Properties of Triangle

Important Results — Section - 1

Sides of $\Delta \equiv a, b, c$

Angles of $\Delta \equiv A, B, C$

1.1 Standard Results - I

(a) Sine Rule : $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ [where R is the circumradius of ΔABC]

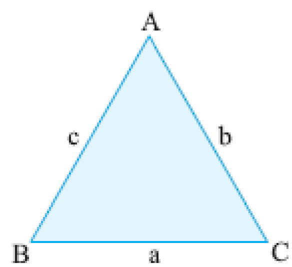
$\Rightarrow$ $a = 2R \sin A,\ b = 2R \sin B,\ c = 2R \sin C$

(b) Consine Rule :

$a^2 = b^2 + c^2 - 2bc \cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$b^2 = c^2 + a^2 - 2ac \cos B$ or $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

$c^2 = a^2 + b^2 - 2ab \cos C$ or $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



(c) Projection Formula :

$a = b \cos C + c \cos B$

$b = c \cos A + a \cos C$

$c = a \cos B + b \cos A$

(d) Napier Analogy :

$\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$; $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$ and $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

Illustration - 1 *In* ΔABC, *if* $a = 2,\ b = 3,\ c = 4$, *then* $\cos A$ *is* :

(A) 7/8 (B) 5/7 (C) 6/7 (D) 5/8

SOLUTION : (A)

Here $a = 2,\ b = 3,\ c = 4$ hence by using cosine rule

We get : $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{7}{8}$

Illustration - 2 *If the angles of a triangle ABC are in A. P.,* $a = 2,\ c = 4,$ *then b is :*

(A) $2\sqrt{3}$ **(B)** $\sqrt{21}$ **(C)** 8 **(D)** 11

SOLUTION : (A)

$\because$ Angles of triangle are in AP hence $2B = A + C \Rightarrow \quad B = 60^\circ \Rightarrow \quad b^2 = a^2 + c^2 = 2ac\cos B$

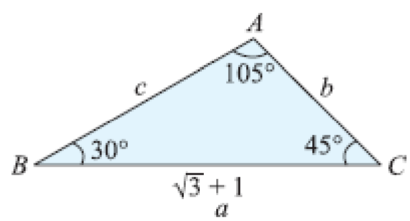
$\Rightarrow \quad b = 2\sqrt{3}$

Illustration - 3 *If the angles of a triangle ABC are* $30^\circ,\ 45^\circ$ *and the included side is* $\sqrt{3}+1.$ *then the remaining sides are :*

(A) $2\sqrt{2}$ **(B)** $2, 2\sqrt{2}$ **(C)** $\sqrt{2}, 4$ **(D)** $2, 4\sqrt{3}$

SOLUTION : (A)

Let $\angle B = 30^\circ \quad \angle C = 45^\circ \quad \angle A = 105^\circ$



$$\Rightarrow \quad \frac{\sqrt{3}+1}{\sin 105^\circ} = \frac{b}{\sin 30^\circ}$$

$$\Rightarrow \quad b = \frac{\left(\sqrt{3}+1\right)\sin 30^\circ}{\sin 105^\circ} = \frac{\left(\sqrt{3}+1\right)}{\dfrac{\sqrt{3}+1}{2\sqrt{2}}} \times \frac{1}{2} = \sqrt{2}$$

and $\dfrac{c}{\sin 40^\circ} = \dfrac{\sqrt{2}}{\sin 30^\circ} \quad \Rightarrow \quad c = 2$

Illustration - 4 *If two angles of a* ΔABC *are* 45° *and* 60°, *then the ratio of the smallest and the greatest sides are :*

(A) $\left(\sqrt{3}-1\right):1$ **(B)** $\sqrt{3}:\sqrt{2}$ **(C)** $1:\sqrt{3}$ **(D)** $\sqrt{3}:1$

SOLUTION : (A)

Angle are $45^\circ, 60^\circ$ and 75°

Ratio of smallest and greatest sides $= \sin 45^\circ = \sin 75^\circ = \dfrac{1}{\sqrt{2}} : \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \dfrac{2}{\sqrt{3}+1} : 1 = \sqrt{3}-1:1$

Illustration - 5 *In* $\Delta ABC, \dfrac{a}{\cos A}=\dfrac{b}{\cos B}=\dfrac{c}{Cos\,C}$; *if* $b=2$, *then the area of the triangle is :*

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** 3

SOLUTION : (B)

$$\frac{a}{\cos A}=\frac{b}{\cos B}=\frac{c}{\cos C}$$

$$\Rightarrow \quad \frac{\sin A}{\cos A}=\frac{\sin B}{\cos B}=\frac{\sin C}{\cos C}$$

$$\Rightarrow \quad \tan A=\tan B=\tan C$$

$$\Rightarrow A=B=C=60^\circ$$

$$\text{Area } = \frac{\sqrt{4}}{3}(2)^2=\sqrt{3}$$

Illustration - 6 *If a flag staff of* 6 *m high placed on the top of a tower throws a shadow of* $2\sqrt{3}\,m$ *along the ground then the angle that the Sun makes with the ground is :*

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** 3

SOLUTION : (D)

$$\tan\theta=\frac{x}{y}=\frac{6+x}{2\sqrt{3}+y} \Rightarrow 2\sqrt{3}\,x+xy=6y+xy \quad \Rightarrow \quad 2\sqrt{3}\,x=6y \quad \Rightarrow \quad \sqrt{3}\,x=3y$$

$$\Rightarrow \frac{x}{y}=\frac{3}{\sqrt{3}}=\sqrt{3} \quad \Rightarrow \tan\theta=\frac{x}{y}=\sqrt{3} \quad \Rightarrow \theta=60^\circ$$

Illustration - 7 *A person walking along a straight road towards a hill observes at two point distance* $\sqrt{3}\,km$, *the angles of elevation of the hill to* 30° *and* 60°. *The height of the hill is :*

(A) $\dfrac{3}{2}km$ **(B)** $\dfrac{\sqrt{2}}{3}km$ **(C)** $\dfrac{\sqrt{3}+1}{2}km$ **(D)** $\sqrt{3}\,km$

SOLUTION : (A)

$$\tan 60^\circ=\frac{h}{x} \quad \Rightarrow \quad h=\sqrt{3}\,x$$

$$\tan 30^\circ=\frac{h}{\left(\sqrt{3}+1\right)} \quad \Rightarrow \quad h\sqrt{3}=\sqrt{3}+x \quad \Rightarrow \quad h\sqrt{3}=\sqrt{3}+\frac{h}{\sqrt{3}}$$

$$\Rightarrow 3h=3+h \quad \Rightarrow \quad 2h=3 \quad \Rightarrow \quad 3/2\,km$$

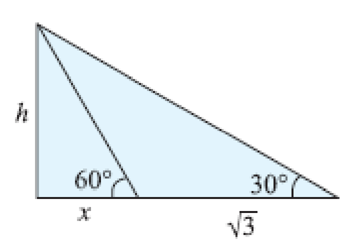


Illustration - 8 *From a point on the level ground, the angle of elevation of the top of a pole is* $30°$. *On moving 20 meters, the angle of elevation is* $45°$. *Then the height of the pole, is meters, is :*

(A) $10\left(\sqrt{3}-1\right)$ **(B)** $10\left(\sqrt{3}+1\right)$ **(C)** 15 **(D)** 20

SOLUTION : (B)

$$\tan 45° = \frac{h}{x} \Rightarrow x = h$$

$$\tan 30° = \frac{h}{x+20} \Rightarrow x+20 = h\cot 30° \Rightarrow h+20 = h\sqrt{3}$$

$$\Rightarrow \left(\sqrt{3}-1\right)h = 20 \Rightarrow h = \frac{20}{\sqrt{3}-1} = 10\left(\sqrt{3}-1\right)$$

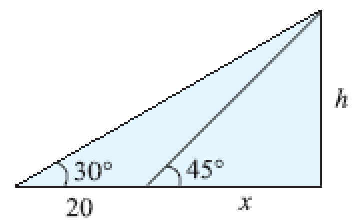


Illustration - 9 *A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is* $60°$ *and when he retires 40 meters away from the tree the angle of elevation becomes* $30°$. *The breadth of the river is :*

(A) 20 *m* **(B)** 60 *m* **(C)** 40 *m* **(D)** 30 *m*

SOLUTION : (A)

$$\tan 60° = \frac{h}{x} \Rightarrow h = x\sqrt{3}$$

$$\tan 30° = \frac{h}{x+40} \Rightarrow h\sqrt{3} = x+40 \Rightarrow 3x = x+40$$

$$\Rightarrow 2x = 40 \Rightarrow x = 20$$

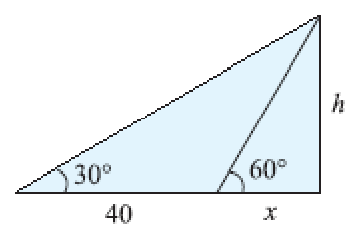


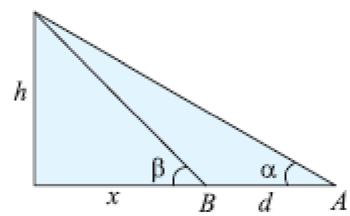
Illustration - 10 *The angles of elevation measured from two points A and B on a horizontal line from the foot of a tower are* α *and* β. *If AB = d, then the height of the tower is :*

(A) $\left|\frac{d \sin\alpha \sin\beta}{\sin(\alpha-\beta)}\right|$ **(B)** $\left|\frac{d \sin\alpha \sin\beta}{\sin(\alpha+\beta)}\right|$

(C) $\left|\frac{d \sin\alpha + \sin\beta}{\sin(\alpha-\beta)}\right|$ **(D)** $\left|\frac{d \sin\alpha - \sin\beta}{\sin(\alpha-\beta)}\right|$

SOLUTION : (A)

$$\tan\beta = \frac{h}{x} \Rightarrow h = h\cot\beta$$



$$\tan\alpha = \frac{h}{x+d} \Rightarrow x+d = h\cot\alpha \Rightarrow h\cot\beta + d = h\cot\alpha$$

