SEQUENCE & SERIES

1. DEFINITION OF SEQUENCE & SERIES :

1.1 Sequence :

A succession of terms a_1, a_2, a_3, a_4........ formed according to some rule or law.

Examples are : 1, 4, 9, 16, 25

–1, 1, –1, 1,........

$\frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}$, ...

1.1.1 Real Sequence

A sequence whose range is a subset of R is called a real sequence.

e.g. (i) 2, 5, 8, 11, ...

(ii) 4, 1, – 2, – 5, ...

(iii) 3, –9, 27, – 81, ...

Note : A finite sequence has a finite (i.e. limited) number of terms, as in the first example above. An infinite sequence has an unlimited number of terms, i.e. there is no last term, as in the second and third examples.

SOLVED EXAMPLE

Example # 1 : Write down the sequence whose n^{th} term is

(i) $\frac{2^n}{n}$ (ii) $\frac{3+(-1)^n}{3^n}$

Solution : (i) Let $t_n = \frac{2^n}{n}$

put n = 1, 2, 3, 4, ... we get

$t_1 = 2, t_2 = 2, t_3 = \frac{8}{3}, t_4 = 4$

so the sequence is $2, 2, \frac{8}{3}, 4$, ...

(ii) Let $t_n = \frac{3+(-1)^n}{3^n}$

put n = 1, 2, 3, 4, ...

so the sequence is $\frac{2}{3}, \frac{4}{9}, \frac{2}{27}, \frac{4}{81}$,............

1.2 Series :

The indicated sum of the terms of a sequence. In the case of a finite sequence a_1, a_2, a_3,................, a_n the corresponding series is $a_1 + a_2 + a_3 + \ldots\ldots + a_n = \sum_{k=1}^{n} a_k$. This series has a finite or limited number of terms and is called a finite series.

e.g. (i) 1 + 2 + 3 + 4 + ... + n

(ii) 2 + 4 + 8 + 16 + ...

(iii) – 1 + 3 – 9 + 27 – ....

2. ARITHMETIC PROGRESSION (A.P.) :

A.P. is a sequence whose terms differ by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then A.P. can be written as

a, a + d, a + 2d, ..., a + (n – 1) d , ...

(a) n^{th} term of AP $T_n = a + (n-1)d$, where d = $t_n - t_{n-1}$

(b) The sum of the first n terms : $S_n = \frac{n}{2}[a + \ell] = \frac{n}{2}[2a + (n-1)d]$

where ℓ is n^{th} term.

Note :

(i) n^{th} term of an A.P. is of the form An + B i.e. a linear expression in 'n', in such a case the coefficient of n is the common difference of the A.P. i.e. A.

(ii) Sum of first 'n' terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in 'n', in such case the common difference is twice the coefficient of n^2. i.e. 2A

(iii) Also n^{th} term $T_n = S_n - S_{n-1}$

SOLVED EXAMPLE

Example # 2 : Find the number of terms in the sequence 4, 12, 20, ...,108.

Solution : a = 4, d = 8 so 108 = 4 + (n – 1)8

$\Rightarrow$ n = 14

Example # 3 : If (x + 1), 3x and (4x + 2) are first three terms of an A.P. then find its 5^{th} term

Solution : (x + 1), 3x, (4x + 2) are in AP

$\Rightarrow$ 3x – (x + 1) = (4x + 2) – 3x

$\Rightarrow$ x = 3

$\therefore$ a = 4, d = 9 – 4 = 5

$\Rightarrow$ $T_5 = 4 + (4)5 = 24$

Example # 4 : The sum of first four terms of an A.P. is 56 and the sum of it's last four terms is 112. If its first term is 11 then find the number of terms in the A.P.

Solution : a + a + d + a + 2d + a + 3d = 56

4a + 6d = 56

44 + 6d = 56 (as a = 11)

6d = 12 hence d = 2

Let total number of terms = n

Now sum of last four terms.

$a + (n - 1)d + a + (n - 2)d + a + (n - 3)d + a + (n - 4)d = 112$

$\Rightarrow \quad 4a + (4n - 10)d = 112$

$\Rightarrow \quad 44 + (4n - 10)2 = 112$

$\Rightarrow \quad 4n - 10 = 34$

$\Rightarrow \quad n = 11$

Example # 5 : Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.

Solution : All these numbers are 101, 108, 115, ..., 997

$997 = 101 + (n - 1)\,7$

$\Rightarrow \quad n = 129$

Example # 6 : The sum of n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11^{th} terms.

Solution : Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively,

then $\dfrac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \dfrac{7n+1}{4n+27}$

$$\Rightarrow \quad \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11^{th} terms

$\frac{n-1}{2} = 10$

$\Rightarrow \quad n = 21$

so ratio of 11^{th} terms is $= \dfrac{7(21)+1}{4(21)+27} = \dfrac{148}{111} = \dfrac{4}{3}$

Example # 7 : If sum of n terms of a sequence is given by $S_n = 2n^2 + 3n$, find its 50^{th} term.

Solution : Let t_n is n^{th} term of the sequence so $t_n = S_n - S_{n-1}$.

$= 2n^2 + 3n - 2(n - 1)^2 - 3(n - 1)$

$= 4n + 1$

so $\quad t_{50} = 201$.

Problems for Self Practice -1:

(1) Write down the sequence whose n^{th} terms is $\frac{2^n}{n}$

(2) For an A.P, show that $t_m + t_{2n+m} = 2t_{m+n}$

(3) If the sum of p terms of an A.P. is q and the sum of its q terms is p, then find the sum of its (p + q) term.

