* $f(x) = x^2 + 4x + 4$, *then solve the following inequalities :*

(i) $f(x) \geq 0$ **(ii)** $f(x) \leq 0$ **(iii)** $f(x) > 0$ **(iv)** $f(x) < 0$

$f(x) = x^2 + 4x + 4 = (x + 2)^2$



Let us find D

$\Rightarrow$ $D = b^2 - 4ac = (4)^2 - 4(1)(4) = 0 \Rightarrow D = 0$

$\Rightarrow$ roots of the corresponding equation ($f(x) = 0$) are real and equal.

Also observe that a = coefficient of $x^2 = 1 > 0$

As $D = 0$ and $a > 0$, we get :

$\Rightarrow$ $f(x) \geq 0 \ \forall \ x \in R$ **[Using 3.5 (d)]**

(i) $f(x) \geq 0$ is true $\forall \ x \in R$. (ii) $f(x) \leq 0$ is true $\forall$ $x \in \{-2\}$

(iii) $f(x) > 0$ is true $\forall \ x \in R - \{-2\}$ (iv) $f(x) < 0$ is true for no value of x *i.e.*, $x \in \{\ \}$.

[Using results given in section3.5]

Illustration - 7 *Solve the following quadratic inequality* : $x^2 - 2x - 3 < 0$.

(A) $x \in [-1,3]$ **(B)** $x \in (-1, 3)$ **(C)** $x \in [3,4]$ **(D)** $(-\infty,-1) \cup (3,\infty)$

SOLUTION : (B)

$x^2 - 2x - 3 < 0$

Let us find $D \Rightarrow D = b^2 - 4ac = (-2)^2 - 4(1)(-3) = 16 > 0 \Rightarrow D > 0$

Now factorize LHS using 'Spliting the middle term' method *i.e.*,

$x^2 - 3x + x - 3 < 0 \quad \Rightarrow \quad (x-3)(x+1) < 0$

$\Rightarrow \quad x \in (-1, 3)$ **[Using result mentioned in section 3.5 (*a*)]**

Illustration - 8 *Solve the following quadratic inequality :* $x^2 + x - 1 \geq 0$

(A) $x \in R$

(B) $x \in \left[\frac{-1-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right]$

(C) $x \in \left(-\infty, \frac{-1-\sqrt{5}}{2}\right] \cup \left[\frac{\sqrt{5}-1}{2}, \infty\right)$

(D) $x \in \phi$

SOLUTION : (C)

$x^2 + x - 1 \geq 0$ ...... **(i)**

Let us find D

$\Rightarrow \quad D = b^2 - 4ac = 1^2 - 4(1)(-1) = 5 > 0 \Rightarrow D > 0$.

LHS cannot be factorized using spliting the middle term method. We will find roots of the corresponding equation (say α and β) then use result $ax^2 + bx + c = a(x-\alpha)(x-\beta)$, where α and β are roots of $ax^2 + bx + c = 0$. ...... **(ii)**

Consider $x^2 + x - 1 = 0$

Using $x = \dfrac{-b \pm \sqrt{D}}{2a}$ formula to find roots, we get $x = \dfrac{-1 \pm \sqrt{5}}{2}$

Using (i), we get : $x^2 + x - 1 = \left[x - \left(\dfrac{-1+\sqrt{5}}{2}\right)\right]\left[x - \left(\dfrac{-1-\sqrt{5}}{2}\right)\right]$ ...... **(iii)**

Combining (i) and (iii), we get : $x^2 + x - 1 = \left[x - \left(\dfrac{-1+\sqrt{5}}{2}\right)\right]\left[x - \left(\dfrac{-1-\sqrt{5}}{2}\right)\right] \geq 0$

$x \in \left(-\infty, \dfrac{-1-\sqrt{5}}{2}\right] \cup \left[\dfrac{\sqrt{5}-1}{2}, \infty\right)$ **[Using result given in section 3.5 (*a*)]**

4.5 Maximum and Minimum values of a Quadratic Polynomial

Let $f(x) = ax^2 + bx + c,\ a \neq 0$.

Case : I ($a > 0$) When $a > 0$, parabola opens upward.

From graph, vertex (V) is the lowest point on the graph.

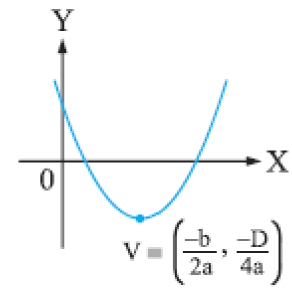
$\Rightarrow y = f(x)$ possesses minimum value at $x = \frac{-b}{2a}$.

$$\Rightarrow y_{min} = f(x)_{min} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}$$

As you can observe on graph,

Maximum value of $f(x)$ is approaching to infinitely large value,

i.e., $y_{max} = f(x)_{max} = \infty$ (not defined).



Case : II ($a < 0$)

When $a < 0$, parabola opens downward.

From graph, vertex (V) is the highest point on the graph.

$y = f(x)$ possesses maximum value at $x = \frac{-b}{2a}$

$$\Rightarrow \quad y_{max} = f(x)_{max} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}$$

As you can observe from graph, minimum value of $f(x)$ is approaching to infinitely small value.

i.e., $y_{min} = f(x)_{min} = -\infty$ (not defined).

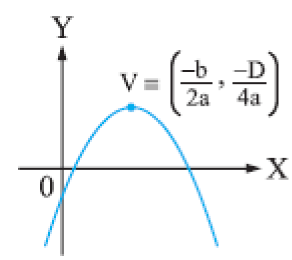


Illustration the Concept :

Find f (max) or f (min) in the following polynomials over $x \in R$.

(i) $f(x) = 4x^2 - 12x + 15$ **(ii)** $f(x) = -3x^2 + 5x - 4$

(i) $f(x) = 4x^2 - 12x + 15$

As $a = 4 > 0$

$f(x)$ has minimum value at vertex.

$D = (12)^2 - 4 \times 4 \times 15 = 144 - 240 = -96$

$$f_{min} = \frac{-D}{4a} \quad \text{at} \quad x = \frac{-b}{2a}$$

(ii) $f(x) = -3x^2 + 5x - 4$

As $a = -3 < 0$

$f(x)$ has maximum value at vertex.

$D = (5)^2 - 4(-3)(-4) = 25 - 48 = -23$

$$f_{max} = \frac{-D}{4a} \text{ at } x = -\frac{b}{2a}$$

$$\Rightarrow \quad f_{min} = \frac{-(-96)}{4\times 4} = \frac{96}{16} = 6 \text{ at } x = -\frac{-12}{2\times 4} = \frac{3}{2}$$

$$\therefore \quad f_{min} = 6 \quad \text{at} \quad x = \frac{3}{2}$$

$$f_{max} = \infty$$

$$f_{max} = -\frac{(-23)}{4(-3)} = -\frac{23}{12} \text{ at } x = \frac{-(5)}{2(-3)} = \frac{5}{6}$$

$$\therefore \quad f_{max} = \frac{-23}{12} \quad \text{at} \quad x = \frac{5}{6}$$

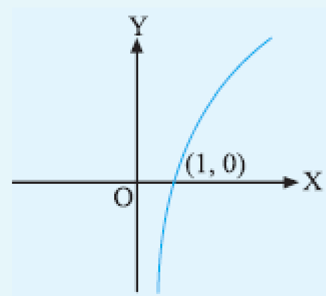
$$f_{min} = -\infty$$

⋆ 4.6 Introduction to Logarithmic Function :

If $x = a^y$ for $x > 0,\ a > 0$ and $a \neq 1$, then logarithm for x with respect to the base a is defined to be y. In symbols $\log_a x = y$.

Note : If any of the condition viz. (i) $x > 0$ (ii) $a > 0$ (iii) $a \neq 1$ is not fulfilled, logarithm is invalid.

Graph of logarithmic Function

$y = \log_a x, a > 1$ or $x = a^y;\ a > 1$	
(i) When $0 < x < 1$ $x = a^y$ We have to choose those values of y for which $0 < a^y < 1$ Since $a > 1,\ y < 0 \quad \Rightarrow \quad y \in (-\infty, 0)$. **(ii)** When $x = 1$, $x = a^y$ We have to choose those values of y for which x becomes 1 $\Rightarrow$ $y = 0$. Since **(iii)** When $x > 1$, $x = a^y$ We have to choose those values of y for which $x > 1$. Since $a > 1,\ 0 < y < \infty$.	Graph of $\log_a x$, $a > 1$
$y = \log_a x,\ 0 < a < 1$ or $x = a^y,\ 0 < a < 1$	
(iv) When $0 < x < 1$, We have to choose those values of y for which $0 < a^y < 1$	

Since $0 < a < 1,\ y > 0.$

(v) When $x = 1$,

$x = a^y$

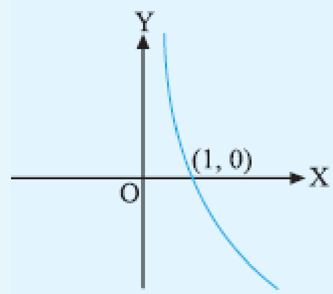
We have to choose those values of y for which x becomes 1

$\Rightarrow \quad y = 0$

(vi) When $x > 1$,

We have to choose those values of y for which $a^y > 1.$

Since $a < 1,\ y < 0$



Graph of $log_a x$, $0 < a < 1$

4.7 General Properties to logarithmic fuction :

(i) $log_a(xy) = log_a x + log_a y.$

(ii) $log_a\left(\frac{x}{y}\right) = log_a x - log_a y.$

(iii) $log_a x^y = y\, log_a x$

(iv) $log_{a^n} x = \frac{1}{n} log_a x$ and $log_{a^{2n}} x = \frac{1}{2n} log_{|a|} x$

(v) $log_a 1 = 0$

(vi) $log_a a = 1$

(vii) $log_y x = \frac{1}{log_x y}$, *where* $x, y > 0,\ x \neq 1,\ y \neq 1$

(viii) $log_y x = \frac{log_z x}{log_z y}$, *where* $x, y, z > 0;\ x \neq 1,\ y \neq 1$

(ix) $a^{log_a x} = x$

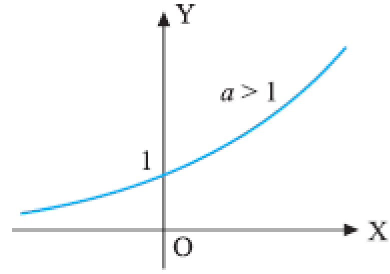
(x) $x^{log_a y} = y^{log_a x}$

(xi) If $a > 1$, and $m > n \Leftrightarrow log_a m > log_a n$

(xii) If $0 < a < 1$, then $m > n \Leftrightarrow log_a m > log_a n$.

Note : Whenever the base of a logarithmic term is not written, its base is assumed to be 10. Logarithm of x to the base e is usually written in $ln\, x$.