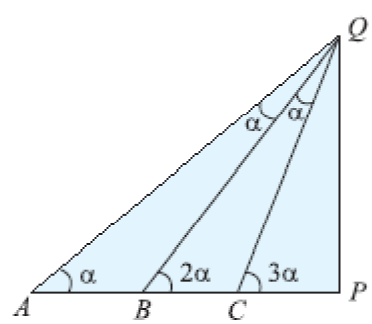
$$\Rightarrow h = \frac{d}{\cot\alpha - \cot\beta} = \frac{d\cdot\sin\alpha\ \sin\beta}{\sin(\beta-\alpha)}$$

Illustration - 11 *A tower subtends angles* $\alpha, 2\alpha$, and 3α *respectively at points A, B and C, all lying on a horizontal line through the foot of the tower. Then AB/BC is :*

(A) $\frac{\sin 3\alpha}{\sin 2\alpha}$ **(B)** $1+2\cos 2\alpha$ **(C)** $2+2\cos 2\alpha$ **(D)** $\frac{\sin 2\alpha}{\sin\alpha}$

SOLUTION : (A)

From $\Delta ABQ \frac{AB}{\sin\alpha} = \frac{BQ}{\sin\alpha}$



From $\Delta BCQ \frac{BC}{\sin\alpha} = \frac{BQ}{\sin\alpha}$

$$\Rightarrow \frac{AB}{BC} = \frac{\sin 3\alpha}{\sin\alpha} = \frac{3\sin\alpha - 4\sin^3\alpha}{\sin\alpha}$$

$$= 3 - 4\sin^2\alpha = 1 + 2\left(1 - 2\sin^2\alpha\right) = 1 + 2\cos 2\alpha$$

1.2★★ Standard Results - II

(a) Semi -Permeter of ΔABC (s) :

$$s = \frac{a+b+c}{2} \quad \Rightarrow \quad 2s = a + b + c$$

$$2s - 2a = b + c - a$$

$$2s - 2b = c + a - b$$

$$2s - 2c = a + b - c$$

(b) Half-angle formulae :

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \frac{s(s-a)}{\Delta}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \qquad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{\Delta}$$

The expressions for $\sin\frac{B}{2}, \cos\frac{B}{2}, \tan\frac{B}{2}, \cot\frac{B}{2}, \sin\frac{C}{2}, \cos\frac{C}{2}, \tan\frac{C}{2}, \cot\frac{C}{2}$ can be derived using symmetry.

$$\Delta = \text{area of triangle } ABC = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

$$\Delta = \frac{abc}{4R} = \text{rs}$$

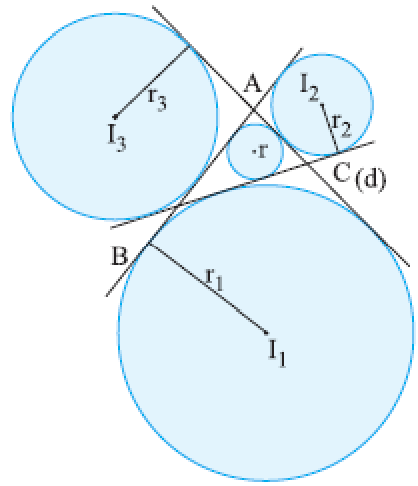
(c) Vlaues of $\sin A$, $\cos A$, $\cot A$:

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{4\Delta}$$

(d) Relation between inradius, sides, semi-perimeter and area of the triangle :



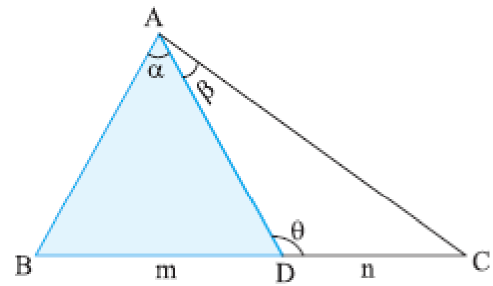
In radius	r	$\frac{\Delta}{s}$	$(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$	$r=\frac{a\sin B/2\cdot\sin C/2}{\cos A/2}$
Ex radius (opposite to A)	r_1	$r_1=\frac{\Delta}{s-a}$	$s\tan\frac{A}{2}$	$\frac{a\cos B/2\cdot\cos C/2}{\cos A/2}$
Ex radius (opposite to B)	r_2	$r_2=\frac{\Delta}{s-b}$	$s\tan\frac{B}{2}$	$\frac{b\cos A/2\cdot\cos C/2}{\cos B/2}$
Ex radius (opposite to C)	r_3	$r_3=\frac{\Delta}{s-c}$	$s\tan\frac{C}{2}$	$\frac{c\cos A/2\cdot\cos B/2}{\cos C/2}$

1.3 m-n Theorem

Consider a triangle ABC where D is a point dividing BC internally in the ratio $m : n$.

$$\Rightarrow \quad \frac{BD}{DC} = \frac{m}{n}$$

The segment AD makes angles α and β with sides AB and AC respectively.



Theorem: (1) $(m + n) \cot \theta = m \cot \alpha - n \cot \beta$

(2) $(m + n) \cot \theta = n \cot B - m \cot C$

Proof: Apply sine rule in ΔABD and ΔACD:

$$\Rightarrow \quad \frac{BD}{AD} = \frac{\sin \alpha}{\sin B} \quad \text{and} \quad \frac{CD}{AD} = \frac{\sin \beta}{\sin C}$$

Divide the two equations to get :

$$\Rightarrow \quad \frac{BD}{CD} = \frac{\sin \alpha \sin C}{\sin B \sin \beta} = \frac{m}{n} \qquad \text{. . . (i)}$$

1. Put $B = \theta - \alpha$ and $C = \pi - (\theta + \beta)$

$$\Rightarrow \quad \frac{m}{n} = \frac{\sin \alpha \sin \left(\pi - \overline{\theta + \beta}\right)}{\sin (\theta - \alpha) \sin \beta} \quad \Rightarrow \quad \frac{m \sin (\theta - \alpha)}{\sin \alpha} = \frac{n \sin (\theta + \beta)}{\sin \beta}$$

$$\Rightarrow \quad \frac{m \sin (\theta - \alpha)}{\sin \alpha \sin \theta} = \frac{n \sin (\theta + \beta)}{\sin \theta \sin \beta} \quad \Rightarrow \quad m (\cot \alpha - \cot \theta) = n (\cot \beta + \cot \theta)$$

$$\Rightarrow \quad (m + n) \cot \theta = m \cot \alpha - n \cot \beta$$

2. Put $\alpha = \theta - B$ and $\beta = \pi - (\theta + C)$

$$\Rightarrow \quad \frac{m}{n} = \frac{\sin (\theta - B) \sin C}{\sin B \sin (\pi - \theta + C)} \quad \Rightarrow \quad \frac{m \sin (\theta + C)}{\sin C} = \frac{n \sin (\theta - B)}{\sin B}$$

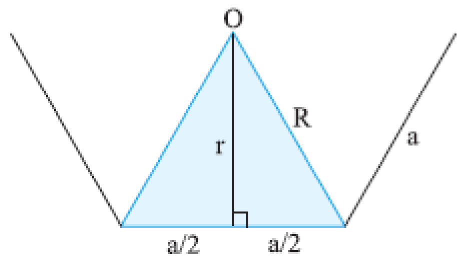
$$\Rightarrow \quad \frac{m \sin (\theta + C)}{\sin \theta \sin C} = \frac{n \sin (\theta - B)}{\sin \theta \sin B} \quad \Rightarrow \quad m (\cot C + \cot \theta) = n (\cot B - \cot \theta)$$

$$\Rightarrow \quad (m + n) \cot \theta = n \cot B - m \cot C$$

1.4 Regular *n* sides Polygon

If the polygon has 'n' sides, Sum of the internal angles is $(n-2)\,\pi$ and each angle is $\dfrac{(n-2)\,\pi}{n}$.

a = side length ; r = in-radius ; R = circum-radius



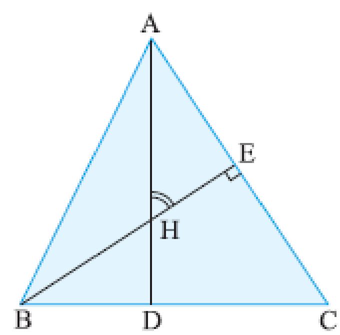
$$r = \frac{a}{2\tan\frac{\pi}{n}} \quad \text{and} \quad R = \frac{a}{2\sin\frac{\pi}{n}}$$

$$\text{Area of polygon} = \frac{1}{4}na^2 \,.\cot\left(\frac{\pi}{n}\right) = nr^2\tan\left(\frac{\pi}{n}\right) = \frac{n}{2}R^2\sin\frac{2\pi}{n}.$$

1.5 More Results

1.5.1 Distance of orthocentre from vertices of triangle

AD, BE are altitudes and H is the orthocentre of a triangle ΔABC as shown.



As quadrilateral $CEHD$ is cyclic,

angle EHA = angle C

from ΔAHE, $AH\sin C = AE$

$\Rightarrow$ $AH\sin C = AB\cos A$ [using Δ ABE]

$\Rightarrow$ $AH = \dfrac{c\cos A}{\sin C} = \left(\dfrac{c}{\sin C}\right)\cos A$

$\Rightarrow$ $AH = 2R\cos A$

$\Rightarrow$ distances of orthocentre (H) from the vertices A, B & C are :

$2R\cos A$, $2R\cos B$ and $2R\cos C$ respectively.