(4) Which number of term of the sequence 2005, 2000, 1995, 1990, 1985, ... is the first negative term

(5) Find the maximum sum of the A.P. 40 + 38 + 36 + 34 + 32 + ...

Answers : (1) $\frac{2}{1},\frac{4}{2},\frac{8}{3},\frac{16}{4},.......$, (3) –(p + q) (4) 403 (5) 420

2.1 Properties of A.P. :

(a) If each term of an A.P. is increased, decreased, multiplied or divided by the some nonzero number, then the resulting sequence is also an A.P.

(b) In general assume

Three numbers in A.P.	:	a – d, a, a + d
Four numbers in A.P.	:	a – 3d, a – d, a + d, a + 3d
Five numbers in A.P.	:	a – 2d, a – d, a, a + d, a + 2d

(c) The common difference can be zero, positive or negative.

(d) k^{th} term from the last = $(n – k +1)^{th}$ term from the beginning (If total number of terms = n).

(e) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms. $\Rightarrow T_k + T_{n-k+1}$ = constant = $a + \ell$.

(f) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it.

$$a_n = (1/2)(a_{n-k} + a_{n+k}),\ k < n$$

For $k = 1,\ a_n = (1/2)(a_{n-1} + a_{n+1})$; For $k = 2,\ a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.

(g) If a, b, c are in AP, then 2b = a + c.

SOLVED EXAMPLE

Example # 8 : If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference ≠ 0, then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.

Solution : As a_1, a_2, a_3, a_4, a_5 are in A.P., we have $a_1 + a_5 = a_2 + a_4 = 2a_3$.

Hence $\sum_{i=1}^{5} a_i = 10$.

Example # 9 : If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P., prove that a^2, b^2, c^2 are also in A.P.

Solution : $\because \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$\Rightarrow \quad \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)} \qquad \Rightarrow \quad \frac{b-a}{b+c} = \frac{c-b}{a+b}$

$\Rightarrow \quad b^2 - a^2 = c^2 - b^2 \qquad \Rightarrow \quad a^2, b^2, c^2$ are in A.P.

Example # 10 : If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., then prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

Solution : Given $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

Add 2 to each term

$\Rightarrow \quad \frac{b+c+a}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c}$ are in A.P.

divide each by a + b + c $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Example # 11 : Four numbers are in A.P. If their sum is 20 and the sum of their squares is 120, then find the middle two terms

Solution : Let the numbers are a – 3d, a – d, a + d, a + 3d

given, $a - 3d + a - d + a + d + a + 3d = 20 \quad \Rightarrow \quad 4a = 20 \Rightarrow a = 5$

and $(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120 \quad \Rightarrow \quad 4a^2 + 20d^2 = 120$

$\Rightarrow \quad 4 \times 5^2 + 20d^2 = 120 \quad \Rightarrow \quad d^2 = 1 \Rightarrow d = \pm 1$

Hence numbers are 2, 4, 6, 8 or 8, 6, 4, 2 $\Rightarrow$ Ans is 4 & 6

Example # 12 : If $a_1, a_2, a_3, \ldots\ldots\ldots, a_n$ are in A.P. where $a_i > 0$ for all i, show that :

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution : L.H.S. = $\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$

$$= \frac{1}{\sqrt{a_2}+\sqrt{a_1}} + \frac{1}{\sqrt{a_3}+\sqrt{a_2}} + \ldots\ldots + \frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{(a_2-a_1)} + \frac{\sqrt{a_3}-\sqrt{a_2}}{(a_3-a_2)} + \ldots\ldots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

Let 'd' is the common difference of this A.P.

then $a_2 - a_1 = a_3 - a_2 = \ldots\ldots. = a_n - a_{n-1} = d$

Now L.H.S.

$$= \frac{1}{d}\left\{\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots\ldots+\sqrt{a_{n-1}}-\sqrt{a_{n-2}}+\sqrt{a_n}-\sqrt{a_{n-1}}\right\} = \frac{1}{d}\left\{\sqrt{a_n}-\sqrt{a_1}\right\}$$

$$= \frac{a_n-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{a_1+(n-1)d-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{1}{d}\frac{(n-1)d}{(\sqrt{a_n}+\sqrt{a_1})} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}} = \text{R.H.S.}$$

Problems for Self Practice -2:

(1) Find the sum of first 24 terms of the A.P. a_1, a_2, a_3......, if it is know that
$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

(2) Find the number of terms common to the two A.P.'s 3, 7, 11, ... 407 and 2, 9, 16, ..., 709

Answers : (1) 900 (2) 14

2.2 Arithmetic Mean :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c. So A.M. of a and c = $\frac{a+c}{2} = b$.

Let a_1, a_2,........ ,a_n be n positive real numbers, then we define their arithmetic mean (A) as

$$A = \frac{a_1 + a_2 + ... + a_n}{n}$$

n-Arithmetic means Between Two Numbers :

If a,b be any two given numbers & a, A_1, A_2, ..., A_n, b are in AP, then A_1, A_2,........A_n are the 'n' A.M's between a & b then. A_1 = a + d , A_2 = a + 2d ,......, A_n= a + nd or b – d, where $d = \frac{b-a}{n+1}$

$$\Rightarrow A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1}, ...$$

Note : Sum of n A.M's inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

SOLVED EXAMPLE

Example # 13 : Between two numbers whose sum is $\frac{13}{6}$, an even number of A.M.s is inserted, the sum of these means exceeds their number by unity. Find the number of means.

Solution : Let a and b be two numbers and 2n A.M.s are inserted between a and b, then

$\frac{2n}{2}$ (a + b) = 2n + 1.