(b) Exponential Function

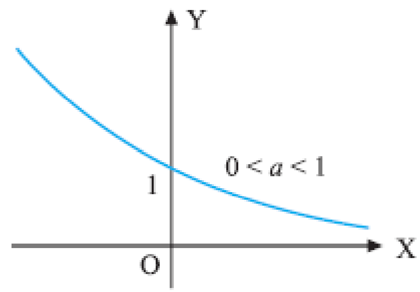


$y = a^x$ where $a > 1$ or $0 < a < 1$ is an exponential fuciton of x.

This function is the inverse of lagarithmic function *i.e.*

it can be obtainned by interchanging x and y in $y = log_a x$.

As observed from the graph, if $a > 1$, then y increases as x increases.
If $0 < a < 1$, then y decreases as x increases.



Continuity :

The graph of $f(x) = a^x$ is continuous (*i.e.* no break in the curve) everywhere.

Domain and Range :

The domain of the function $f(x)$ is $x \in R$ and range in $y > 0$.

Illustration the Concepts :

Prove that : $\left(\sqrt[3]{9}\right)^{\frac{1}{5 log_5 3}} = 25^{\frac{1}{15}}$

L.H.S

$$= \left[\sqrt[3]{9}\right]^{\frac{1}{5 log_5 3}}$$

$$= \left[9^{1/3}\right]^{\frac{1}{5} log_3 5} = \left[\left(3^2\right)^{1/3}\right]^{\frac{1}{5} log_3 5}$$

$$= \left[3^{2/3}\right]^{\frac{1}{5} log_3 5}$$

$$= (3)^{\frac{2}{15} log_3 5}$$

$$= 3^{log_3 5^{\frac{2}{15}}}$$

$$= 5^{\frac{2}{15}} = \left(5^2\right)^{\frac{1}{15}} = 25^{\frac{1}{15}} = \text{R.H.S}$$

★

Illustration - 9 *If $a > 0$, $a \neq 1$, then the equation $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0$ has*

(A) *exactly one real root* **(B)** *two real roots*

(C) *no real roots* **(D)** *infinite number of real roots*

SOLUTION : (B)

The equation can be written as

$$\frac{2\log a}{\log x} + \frac{\log a}{\log(ax)} + \frac{3\log a}{\log(a^2x)} = 0 \quad \text{.......(i)} \quad \left[\text{Using } \log_a b = \frac{\log b}{\log a}\right]$$

As $a > 0$ and $a \neq 1$, $\log a \neq 0$, (i) can be written as

$$\frac{2}{y} + \frac{1}{b+y} + \frac{3}{2b+y} = 0 \qquad (\text{where } b = \log a \text{ and } y = \log x)$$

$\Rightarrow$ $2(b+y)(2b+y) + y(2b+y) + 3y(b+y) = 0$ $\Rightarrow$ $4b^2 + 11by + 6y^2 = 0$

Above equation is a quadratic in y. On solving, we get :

$$\Rightarrow \quad y = \frac{-11b \pm \sqrt{121b^2 - 96b^2}}{12} = -\frac{4b}{3}, -\frac{b}{2}$$

As $y = \log x$ and $b = \log a$

$\Rightarrow$ $\log x = -\frac{4}{3}\log a$ or $-\frac{1}{2}\log a \Rightarrow x = a^{-4/3}, a^{-1/2}$ **[Using : $\log_a b = c \Rightarrow b = a^c$]**

$\Rightarrow$ Two real roots can exist.

Illustration - 10 *If the graph of the polynomial : $y = x^2 + kx - x + 9$ is above X-axis, then the possible values of k are :*

(A) $k \in R$ **(B)** $k \in (-5, 7)$ **(C)** $k \in \phi$ **(D)** $k \in (7, 9)$

SOLUTION : (B)

$y = ax^2 + bx + c$ has its graph above x-axis if :

$a > 0$ and $D < 0$

Given $y = x^2 + (k-1)x + 9$

Coefficient of $x^2 = a = 1$ *i.e.* positive.

$D = (k-1)^2 - 36 < 0$, for graph to lie above x-axis

$(k-7)(k+5) < 0$

$\Rightarrow$ $-5 < k < 7$ {For graph to lie above x-axis}

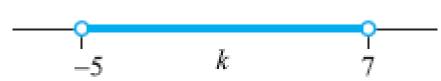


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Illustration - 11 *If* $\log_2(ax^2 + x + a) \geq 1 \ \forall \ x \in R$, *then exhaustive set of values of 'a' is :*

(A) $\left(0, 1+\frac{\sqrt{5}}{2}\right)$ **(B)** $\left(1-\frac{\sqrt{5}}{2}, 1+\frac{\sqrt{5}}{2}\right)$ **(C)** $\left(0, 1-\frac{\sqrt{5}}{2}\right)$ **(D)** $\left[1+\frac{\sqrt{5}}{2}, \infty\right)$

SOLUTION : (D)

$\log_2(ax^2 + x + a) \geq 1 \ \forall \ x \in R$ $\Rightarrow$ $a > 0$ and $4a^2 - 8a - 1 \geq 0$

$\Rightarrow$ $ax^2 + x + a \geq 2 \ \forall \ x \in R$ $\Rightarrow$ $a > 0$ and $a \in \left(-\infty, 1-\frac{\sqrt{5}}{2}\right] \cup \left[1+\frac{\sqrt{5}}{2}, \infty\right)$

$\Rightarrow$ $ax^2 + x + (a - 2) \geq 0 \ \forall \ x \in R$

$\Rightarrow$ coefficient of $x^2 > 0$ and $D \leq 0$ $\Rightarrow$ $a \in \left[1+\frac{\sqrt{5}}{2}, \infty\right)$.

$\Rightarrow$ $a > 0$ and $1 - 4a(a-2) \leq 0$

Illustration - 12 *The least Integral value of 'k' for which* $(k - 2)x^2 + 8x + k + 4 > 0$ *for all* $x \in R$, *is :*

(A) 5 **(B)** 2 **(C)** 3 **(D)** *None of these*

SOLUTION : (A)

Let $f(x) = (k - 2)x^2 + 8x + k + 4$

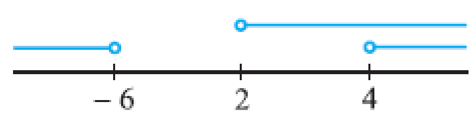
$f(x) > 0 \Rightarrow a > 0$ and $D < 0$ **[By using result 3.5 (*b*)]**

$k - 2 > 0$ and $64 - 4(k - 2)(k + 4) < 0$

$k > 2$ and $16 - (k^2 + 2k - 8) < 0$

$k > 2$ and $k^2 + 2k - 24 > 0$

$k > 2$ and $(k < -6$ or $k > 4)$



As it can be observed that k can take value greater than 4

$\Rightarrow$ $k > 4$.

$\therefore$ least integral value of $k = 5$.

Illustration - 13 *If* $a < b$, *then solution of* $x^2 + (a + b)x + ab < 0$ *is given by*

(A) $x < b$ or $x < a$ **(B)** $a < x < b$ **(C)** $x < a$ or $x > b$ **(D)** $-b < x < -a$

SOLUTION : (D)

$x^2 + (a + b)x + ab < 0$

$\Rightarrow$ $(x + a)(x + b) < 0$ $\Rightarrow$ $-b < x < -a$

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Illustration - 14 *If a, b, c $\in R$ and $(a + b + c)\, c < 0$, then the quadratic equation $p(x) = ax^2 + bx + c = 0$ has:*

(A) *A negative root* **(B)** *Two real root* **(C)** *Two imaginary root* **(D)** *None of these*

SOLUTION : (B)

$p(x) = ax^2 + bx + c = 0$

Now , $a + b + c = p(1)$ and $c = p(0)$

According to question

$(a + b + c)\, c < 0 \quad \Rightarrow \quad p(1)\, p(0) < 0$

$\Rightarrow \quad p(x) = 0$ has atleast one root in (0, 1).

$p(x) = 0$ has two real roots because if coefficients and any one root are real, then other root would also be real.

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Illustration - 15 *Let a, b, c be three distinct real numbers such that each of the expression $ax^2 + bx + c$, $bx^2 + cx + a$ and $cx^2 + ax + b$ is positive for all $x \in R$ and let $\alpha = \dfrac{bc + ca + ab}{a^2 + b^2 + c^2}$ then*

(A) $\alpha < 4$ **(B)** $\alpha < 1$ **(C)** $\alpha > 1/4$ **(D)** $\alpha > 1$

SOLUTION : (B) & (C)

According to the given conditions $a > 0,\ b^2 < 4ac$; $b > 0,\ c^2 < 4ab$; $c > 0,\ a^2 < 4bc$

$\therefore \quad a^2 + b^2 + c^2 < 4(bc + ca + ab)$ ...... **(i)**

$$\Rightarrow \quad \frac{1}{4} < \frac{bc + ca + ab}{a^2 + b^2 + c^2} \quad \Rightarrow \quad \frac{1}{4} < \alpha.$$

Also $a^2 + b^2 + c^2 - (bc + ca + ab) = \frac{1}{2}[(b - c)^2 + (c - a)^2 + (a - b)^2] > 0$ ...... **(ii)**

But $\frac{1}{2}\left[(b-c)^2 + (c-a)^2 + (a-b)^2\right] > 0 \Rightarrow a^2 + b^2 + c^2 - (bc + ca + ab) > 0$ **[Using (ii)]**

$$\Rightarrow \quad \frac{bc + ca + ab}{a^2 + b^2 + c^2} < 1 \quad \Rightarrow \quad \alpha < 1.$$

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Illustration - 16 $a, b, c \in R$, $a \neq 0$ and the quadratic equation $ax^2 + bx + c = 0$ has no real roots, then

(A) $a + b + c > 0$ **(B)** $a(a + b + c) > 0$ **(C)** $b(a + b + c) > 0$ **(D)** $c(a + b + c) > 0$

SOLUTON : (B) & (D)

Let $f(x) = ax^2 + bx + c$. It is given that $f(x) = 0$ has no real roots. So, either $f(x) > 0$ for all $x \in R$ or

$f(x) < 0$ for all $x \in R$ *i.e.* $f(x)$ has same sign for all values of x.

$\therefore \quad f(0)\,f(1) > 0 \quad \Rightarrow \quad c(a+b+c) > 0.$

Also, $a\,f(1) > 0 \quad \Rightarrow \quad a(a+b+c) > 0.$

Illustration - 17 *Let* $f(x) = x^2 + 4x + 1$, *then*

(A) $f(x) > 0$ *for all* x **(B)** $f(x) \geq 1$ *when* $x \geq 0$

(C) $f(x) \geq 1$ when $x \leq -4$ **(D)** $f(x) = f(-x)$ for all x

SOLUTION : (B) & (C)

Since $f(x)$ is a quadratic expression having real roots. Therefore $f(x)$ does not have the same sign for all x.