1.5.2 Distance of orthocentre from sides of triangle

$DH = AD - AH$

$\Rightarrow$ $DH = AB\sin B - 2R\cos A$

$\Rightarrow$ $DH = c\sin B - 2R\cos A$

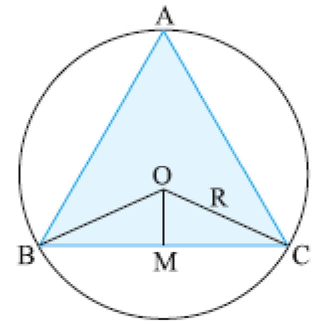
$\Rightarrow$ $DH = 2R\sin C\sin B + 2R\cos(B+C)$ $\quad (\because A = \pi - (B+C))$

$\Rightarrow$ $DH = 2R\cos B\cos C$

$\Rightarrow$ The distances of orthocentre (H) from the sides BC, CA & AB are :

$2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$ respectively.

1.5.3 Distance of circumcentre *O* from sides :



$\angle BOC = 2A$

$\Rightarrow \quad \angle COM = A$

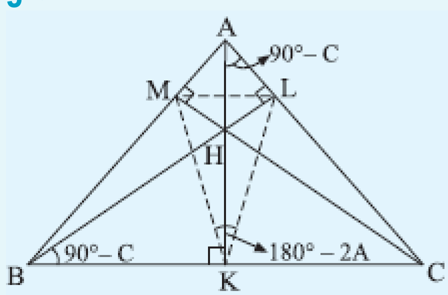
$\Rightarrow \quad OM = R\cos A$

$\Rightarrow$ distances of circumcentre from sides *BC*, *CA* and *AB* are $R\cos A$, $R\cos B$ and $R\cos C$ respectively.

1.6 Pedal Triangle

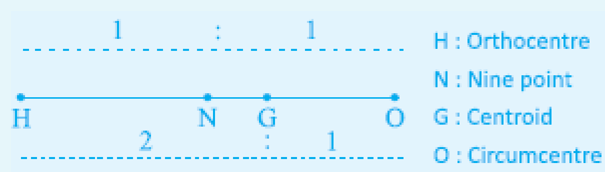
Let *ABC* be any triangle, and let *AK*, *BL* and *CM* be the perpendicular from *A*, *B* and *C* upon the opposite sides of the triangle. These three perpendiculars meet at a point '*H*' which is called the orthocentre of the triangle *ABC*. The triangle *KLM*, formed by joining the feet of these perpendiculars is called the pedal triangle of *ABC*.

Important Points for Pedal Triangle KLM :



ΔKLM is the pedal triangle of ΔABC

- Angles of pedal triangle : K = 180° – 2A, L = 180° – 2B, M = 180° – 2C
- Sides of pedal triangle : LM = $a\cos A$, MK = $b\cos B$, KL = $c\cos C$
- Area of the pedal triangle : $\frac{1}{2}R^2 . \sin 2A . \sin 2B . \sin 2C$
- Circum radius of pedal triangle : $\frac{R}{2}$
- In-radius of pedal triangle : $2R\cos A . \cos B . \cos C$
- Orthocentre of ΔABC is the incentre of the pedal ΔKLM
- Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocentre of the given triangle. This circle is known as nine-point circle.
- Nine point $\equiv$ Circum - centre of the dedal triangle
- Circum-centre of the pedal triangle of a given tri angle bisects the line joining the circumcentre of the triangle to the orthocentre.



1.7 Important Theorem

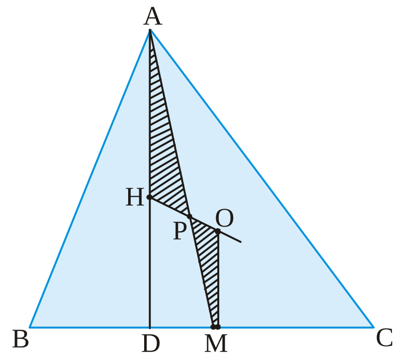
The centroid , circumcentre & orthocentre in any triangle are collinear . The centroid divides the line joining orthocentre and circumcentre in 2 : 1 internally .

Proof : Let M be the mid-point of BC and AD be the altitude of ΔABC as shown.

O is the circumcentre and hence OM is perpendicular to side BC.

We have :

$AH = 2R\cos A$; $OM = R\cos A$; $\Delta APH \sim \Delta MPO$ (equiangular)



$$\Rightarrow \quad \frac{AP}{MP} = \frac{PH}{PO} = \frac{AH}{MO}$$

$$\Rightarrow \quad \frac{AP}{MP} = \frac{PH}{PO} = \frac{2R\cos A}{R\cos A}$$

$$\Rightarrow \quad \frac{AP}{MP} = \frac{2}{1} \quad \text{and} \quad \frac{PH}{PO} = \frac{2}{1}$$

As P divides median AM in 2 : 1 , the point P is the centroid and hence H, P, O are collinear.

$$\Rightarrow \quad \frac{PH}{PO} = \frac{2}{1} \quad \Rightarrow$$ The centroid divides HO in 2 : 1

1.8.1 Ambiguous Case

If two sides and an angle opposite to one of these sides is given, two triangles can be drawn. This situation is known as ambiguons case. This possibility can arise if :

a, b and A (or B) are given.

or if : b, c *and* B (or C) are given.

or if : c, a *and* C (or A) are given.

1.8.1 a, b and A (or B) are given $(A < \pi/2 \ \& \ a < b)$

(i) The two values of the third side c can be calculated from the cosine rule :

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$c^2 - (2b\cos A)c + b^2 - a^2 = 0 \qquad \ldots (i)$$

The two roots of this quadratic equation are two values of the third side i.e. c_1 *and* c_2

$\Rightarrow \quad c_1 + c_2 = 2b\cos A \quad \& \quad c_1 \cdot c_2 = b^2 - a^2$

(ii) The triangle is possible only if the above quadratic has real roots.

$\Rightarrow \quad 4b^2\cos^2 A - 4(b^2 - a^2) \geq 0 \quad \Rightarrow \quad c_1 \cdot c_2 = b^2 - a^2$

$\Rightarrow \quad a \geq b\sin A$ [condition for the triangle to be possible]

For $a = b\,sinA$, there is only one triangle (right angled) and for $a > b\,sinA$, there are two triangles.

(iii) The angles B and C can be found using the sine rule :

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \quad \sin B = \frac{b\sin A}{a}$$

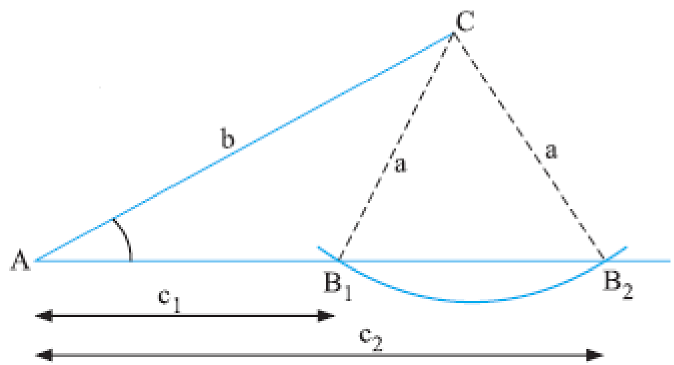
$$\Rightarrow \quad B_1 = \sin^{-1}\left(\frac{b\sin A}{a}\right) \text{ and } B_2 = \pi - \sin^{-1}\left(\frac{b\sin A}{a}\right)$$

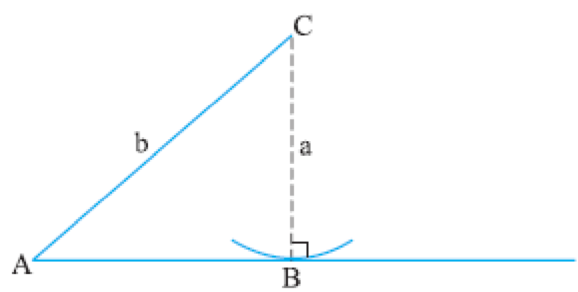
The corresponding values of angle C are :

$C_1 = \pi - B_1 - A$ and $C_2 = \pi - B_2 - A$

For $a > b$ sinA, two triangles :

ΔAB_1C and ΔAB_2C are possible

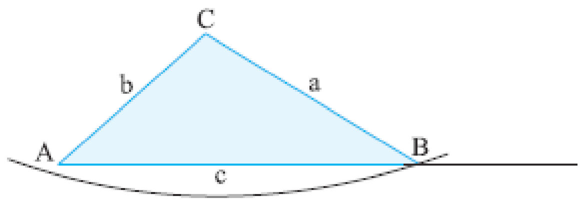




For $a = b\,sinA$, only one triangle ΔABC is possible.

1.8.2 a, b and A are given $(A < \pi/2 \ \& \ \ a > b)$:

The quadratic (i) in this case has one positive and one negative root. The value of third side c is equal to the positive root of the quadratic. Hence there is only one triangle.



1.8.3 a, b and A are given $(A > \pi/2)$:

There will be only one triangle and that is possible only if $a > b.$ The quadratic (i) has one positive and one negative root. The value of third side c is equal to the positive root of the quadratic.

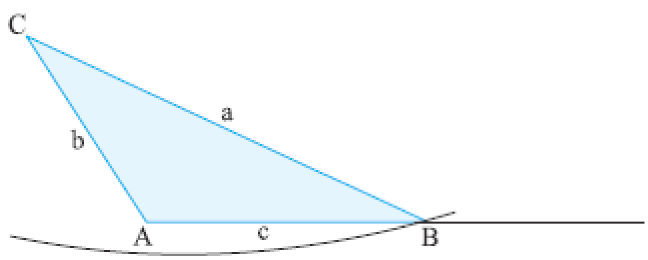


Illustration - 12 *Prove the following results :*

(i) $r = (s-a)\, tan\,\dfrac{A}{2} = (s-b)\, tan\,\dfrac{B}{2} = (s-c)\, tan\,\dfrac{C}{2}$

(ii) $r_1 = s = tan\dfrac{A}{2}, r_2 = s\ tan\dfrac{B}{2}, r_3 = s\ tan\ \dfrac{C}{2}$ (iii) $r = 4R\ sin\ \dfrac{A}{2}\ sin\dfrac{B}{2}\ sin\ \dfrac{C}{2}$

SOLUTION :

(i) $r = \dfrac{\Delta}{s} = (s-a)\dfrac{\Delta}{s(s-a)}$

$\Rightarrow \quad r = (s-a)\tan\dfrac{A}{2} \qquad \left(\text{using } \cot\dfrac{A}{2} = \dfrac{s(s-a)}{\Delta}\right)$

The other results follow by symmetry.