$n\left(\frac{13}{6}\right)$ = 2n + 1. $\left[\text{given } a + b = \frac{13}{6}\right]$

$\Rightarrow$ n = 6.

$\therefore$ Number of means = 12.

Example # 14 : Insert 20 A.M. between 2 and 86.

Solution : Here 2 is the first term and 86 is the 22^{nd} term of A.P. so 86 = 2 + (21)d

$\Rightarrow$ d = 4

so the series is 2, 6, 10, 14,......., 82, 86

$\therefore$ required means are 6, 10, 14,...,82.

Problems for Self Practice -3:

(1) If A.M. between p^{th} and q^{th} terms of an A.P. be equal to the A.M. between r^{th} and s^{th} terms of the A.P., then prove that p + q = r + s.

(2) If n A.M.s are inserted between 20 and 80 such that first mean : last mean = 1 : 3, find n.

(3) For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$, $a \neq b$ is the A.M. of a and b.

Answers : (2) n = 11 (3) n = 0

3. GEOMETRIC PROGRESSION (G.P.) :

G.P. is a sequence of non zero numbers each of the succeeding term is equal to the preceeding term multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the COMMON RATIO of the sequence & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2, ar^3, ar^4 , ... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term ; $T_n = a\,r^{n-1}$

(b) Sum of the first n terms; $S_n = \frac{a(r^n-1)}{r-1},\ \text{if } r \neq 1$

(c) Sum of infinite G.P. , $S_\infty = \frac{a}{1-r};\ 0 < |r| < 1$

3.1 Properties of GP :

(a) If each term of a G.P. be multiplied or divided by the some non-zero quantity, then the resulting sequence is also a G.P.

(b) In general assume
Three consecutive terms of a GP : a/r, a, ar ;
Four consecutive terms of a GP : a/r^3, a/r, ar, ar^3 & so on.

(c) If a, b, c are in G.P. then $b^2 = ac$.

(d) If in a G.P, the product of two terms which are equidistant from the first and the last term, is constant and is equal to the product of first and last term. $\Rightarrow T_k \cdot T_{n-k+1}$ = constant = $a.\ell$

(e) If each term of a G.P. be raised to the same power, then resulting sequence is also a G.P.

(f) In a G.P., $T_r^2 = T_{r-k} \cdot T_{r+k}$, $k < r$, $r \neq 1$

(g) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(h) If $a_1, a_2, a_3 \ldots\ldots a_n$ is a G.P. of positive terms, then $\log a_1, \log a_2, \ldots\ldots \log a_n$ is an A.P. and vice-versa.

(i) If a_1, a_2, a_3..... and b_1, b_2, b_3..... are two G.P.'s then a_1b_1, a_2b_2, a_3b_3..... & $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$.......... is also in G.P.

SOLVED EXAMPLE

Example # 15 : If a, b, c, d and p are distinct real numbers such that

$\left(a^2+b^2+c^2\right)p^2-2p\left(ab+bc+cd\right)+\left(b^2+c^2+d^2\right)\le 0$ then a, b, c, d are in

(A) A.P. (B) G.P. (C) H.P. (D) none of these

Solution : Here, the given condition $\left(a^2+b^2+c^2\right)p^2-2p\left(ab+bc+ca\right)+b^2+c^2+d^2\le 0$

$\Rightarrow \left(ap-b\right)^2+\left(bp-c\right)^2+\left(cp-d\right)^2\le 0$

$\because$ a square can not be negative

$\therefore \quad ap-b=0, bp-c=0, cp-d=0 \Rightarrow p=\frac{b}{a}=\frac{c}{b}=\frac{d}{c}\Rightarrow$ a, b, c, d are in G.P. **Ans. (B)**

Example # 16 : If a, b, c are in G.P., then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in -

(A) A.P. (B) G.P. (C) H.P. (D) none of these

Solution : a, b, c are in G.P $\Rightarrow b^2=ac$

Now the equation $ax^2+2bx+c=0$ can be rewritten as $ax^2+2\sqrt{ac}x+c=0$

$\Rightarrow\left(\sqrt{a}x+\sqrt{c}\right)^2=0\Rightarrow x=-\sqrt{\frac{c}{a}},-\sqrt{\frac{c}{a}}$

If the two given equations have a common root, then this root must be $-\sqrt{\frac{c}{a}}$.

Thus $d\frac{c}{a}-2e\sqrt{\frac{c}{a}}+f=0 \quad \Rightarrow \quad \frac{d}{a}+\frac{f}{c}=\frac{2e}{c}\sqrt{\frac{c}{a}}=\frac{2e}{\sqrt{ac}}=\frac{2e}{b} \quad \Rightarrow \frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in A.P. **Ans. (A)**

Example # 17 : A number consists of three digits which are in G.P. the sum of the right hand and left hand digits exceeds twice the middle digit by 1 and the sum of the left hand and middle digits is two third of the sum of the middle and right hand digits. Find the numbers.