Now, $f(x) \geq 1 \quad \Rightarrow \quad x^2 + 4x + 1 \geq 1 \quad \Rightarrow \quad x^2 + 4x \geq 0$

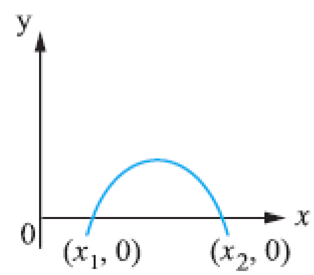
$\Rightarrow \quad x \leq -4 \quad$ or $\quad x \geq 0 \quad \Rightarrow$ (B) and (C) are correct.

$f(-x) = x^2 - 4x + 1 \quad \Rightarrow \quad f(-x) \neq f(x) \quad \Rightarrow$ (D) is wrong

Illustration - 18 *The adjoining figure shows the graph of* $y = ax^2 + bx + c$. *Then*

(A) $a < 0$ **(B)** $b^2 < 4\,ac$

(C) $c > 0$ **(D)** *a and b are of opposite signs.*



SOLUTION : (A) & (D)

As it is clear from the figure that it is a parabola opening downwards i.e. $a < 0$.

$\Rightarrow$ (A) is correct.

$\Rightarrow$ It is $y = ax^2 + bx + c$ *i.e.* degree two polynomial.

Now, if $ax^2 + bx + c = 0 \Rightarrow$ it has two roots x_1 and x_2, as it cuts the axis at two distinct point x_1 and x_2.

Now, from the figure it is also clear that $x_1 + x_2 > 0$. (*i.e.* sum of roots are positive)

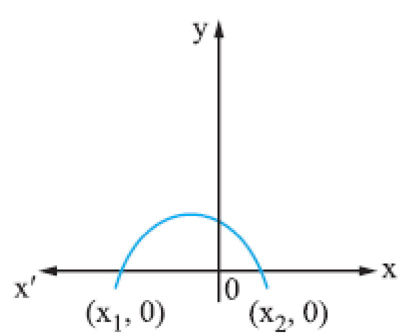
$$\Rightarrow \quad \frac{-b}{a} > 0 \quad \Rightarrow \quad \frac{b}{a} > 0 \quad \Rightarrow \quad a \text{ and } b \text{ are of opposite sings.} \quad \Rightarrow \quad (D) \text{ is correct}$$

As $D > 0$ and $f(0) = c < 0$, both (B) and (C) are wrong.

Illustration - 19 *The diagram shows the graph of* $y = ax^2 + bx + c$. *Then,*

(A) $a > 0$ **(B)** $b < 0$

(C) $c > 0$ **(D)** $b^2 - 4\,ac = 0$



SOLUTION : (B) & (C)

As it is clear from the figure that it is a parabola opening downwards i.e. $a < 0$.

$\Rightarrow$ It is $y = ax^2 + bx + c$ *i.e.* degree two polynomial.

Now, if $ax^2 + bx + c = 0 \quad \Rightarrow \quad$ it has two roots x_1 and x_2 as it cuts the axis at two distinct point x_1 and x_2.

Now from the figure it is also clear that $x_1 + x_2 < 0$. (*i.e.* sum of roots are negative)

$$\Rightarrow \quad \frac{-b}{a} < 0 \Rightarrow \quad \frac{b}{a} > 0 \quad b < 0 \Rightarrow (B)$$

As the graph of $y = f(x)$ cuts the + y-axis at (0,c) where c > 0 $\Rightarrow$ (C) is correct.

RATIONAL FUNCTION & RATIONAL INEQUATION — Section - 5

5.1 Introduction to Rational Functions

Rational function of x is defined as ratio of two polynomial of x, say $P(x)$ and $Q(x)$ where $Q(x) \neq 0$. *i.e.*

If $f(x) = \dfrac{P(x)}{Q(x)}$; $Q(x) \neq 0$,

then $f(x)$ is a rational function of x.

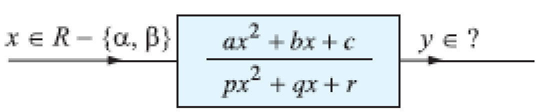
Following are some examples of rational functions of x.

$$f(x) = \frac{x+1}{x^2+x+1} \ ; \ f(x) = \frac{x^2 - x + 2}{x^2 - 5x + 6} \ ; \ x \neq 2, x \neq 3 \ ; \ f(x) = \frac{x^4 + x^3 + x + 1}{(x-1)^2} \ ; \ x \neq 1$$

5.2 Maximum and Minimum values of a Rational Function of *x*

Consider : $f(x) = y = \dfrac{ax^2 + bx + c}{px^2 + qx + r}$ where $x \in R - \{\alpha, \beta\}$,

where $\alpha\beta$ are roots of $px^2 + qx + r = 0$ .....(i)



We will find maximum and minimum values $f(x)$ can take.

Cross Multiply in (i) to get :

$$y(px^2 + qx + r) = ax^2 + bx + c \quad \Rightarrow \quad (a - py)x^2 + (b - qy)x + (c - ry) = 0$$

As x is real, $D \geq 0$

$$\Rightarrow \quad (b - qy)^2 - 4(a - py)(c - ry) \geq 0$$

Above relationship is an inequality in y. On solving the inequality we will get values y can take.

Case - I : $y \in [A, B]$

If y can take values between A and B, then,

Maximum value of $y = y_{max} = B$, Minimum value of $y = y_{min} = A$.

Case - II : $y \in (-\infty, A] \cup [B, \infty)$

If y can take values outside A and B, then

Maximum value of $y = y_{max} = \infty$ *i.e.* not defined.

Minimum value of $y = y_{min} = -\infty$. *i.e.* not defined.

Case - III : $y \in (-\infty, \infty)$ i.e. $y \in R$

If y can take all values, then

Maximum value of $y = y_{max} = \infty$ *i.e.* not defined.

Minimum value of $y = y_{min} = -\infty$ *i.e.* not defined.

★

Illustration - 20

If $f(x) = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$. $x \in R$, then $f(x)$ can take values :

(A) $(5, 9)$ **(B)** $(-\infty, 5] \cup [9, \infty)$ **(C)** $[5, 9]$ **(D)** None of these

SOLUTION : (B)

Let $\dfrac{x^2 + 34x - 71}{x^2 + 2x - 7} = k$

By cross multiply and making a quadratic equation in x, we get :

$\Rightarrow \quad x^2(1 - k) + (34 - 2k)x + 7k - 71 = 0$

As $x \in R$, discriminant ≥ 0

$\Rightarrow \quad (34 - 2k)^2 - 4(1 - k)(7k - 71) \geq 0$

$\Rightarrow \quad (17 - k)^2 - (1 - k)(7k - 71) \geq 0$

$\Rightarrow \quad 8k^2 - 112k + 360 \geq 0$

$\Rightarrow \quad k^2 - 14k + 45 \geq 0$

$\Rightarrow \quad (k - 5)(k - 9) \geq 0$

$\Rightarrow \quad k \in (-\infty, 5] \cup [9, \infty)$ **[Using result 3.5 (*a*)]**

Hence k can never lie between 5 and 9

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Illustration - 21

The values of m for which the expression : $\dfrac{2x^2-5x+3}{4x-m}$ *can take all real values for* $x \in R.-\left\{\dfrac{m}{4}\right\}$

(A) $m \in [4, 6]$ **(B)** $m \in [6, 8]$ **(C)** $m \in [-6,-4]$ **(D)** $m \in [-4,-2]$

SOLUTION : (A)

Let $\dfrac{2x^2-5x+3}{4x-m} = k \quad \Rightarrow \quad 2x^2 - (4k + 5)\, x + 3 + mk = 0$

$\Rightarrow$ As $x \in R$, discriminant ≥ 0

$\Rightarrow \quad (4k + 5)^2 - 8\,(3 + mk) \geq 0 \quad \Rightarrow \quad 16k^2 + (40 - 8\,m)\,k + 1 \geq 0$

A quadratic in k is positive for all values of k if coefficient of k^2 *is positive and discriminant* ≤ 0.

$\Rightarrow \quad (40 - 8\,m)^2 - 4\,(16)\,(1) \leq 0 \quad \Rightarrow \quad (5 - m)^2 - 1 \leq 0$

$\Rightarrow \quad (m - 5 - 1)\,(m - 5 + 1) \leq 0 \quad \Rightarrow \quad (m - 6)\,(m - 4) \leq 0$

$\Rightarrow \quad m \in [4, 6]$ **[By using 3.5 (*a*)]**

So for the given expression to take all real values, *m* should take values : $\boldsymbol{m \in [4, 6]}$.

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Illustration - 22

The values of m so that the inequality : $\left|\dfrac{x^2+mx+1}{x^2+x+1}\right| < 3$ *holds for all* $x \in R$.

(A) $m \in (-1,8)$ **(B)** $m \in (-\infty,-1)\cup(5,\infty)$ **(C)** $m \in (-1,5)$ **(D)** None of these

SOLUTION : (C)

We know that $|\,a\,| < b \quad \Rightarrow \quad -b < a < b$

Hence $\left|\dfrac{x^2+mx+1}{x^2+x+1}\right| < 3. \quad \Rightarrow \quad -3 < \dfrac{x^2+mx+1}{x^2+x+1} < 3$

Case I : $\dfrac{x^2+mx+1}{x^2+x+1} < 3$

$\Rightarrow \quad \dfrac{(x^2+mx+1)-3\,(x^2+x+1)}{x^2+x+1} < 0 \quad \Rightarrow \quad \dfrac{-2x^2+(m-3)x-2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} < 0$

Multiplying both sides by denominator, we get :

$\Rightarrow \quad -2x^2 + (m-3)x - 2 < 0$ ***(because denominator is always positive)***

$\Rightarrow \quad 2x^2 - (m-3)x + 2 > 0$

A quadratic expression in x is always positive if coefficient of $x^2 > 0$ and $D < 0$.

$\Rightarrow \quad (m-3)^2 - 4(2)(2) < 0 \qquad \Rightarrow \quad m^2 - 6m - 7 < 0$

$\Rightarrow \quad (m-7)(m+1) < 0 \qquad \Rightarrow \quad m \in (-1, 7)$ ......(i)

Case II : $-3 < \dfrac{x^2 + mx + 1}{x^2 + x + 1} \qquad \Rightarrow \quad < 0 \; \dfrac{(x^2 + mx + 1) + 3(x^2 + x + 1)}{x^2 + x + 1}$

$\Rightarrow \quad 4x^2 + (m+3)x + 4 > 0$

For this to be true for all $x \in R, D < 0$

$\Rightarrow \quad (m+3)^2 - 4(4)(4) < 0$

$\Rightarrow \quad (m+3-8)(m+3+8) < 0$ **[Using $a^2 - b^2 = (a+b)(a-b)$]**

$\Rightarrow \quad (m-5)(m+11) < 0$

$\Rightarrow \quad m \in (-11, 5)$ ....(ii)

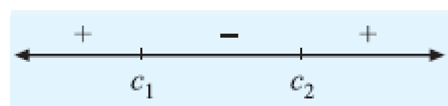
We will combine (i) and (ii) because must be satisfied.