(ii) $r_1 = \dfrac{\Delta}{s-a} = \dfrac{s\Delta}{s(s-a)} = s\tan\dfrac{A}{2}$

The other results follow by symmetry.

(iii) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$; $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$; $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ba}}$

Multiply the three results to get :

$\Rightarrow \quad \sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{(s-a)(s-b)(s-c)}{abc} \quad \Rightarrow \quad \sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \left(\dfrac{\Delta^2}{s}\right)\left(\dfrac{1}{4R\Delta}\right)$

$\Rightarrow \quad \sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \left(\dfrac{\Delta}{s}\right)\left(\dfrac{1}{4R}\right) \quad \Rightarrow \quad r = \dfrac{\Delta}{s} = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

Illustration - 13 *Show that in a triangle* $\Delta ABC : a\cot A + \cot B + c\cot C = 2(R+r)$.

SOLUTION :

$$\text{L.H.S.} = \sum 2r\sin A\cot A = 2R\sum \cos A$$

$$\Rightarrow \quad \text{L.H.S.} = 2R\sum \cos A$$

$$\Rightarrow \quad \text{L.H.S.} = 2R\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) \quad (\because A+B+C=\pi)$$

$$\Rightarrow \quad \text{L.H.S.} = 2R + 8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$\Rightarrow \quad \text{L.H.S.} = 2R + 2r = \text{R.H.S.}$ [using the result of last Illustration]

Illustration - 14 *Show that :* $\dfrac{r_1}{bc} + \dfrac{r_2}{ca} + \dfrac{r_3}{ba} = \dfrac{1}{r} - \dfrac{1}{2R}$.

SOLUTION :

$$\text{L.H.S.} = \frac{\Delta}{abc}\left(\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-C}\right)$$

$$= \frac{\Delta}{abc}\left(\frac{a}{s-a}+1+\frac{b}{s-b}+1+\frac{c}{s-c}\right)-\frac{1}{2R}=\frac{\Delta}{abc}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right)-\frac{1}{2R}$$

$$= \frac{\Delta}{abc}\left(\frac{s(2s-a-b)}{(s-a)(s-b)}+\frac{c}{s-c}\right)-\frac{1}{2R}=\frac{\Delta}{abc}\left(\frac{s}{(s-a)(s-b)}+\frac{1}{s-c}\right)-\frac{1}{2R}$$

$$= \frac{\Delta}{abc}\left(\frac{s(2s-a-b)}{(s-a)(s-b)}+\frac{c}{s-c}\right)-\frac{1}{2R}=\frac{\Delta}{abc}\left(\frac{s}{(s-a)(s-b)}+\frac{1}{s-c}\right)-\frac{1}{2R}$$

$$= \frac{\Delta}{abc}\left(\frac{2s^2-s(2s)+ab}{(s-a)(s-b)(s-c)}\right)-\frac{1}{2R}=\frac{\Delta}{(s-a)(s-b)(s-c)}-\frac{1}{2R}$$

$$\text{L.H.S.} = \frac{\Delta s}{\Delta^2}-\frac{1}{2R}=\frac{1}{r}-\frac{1}{2R}=\text{R.H.S.}$$

Illustration - 15

In a ΔABC, show that :

(i) $c^2=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}$

(ii) $a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}$

(iii) $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$

SOLUTION :

(i) $\text{R.H.S.} = (a-b)^2\left(\frac{1+\cos C}{2}\right)+(a+b)^2\left(\frac{1-\cos C}{2}\right)$

$\text{R.H.S.} = \frac{1}{2}\left[(a-b)^2+(a+b)^2\right]+\frac{1}{2}\cos C\left[(a-b)^2-(a+b)^2\right]$

$\text{R.H.S.} = a^2+b^2+\frac{1}{2}\cos C-(-4ac)=c^2$ [using cosine rule]

(ii) $\text{L.H.S.} = a\sin\left(\frac{A}{2}+B\right)=2R\sin A\sin\left(\frac{A}{2}+B\right)$ [using sine rule]

$$\text{L.H.S.} = 2R\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\sin\left(\frac{A}{2}+B\right) = 2R\sin\frac{A}{2}\left[2\cos\frac{A}{2}\sin\left(\frac{A}{2}+B\right)\right]$$

$$\text{L.H.S.} = 2R\ \sin\frac{A}{2}\left[\sin(A+B)-\sin(-B)\right] = 2R\sin\frac{A}{2}\left[\sin C+\sin B\right]$$

$$\text{L.H.S.} = \sin\frac{A}{2}\left[2R\sin C+2R\sin B\right]$$

$$\text{L.H.S.} = \sin\frac{A}{2}(c+b) = \text{R.H.S.}$$

Note : Try to prove the same identity using R.H.S.

(iii) $\text{L.H.S.} = (b+c)\cos A+(c+a)\cos B+(a+b)\cos C$

$\text{L.H.S.} = [c\cos B+b\cos C]+[a\cos C+c\cos A]+[b\cos A+a\cos B]$

$\text{L.H.S.} = a+b+c = \text{R.H.S.}$

Illustration - 16 *In a Δ ABC, prove that :* $\left(b^2-c^2\right)\cot A+\left(c^2-a^2\right)\cot B+\left(a^2-b^2\right)\cot C = 0.$

SOLUTION :

Starting from L.H.S.

$$= \sum\left(b^2-c^2\right)\cot A$$

$$= 4R^2\sum\left(\sin^{2'}B-\sin^2 C\right)\cot A$$

[using sine rule]

$$= 4R^2\sum\sin(B+C)\sin(B-C)\ \cot A$$

$$= 4R^2\sum\sin A\sin(B-C)\frac{\cos A}{\sin A}$$

$$= -2R^2\sum 2\cos(B+C)\sin(B-C)$$

[using $\cos A = -\cos(B+C)$]

$$= -2R^2\sum(\sin 2B-\sin 2C)$$

$$= -2R^2[(\sin 2B-\sin 2C)+(\sin 2C-\sin 2A)+(\sin 2A-\sin 2B)]$$

$$= 0 = \text{R.H.S.}$$

Illustration - 17 *In a Δ ABC, show that :* $(a+b+c)\left[\tan\frac{A}{2}+\tan\frac{B}{2}\right]=2c\cot\frac{C}{2}$

SOLUTION :

Starting From L.H.S.

$= (a+b+c)$

$\left[\frac{(s-b)(s-c)}{\Delta}+\frac{(s-c)(s-a)}{\Delta}\right]$

$=\left(\frac{a+b+c}{\Delta}\right)(s-c)[s-b+s-a]$

$=\left(\frac{s-c}{\Delta}\right)(a+b+c)(c)$

$=\frac{(s-c)\,2sc}{\Delta}=2c\left[\frac{s(s-c)}{\Delta}\right]$

$=2c\cot\frac{C}{2}=$ R.H.S.

Illustration - 18 *In a Δ ABC, prove that :*

(i) $r_1+r_2+r_3-r=4R$ (ii) $rr_1+rr_2+rr_3=ab+bc+ca-s^2$.

SOLUTION :

(i) Starting from L.H.S.

$=\left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right)+\left(\frac{\Delta}{s-c}-\frac{\Delta}{s}\right)$

$=\Delta\frac{(2s-\overline{a+b})}{(s-a)}+\frac{\Delta(s-\overline{s-c})}{s(s-c)}$

$=\frac{\Delta c}{(s-a)(s-b)}+\frac{\Delta c}{s(s-c)}$

$=\frac{\Delta c}{s(s-a)(s-b)(s-c)}$

$[s(s-c)+(s-a)(s-b)]$

$=\frac{c}{\Delta}\left[2s^2-2s^2+ab\right]=\frac{abc}{\Delta}=4\left(\frac{abc}{4\Delta}\right)$

$=4R$

(i) Starting from L.H.S.

$=\frac{\Delta^2}{s}\left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right]$

$=\frac{\Delta^2}{s}\left[\frac{\sum(s-b)(s-c)}{(s-a)(s-b)(s-c)}\right]$

$=3x^2-2x(a+b+c)+bc+ca+ab$

$=3x^2-4s^2+bc+ca+ab$

$=ab+bc+ca-s^2=$ R.H.S

Illustration - 19

In a ΔABC, show that : $\dfrac{(a+b+c)^2}{a^2+b^2+c^2} = \dfrac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$

SOLUTION :

Starting from R.H.S.

$$= \frac{\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}}{\frac{b^2+c^2-a^2}{4\Delta}+\frac{c^2+a^2-b^2}{4\Delta}+\frac{a^2+b^2-c^2}{4\Delta}}$$

$$= \frac{4s[s-a+s-b+s-c]}{b^2+c^2+a^2}$$

$$= \frac{4s(3s-2s)}{a^2+b^2+c^2} = \frac{4s^2}{a^2+b^2+c^2} = \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

= L.H.S.

Illustration - 20 *If a^2, b^2, c^2 in a Δ ABC are in A.P. Prove that cot A , cot B and cot C are also in A.P.*

SOLUTION :

$\cot A$, $\cot B$ and $\cot C$ are in A.P.

if $\quad \cot A - \cot B = \cot B - \cot C$

if $\quad \dfrac{\cos A}{\sin A} - \dfrac{\cos B}{\sin B} = \dfrac{\cos B}{\sin B} - \dfrac{\cos C}{\sin C}$

if $\quad \dfrac{\sin(B-A)}{\sin A \sin B} = \dfrac{\sin(C-B)}{\sin B \sin C}$

if $\quad \sin(B-A)\sin C = \sin(C-B)\sin A$

if $\quad \sin(B-A)\sin(B+A) = \sin(C-B)\sin(C+B)$

if $\quad \sin^2 B - \sin^2 A = \sin^2 C - \sin^2 B$

if $\quad \dfrac{b^2}{4R^2} - \dfrac{a^2}{4R^2} = \dfrac{c^2}{4R^2} - \dfrac{b^2}{4R^2}$ [using sine rule]

if $\quad b^2 - a^2 = c^2 - b^2 \quad \Rightarrow \quad 2b^2 = a^2 + c^2$

if $\quad a^2, b^2, c^2$ are in A.P.