Solution : Let the three digits be a, ar and ar^2 then number is

$100a + 10ar + ar^2$ ....(i)

Given, $a + ar^2 = 2ar + 1$

or $a(r^2 - 2r + 1) = 1$

or $a(r-1)^2 = 1$ ....(ii)

Also given $a + ar = \frac{2}{3}(ar + ar^2)$

$\Rightarrow$ $3 + 3r = 2r + 2r^2 \Rightarrow 2r^2 - r - 3 = 0 \Rightarrow (r + 1)(2r - 3) = 0$

$\therefore$ $r = -1, 3/2$

for $r = -1$, $a = \frac{1}{(r-1)^2}=\frac{1}{4}\notin I$ $\therefore r \ne -1$

for r = 3/2, a = $\dfrac{1}{\left(\frac{3}{2}-1\right)^2} = 4$ {from (ii)}

From (i), number is 400 + 10.4.$\dfrac{3}{2} + 4.\dfrac{9}{4} = 469$ **Ans.**

Example # 18 : Find the value of $0.32\overline{58}$

Solution : Let R = $0.32\overline{58}$ $\Rightarrow$ R = 0.32585858.... .... (i)

Here number of figures which are not recurring is 2 and number of figures which are recurring is also 2.

then 100 R = 32.585858...... ....(ii)

and 10000 R = 3258.5858..... ....(iii)

Subtracting (ii) from (iii) , we get

9900 R = 3226 $\Rightarrow$ $R = \dfrac{1613}{4950}$

Aliter Method : R = .32 + .0058 + .0058 + .000058 +..........

$$= .32 + \frac{58}{10^4}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + ...\infty\right)$$

$$= .32 + \frac{58}{10^4}\left(\frac{1}{1 - \frac{1}{100}}\right)$$

$$= \frac{32}{100} + \frac{58}{9900} = \frac{3168 + 58}{9900} = \frac{3226}{9900} = \frac{1613}{4950}$$

<u>Problems for Self Practice</u> -4:

(1) If the p^{th}, q^{th}, r^{th} terms of a G.P. be a, b, c respectively, prove that $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$.

(2) The sum of infinite number of terms of a G.P. is 4, and the sum of their cubes is 192, find the series.

(3) Find a three digit number whose consecutive digits form a G.P. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now, if we increase the second digit of the required number by 2, then the resulting digits will form an A.P.

(4) If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of the given G.P., then show that (q – r) log a + (r – p) log b + (p – q)log c = 0, where a, b, c > 0.

(5) Find three numbers in G.P., whose sum is 52 and the sum of whose products in pairs is 624.

(6) A G.P. consist of 2n terms. If the sum of the terms occupying the odd places is S_1 and that of the terms occupying the even places is S_2 , then find the common ratio of the progression.

(7) The sum of three numbers in G.P. is 70, if the two extremes be multiplied each by 4 and the mean by 5, the products are in A.P. Find the numbers.

(8) If a = $\underbrace{111..........1}_{55}$, b = 1 + 10 + 10^2 + 10^3 + 10^4 and c = 1 + 10^5 + 10^{10} + ... + 10^{50}, then prove that

(i) 'a' is a composite number (ii) a = bc.

Answers : (2) 6, – 3, $\dfrac{3}{2}$,........ (3) 931 (5) 4, 12, 36 (6) $\dfrac{S_2}{S_1}$ (7) 10, 20, 40

3.2 Geometric Mean :

If a, b, c are in G.P., then b is the G.M. between a & c, $b^2 = ac$. So G.M. of a and c = $\sqrt{ac} = b$

Let $a_1, a_2, ..., a_n$ be n positive real numbers, then we define their geometric mean (G) as

$G = (a_1 a_2 ....a_n)^{1/n}$

n-Geometric Means Between Two Numbers :

If a, b are two given positive numbers & a, $G_1, G_2, ..., G_n$, b are in G.P. Then $G_1, G_2, G_3, ...G_n$ are 'n' G.Ms between a & b. where $b = ar^{n+1} \Rightarrow r = (b/a)^{1/n+1}$

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$ ...., $G_n = a(b/a)^{n/n+1}$

= ar, = ar^2, ... = ar^n = b/r

Note : The product of n G.Ms between a & b is equal to n^{th} power of the single G.M. between a & b i.e.

$\prod_{r=1}^{n} G_r = (G)^n$ where G is the single G.M. between a & b

SOLVED EXAMPLE

Example # 19 : Insert 4 G.M.s between 2 and 486.

Solution : Common ratio of the series is given by r = $\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ = $(243)^{1/5}$ = 3

Hence four G.M.s are 6, 18, 54, 162.

Example # 20 : If the third term of G.P. is 4, then find the product of first five terms.

Solution : $T_1T_2T_3T_4T_5 = (T_3)^5 = 4^5 = 1024$

4. HARMONIC PROGRESSION (H.P.) :

A sequence is said to be in H.P. if the reciprocal of its terms are in AP.

If the sequence $a_1, a_2, a_3, ..., a_n$ is an HP then $1/a_1, 1/a_2, ..., 1/a_n$ is an AP . Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ... \frac{1}{a+(n-1)d}$$

Note : No term of any H.P. can be zero.

(i) If a, b, c are in HP, then $b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$

SOLVED EXAMPLE

Example # 21 : If m^{th} term of H.P. is n, while n^{th} term is m, find its $(m + n)^{th}$ term.

Solution : Given T_m = n or $\frac{1}{a+(m-1)d}$ = n; where a is the first term and d is the common difference of the corresponding A.P.

so $a + (m - 1)d = \frac{1}{n}$

and $a + (n-1)d = \frac{1}{m}$ $\Rightarrow$ $(m-n)d = \frac{m-n}{mn}$

or $d = \frac{1}{mn}$ so $a = \frac{1}{n} - \frac{(m-1)}{mn} = \frac{1}{mn}$

Hence $T_{(m+n)} = \frac{1}{a+(m+n-1)d} = \frac{mn}{1+m+n-1} = \frac{mn}{m+n}$.