$\Rightarrow$ The common solution is $m \in (-1, 5)$.

5.3 Rational Algebraic Inequalities :

(a) Solving Quadratic Inequality :

A simple and quick method of solving quadratic inequarions is as follows :



- Make the coefficient of x^2 positive if necessary.
- Check for $b^2 - 4ac$. If it is ngative then the solution is either all real x or no real x depending on the inequality sign. If $b^2 - 4ac > 0$, then solve the given quardratic to get the real roots c_1 and c_2 where $c_1 < c_2$.
- If the final sign of the inequation is ' > ' then the soluiton set is $(-\infty, c_1) \cup (c_2, \infty)$ and if the final sign is '>' then the solution set is (c_1, c_2).

(b) Equivalence in Inequality :

The inequations are said to be equivalent if every solution of one is a solution of the other. For instance, the inequations $(x-2)(x-3) > 0$ and $\dfrac{x-2}{x-3} > 0$ are equivalent. The solution set to both inequations is $(-\infty,2)\cup(3,\infty)$.

PROOF

The inequation $\dfrac{x-2}{x-3}>0$ makes sense if $x\neq 3$. Multiplying by $(x-3)^2>0$ on both sides, we get $(x-2)(x-3)>0$ whose soluiton set is easily seen to be $(-\infty,2)\cup(3,\infty)$.

The student must note carefully that the inequations $(x-2)(x-3)\geq 0$ and $\dfrac{x-2}{x-3}\geq 0$ are NOT equivalent. The former has solution set $(-\infty,\,2]\cup[3,\infty)$, while the latter has solution set $(-\infty,\,2]\cup[3,\infty)$.

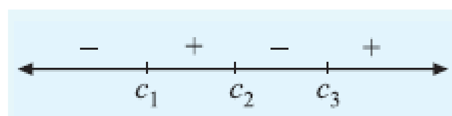
Similarly the inequations

$$(x-1)(x-2)(x-3)>0\ ;\ \frac{(x-1)(x-2)}{(x-3)}>0 \text{ and } \frac{(x-1)}{(x-2)(x-3)}>0$$

are all equivalent, (See the section for solutions to the cubic inequations)

(c) Cubic inequations :

Suppose the inequation can be written as



$(x-c_1)(x-c_2)(x-c_3)>0$, where $c_1<c_2<c_3$.

The solution set $=(c_1,c_2)\cup(c_3,\infty)$

If the sign in the above inequation is ' < ', then the solution set is $(-\infty,\,c_1)\cup(c_2,c_3)$

(d) Generalization :

The solution set to the inequation

$(x-c_1)(x-c_2).....(x-c_n)>0$, where $c_1<c_2<......<c_n$ is

The solution set is : $(c_1,\,c_2)\cup(c_3,c_4)\cup......\cup(c_n,\,\infty)$, if n is odd

and $(-\infty,\,c_1)\cup(c_2,c_3)\cup......\cup(c_n,\infty)$, if n is even.

Note : Dealing with inequations in an immatured manner leads to serious errors. The consequences are generally

(1) Allowing fake solutions. **(2)** Discarding correct ones.

(e) Rational Algebraic Inequalities

Consider the following types of rational algebraic inequalities.

$$\frac{P(x)}{Q(x)} > 0, \quad \frac{P(x)}{Q(x)} < 0, \quad \frac{P(x)}{Q(x)} \geq 0, \quad \frac{P(x)}{Q(x)} \leq 0$$

where $P(x)$ and $Q(x)$ are polynomials in x.

- These inequalities can be solved by the *method of intervals* also known as *sign method* or *wavy curve method*.

How to solve Rational Algebraic Inequality :

(a) Factorise $P(x)$ and $Q(x)$ into linear factors.

(b) Make coefficient of x positive in all factors.

(c) Equate all the factors to zero and find corresponding values of x. These values are known as critical points.

(d) Plot the critical points on a number line. n critical points will divide the number line $(n + 1)$ regions.

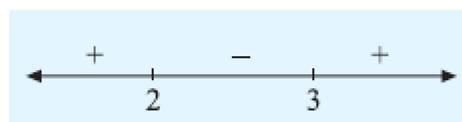
(e) In right most region, the expression bears positive sign and in other regions the expression bears alternate positive and negative signs.

Illustration the Concepts :

(i) Solve $x^2 - 5x + 6 > 0$.

It is easy to see that $x^2 - 5x + 6 = (x-2)(x-3)$.

Thus, the critical points are 2 and 3 and since the sign is '$>$', the solution set is $(-\infty, 2) \cup (3, \infty)$

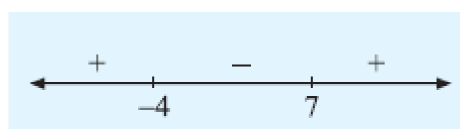


(ii) Solve $28 + 3x - x^2 > 0$

Multiplying with –1, we get $x^2 - 3x - 28 < 0$

$\Rightarrow \quad (x+4)(x-7) < 0$

$\Rightarrow \quad x \in (-4, 7)$



(iii) Solve $5 - 2x - 3x^2 \leq 0$

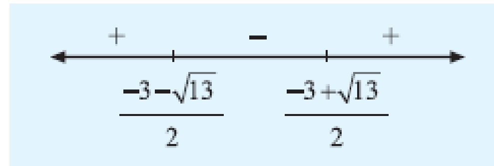
Multiplying with –1, we get $3x^2 + 2x - 5 \geq 0$.

$\Rightarrow \quad (x-1)(3x+5) \geq 0$

$\Rightarrow \quad x \in \left(-\infty, \frac{-5}{3}\right] \cup [1, \infty)$

(iv) Solve $x^2 + 3x - 1 < 0$

Now the factors of $x^2 + 3x - 1$ are not possible by inspection although they are real



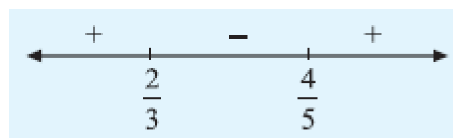
Solving , $x^2 + 3x - 1 = 0$, we get $\frac{-3 \pm \sqrt{13}}{2}$ as critical points. Thus, the inequation can be written as

$$\left[x - \frac{-3-\sqrt{13}}{2}\right]\left[x - \frac{-3+\sqrt{13}}{2}\right] < 0$$

Hence the solution set $\left(\frac{-3-\sqrt{13}}{2}, \frac{-3+\sqrt{13}}{2}\right)$

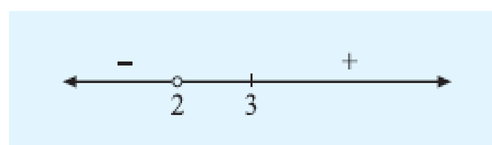
(v) Solve $\frac{3x-2}{5x-4} < 0$

The critical point are 2/3 and 4/5 and therefore the solution is $\left(\frac{2}{3}, \frac{4}{5}\right)$



(vi) Solve $(x-3)(x-2)^2 > 0$

Since $(x-2)^2 > 0$ for all x except at $x = 2$ for which $(x-2)^2 = 0$.

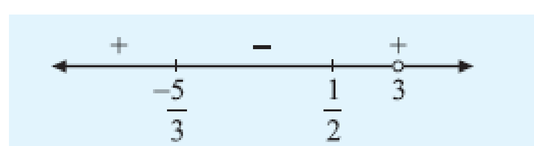


The given inequation is equivalent to $x - 3 > 0$ and $x \neq 2$.

Now $x - 3 > 0$ is obviously satisfied in $(3, \infty)$ which does not include 2. Thus the required solution is $(3, \infty)$

(vii) Solve $(x-3)^2(2x-1)(3x+5) > 0$

Since $(x-3)^2 > 0$ for all x except at $x = 3$ hence the inequation can be written as $(2x-1)(3x+5) > 0$ if $x \neq 3$.



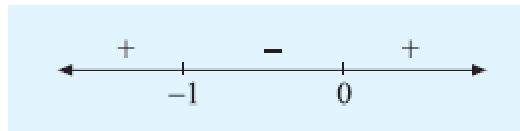
$$\Rightarrow \quad x \in \left(-\infty, \frac{-5}{3}\right) \cup \left(\frac{1}{2}, \infty\right), \text{ where } x \neq 3,$$

Hence, the required solution is :

$$\left(-\infty, \frac{5}{3}\right) \cup \left(\frac{1}{2}, 3\right) \cup (3, \infty)$$

(viii) Solve $1/x > -1$

$\frac{1}{x} + 1 > 0$; $\frac{x+1}{x} > 0$

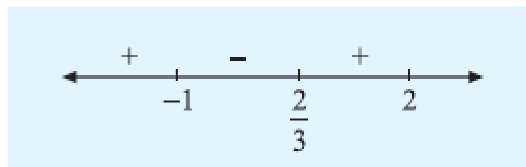


The critical points are –1 and 0

Hence, the required solution $= (-\infty, -1) \cup (0, \infty)$

(ix) Solve $(x-2)(3x-2)(x+1) < 0$

The critical pointsin the ascending order are –1, 2/3 and 2 using our algorithm for cubic inequation.



Hence, the required solution is $(-\infty, -1) \cup \left(\frac{2}{3}, 2\right)$

Illustration - 23

Solve for x : $\frac{8x^2 + 16x - 51}{(2x-3)(x+4)} > 3$

SOLUTION :

$$\frac{8x^2 + 16x - 51}{(2x-3)(x+4)} > 3$$

$$\Rightarrow \quad \frac{8x^2 + 16x - 51 - 3(2x-3)(x+4)}{(2x-3)(x+4)} > 0 \quad \Rightarrow \quad \frac{2x^2 + x - 15}{(2x-3)(x+4)} > 0$$

$$\Rightarrow \quad \frac{(2x-5)(x+3)}{(2x-3)(x+4)} > 0$$

Critical points are : $x = -4, -3, 3/2, 5/2$

The solution from the number line is :

$$x \in (-\infty, -4) \cup \left(-3, \frac{3}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$$

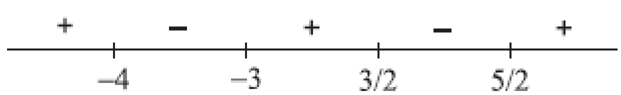


Illustration - 24

Solve for x : $\dfrac{4}{1+x}+\dfrac{2}{1-x}<1.$

SOLUTION :

$$\frac{4}{1+x}+\frac{2}{1-x}<1$$

On solving the above inequality, we get : $\dfrac{x^2-2x+5}{(1+x)(1-x)}<0$

$\Rightarrow \quad \dfrac{1}{(1+x)(1-x)}<0$ **[As $x^2 - 2x + 5 > 0$ for all $x \in R$ (*because $D < 0$, $a > 0$*)]**