Hence $\cot A$, $\cot B$ and $\cot C$ are in A.P.

Illustration - 21 *If x, y, z are respectively the perpendiculars from circumcentre to the sides BC, CA, AB of the triangle ABC, Prove that :* $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z} = \dfrac{abc}{4xyz}$.

SOLUTION :

We know that : $x = R\cos A$, $y = R\cos B$, $z = R\cos C$

Consider L.H.S. :

$$= \frac{a}{R\cos A} + \frac{b}{R\cos B} + \frac{c}{R\cos C} = \frac{2R\sin A}{R\cos A} + \frac{2R\sin B}{R\cos B} + \frac{2R\cos C}{R\cos C}$$

$$= 2(\tan A + \tan B + \tan C) = 2(\tan A \tan B \tan C) \qquad [\text{as } A + B + C = \pi]$$

$$= 2\left[\frac{\sin A}{\cos A}\frac{\sin B}{\cos B}\frac{\sin C}{\cos C}\right] = \frac{2}{8R^3}\left[\frac{abc}{\cos A\cos B\cos C}\right] \qquad [\text{using sine rule}]$$

$$= \frac{1}{4}\left[\frac{abc}{(R\cos A)(R\cos B)(R\cos C)}\right] = \frac{1}{4}\frac{abc}{xyz} = \text{R.H.S.}$$

Illustration - 22 *I is the incentre of ΔABC and P_1, P_2, P_3 are respectively the radii of the circumcircles of ΔIBC, ΔICA and ΔIAB, prove that : $P_1 P_2 P_3 = 2R^2 r$.*

SOLUTION :

$$\angle BIC = \pi - \frac{1}{2}(B + C)$$

$$= \pi - \frac{1}{2}(\pi - A) = \frac{\pi}{2} + \frac{A}{2}$$

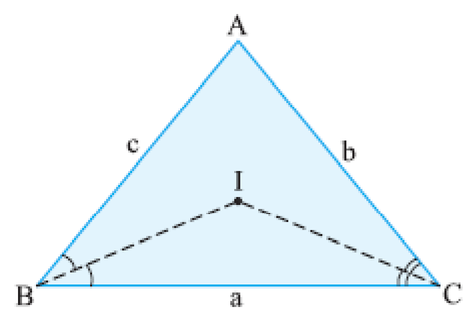
Circumradius of ΔIBC is :

$$P_1 = \frac{BC}{2\sin\angle BIC} = \frac{BC}{2\sin\left(\frac{\pi}{2} + \frac{A}{2}\right)} = \frac{a}{2\cos\frac{A}{2}}$$

Similarly we can show that :

$$P_2 = \frac{b}{2\cos\frac{B}{2}} \quad \text{and} \quad P_3 = \frac{c}{2\cos\frac{C}{2}}$$

$$\Rightarrow \quad P_1 P_2 P_3 = \frac{abc}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$



$$= \frac{8R^3\sin A\sin B\sin C}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= \frac{8R^3\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{A}{2}\cos\frac{C}{2}}$$

$$= 8R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2R^2 r = \text{R.H.S.}$$

Illustration - 23 *If ABCD is a cyclic quadrilateral, show that AC . BD = AB . CD + BC . AD*

SOLUTION :

Let $AB = a$, $BC = b$, $CD = c$, $DA = d$

Using cosine rule in ΔABC and ΔADC :

$AC^2 = a^2 + b^2 - 2ab \cos B$

$AC^2 = c^2 + d^2 - 2cd \cos D$

and $B + D = \pi$

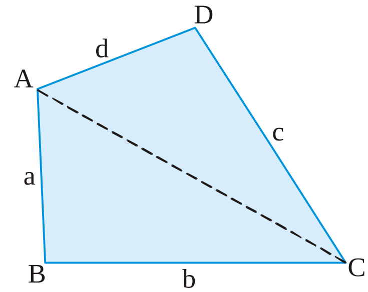
$\Rightarrow \cos B + \cos D = 0$

$\Rightarrow AC^2 (cd + ab) = (a^2 + b^2)\, cd + (c^2 + d^2)\, ab$

$$\Rightarrow AC^2 = \frac{(a^2 cd + c^2 ab) + (b^2 cd + d^2 ab)}{cd + ab}$$

$$\Rightarrow AC^2 = \frac{(ad + bc)(ac + bd)}{cd + ab}$$

Similarly by taking another diagonal BD, we can show that :



$$BD^2 = \frac{(ba + cd)(bd + ca)}{da + bc}$$

Multiplying the two equations:

$\Rightarrow (AD . BD)^2 = (ac + bd)^2$

$\Rightarrow AC . BD = ac + bd$

$\Rightarrow AC . BD = AB . CD + BC . AD$

Illustration - 24 *Show that :* $\left[\cot\frac{A}{2} + \cot\frac{B}{2}\right]\left[a\sin^2\frac{B}{2} + b\sin^2\frac{A}{2}\right] = c\cot\frac{C}{2}$

SOLUTION :

Taking L.H.S. : $= \left[\frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta}\right]\left[\frac{a(s-c)(s-a)}{ca} + \frac{b(s-b)(s-c)}{bc}\right]$

$$= \frac{s}{\Delta}[2s - a - b]\left(\frac{s-c}{c}\right)(2s - a - b)$$

$$= \frac{s(s-c)}{\Delta c}c^2 = c\frac{s(s-c)}{\Delta} = c\cot\frac{C}{2} = \text{R.H.S.}$$

Illustration - 25 *In a ΔABC, show that $a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3abc$.*

SOLUTION :

$$= \sum a^3 \cos (B-C) = \sum a^2 (2R \sin A) \cos (B-C)$$

$$= R \sum a^2 \left(2 \sin \overline{B+C} \cos \overline{B-C}\right) = R \sum a^2 (\sin 2B + \sin 2C)$$

$$= 2R \sum a^2 (\sin B \cos B + \sin C \cos C) = \sum a^2 (b \cos B + c \cos C)$$

$$= a^2 \left(\underline{b \cos B} + \overline{c \cos C}\right) + b^2 \left(c \cos C + \underline{a \cos A}\right) + c^2 \left(\overline{a \cos A} + b \cos B\right)$$

$$= ab (a \cos B + b \cos A) + ac (a \cos C + c \cos A) + bc (b \cos C + c \cos B)$$

$= abc + acb + bca$ [using projection formula]

$= 3abc =$ R.H.S.

Illustration - 26 *If the sides a , b, c of a ABC are in A.P. , then prove that cot A/2 , cot B/2 and cot C/2 are also in A.P.*

SOLUTION :

a, b, c are in A.P. $\Rightarrow a - b = b - c$

$$\Rightarrow \sin A - \sin B = \sin B - \sin C$$

$$\Rightarrow 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$= 2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}$$

$$\Rightarrow \sin \frac{C}{2} \sin \frac{A-B}{2} = \sin \frac{A}{2} \sin \frac{B-C}{2}$$

$$\Rightarrow \frac{\sin\left(\frac{A}{2} - \frac{B}{2}\right)}{\sin \frac{A}{2} \sin \frac{B}{2}} = \frac{\sin\left(\frac{B}{2} - \frac{C}{2}\right)}{\sin \frac{B}{2} \sin \frac{C}{2}}$$

$$\Rightarrow \cot \frac{B}{2} - \cot \frac{A}{2} = \cot \frac{C}{2} - \cot \frac{B}{2}$$

$$\Rightarrow \cot \frac{A}{2}, \cot \frac{B}{2} \text{ and } \cot \frac{C}{2} \text{ are in A.P.}$$

Illustration - 27

In a ΔABC, prove that $A = B$ if : $a \tan A + b \tan B = (a+b)\tan\left(\frac{A+B}{2}\right)$.

SOLUTION :

Rearranging the terms of the given expression as follows:

$$\Rightarrow \quad a \tan A - a \tan\frac{A+B}{2}$$

$$= b \tan\frac{A+B}{2} - b \tan B$$

$$\Rightarrow \quad \frac{a \sin\left(A - \frac{A+B}{2}\right)}{\cos A \cos\frac{A+B}{2}} = \frac{b \sin\left(\frac{A+B}{2} - B\right)}{\cos\frac{A+B}{2} \cos B}$$

$$\Rightarrow \quad \frac{2R \sin A \sin\left(\frac{A-B}{2}\right)}{\cos A}$$

$$= \frac{2R \sin B \sin\left(\frac{A-B}{2}\right)}{\cos B}$$

$$\Rightarrow \quad \sin\left(\frac{A-B}{2}\right)[\tan A - \tan B] = 0$$

$$\Rightarrow \quad \sin\left(\frac{A-B}{2}\right) = 0$$

or $\quad \tan A - \tan B = 0 \Rightarrow \quad A = B$

Illustration - 28

If the sides of a triangle are in A.P. and the greatest angle exceeds the smallest angle by a, show that the sides are in the ratio $1-x : 1 : 1+x$; where $x = \sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}$

SOLUTION :

Let $A > B > C$

$\Rightarrow \quad A - C = a \quad$ and $\quad 2b = a + c$

We will first find the values of $\sin B/2$ and $\cos B/2$.