Example # 22 : If p^{th}, q^{th}, r^{th} terms of an H.P. be a, b, c respectively, prove that $(q-r)bc + (r-p)ac + (p-q)ab = 0$

Solution : Let 'x' be the first term and 'd' be the common difference of the corresponding A.P..

so $\frac{1}{a} = x + (p-1)d$ ....(i)

$\frac{1}{b} = x + (q-1)d$ ....(ii)

$\frac{1}{c} = x + (r-1)d$ ....(iii)

(i) - (ii) $\Rightarrow$ $ab(p-q)d = b-a$ ....(iv)

(ii) - (iii) $\Rightarrow$ $bc(q-r)d = c-b$ ....(v)

(iii) - (i) $\Rightarrow$ $ac(r-p)d = a-c$ ....(vi)

(iv) + (v) + (vi) gives

$bc(q-r) + ac(r-p) + ab(p-q) = 0$.

Example # 23 : The sum of three numbers are in H.P. is 37 and the sum of their reciprocals is 1/4. Find the numbers.

Solution : Three numbers are in H.P. can be taken as

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

then $\frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 37$ ....(i)

and $a - d + a + a + d = \frac{1}{4} \Rightarrow a = \frac{1}{12}$

from (i), $\frac{12}{1-12d} + 12 + \frac{12}{1+12d} = 37 \Rightarrow \frac{12}{1-12d} + \frac{12}{1+12d} = 25$

$\Rightarrow \frac{24}{1-144d^2} = 25 \Rightarrow 1 - 144d^2 = \frac{24}{25} \Rightarrow d^2 = \frac{1}{25 \times 144}$

$\therefore$ $d = \pm\frac{1}{60}$

$\therefore$ $a-d, a, a+d$ are $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$ or $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}$

Hence, three numbers in H.P. are 15, 12, 10 or 10, 12, 15 **Ans.**

Example # 24 : Suppose a is a fixed real number such that $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$

If p, q, r are in A.P., then prove that x, y, z are in H.P.

Solution : $\because$ p, q, r are in A.P.

$\therefore$ q – p = r – q .... (i)

$\Rightarrow$ p – q = q – r = k (let)

given $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz} \Rightarrow \frac{\frac{a}{x}-1}{p}=\frac{\frac{a}{y}-1}{q}=\frac{\frac{a}{z}-1}{r}$

$$\Rightarrow \frac{\left(\frac{a}{x}-1\right)-\left(\frac{a}{y}-1\right)}{p-q}=\frac{\left(\frac{a}{y}-1\right)-\left(\frac{a}{z}-1\right)}{q-r}$$ (by law of proportion)

$$\Rightarrow \frac{\frac{a}{x}-\frac{a}{y}}{k}=\frac{\frac{a}{y}-\frac{a}{z}}{k}$$ {from (i)}

$$\Rightarrow a\left(\frac{1}{x}-\frac{1}{y}\right)=a\left(\frac{1}{y}-\frac{1}{z}\right) \Rightarrow \frac{1}{x}-\frac{1}{y}=\frac{1}{y}-\frac{1}{z}$$

$$\therefore \frac{2}{y}=\frac{1}{x}+\frac{1}{z}$$

$\therefore$ $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ are in A.P.

Hence x, y, z are in H.P.

Problems for Self Practice -5 :

(1) If the 7th term of a H.P. is 8 and the 8th term is 7. Then find the 28th term.

(2) In a H.P., if 5th term is 6 and 3rd term is 10. Find the 2nd term.

(3) If the pth, qth and rth terms of a H.P. are a,b,c respectively, then prove that $\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}=0$.

(4) If a, b, c be in H.P., show that a : a – b = a + c : a – c.

(5) If the ratio of H.M. between two positive numbers 'a' and 'b' (a > b) is to their G.M. as 12 to 13, prove that a : b is 9 : 4.

(6) If H be the harmonic mean of a and b, then find the value of $\frac{H}{2a}+\frac{H}{2b}-1$.

(7) If a, b, c, d are in H.P., then show that ab + bc + cd = 3ad

Answers : (1) 2 (2) 15 (6) 0

4.1 Harmonic Mean :

If a, b, c are in H.P., then b is H.M. between a & c. So H.M. of a and c = $\frac{2ac}{a+c}$ = b.

Let a_1, a_2,........ ,a_n be n positive real numbers, then we define their harmonic mean (H) as

$$H = \frac{n}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+....\frac{1}{a_n}\right)}$$

Insertion of 'n' HM's between a and b :

a, H_1, H_2, H_3 ,........, H_n, b $\rightarrow$ H.P

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, ...\frac{1}{H_n}, \frac{1}{b} \rightarrow$ A.P.

$$\frac{1}{b} = \frac{1}{a} + (n + 1)D \Rightarrow D = \frac{\frac{1}{b}-\frac{1}{a}}{n+1}$$

$$\frac{1}{H_n} = \frac{1}{a} + n\left(\frac{\frac{1}{b}-\frac{1}{a}}{n+1}\right)$$

SOLVED EXAMPLE

Example # 25 : Insert 4 H.M between $\frac{2}{3}$ and $\frac{2}{13}$.

Solution : Let 'd' be the common difference of corresponding A.P..

so $d = \frac{\frac{13}{2}-\frac{3}{2}}{5} = 1.$

$\therefore$ $\frac{1}{H_1} = \frac{3}{2} + 1 = \frac{5}{2}$ or $H_1 = \frac{2}{5}$

$\frac{1}{H_2} = \frac{3}{2} + 2 = \frac{7}{2}$ or $H_2 = \frac{2}{7}$

$\frac{1}{H_3} = \frac{3}{2} + 3 = \frac{9}{2}$ or $H_3 = \frac{2}{9}$

$\frac{1}{H_4} = \frac{3}{2} + 4 = \frac{11}{2}$ or $H_4 = \frac{2}{11}$.