$\Rightarrow \quad \dfrac{1}{(1+x)(x-1)}>0 \quad \Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)$

Illustration - 25

Let $y=\sqrt{\dfrac{2}{x^2-x+1}-\dfrac{1}{x+1}-\dfrac{(2x+1)}{x^3+1}}$ *; find all the real values of x for which y takes real values (i.e. find domain). are :*

(A) $x \in (-1, 0) \cup (1, 2)$ **(B)** $x \in (-\infty, -1) \cup [0, 1]$

(C) $m \in (0, 1) \cup (2, 3)$ **(D)** $m \in (-1, 1)$

SOLUTION : (B)

For y to take real values ; $\dfrac{2}{x^2-x+1}-\dfrac{1}{x+1}-\dfrac{(2x+1)}{x^3+1}\geq 0.$

$\Rightarrow \quad \dfrac{2(x+1)-(x^2+1-x)-(2x+1)}{x^3+1}\geq 0 \quad \Rightarrow \quad \dfrac{-x^2+x}{(x+1)(x^2-x+1)}\geq 0$

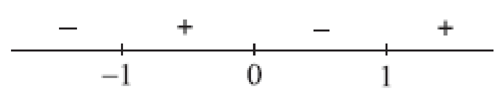
$\Rightarrow \quad \dfrac{x(x-1)}{(x+1)(x^2-x+1)}\leq 0$

Multiply both sides by $x^2 - x + 1$ to get,

[As $x^2 - x + 1 > 0$ for all $x \in R$ (*because $D < 0$, $a > 0$*) we can multiply both sides by x^2-x+1]

$\Rightarrow \quad \dfrac{x(x-1)}{(x+1)}\leq 0$

Critical points are $x = 0$, $x = 1$, $x = -1$.



Expression is negative for $x \in (-\infty, -1) \cup [0, 1]$

So real values of x for which y is real are $x \in (-\infty, -1) \cup [0, 1]$.

POSITION OF ROOTS OF A QUADRATIC EQUATION — Section - 6

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ be a quadratic expression and k, k_1, k_2 be real numbers such that $k_1 < k_2$. Let $\alpha, \beta\left(\alpha \neq \beta\right)$ be the roots of the equation $f(x) = 0$ *i.e.* $ax^2 + bx + c = 0$. Then $\alpha = \dfrac{-b-\sqrt{D}}{2a}$ and $\beta = \dfrac{-b+\sqrt{D}}{2a}$, where D is the discriminant of the equation.

6.1 Conditions for a number *k* to lie between the roots of a quadratic equation

If a number k lies between the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$, then the equation must have real roots and the sign of $f(k)$ is opposite to the sign of 'a' as is evident from Fig. 1 and 2.

Fig. 1

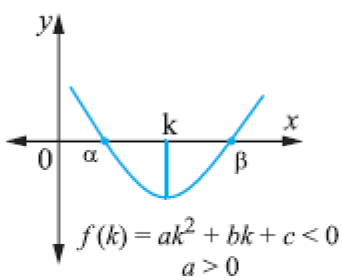
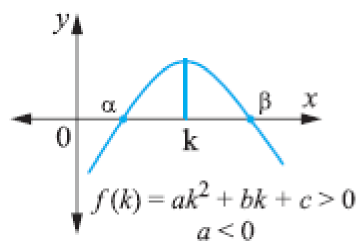


Fig. 2



$af(k) < 0$ ......(i) $af(k) < 0$ .....(ii)

Combining (i) and (ii), $af(k) < 0$ for k to lie between roots.

Note : For roots to be real, $D \geq 0$. There is no need to take this condition as when $af(k) < 0$, then D will always be positive i.e. $D \geq 0$. Hence $af(k) < 0$ is necessary and sufficient condition for k to lie between roots.

Thus, a number k lies between the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$

If $af(k) < 0$.

6.2 Conditions for both k_1 and k_2 to lie between the roots of a quadratic equation

If both k_1 and k_2 lie between the roots α and β of a quadratic equation, then $f(x) = ax^2 + bx + c = 0$, then sign of $f(k_1)$ and $f(k_2)$ should be positive or negative depending upon sign of a as it is evident from figure 3 and 4.

Fig. 3

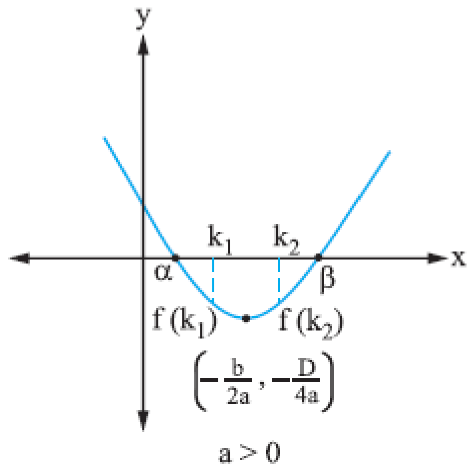
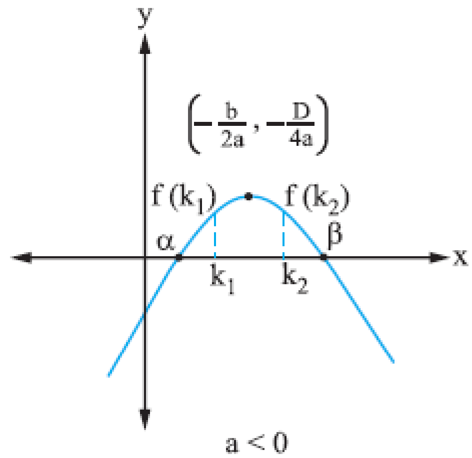


Fig. 4



$f(k_1) < 0$ and $f(k_2) < 0$ ......(i) $\qquad$ $f(k_1) > 0$ and $f(k_2) > 0$ ......(ii)

Combining (i) and (ii), $af(k_1) < 0$ and $af(k_2) < 0$

Hence for k_1 and k_2 to lie between roots, $af(k_1) < 0$ and $af(k_2) < 0$.

6.3 Conditions for a number k to be less than Roots of a Quadratic Equation

If a number k is smaller than the roots of a quadratic equation $f(x) = ax^2 + bx + c$, then the equation must have real and distinct roots and the sign of $f(k)$ is same as the sign of 'a' as is evident from Figs. 5 and 6. Also, k is less than the x-coordinate of the vertex of the parabola $y = ax^2 + bx + c$ *i.e.* $k < -b/2a$.

Fig. 5

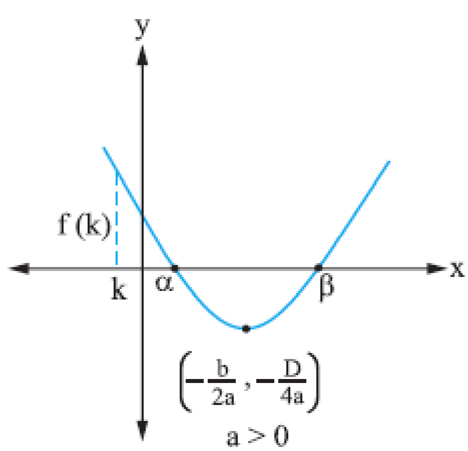
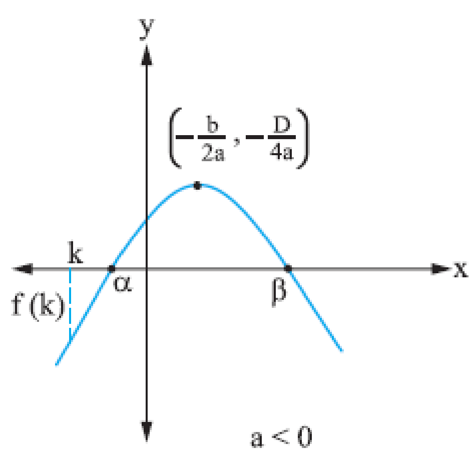


Fig. 6



$f(k) > 0$ and $x_v > k$ ......(i) $\qquad$ $f(k) < 0$ and $x_v > k$ ......(ii)

Combining (i) and (ii) we get : $af(k) > 0$, $x_v > k$ *i.e.* $-\dfrac{b}{2a} > k$ and $D \geq 0$

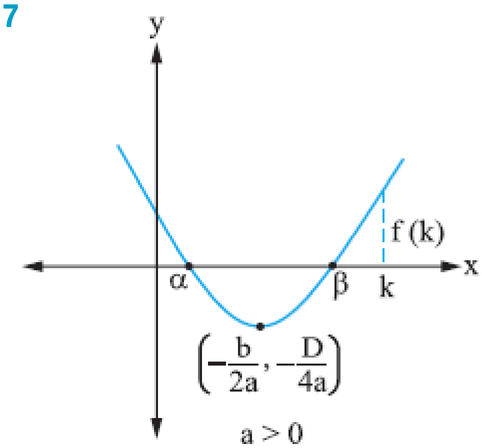
Thus, a number k is smaller than the roots of a quadratic equation $ax^2 + bx + c = 0$, if

(i) $D \geq 0$ **(ii)** $af(k) > 0$ **(iii)** $k < x_v = -b/2a$.

6.4 Conditions for a number *k* to be more than the roots of a quadratic equation

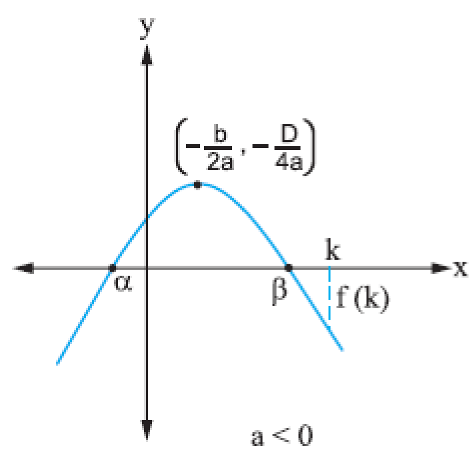
If a number k is larger than the roots of a quadratic equation $f(x) = ax^2 + bx + c$, then the equation must have real and distinct roots and the sign of $f(k)$ is same as the sign of 'a' as is evident from Figs. 7 and 8. Also, k is greater than the x-coordinate of the vertex of the parabola $y = ax^2 + bx + c$ *i.e.* $k > -b/2a$.

Fig. 7



$f(k) > 0$ and $x_v < k$ ......(i)

Fig. 8



$f(k) < 0$ and $x_v < k$ ......(ii)

Combining (i) and (ii) we get : $af(k) > 0, x_v < k$ *i.e.* $-\dfrac{b}{2a} < k$ and $D \geq 0$

Thus, a number k is smaller than the roots of a quadratic equation $ax^2 + bx + c = 0$, if

(i) $D \geq 0$ **(ii)** $af(k) > 0$ **(iii)** $k > -b/2a$.