$$2b = a + c$$

$$\Rightarrow \quad 2 \sin B = \sin A + \sin C$$

$$\Rightarrow \quad 4 \sin\frac{B}{2}\cos\frac{B}{2} = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$$

$$\Rightarrow \quad 4 \sin\frac{B}{2}\cos\frac{B}{2} = 2\cos\frac{B}{2}\cos\frac{\alpha}{2}$$

$$\Rightarrow \quad \sin\frac{B}{2} = \frac{1}{2}\cos\frac{\alpha}{2} \Rightarrow \sin\frac{B}{2} = \frac{\sqrt{1+\cos\alpha}}{2\sqrt{2}}$$

$$\Rightarrow \quad \cos\frac{B}{2} = \sqrt{1-\sin^2\frac{B}{2}} = \frac{\sqrt{7-\cos\alpha}}{2\sqrt{2}} \quad \ldots \text{(i)}$$

Consider :

$$\frac{a}{c} = \frac{\sin A}{\sin C} \qquad \text{[using sine rule]}$$

$$\Rightarrow \frac{a+c}{a-c} = \frac{\sin A + \sin C}{\sin A - \sin C}$$

$$\Rightarrow \frac{a+c}{a-c} = \frac{2\sin B}{2\cos\frac{A+C}{2}\sin\frac{A-C}{2}}$$

$$\Rightarrow \frac{a+c}{a-c} = \frac{2\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)}{2\sin\frac{B}{2}\sin\frac{\alpha}{2}}$$

$$\Rightarrow \frac{a+c}{a-c} = 2\frac{\cos B/2}{\sin\alpha/2}$$

$$\Rightarrow \frac{a+c}{a-c} = \frac{2\left(\frac{\sqrt{7-\cos\alpha}}{2\sqrt{2}}\right)}{\sin\alpha/2} \quad \text{[using (i)]}$$

$$\Rightarrow \frac{a+c}{a-c} = \frac{\sqrt{7-\cos\alpha}}{\sqrt{1-\cos\alpha}} \quad \Rightarrow \frac{a+c}{a-c} = \frac{1}{x}$$

$$\Rightarrow \frac{a}{c} = \frac{1+x}{1-x} \quad \Rightarrow \frac{a}{1+x} = \frac{c}{1-x}$$

$$\Rightarrow \frac{a}{1+x} = \frac{c}{1-x} = \frac{a+c}{2}$$

$$\Rightarrow \frac{a}{1+x} = \frac{c}{1-x} = \frac{2b}{2} \quad \Rightarrow \frac{a}{1+x} = \frac{b}{1} = \frac{c}{1-x}$$

Illustration - 29

D is the mid point of BC in a ΔABC. If AD is perpendicular to AC, show that :

$$\cos A\cos C = \frac{2(c^2-a^2)}{3ac}$$

SOLUTION :

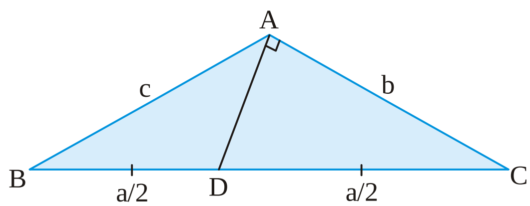
The value of cos C can be found by cosine rule in ΔABC or ΔADC.

From ΔABC: $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

From ΔADC: $\cos C = \dfrac{b}{a/2}$

$$\Rightarrow \frac{2b}{a} = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow b^2 = \frac{a^2-c^2}{3} \quad \ldots \text{(i)}$$



L.H.S. = $\cos A\cos C$

$$= \left(\frac{b^2+c^2-a^2}{2bc}\right)\left(\frac{b}{a/2}\right)$$

$$= \frac{b^2+c^2-a^2}{ac} = \frac{\frac{a^2-c^2}{3}+c^2-a^2}{ac} \quad \text{[using (i)]}$$

$$= \frac{2(c^2-a^2)}{3ac} = \text{R.H.S.}$$

Illustration - 30 *Let O be a point inside a ΔABC such that ∠OAB = ∠OBC = ∠OCA = ω. Show that :*

(i) $\cot\omega = \cot A + \cot B + \cot C$ *(ii)* $\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$

SOLUTION :

(i) Apply the sine rule in $\Delta\, OBC$:

$$\frac{OB}{a} = \frac{\sin(C-\omega)}{\sin[\pi - (\omega + C - \omega)]}$$

$$\Rightarrow \frac{OB}{a} = \frac{\sin(C-\omega)}{\sin C} \qquad \ldots \text{(i)}$$

Applying sine rule in $\Delta\, OAB$ and proceeding similarly:

$$\Rightarrow \frac{OB}{c} = \frac{\sin\omega}{\sin B} \qquad \ldots \text{(ii)}$$

Divide (i) by (ii) to get :

$$\frac{c}{a} = \frac{\sin(C-\omega)\sin B}{\sin\omega \sin C} \qquad [\text{sine rule in } \Delta ABC]$$

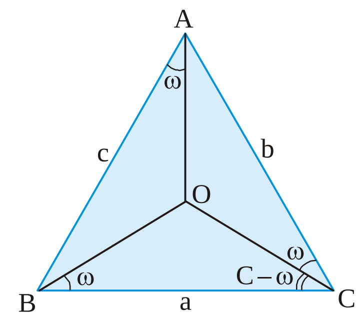
$$\Rightarrow \frac{\sin C}{\sin A \sin B} = \frac{\sin(C-\omega)}{\sin\omega\sin C}$$

$$\Rightarrow \frac{\sin(A+B)}{\sin A \sin B} = \frac{\sin(C-\omega)}{\sin\omega\sin C}$$

$$\Rightarrow \frac{\sin A\cos B + \cos A \sin B}{\sin A \sin B}$$

$$= \frac{\sin C\cos\omega - \cos C\sin\omega}{\sin\omega\sin C}$$

$$\cot A + \cot B + \cot C = \cot\omega$$



(ii) Squaring the above result :

$$\cot^2\omega = (\cot A + \cot B + \cot C)^2$$

$$\Rightarrow \operatorname{cosec}^2\omega - 1 = \sum\cot^2 A + 2\sum\cot A\cot B$$

$$\Rightarrow \operatorname{cosec}^2\omega - 1 = \sum(\operatorname{cosec}^2 A - 1) + 2$$

$$\left[\text{As in a } \Delta, \sum\cot A\cot B = 1\right]$$

$$\Rightarrow \operatorname{cosec}^2\omega - 1 = \sum\operatorname{cosec}^2 A - 3 + 2$$

$$\Rightarrow \operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$$

Illustration - 31 *For a triangle ABC, it is given that : cos A + cos B + cos C = 3/2 . Prove that the triangle is equilateral.*

SOLUTION :

Consider $\cos A + \cos B + \cos C = 3/2 \quad \Rightarrow \quad \frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab} = \frac{3}{2}$

$$\Rightarrow \quad a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2) = 3abc$$

$$\Rightarrow \quad a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) = a^3 + b^3 + c^3 + 3abc$$

$\Rightarrow \quad a(b^2+c^2-2bc)+b(c^2+a^2-2ac)+c(a^2+b^2-2ab)=a^3+b^3+c^3-3abc$

$\Rightarrow \quad a(b-c)^2+b(c-a)^2+c(a-b)^2-1/2(a+b+c)[(b-c)^2+(c-a)^2+(a-b)^2]=0$

$\Rightarrow \quad (b-c)^2(b+c-a)+(c-a)^2(c+a-b)+(a-b)^2(a+b-c)=0$ [as sum of two sides > third side]

$\Rightarrow$ All terms in L.H.S. are non-negative.

Hence each term = 0 $\quad \Rightarrow \quad b-c=c-a=a-b=0 \quad \Rightarrow \quad a=b=c$

$\Rightarrow \quad \Delta ABC$ is a equilateral.

Illustration - 32 *In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.*

SOLUTION :

Here, $\tan\left(\frac{A-B}{2}\right)=\frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ ...(i)

using Napier's analogy

$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cdot\cot\left(\frac{C}{2}\right)$ ...(ii)

from (i) and (ii) ;

$\frac{1}{3}\tan\left(\frac{A+B}{2}\right)=\frac{a-b}{a+b}\cdot\cot\left(\frac{C}{2}\right)$

$\Rightarrow \quad \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{a-b}{a+b}\cdot\cot\left(\frac{C}{2}\right)$

$\left[\text{as } A+B+C=\pi \;\therefore\; \tan\left(\frac{B+C}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}\right]$

$\Rightarrow \quad \frac{a-b}{a+b}=\frac{1}{3}$ or $3a-3b=a+b$

$2a=4b$ or $\frac{a}{b}=\frac{2}{1} \Rightarrow \frac{b}{a}=\frac{1}{2}$

Thus the ratio of the sides opposite to the angles is $b:a=1:2$.

Illustration - 33 *If g, h, k denotes the side of a pedal triangle, prove that :*

$$\frac{g}{a^2}+\frac{h}{b^2}+\frac{k}{c^2}=\frac{a^2+b^2+c^2}{2abc}$$

SOLUTION :

We have, $\quad g=a\cos A,\ h=b\cos B,\ k=c\cos C$ [as sides of pedal Δ]

$$\frac{g}{a^2}+\frac{h}{b^2}+\frac{k}{c^2}=\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$$

$$= \frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$$

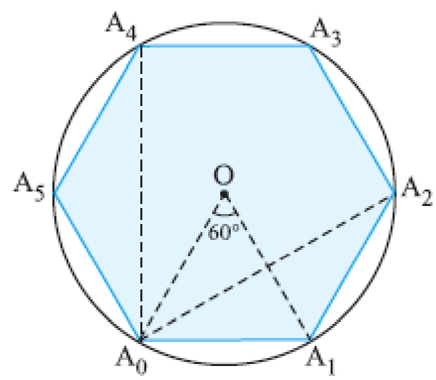
$$= \frac{a^2+b^2+c^2}{2abc}$$

$$\frac{g}{a^2} + \frac{h}{b^2} + \frac{k}{c^2} = \frac{a^2+b^2+c^2}{2abc}$$

Illustration - 34 *If A_0, A_1, A_2, A_4 and A_5 be the consecutive vertices of a regular hexagon inscribed in a unit circle. Then find the product of length of A_0A_1, A_0A_2 and A_0A_4.*

SOLUTION :

We know, in hexagon central angle is $\frac{360}{6} = 60°$ and each angle



$$= \frac{(2n-4)\pi}{2n} = \frac{(6-2)\times 180°}{6} = 120°$$

As the unit circumcircle is unit circle, $\therefore$ radius $OA_0 = 1 = r$

$\because$ $\Delta A_0A_1A_2$,

$$\Rightarrow \cos 120° = \frac{\left(A_0A_1^2\right) + \left(A_1A_2^2\right) - \left(A_0A_2^2\right)}{2A_0A_1 \cdot A_1A_2} = \frac{1+1-\left(A_0A_2^2\right)}{2.1.1}$$

$$\Rightarrow A_0A_2 = \sqrt{3}$$

Similarly in $\Delta A_0A_5A_4$, we have

$$A_0A_4 = \sqrt{3}$$

Thus the value of ; $(A_0A_1).(A_0A_2).(A_0A_4) = 1.\sqrt{3}.\sqrt{3} = 3$ square units.