<u>Problems for Self Practice</u> -6:

(1) If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the G.M. between a & b then find the value of 'n'.

(2) If b is the harmonic mean between a and c, then prove that $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$.

Answers : (1) $\frac{1}{2}$

5. ARITHMETICO-GEOMETRIC SERIES :

A series, each term of which is formed by multiplying the corresponding term of an A.P. & G.P. is called the Arithmetico-Geometric Series , e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5, ... are in A.P. & 1, x, x^2, x^3 .... are in G.P.

(a) Sum of n terms of an Arithmetico-Geometric Series :

Let $S_n = a + (a+d)r + (a+2d)r^2 + ... + [a+(n-1)d]r^{n-1}$

then $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a+(n-1)d]\ r^n}{1-r}, \ r \neq 1$

(b) Sum to infinity :

If $0 < |r| < 1$ & $n \to \infty$, then $\lim\limits_{n\to\infty} r^n = 0, S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

SOLVED EXAMPLE

Example # 26 : Find the sum of the series $1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + ...$ to n terms.

Solution : Let $S = 1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + ... + \dfrac{3n-2}{5^{n-1}}$ ....(i)

$\left(\dfrac{1}{5}\right) S = \dfrac{1}{5} + \dfrac{4}{5^2} + \dfrac{7}{5^3} + ... + \dfrac{3n-5}{5^{n-1}} + \dfrac{3n-2}{5^n}$ ....(ii)

(i) – (ii) $\Rightarrow$

$\dfrac{4}{5} S = 1 + \dfrac{3}{5} + \dfrac{3}{5^2} + \dfrac{3}{5^3} + ... + \dfrac{3}{5^{n-1}} - \dfrac{3n-2}{5^n}.$

$\dfrac{4}{5} S = 1 + \dfrac{\dfrac{3}{5}\left(1-\left(\dfrac{1}{5}\right)^{n-1}\right)}{1-\dfrac{1}{5}} - \dfrac{3n-2}{5^n} = 1 + \dfrac{3}{4} - \dfrac{3}{4} \times \dfrac{1}{5^{n-1}} - \dfrac{3n-2}{5^n}$

$= \dfrac{7}{4} - \dfrac{12n+7}{4.5^n} \quad \therefore \quad S = \dfrac{35}{16} - \dfrac{(12n+7)}{16\,.\,5^{n-1}}.$

Example # 27 : Evaluate $1 + 2x + 3x^2 + 4x^3 + ...$ upto infinity, where $|x| < 1$.

Solution : Let $S = 1 + 2x + 3x^2 + 4x^3 + ...$ ....(i)

$xS = x + 2x^2 + 3x^3 + ...$ ....(ii)

(i) - (ii) $\Rightarrow$ $(1-x)S = 1 + x + x^2 + x^3 + ...$

or $S = \dfrac{1}{(1-x)^2}$

Example # 28 : Find the sum of series $4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$.

Solution : Let $S = 4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$

$-Sx = -4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 + \ldots\ldots\ldots\ldots \infty$

On subtraction, we get

$S(1 + x) = 4 - 5x + 7x^2 - 9x^3 + 11x^4 - 13x^5 + \ldots\ldots\ldots \infty$

$-S(1 + x)x = -4x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + \ldots\ldots\ldots \infty$

On subtraction, we get

$S(1 + x)^2 = 4 - x + 2x^2 - 2x^3 + 2x^4 - 2x^5 + \ldots\ldots\ldots \infty$

$= 4 - x + 2x^2 (1 - x + x^2 - \ldots\ldots\ldots\ldots \infty) = 4 - x + \dfrac{2x^2}{1+x} = \dfrac{4+3x+x^2}{1+x}$

$S = \dfrac{4+3x+x^2}{(1+x)^3}$ **Ans.**

Example # 29 : Find the sum of series upto n terms $\left(\dfrac{2n+1}{2n-1}\right)+3\left(\dfrac{2n+1}{2n-1}\right)^2+5\left(\dfrac{2n+1}{2n-1}\right)^3+\ldots\ldots\ldots$.

Solution : For $x \neq 1$, let

$S = x + 3x^2 + 5x^3 + \ldots\ldots. + (2n - 3)x^{n-1} + (2n - 1)x^n$ .... (i)

$\Rightarrow$ $xS = x^2 + 3x^3 + \ldots\ldots. + (2n - 5)x^{n-1} + (2n - 3)x^n + (2n - 1)x^{n+1}$ .... (ii)

Subtracting (ii) from (i), we get

$(1 - x)S = x + 2x^2 + 2x^3 + \ldots\ldots\ldots + 2x^{n-1} + 2x^n - (2n - 1)x^{n+1} = x + \dfrac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1}$

$= \dfrac{x}{1-x}[1 - x + 2x - 2x^n - (2n - 1)x^n + (2n - 1)x^{n+1}]$

$\Rightarrow$ $S = \dfrac{x}{(1-x)^2}[(2n-1)x^{n+1} - (2n + 1)x^n + 1 + x]$

Thus $\left(\dfrac{2n+1}{2n-1}\right)+3\left(\dfrac{2n+1}{2n-1}\right)^2+\ldots\ldots\ldots+(2n-1)\left(\dfrac{2n+1}{2n-1}\right)^n$

$= \left(\dfrac{2n+1}{2n-1}\right)\left(\dfrac{2n-1}{2}\right)^2\left[(2n-1)\left(\dfrac{2n+1}{2n-1}\right)^{n+1}-(2n+1)\left(\dfrac{2n+1}{2n-1}\right)^n+1+\dfrac{2n+1}{2n-1}\right]$

$= \dfrac{4n^2-1}{4}\cdot\dfrac{4n}{2n-1} = n(2n + 1)$ **Ans.**

Example # 30 : Evaluate : $1 + (1 + b) r + (1 + b + b^2) r^2 + \ldots\ldots\ldots$ to infinite terms for $|br| < 1$.