6.5 Condition for both the roots of a quadratic equation to lie between numbers k_1 and k_2

If both these roots α and β of a quadratic equation $f(x) = ax^2 + bx + c = 0$, lie between number k_1 and k_2, then equation must have real roots, signs of $f(k_1)$ and $f(k_2)$ are same as sign of 'a' is evident from fig 9. and 10. Also $x_v = \dfrac{-b}{2a}$ must be between k_1 and k_2

Fig. 9

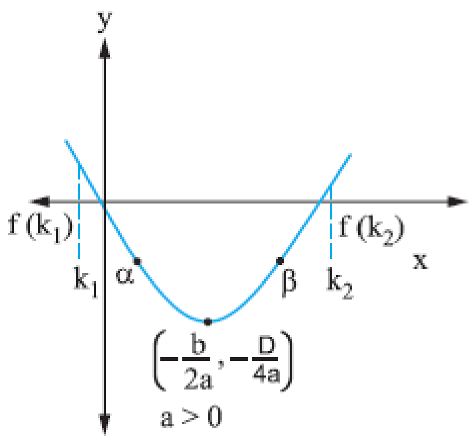
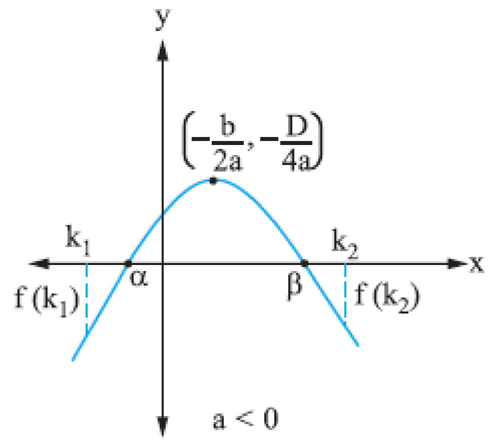


Fig. 10



$f(k_1) > 0,$

$f(k_2) > 0,$

$k_1 < x_v < k_2$ .....(*i*)

$f(k_1) < 0,$

$f(k_2) < 0,$

$k_1 < x_v < k_2$ ......(*ii*)

Combining (i) and (ii) we get : $af\left(k_1\right) > 0,\ af\left(k_2\right) > 0,\ k_1 < x_v < k_2$ and $D \geq 0$

If both the roots of a quadratic equation lie between numbers k_1 and k_2, *then*

(i) $D \geq 0$ **(ii)** $af\ (k_1) > 0,\ af\ (k_2) > 0$ **(iii)** $k_1 < -\dfrac{b}{2a} < k_2.$

6.6 Condition for exactly one root of a quadratic equation to lie in the interval (k_1, k_2), where $k_1 < k_2$

If exactly one root of the equation $ax^2 + bx + c = 0$ lies in the interval (k_1, k_2), then $f(k_1)$ and $f(k_2)$ must be of opposite sign as shown in Figs. 11 and 12.

Fig. 11

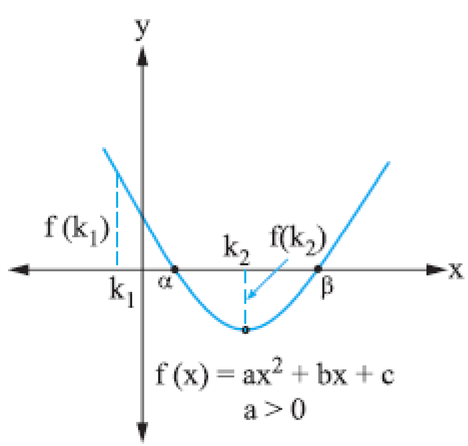
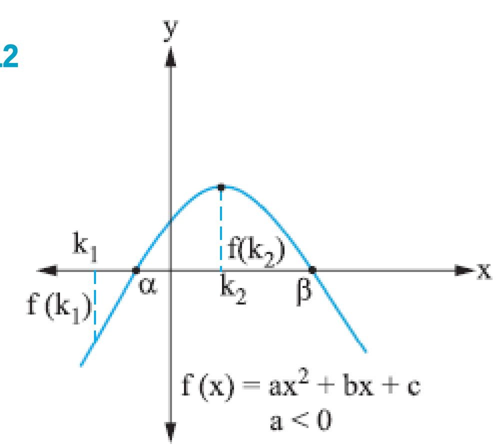


Fig. 12



$f(k_1) > 0$ and $f(k_2) < 0$ ......**(i)**

$f(k_1) < 0$ and $f(k_2) > 0$ ......**(i)**

Combining (i) and (ii), we get : $f(k_1)\, f(k_2) < 0$

Note : Exactly one root lie between k_1 and k_2. Therefore graph of quadratic polynomial will cross x-axis once between k_1 and k_2. This implies signs of $f(k_1)$ and $f(k_2)$ would be different. Hence $f(k_1)\, f(k_2) < 0$

Thus, exactly one root of the equation $ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if

(i) $f(k_1)\, f(k_2) < 0$

6.7 Some More Result on Roots of Quadratic Equation

- Both roots of $f(x) = 0$ are negative,

  if sum of the roots < 0, product of the roots > 0 and $D \geq 0$

  i.e. $-\dfrac{b}{a} < 0,\ \dfrac{c}{a} > 0,\ b^2 - 4ac \geq 0$

- Both roots of $f(x) = 0$ are positive,

  if sum of the roots > 0, product of the roots > 0 and D ≥ 0

  i.e. $-\dfrac{b}{a} > 0,\ \dfrac{c}{a} > 0,\ b^2 - 4ac \geq 0$

- Roots of $f(x) = 0$ are opposite in sign,

  if product of the roots < 0 i.e. $\dfrac{c}{a} < 0$

Illustration - 26 *If the roots of the equations* $x^2 - 2ax + a^2 + a - 3 = 0$ *are real and less than* 3, *then :*

(A) $a > 2$ **(B)** $2 \leq a \leq 3$ **(C)** $3 < a \leq 4$ **(D)** $a > 4$

SOLUTION : (A)

Let $f(x) = x^2 - 2ax + a^2 + a - 3$

As both roots of $f(x) = 0$ are less than 3, we can take a $f(3) > 0$, $-b/2a < 3$ and $D \geq 0$

[Using section 6.4]

Consider $a\, f(3) > 0$:

$\Rightarrow\ 1\left[9 - 6a + a^2 + a - 3\right] > 0 \qquad \Rightarrow\ a^2 - 5a + 6 > 0$

$\Rightarrow\ a \in (-\infty, 2) \cup (3, \infty)$ .... (i)

Consider $-b/2a < 3$: $\dfrac{-(-2a)}{3} < 3$

$\Rightarrow \quad a < 3$ .... **(ii)**

Consider $D \geq 0 : 4a - 4\left(a^2 + a - 3\right) \geq 0 \Rightarrow \quad -4(a-3) \geq 0 \quad \Rightarrow \quad a - 3 \leq 0$

$\Rightarrow \quad a \in (-\infty, 3]$ .... **(iii)**

Combining **(i)**, **(ii)** and **(iii)** on the number line, we get : $a \in (-\infty, 2)$

Illustration - 27 *The values of p for which the roots of the equation* $(p-3)x^2 - 2px + 5p = 0$ *are real and positive are*:

(A) $p \in [2, 3]$ **(B)** $p \in \left(3, \frac{15}{4}\right]$ **(C)** $p \in (-\infty, 0) \cup (3, \infty)$ **(D)** $p \in \left(2, \frac{15}{4}\right)$

SOLUTION : (B)

The roots are real and positive if $D \geq 0$, sum of the roots > 0 and product of the roots > 0.

[Using result 6.7] $D \geq 0$:

$\Rightarrow \quad 4p^2 - 20p(p-3) \geq 0 \quad \Rightarrow \quad -4p^2 + 15p \geq 0 \quad \Rightarrow \quad 4p^2 - 15p \leq 0 \quad \Rightarrow \quad p \in [0, 15/4]$ .. **(i)**

Sum of the roots > 0 :

$$\frac{2p}{p-3} > 0 \quad \Rightarrow \quad \frac{p}{p-3} > 0$$

$\Rightarrow \quad p(p-3) > 0 \quad \Rightarrow \quad p \in (-\infty, 0) \cup (3, \infty)$ .... **(ii)**

Product of the roots > 0 :

$\frac{5p}{p-3} > 0 \quad \Rightarrow \quad \frac{p}{p-3} > 0 \quad \Rightarrow \quad p(p-3) > 0 \quad \Rightarrow \quad p \in (-\infty, 0) \cup (3, \infty)$ .... **(iii)**

Combining **(i)**, **(ii)** and **(iii)** on the number line, we get :

$p \in (3, 15/4]$.

Illustration - 28 *The values of a for which* $2x^2 - 2(2a+1)x + a(a+1) = 0$ *may have one root less than a and other root greater than 'a' are given by*

(A) $1 > a > 0$ **(B)** $-1 < a < 0$ **(C)** $a \geq 0$ **(D)** $a > 0$ or $a < -1$

SOLUTION : (D)

The given condition suggests that a lies between the roots. Let $f(x) = 2x^2 - 2(2a+1)x + a(a+1)$. For a to lie between the roots, we must have $f(a) < 0$

$\Rightarrow \quad 2a^2 - 2a(2a+1) + a(a+1) < 0$

$\Rightarrow \quad -a^2 - a < 0 \quad \Rightarrow \quad a^2 + a > 0 \quad \Rightarrow \quad a > 0$ or $a < -1$

Illustration - 29 *The values of 'a' for which both the roots of* $x^2 - 4ax + 2a^2 - 3a + 5 = 0$ *is greater than 2, are :*

(A) $a \in (1, \infty)$ **(B)** $a = 1$ **(C)** $a \in (-\infty, 1)$ **(D)** $a \in (9/2, \infty)$

SOLUTION : (D)

Let $f(x) = x^2 - 4ax + 2a^2 - 3a + 5$. The conditions for both the roots to exceed 2 are

(i) $D \geq 0$ **(ii)** $f(2) > 0$ and **(iii)** $x_v > 2$

Now consider $D \geq 0$

$\Rightarrow \quad 16a^2 - 4\left(2a^2 - 3a + 5\right) \geq 0 \Rightarrow \quad 2a^2 + 3a - 5 \geq 0$

$\Rightarrow \quad (2a + 5)(a - 1) \geq 0$

$\Rightarrow \quad a \in (-\infty, 5/2] \cup [1, \infty)$ ...... **(i)**

Now consider $f(2) > 0$

$\Rightarrow \quad 4 - 8a + \left(2a^2 - 3a + 5\right) > 0 \quad \Rightarrow \quad 2a^2 - 11a + 9 > 0$

$\Rightarrow \quad (2a - 9)(a - 1) > 0$

$\Rightarrow \quad a \in (-\infty, 1) \cup \left(\frac{9}{2}, \infty\right)$ ......**(ii)**

Now consider $x_v > 2$

$\Rightarrow \quad \frac{4a}{2} > 2$ constant

$\Rightarrow \quad a > 1$ ......**(iii)**

On combining (i), (ii) and (iii) , we get : $a \in \left(\frac{9}{2}, \infty\right)$

★★

TRANSFORMATION OF EQUATIONS — Section - 7

7.1. Transformaiton of an equation into another equation whose roots are the reciprocals of the roots of the given equation

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots.. + a_{n-1}x + a_n = 0$ ...... **(i)**

be the given equation. Let x and y be respectively the roots of given equation and that of the transformed equation.