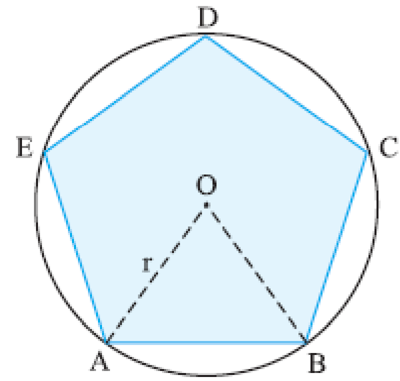
Illustration - 35 *If the area of circle is A_1 and area of regular pentagon inscribed in the circle is A_2, find the ratio of area of two.*

SOLUTION :

In ΔOAB, $\quad OA = OB = r$ and $\angle AOB = \dfrac{360°}{5} = 72°$

$\therefore \quad$ area of $\Delta AOB = \dfrac{1}{2} . r . r . \sin 72°$

$\therefore \quad$ area of $(\Delta AOB) = \dfrac{1}{2} r^2 \cos 18° \quad$ . . . (i)



$\Rightarrow$ area of pentagon = 5 (area of ΔAOB)

$\Rightarrow \quad A_2 = 5\left\{\dfrac{1}{2} r^2 \cos 18°\right\} \quad$ . . . (ii)

Also we know,

Area of circle = πr^2

$\Rightarrow \quad A_1 = \pi r^2$

Thus, $\quad \dfrac{A_1}{A_2} = \dfrac{\pi r^2}{\frac{5}{2} r^2 \cos 18°} = \dfrac{2\pi}{5} \sec\left(\dfrac{\pi}{10}\right)$

Note : The following 3 Illustrations are based on Ambigous Case (explained on Page Number 6,7)

Illustration - 36 *If a = 100 , c = 100 √2 and A = 30°, solve the triangle .*

SOLUTION :

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 - 2b\,(100\sqrt{2})\cos 30° + (100\sqrt{2})^2 - 100^2 = 0$

$b^2 - 100\sqrt{6}\, b + 10000 = 0$

$\Rightarrow \quad b = \dfrac{100\sqrt{6} \pm 100\sqrt{2}}{2} = 50\sqrt{2}\,(\sqrt{3} \pm 1)$

$\Rightarrow \quad b_1 = 50\sqrt{2}\,(\sqrt{3} - 1)\ ; \qquad b_2 = 50\sqrt{2}\,(\sqrt{3} + 1)$

$\Rightarrow \quad \sin C = \dfrac{c \sin A}{a} = \dfrac{100\sqrt{2} \sin 30°}{100} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad C_1 = 135° \quad$ and $\quad C_2 = 45°$

$\Rightarrow \quad B_1 = 180° - (135° + 30°) = 15°$

$\Rightarrow \quad B_1 = 180° - (45° + 30°) = 105°.$

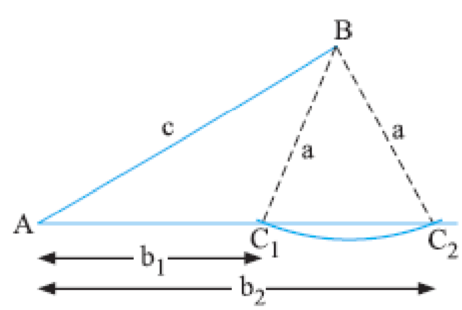


Illustration - 37 *In the ambiguous case, if the remaining angles of the triangle formed with a , b and A be B_1 , C_1 and B_2, C_2, then prove that :*

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = 2\cos A$$

SOLUTION :

$$\sin B_1 = \sin B_2 = \frac{b \sin A}{a} \quad \text{[using sine rule]}$$

$$\sin C_1 = \frac{c_1 \sin A}{a} \quad \text{and} \quad \sin C_2 = \frac{c_2 \sin A}{a}$$

$$\Rightarrow \text{L.H.S.} = \frac{\dfrac{c_1 \sin A}{a}}{\dfrac{b \sin A}{a}} + \frac{\dfrac{c_2 \sin A}{a}}{\dfrac{b \sin A}{a}}$$

$$\Rightarrow \text{L.H.S.} = \frac{c_1 + c_2}{b} = \frac{2b\cos A}{b} = 2\cos A$$

Illustration - 38 *In a Δ ABC ; a, c , A are given and $b_1 = 2b_2$, where b_1 and b_2 are two values of the third side: then prove that :* $3a = c\sqrt{1 + 8\sin^2 A}$

SOLUTION :

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Consider this equation as a quadratic in b.

$$\Rightarrow b^2 - (2c\cos A)\, b + c^2 - a^2 = 0$$

$$\Rightarrow b_1 + b_2 = 2c\cos A \;\&\; b_1 \,.\, b_2 = c^2 - a^2$$

$$\& \quad b_1 = 2b_2$$

$$\Rightarrow 3b_1 = 2c\cos A \quad \& \quad 2b_2{}^2 = c^2 - a^2$$

$$\Rightarrow 2\left(\frac{2c\cos A}{3}\right)^2 - (c^2 - a^2) = 0$$

$$\Rightarrow 8c^2\cos^2 A = 9c^2 - 9a^2$$

$$\Rightarrow 8c^2(1 - \sin^2 A) = 9c^2 - 9a^2$$

$$\Rightarrow 9a^2 = c^2 + 8c^2\sin^2 A$$

$$\Rightarrow 3a = c\sqrt{1 + 8\sin^2 A}$$

IN-CHAPTER EXERCISE - A

1. *In any triangle ABC, prove the following :*

(i) $a \cos\left(\frac{1}{2}(B-C)\right) = (b+c)\sin\left(\frac{1}{2}A\right)$

(ii) $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = \frac{a^2+b^2}{a^2+c^2}$

(iii) $a(\cos B \cos C + \cos A) = b(\cos C \cos A + \cos B) = c(\cos A \cos B + \cos C)$

(iv) (a) $\frac{c}{a-b} = \frac{\tan A/2 + \tan B/2}{\tan A/2 - \tan B/2}$

(b) $\frac{c}{a+b} = \frac{1-\tan A/2 \tan B/2}{1+\tan A/2 \tan B/2}$

(v) $\frac{a-b}{a+b} = \cot\frac{A+B}{2}\tan\frac{A-B}{2}$ *also show that the area of triangle is :* $\frac{1}{2}a^2\frac{\sin B \sin C}{\sin A}$

(vi) $\frac{\cos^2\left(\frac{B-C}{2}\right)}{(b+c)^2} + \frac{\sin^2\left(\frac{B-C}{2}\right)}{(b-c)^2} = \frac{1}{a^2}$

(vii) $\frac{a^2\sin(B-C)}{\sin B+\sin C} + \frac{b^2\sin(C-A)}{\sin C+\sin A} + \frac{c^2\sin(A-B)}{\sin A+\sin B} = 0$

(viii) $a \sin A/2 \sin (B-C)/2 + b \sin B/2 \sin (C-A)/2 + c \sin C/2 \sin (A-B)/2 = 0$

(ix) $(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0$

(x) $a^3 \sin(B-C) + b^3 \sin(C-A) + c^3 \sin(A-B) = 0$

(xi) $1-\tan\frac{A}{2}\tan\frac{B}{2} = \frac{2c}{(a+b+c)}$

(xii) $2\cos\frac{A-C}{2} = \frac{a+c}{\sqrt{a^2-ac+c^2}}$ *if angles A, B, C are in A.P.*

(xiii) $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$ *if* $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$

2. *If in a ΔABC, $\cot A + \cot B + \cot C = \sqrt{3}$. Prove that triangle is equilateral.*

3. *If $b + c = 3a$, prove that $\cot B/2 \cot C/2 = 2$.*

4. *If p_1, p_2 and p_3 are the altitudes of a triangle from the vertices of a ΔABC and Δ is the area of triangle, prove that:*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{2ab}{(a+b+c)\Delta} \cos^2 \frac{C}{2}$$

5. *If α, β and γ are the lengths of altitudes of a triangle ABC and Δ be its area, prove that :*

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{(\cot A + \cot B + \cot C)}{\Delta}$$

6. *Prove the following :*

(i) $a(r r_1 + r_2 r_3) = b(r r_2 + r_3 r_1) = c(r r_3 + r_1 r_2)$

(ii) $(r_1 - r)(r_2 - r)(r_3 - r) = 4r^2 R$

(iii) $(r_1 - r)(r_2 + r_3) = a^2$

(iv) $\dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} + \dfrac{1}{r^2} = \dfrac{a^2 + b^2 + c^2}{\Delta^2}$

(v) $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$

(vi) $\dfrac{r_2 + r_1}{1 + \cos A} = \dfrac{r_3 + r_1}{1 + \cos B} = \dfrac{r_1 + r_2}{1 + \cos C}$

7. *If p_1, p_2 and p_3 are respectively the bring perpendiculars from the vertices of a triangle to the opposite sides, prove that:*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$$

8. *Prove that the distance of the incentre of ΔABC from A is $4R \sin B/2 \sin C/2$.*

9. *Draw the graphs of the following functions :*

(i) $y = 1/\sqrt{2}(\sin x + \cos x)$; from $x = -\pi/2$ to $x = \pi/2$

(ii) $y = \tan x$; $0 \le x \le 2$

(iii) $y = \operatorname{cosec} x$; $-\pi \le x \le \pi$

(iv) $y = |\sin x|$; $-2\pi \le x \le 2\pi$

(v) $y = \sin(3x + n/4)$; $-\pi/3 \le x \le \pi/3$

10. *If A, A_1, A_2, A_3 are respectively areas of the inscribed and escribed circles, prove that :*

$$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$$

11. *Prove that a triangle is right angled if :* $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2.$

12. *In a triangle ΔABC, prove the follownig :*

(i) $2\,abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = (a+b+c)\Delta$

(ii) $\frac{\tan A/2}{(a-b)(a-c)} + \frac{\tan B/2}{(b-c)(b-a)} + \frac{\tan C/2}{(c-a)(c-b)} = \frac{1}{s}$

13. *If in any triangle the ratio of angles be* 1 : 2 : 3, *prove that the corresponding sides are in the ratio* 1 : √3 : 2.