Solution : Let $S = 1 + (1 + b)r + (1 + b + b^2) r^2 + \ldots\ldots$ ....(i)

$rS = r + (1 + b) r^2 + \ldots\ldots\ldots$ ....(ii)

(i) - (ii)

$\Rightarrow$ $(1 - r)S = 1 + br + b^2r^2 + b^3r^3 + \ldots\ldots$

$\Rightarrow$ $S = \dfrac{1}{(1-br)(1-r)}$

Problems for Self Practice -7:

(1) Evaluate : $1.2 + 2.2^2 + 3.2^3 + ... + 100.\ 2^{100}$

(2) Evaluate : $1 + 3x + 6x^2 + 10x^3 + ...$ upto infinite term, where $|x| < 1$.

(3) Sum to n terms of the series : $1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + ...$

(4) Find sum to n terms of the series $3+5\times\frac{1}{4}+7\times\frac{1}{4^2}+........$

(5) If the sum to the infinity of the series $3 + 5r + 7r^2 + ...$ is $\frac{44}{9}$, then find the value of r.

(6) If the sum to infinity of the series $3+(3+d).\frac{1}{4}+(3+2d).\frac{1}{4^2}+.........$ is $\frac{44}{9}$ then find d.

Answers : (1) $99.2^{101} + 2$. (2) $\frac{1}{(1-x)^3}$ (3) n^2

(4) $4+\frac{8}{9}\left(1-\frac{1}{4^{n-1}}\right)-\left(\frac{2n+1}{3\times4^{n-1}}\right)$ (5) $\frac{1}{4}$ (6) 2

6. RELATION BETWEEN MEANS

(i) If A, G, H, are respectively A.M., G.M., H.M. between two positive number a & b then
(a) $G^2 = AH$ (A, G, H constitute a GP) (b) $A \ge G \ge H$ (c) $A = G = H \Leftrightarrow a = b$

(ii) Let $a_1, a_2, ..., a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \frac{a_1+a_2+.....+a_n}{n}$

$G = (a_1 a_2 ....a_n)^{1/n}$ and $H=\frac{n}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+....\frac{1}{a_n}\right)}$

It can be shown that $A \ge G \ge H$. Moreover equality holds at either place if and only if $a_1 = a_2 = ... = a_n$

SOLVED EXAMPLE

Example # 31 : If a, b, c > 0, prove that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3$

Solution : Using the relation A.M. ≥ G.M. we have

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \ge \left(\frac{a}{b}.\frac{b}{c}.\frac{c}{a}\right)^{\frac{1}{3}} \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3$$

Example # 32 : If x,y,z are positive, then prove that $(x + y + z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$

Solution : Using the relation A.M. ≥ H.M.

$$\frac{x+y+z}{3} \geq \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \Rightarrow (x + y + z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$$

Example # 33 : If $a_i > 0$ $\forall$ $i \in N$ such that $\prod_{i=1}^{n} a_i = 1$, then prove that $(1 + a_1)(1 + a_2)(1 + a_3) ....(1 + a_n) \geq 2^n$

Solution : Using A.M. ≥ G.M.

$1 + a_1 \geq 2\sqrt{a_1}$

$1 + a_2 \geq 2\sqrt{a_2}$

$\vdots$

$1 + a_n \geq 2\sqrt{a_n}$

$\Rightarrow$ $(1 + a_1)(1 + a_2) ....(1 + a_n) \geq 2^n(a_1a_2a_3.....a_n)^{1/2}$

As $a_1\, a_2\, a_3 \,.....\, a_n = 1$

Hence $(1 + a_1)(1 + a_2) .... (1 + a_n) \geq 2^n$.

Example # 34 : If n > 0, prove that $2^n > 1 + n\sqrt{2^{n-1}}$

Solution : Using the relation A.M. ≥ G.M. on the numbers $1, 2, 2^2, 2^3, ..., 2^{n-1}$, we have

$$\frac{1+2+2^2+.......+2^{n-1}}{n} > (1.2.\,2^2.\,2^3.\,.........\,.2^{n-1})^{1/n}$$

Equality does not hold as all the numbers are not equal.

$$\Rightarrow \frac{2^n-1}{2-1} > n\left(2^{\frac{(n-1)n}{2}}\right)^{\frac{1}{n}} \Rightarrow 2^n - 1 > n.\,2^{\frac{(n-1)}{2}} \Rightarrow 2^n > 1 + n.\,2^{\frac{(n-1)}{2}}$$

Example # 35 : Find the greatest value of xyz for positive value of x, y, z subject to the condition xy + yz + zx = 12.