Then, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

Putting $x = \frac{1}{y}$ in **(i)**, we get :

$$\frac{a_0}{y^n} + \frac{a_1}{y^{n-1}} + \frac{a_2}{y^{n-2}} + \ldots.. + \frac{a_{n-1}}{y} + a_n = 0 \Rightarrow a_ny^n + a_{n-1}y^{n-1} + \ldots.. + a_1y + a_0 = 0$$

This is the required equation.

Note : Thus, to obtain an equation whose roots are reciprocals of the roots of a given equaiton is obtained by replacing x by $1/x$ in the given equation.

Illustration - 30 *Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ be in H.P.*

(A) $27r^2 + 9pqr + 2q^3 = 0$ *(B)* $27r^2 - 9pqr + 2q^3 = 0$

(C) $27r^2 + 9pqr + q^3 = 0$ (D) $27r^2 - 9pqr + q^3 = 0$

SOLUTION : (B)

The equation whose roots are reciprocals of the roots of the given equation is given by

$$\frac{1}{x^3} - \frac{p}{x^2} + \frac{q}{x} - r = 0 \quad \text{or} \quad rx^3 - qx^2 + px - 1 = 0 \qquad \ldots\ldots \textbf{(i)}$$

Since the roots of the given equation are in H.P. so, the roots of this equation are in A.P. Let its roots be $a - d$, a and $a + d$. Then, $(a - d) + a + (a + d) = -\left(-\frac{q}{r}\right) \Rightarrow 3a = \frac{q}{r} \Rightarrow a = \frac{q}{3r}$

Since a is a root of **(i)**, so,

$$ra^3 - qa^2 + pa - 1 = 0 \Rightarrow r\left(\frac{q}{3r}\right)^3 - q\left(\frac{q}{3r}\right)^2 + p\left(\frac{q}{3r}\right) - 1 = 0$$

$$\Rightarrow \quad \frac{q^3}{27r^2} - \frac{q^3}{9r^2} + \frac{pq}{3r} - 1 = 0 \quad \Rightarrow \quad q^3 - 3q^3 + 9pqr - 27r^2 = 0 \quad \Rightarrow \quad 27r^2 - 9pqr + 2q^3 = 0$$

7.2 Transformation of an equaiton into another equation whose roots are negatives of the given equation.

Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$$

Note : Let x be root of the given equation and y be a root of the transformed equation. Then $y = -x$ or $x = -y$. Thus the transformed equation is obtained by putting $x = -y$ in $f(x) = 0$ and is therefore $f(-y) = 0$

or $\quad a_0y^n - a_1y^{n-1} + a_2y^{n-2} + ... + (-1)^n a_n = 0$

Illustrating the Concepts :

The equation whose roots are negative of the roots of the equation : $x^3 - 5x^2 - 7x - 3 = 0$

$$(-x)^3 - 5(-x)^2 - 7(-x) - 3 = 0 \text{ or } -x^3 - 5x^2 + 7x - 3 = 0 \text{ or } x^3 + 5x^2 - 7x + 3 = 0.$$

7.3 Transformation of an equation of another whose roots are squre of the roots of a given equation

Let x be a root of the given equation and y be that of the trasformed equation. Then,

$$y = x^2 \Rightarrow \quad x = \sqrt{y}.$$

Note : Thus, an equation whose roots are squares of the roots of a given equation is obtained by replacing x by $\sqrt{x}$ in the given equaiton

Illustrating the Concepts :

Form an equation whose roots are squares of the roots of the equation : $x^3 - 6x^2 + 11x - 6 = 0.$

Replacing x by $\sqrt{x}$ in the given equation, we get :

$$(\sqrt{x})^3 - 6(\sqrt{x})^2 + 11\sqrt{x} - 6 = 0 \quad \Rightarrow \quad x^{3/2} + 11\sqrt{x} = 6x + 6 \quad \Rightarrow \quad \sqrt{x}\,(x + 11) = 6\,(x + 1)$$

$$\Rightarrow \quad x\,(x + 11)^2 = 36\,(x + 1)^2 \quad \Rightarrow \quad x^3 - 14x^2 + 49x - 36 = 0$$

7.4 Transformation of an equation into another equation whose roots are cubes of the roots of the given equation.

Let x be a root of the given equation and y be that of the trasformed equation. Then, $y = x^3 \Rightarrow x = y^{1/3}$

Note : Thus, an equation whose roots are cubes of the roots of a given equation is obtained by replacing x by $x^{1/3}$ in the given equaiton

Illustrating the Concepts :

Form an equation whose roots are cubes of the roots of equation : $ax^3 + bx^2 + cx + d = 0.$

Replacing x by $x^{1/3}$ in the given equation, we get

$$a(x^{1/3})^3 + b(x^{1/3})^2 + c(x^{1/3}) + d = 0 \quad \Rightarrow \quad ax + d = -(bx^{2/3} + cx^{1/3})$$

$$\Rightarrow \quad (ax + d)^3 = -(bx^{2/3} + cx^{1/3})^3$$

$$\Rightarrow \quad a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 = -\{b^3x^2 + c^3x + 3bcx(bx^{2/3} + cx^{1/3})\}$$

$$\Rightarrow \quad a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 = -\{b^3x^2 + c^3x - 3bcx(ax + d)\}$$

$$\Rightarrow \quad a^3x^3 + x^2(3a^2d - 3abc + b^3) + x(3ad^2 - 3bcd + c^3) + d^3 = 0$$

This is the required equation.

7.5 Relations between Roots and Coefficients

If $\alpha_1, \alpha_2, \alpha_3, \ldots. \alpha_n$ are roots of the equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0, \quad \text{then}$$

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n)$$

$$\therefore \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$$

Comparing the coefficients of x^{n-1} on both sides, we get :

$$S_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_n = \sum \alpha_i = \frac{-a_1}{a_0} \quad \text{or,} \quad S_1 = -\frac{\text{coeff. of } x^{n-1}}{\text{coeff. of } x^n}$$

Comparing the coefficients of x^{n-2} on both sides, we get :

$$S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots = \sum_{i \neq j} \alpha_i a_j = (-1)^2\frac{a_2}{a_0} \quad \text{or,} \quad S_2 = \frac{(-1)^2 \text{ coeff. of } x^{n-2}}{\text{coeff. of } x^n}$$

Comparing the coefficients of x^{n-3} on both sides, we get :

$$S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \ldots = \sum_{i \neq j \neq k} \alpha_i\alpha_j\alpha_k = (-1)^3\frac{a_3}{a_0}$$

$$\text{or,} \quad S_3 = \frac{(-1)^3 \text{ coeff. of } x^{n-3}}{\text{coeff. of } x^n}$$

… … … …
… … … …

$$S_n = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{const term}}{\text{coeff of } x^n}$$

Here, S_k denotes the sum of the products of the roots taken k at a time.

Particular Cases :

Quaratic Equation : If α, β are roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Cubic Equation : If α, β, γ are roots of a cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad \text{then} \quad \alpha + \beta + \gamma = -b/a, \quad \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

and $\alpha\beta\gamma = (-1)^3 \frac{d}{a} = -\frac{d}{a}$.

Biquadratic Equation : If α, β, γ, δ are roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$S_2 = \alpha\beta + \beta\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

or, $S_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

or, $S_3 = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a}$ and, $S_4 = \alpha\beta\gamma\delta = (-1)^4 \frac{e}{a} = \frac{e}{a}$.

Illustration - 31 *If the sum of two roots of the equation $x^3 - p x^2 + qx - r = 0$ is zero, then*

(A) $pq = r$ **(B)** $pr = q$ **(C)** $qr = p$ **(D)** *None of these*

SOLUTION : (A)

Let the roots of the given equation be α, β, γ such that α + β = 0. Then,

$$\alpha + \beta + \gamma = -\frac{(-p)}{1} \quad \Rightarrow \quad \alpha + \beta + \gamma = p \quad \Rightarrow \quad \gamma = p \quad [\because \alpha + \beta = 0]$$

But γ is a root of the given equation. Therefore,

$$\gamma^3 - p\gamma^2 + q\gamma - r = 0 \quad \Rightarrow \quad p^3 - p^3 + qp - r = 0 \quad \Rightarrow \quad pq = r$$

Illustration - 32 *Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P.*

(A) $2p^3 + 9pq + 27r = 0$ (B) $p^3 + 9pq + 27r = 0$

(C) $2p^3 - 9pq + 27r = 0$ (D) $p^3 - 9pq + 27r = 0$

SOLUTION : (C)

Let the roots of the given equation be $a - d, a, a + d$.

Then,

$$(a - d) + a + (a + d) = \frac{-(-p)}{1} \quad \Rightarrow \quad a = p/3$$

Since a is a root of the given equation. Therefore,

$$a^3 - pa^2 + qa - r = 0 \quad \Rightarrow \quad \frac{p^3}{27} - \frac{p^3}{9} + \frac{qp}{3} - r = 0 \quad \Rightarrow \quad 2p^3 - 9pq + 27r = 0$$

This is the required condition.

THINGS TO REMEMBER

1. If $ax^2 + bx + c = 0$ is a quadratic equation and α, β are its roots then

Roots of a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0, a, b, c \in R)$ are given by : $\alpha, \beta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- Sum of the roots $= \alpha + \beta = -\dfrac{b}{a}$
- Product of roots $= \alpha\beta = \dfrac{c}{a}$
- $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.

2. If S be the sum and P be the product of roots, then quadratic equation is : $x^2 - Sx + P = 0$

3. **Nature of Quadratic Equation :**

- (a) **If $D < 0$** $(b^2 - 4ac < 0)$, then the roots of the quadratic equation are non – real *i.e.* complex root.

  (b) **If $D = 0$** $(b^2 - 4ac = 0)$, then the roots are real and equal.

  Equal root $= -\dfrac{b}{2a}$

  (c) **If $D > 0$** $(b^2 - 4ac > 0)$, then the roots are real and unequal.

- If D *i.e.* $(b^2 - 4ac)$ is a perfect square and a, b and c are rational, *roots are rational*.

- If D *i.e.* $(b^2 - 4ac)$ is not a perfect square and a, b and c are rational, then roots are of the form $m + \sqrt{n}$ & $m - \sqrt{n}$.
- If $a = 1$, $b, c \in I$ and the roots are rational numbers, then the roots must be integer.
- If a quadratic equation in x has more than two roots, then it is an identity in x (*i.e.* true for all real values of x) and $a = b = c = 0$.