14. *If a cos A = b cos B, prove that the ΔABC is either isosceles or right angled.*

15. *If in a* ΔABC, *c (a + b) cos B/2 = b (a + c) cos C/2, prove that the triangle is isosceles.*

16. *Let A, B be two points on one bank of a straight river, and C, D two points on the other bank, the directions from A to B along the river being the same as from C to D.*

If AB = a, ∠CAD = α, ∠DAB = β, ∠CBA = γ, then prove that AB || CD : $CD = \frac{a\sin\alpha\sin\gamma}{\sin\beta\sin(\alpha+\beta+\gamma)}$

17. *The sides of a triangle are* $x^2 + x + 1$, $2x + 1$, $x^2 - 1$ *; prove that the greatest angle is* 120°.

18. *In the ambiguous case, if two triangles are formed with a, b, A ; then prove that the sum of the areas of these triangles is* $1/2\, b^2 \sin 2A$.

19. *The sides of a triangle are in the ratio* 2 : √6 : (√3 + 1) *; find its angle.*

For Q. No. 20 - 21

In each of the following questions two statements are given as Statement-1 and Statement-2. Examine the statements carefully and answer the questions according to the instructions given below :

(A) If Statement-I is True, Statement-II is True; Statement-II is a correct explanation for Statement-I

(B) If Statement-II is True, Statement-II is True; Statement-II is NOT a correct explanation for Statement-I

(C) If Statement-I is True, Statement-II is False

(D) If Statement-I is False, Statement-II is True

22. **Statement 1 :** *In a DABC, if a < b < c and r is inradius and r_1, r_2, r_3 are the exradii opposite to angle A, B, C respectively then* $r < r_1 < r_2 < r_3$

Statement 2 : For, $\Delta ABC,\ r_1 r_2 + r_2 r_3 + r_3 r_1 = \dfrac{r_1 r_2 r_3}{r}$

23. **Statement 1 :** *If the sides of a triangle are 13, 14, 15 then the radius of incircle = 4*

Statement 2 : *In a* $\Delta ABC,\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ *where* $s = \dfrac{a+b+c}{2}$ *and* $r = \dfrac{\Delta}{s}$

24. **Statement 1 :** *In a* $\Delta ABC, \sum \dfrac{\cos^2 \frac{A}{2}}{a}$ *has the value equal to* $\dfrac{s^2}{abc}$

Statement 2 : *In a* ΔABC

$$\cos\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ \cos\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}},\ \cos\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

THINGS TO REMEMBER

1. Standard Results - I

(a) Sine Rule : $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ [where R is the circumradius of ΔABC.]

$\Rightarrow \quad a = 2R \sin A,\ b = 2R \sin B,\ c = 2R \sin C$

(b) Consine Rule :

$a^2 = b^2 + c^2 - 2bc \cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$b^2 = c^2 + a^2 - 2ac \cos B$ or $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

$c^2 = a^2 + b^2 - 2ab \cos C$ or $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

(c) Projection Formula : $a = b \cos C + c \cos B$

$b = c \cos A + a \cos C$

$c = a \cos B + b \cos A$

(d) Napier Analogy :

$\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$; $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$ and $\tan \dfrac{C-A}{2} = \dfrac{c-a}{c+a} \cot \dfrac{B}{2}$

2. Standard Results - II

(a) Semi -Permeter of ΔABC (s) :

$s = \dfrac{a+b+c}{2}$ $\qquad 2s - 2a = b + c - a$

$\Rightarrow \quad 2s = a + b + c$ $\qquad 2s - 2b = c + a - b$

$2s - 2c = a + b - c$

(b) Half-angle formulae :

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \cot\frac{A}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\frac{s(s-a)}{\Delta}$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}} \qquad \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{(s-b)(s-c)}{\Delta}$$

The expressions for $\sin\frac{B}{2}, \cos\frac{B}{2}, \tan\frac{B}{2}, \cot\frac{B}{2}, \sin\frac{C}{2}, \cos\frac{C}{2}, \tan\frac{C}{2}, \cot\frac{C}{2}$ can be derived using symmetry.

Δ = area of triangle $ABC=\sqrt{s(s-a)(s-b)(s-c)}$

$$\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$$

$$\Delta=\frac{abc}{4R}=rs$$

(c) Values of sin A, cos A, cot A :

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2\Delta}{bc}$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\cot A=\frac{\cos A}{\sin A}=\frac{b^2+c^2-a^2}{4\Delta}$$

(d) Relation between inradius, sides, semi-perimeter and area of the triangle :

in radius	r	$\frac{\Delta}{s}$	$(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$	$r=\frac{a\sin B/2\cdot\sin C/2}{\cos A/2}$
ex radius (opposite to A)	r_1	$r_1=\frac{\Delta}{s-a}$	$s\tan\frac{A}{2}$	$\frac{a\cos B/2\cdot\cos C/2}{\cos A/2}$
ex radius (opposite to B)	r_2	$r_2=\frac{\Delta}{s-b}$	$s\tan\frac{B}{2}$	$\frac{b\cos A/2\cdot\cos C/2}{\cos B/2}$
ex radius (opposite to C)	r_3	$r_3=\frac{\Delta}{s-c}$	$s\tan\frac{C}{2}$	$\frac{c\cos A/2\cdot\cos B/2}{\cos C/2}$

3. m-n Theorem

Theorem : (1) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(2) $(m+n)\cot\theta = n\cot B - m\cot C$

4. Results *n* sides Polygon

If the polygon has 'n' sides, Sum of the internal angles is $(n-2)\pi$ and each angle is $\dfrac{(n-2)\pi}{n}$.

a = side length ; r = in-radius ; R = circum-radius

$$r = \frac{a}{2\tan\frac{\pi}{n}} \quad \text{and} \quad R = \frac{a}{2\sin\frac{\pi}{n}}$$

$$\text{Area of polygon} = \frac{1}{4}na^2 \cdot \cot\left(\frac{\pi}{n}\right) = nr^2\tan\left(\frac{\pi}{n}\right) = \frac{n}{2}R^2\sin\frac{2\pi}{n}$$

5. More results

5.1 Distance of orthocentre from vertices of triangle is $2R\cos A$, $2R\cos B$ and $2R\cos C$ respectively.

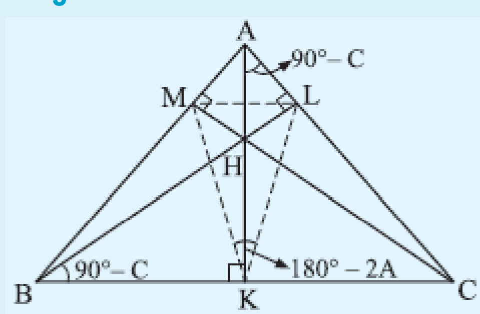
5.2 Distance of orthocentre from sides of triangle $2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$ respectively.

5.3 Distance of circumcentre O from sides : BC, CA and AB are $R\cos A$, $R\cos B$ and $R\cos C$ respectively.

6. Pedal Triangle

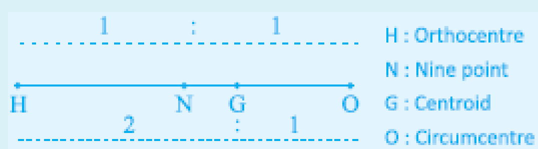
Let ABC be any triangle, and let AK, BL and CM be the perpendicular from A, B and C upon the opposite sides of the triangle. These three perpendiculars meet at a point 'O' which is called the orthocentre of the triangle ABC. The triangle KLM, formed by joining the feet of these perpendiculars is called the pedal triangle of ABC.

Important Point for Pedal Triangle KLM :



ΔKLM is the pedal triangle of ΔABC

- Angles of pedal triangle : K = 180° – 2A, L = 180° – 2B, M = 180° – 2C
- Sides of pedal triangle : LM = $a \cos A$, MK = $b \cos B$, KL = $c \cos C$
- Area of the pedal triangle : $\frac{1}{2} R^2 . \sin 2A . \sin 2B . \sin 2C$
- Circum radius of pedal triangle : $\frac{R}{2}$
- In-radius of pedal triangle : $2R \cos A . \cos B . \cos C$
- Orthocentre of ΔABC is the incentre of the pedal ΔKLM
- Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocentre of the given triangle. This circle is known as nine-point circle.
- Nine point ≡ Circum - centre of the pedal triangle
- Circum-centre of the pedal triangle of a given triangle bisects the line joining the circumcentre of the triangle to the orthocentre.



7. Important Theorem

The centroid, circumcentre & orthocentre in any triangle are collinear . The centroid divides the line joining orthocentre and circumcentre in 2 : 1 internally .