4. Condition of Common Roots :

Consider two quadratic equations :

$$ax^2 + bx + c = 0 \quad \text{and} \quad a'x^2 + b'x + c' = 0$$

(a) For two common roots : $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

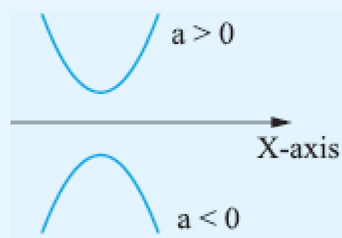
(b) For one common root : $\Rightarrow \quad (bc' - b'c)(ab' - a'b) = (a'c - ac')^2$

5. How to draw quadratic polynomial $y = ax^2 + bx + c$

Graph of a Quadratic Polynomial

$$f(x) = ax^2 + bx + c \qquad (a \neq 0)$$

To draw the graph of f (x), proceed according to following steps :



I. The shape of the curve $y = f(x)$ is **parabolic**.

II. For $a > 0$, the parabola opens upwards.

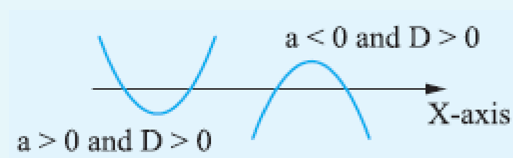
For $a < 0$, the parabola opens downwards.

III. Intersection with axes :

(i) with X-axis

- **For $D > 0$**

  Parabola cuts X-axis in two points.

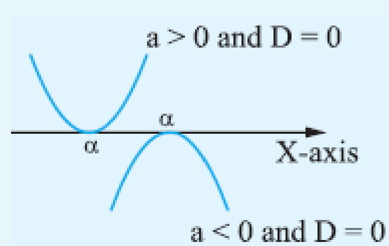


The points of intersection are $\alpha, \beta = \dfrac{-b \pm \sqrt{D}}{2a}$.

- **For $D = 0$**

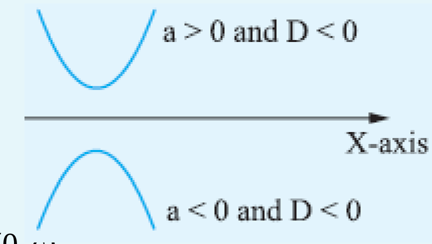
  Parabola touches X-axis in one point.

  The points of intersection is $\alpha = \dfrac{-b}{2a}$.



- **For $D < 0$**

Parabola does not cut X-axis at all *i.e.* no point of intersection with X-axis.



(ii) with Y-axis

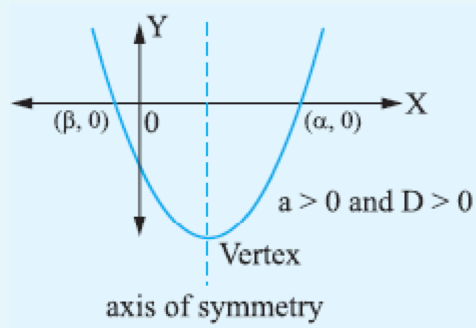
The points of intersection with Y-axis is $f(0) = c$. i.e. $(0, c)$

{put $x = 0$ in the quadratic polynomial}

IV. Obtain V where V is called as vertex of parabola.

The coordinates of $V \equiv \left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$

The line passing through vertex and parallel to the Y-axis is called as axis of symmetry.



6. Maximum and Minimum value of *f* (*x*) :

- $f(x)$ has minimum value at vertex if $a > 0$ and $f_{min} = -\dfrac{D}{4a}$ at $x = -\dfrac{b}{2a}$.
- $f(x)$ has minimum value at vertex if $a < 0$ and $f_{max} = -\dfrac{D}{4a}$ at $x = -\dfrac{b}{2a}$.

7. Quadratic Inequation :

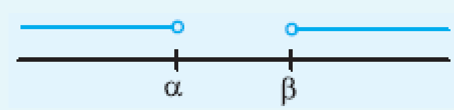
Let $f(x) = ax^2 + bx + c$ where $a, b, c \in R$ and $a \neq 0$. To solve the inequations of type :

$\{ f(x) \leq 0 \quad ; \quad f(x) < 0 \quad ; \quad f(x) \geq 0 \quad ; \quad f(x) > 0 \}$

(a) $D > 0$

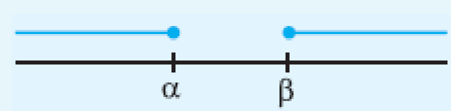
- Make the coefficient of x^2 positive
- Factorise the expression and represent the left hand side of inequality in the form $(x - \alpha)(x - \beta)$.
- If $(x - \alpha)(x - \beta) > 0$, then x lies outside α and β.

  $\Rightarrow \quad x \in (-\infty, \alpha) \cup (\beta, \infty)$



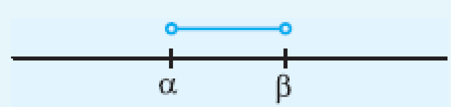
- If $(x - \alpha)(x - \beta) \geq 0$, then x lies on and outside α and β.

  $\Rightarrow \quad x \in (-\infty, \alpha] \cup [\beta, \infty]$



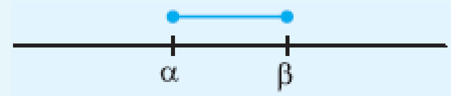
- If $(x - \alpha)(x - \beta) < 0$, then x lies inside α and β.

  $\Rightarrow \quad x \in (\alpha, \beta)$



- If $(x-\alpha)(x-\beta) \le 0$, then x lies on and inside α and β.

  $\Rightarrow \quad x \in [\alpha, \beta]$



(b) $D < 0$ and $a > 0$: $f(x) > 0$ for all $x \in R$.

(c) $D < 0$ and $a < 0$: $f(x) < 0$ for all $x \in R$.

(d) $D = 0$ and $a > 0$: $f(x) \ge 0$ for all $x \in R$.

(e) $D = 0$ and $a < 0$: $f(x) \le 0$ for all $x \in R$.

(f) $D \le 0$, $a > 0$: $f(x) \ge 0$ for all $x \in R$.

(g) $D \le 0$, $a < 0$: $f(x) \le 0$ for all $x \in$ R

8. Rational Algebraic Inequalities

Consider the following types of rational algebraic inequalities.

$$\frac{P(x)}{Q(x)} > 0, \quad \frac{P(x)}{Q(x)} < 0, \quad \frac{P(x)}{Q(x)} \ge 0, \quad \frac{P(x)}{Q(x)} \le 0$$

where $P(x)$ and $Q(x)$ are polynomials in x.

- These inequalities can be solved by the *method of intervals* also known as *sign method* or *wavy curve method*.

How to solve Rational Algebraic Inequality :

(a) Factorise $P(x)$ and $Q(x)$ into linear factors.

(b) Make coefficient of x positive in all factors.

(c) Equate all the factors to zero and find corresponding values of x. These values are known as critical points.

(d) Plot the critical points on a number line. n critical points will divide the number line $(n+1)$ regions.

(e) In right most region, the expression bears positive sign and in other regions the expression bears alternate positive and negative signs.

9. Maximum and Minimum values of a Rational Function of x

Consider : $f(x) = y = \dfrac{ax^2 + bx + c}{px^2 + qx + r}$ where $x \in R$.

We will find maximum and minimum values $f(x)$ can take by observing the following cases.

Case - I : $y \in [A, B]$

If y can take values between A and B, then,

Maximum value of $y = y_{max} = B$,

Minimum value of $y = y_{min} = A$.

Case - II : $y \in (-\infty, A] \cup [B, \infty)$

If y can take values outside A and B, then

Maximum value of $y = y_{max} = \infty$ *i.e.* not defined.

Minimum value of $y = y_{min} = -\infty$. *i.e.* not defined.

Case - III : $y \in (-\infty, \infty)$ **i.e.** $y \in R$

If y can take all values, then

Maximum value of $y = y_{max} = \infty$ *i.e.* not defined.

Minimum value of $y = y_{min} = -\infty$ *i.e.* not defined.

10. POSITION OF ROOTS OF A QUADRATIC EQUATION $ax^2 + bx + c = 0$

I. Conditions for a number k to lie between the Roots of a Quadratic Equation is

(i) $af(k) < 0$

II. Conditions for both k_1 and k_2 to lie between the roots of a quadratic equation is

(i) $af(k_1) < 0$ **(ii)** $af(k_2) < 0$.

III. Conditions for a number k to be less than Roots of a Quadratic Equation is

(i) $D \geq 0$ **(ii)** $af(k) > 0$ **(iii)** $k < x_v = -b/2a$.

IV. Conditions for a number k to be more than the roots of a quadratic equation is

(i) $D \geq 0$ **(ii)** $af(k) > 0$ **(iii)** $k > -b/2a$.

V. Condition for both the Roots of a Quadratic Equation to lie between numbers k_1 and k_2 is

(i) $D > 0$ **(ii)** $af(k_1) > 0$, $af(k_2) > 0$ **(iii)** $k_1 < -\dfrac{b}{2a} < k_2$.

VI. Condition for exactly one root of a quadratic equation to lie in the interval (k_1, k_2), where $k_1 < k_2$ is

(i) $f(k_1) f(k_2) < 0$

VII. Both roots of $f(x) = 0$ are negative,

(i) $-\dfrac{b}{a} < 0$ **(ii)** $\dfrac{c}{a} > 0$ **(iii)** $b^2 - 4ac \geq 0$

VIII. Both roots of $f(x) = 0$ are positive,

(i) $-\dfrac{b}{a} > 0$ **(ii)** $\dfrac{c}{a} > 0$ **(iii)** $b^2 - 4ac \geq 0$

IX. Roots of $f(x) = 0$ are opposite in sign,

(i) $\dfrac{c}{a} < 0$

11. Transformationations of Equation

I. To obtain an equation whose roots are reciprocals of the roots of a given equation is obtained by replacing x *by* $1/x$ in the given equation.

II. To obtain an equation whose roots are negative of the roots of a given equation is obtain by replacing $x = -y$ in $f(x) = 0$

III. To obtain an equation whose roots are square of the roots of a given equation is obtain by replacing x by $\sqrt{x}$ in the given equation.

IV. To obtain an equation whose roots are cubes of the roots of a give equation is obtained by replacing x by $x^{1/3}$ in the given equation.

12. Some more Results (Relation between the roots) :

I. Quaratic Equation : If α, β are roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

II. Cubic Equation : If α, β, γ are roots of a cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad \text{then} \quad \alpha + \beta + \gamma = -b/a, \quad \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

$$\text{and } \alpha\beta\gamma = (-1)^3 \frac{d}{a} = -\frac{d}{a}.$$

III. Biquadratic Equation : If α, β, γ, δ are roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$S_2 = \alpha\beta + \beta\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

or, $$S_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

or, $$S_3 = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a}$$

and, $$S_4 = \alpha\,\beta\,\gamma\,\delta = (-1)^4 \frac{e}{a} = \frac{e}{a}$$