Solution : Using the relation A.M. ≥ G.M.

$$\frac{xy+yz+zx}{3} \geq (x^2\,y^2\,z^2)^{1/3} \Rightarrow 4 \geq (x\,y\,z)^{2/3} \Rightarrow xyz \leq 8$$

Example # 36 : If a, b, c are in H.P. and they are distinct and positive, then prove that $a^n + c^n > 2b^n$

Solution : Let a^n and c^n be two numbers

then $\frac{a^n+c^n}{2} > (a^n\,c^n)^{1/2}$

$a^n + c^n > 2\,(ac)^{n/2}$ ....(i)

Also G.M. > H.M.

i.e. $\sqrt{ac} > b$, $(ac)^{n/2} > b^n$ ....(ii)

hence from (i) and (ii), we get $a^n + c^n > 2b^n$

Problems for Self Practice -8:

(1) If a, b, c are real and distinct, then show that $a^2(1 + b^2) + b^2(1 + c^2) + c^2(1 + a^2) > 6abc$

(2) If a, b, c, d be four distinct positive quantities in G.P., then show that

(i) $a + d > b + c$

(ii) $\frac{1}{ab} + \frac{1}{cd} > 2\left(\frac{1}{bd} + \frac{1}{ac} - \frac{1}{ad}\right)$

(3) Prove that ΔABC is an equilateral triangle iff $\tan A + \tan B + \tan C = 3\sqrt{3}$

(4) If a, b, c > 0, prove that $[(1 + a)(1 + b)(1 + c)]^7 > 7^7 a^4 b^4 c^4$

7. SIGMA NOTATIONS (Σ)

Theorems :

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$ **(b)** $\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$

(c) $\sum_{r=1}^{n} k = nk$; where k is a constant.

7.1. Results

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

Note :

If n^{th} term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$ where a, b, c, d are constants, then sum of n terms $S_n = \Sigma T_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + \Sigma d$

This can be evaluated using the above results.

SOLVED EXAMPLE

Example # 37 : Find the sum of the series to n terms whose general term is 2n + 1.

Solution : $S_n = \Sigma T_n = \Sigma(2n + 1) = 2\Sigma n + \Sigma 1 = \frac{2(n+1)\,n}{2} + n = n^2 + 2n$

Example # 38 : $T_k = k^2 + 2^k$, then find $\sum_{k=1}^{n} T_k$.

Solution : $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k = \frac{n(n+1)(2n+1)}{6} + \frac{2(2^n - 1)}{2-1} = \frac{n(n+1)(2n+1)}{6} + 2^{n+1} - 2.$

Example # 39 : Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$

Solution : $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j$

$= \sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right] = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{n(n+1)}{12}[2n + 1 + 3] = \frac{n(n+1)(n+2)}{6}$.

Example # 40: Find the sum up to 16 terms of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$

Solution : $t_n = \frac{1^3+2^3+3^3+....+n^3}{1+3+5+....(2n-1)} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}} = \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{(n+1)^2}{4} == \frac{n^2}{4} + \frac{n}{2} + \frac{1}{4}$

$\therefore \quad S_n = \Sigma t_n = \frac{1}{4}\Sigma n^2 + \frac{1}{2}\Sigma n + \frac{1}{4}\Sigma 1 \quad = \frac{1}{4}.\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}.\frac{n(n+1)}{2} + \frac{1}{4}.n$

$\therefore \quad S_{16} = \frac{16.17.33}{24} + \frac{16.17}{4} + \frac{16}{4} = 446$

<u>Problems for Self Practice</u> -9:

(1) Find the sum of the series upto n terms $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \frac{1+2+3+4}{4} + ...$

(2) Find the sum of 'n' terms of the series whose n^{th} term is $t_n = 3n^2 + 2n$.

Answers : (1) $\frac{n(n+3)}{4}$ (2) $\frac{n(n+1)(2n+3)}{2}$

8. METHOD OF DIFFERENCE FOR FINDING T_n (N^{TH} TERM):

Some times the n^{th} term of a sequence or a series can not be determined by the method, we have discussed earlier. So we compute the difference between the successive terms of given sequence for obtained the n^{th} terms.

If $T_1, T_2, T_3, \ldots, T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2, \ldots$ constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.

Case 1 :

(a) If difference series are in A.P., then
Let $T_n = an^2 + bn + c$, where a, b, c are constant

(b) If difference of difference series are in A.P.
Let $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constant

Case 2 :

(a) If difference are in G.P., then
Let $T_n = ar^n + b$, where r is common ratio & a, b are constant

(b) If difference of difference are in G.P., then
Let $T_n = ar^n + bn + c$, where r is common ratio & a, b, c are constant
Determine constant by putting n = 1, 2, 3 .... n and putting the value of T_1, T_2, T_3 ...... and sum of series $(S_n) = \sum T_n$

Note : The above method can be generalised as follows :
Let $u_1, u_2, u_3, \ldots$ be a given sequence.
The first differences are $\Delta_1 u_1, \Delta_1 u_2, \Delta_1 u_3, \ldots$ where $\Delta_1 u_1 = u_2 - u_1$, $\Delta_1 u_2 = u_3 - u_2$ etc.
The second differences are $\Delta_2 u_1, \Delta_2 u_2, \Delta_2 u_3, \ldots$, where $\Delta_2 u_1 = \Delta_1 u_2 - \Delta_1 u_1$, $\Delta_2 u_2 = \Delta_1 u_3 - \Delta_1 u_2$ etc.
This process is continued untill the k^{th} differences $\Delta_k u_1, \Delta_k u_2, \ldots$ are obtained, where the k^{th} differences are all equal or they form a GP with common ratio different from 1.

Case - 1 : The k^{th} differences are all equal.
In this case the n^{th} term, u_n is given by $u_n = a_0 n^k + a_1 n^{k-1} + \ldots + a_k$, where $a_0, a_1, \ldots, a_k$ are calculated by using first 'k + 1' terms of the sequence.

Case - 2 : The k^{th} differences are in GP with common ratio r $(r \neq 1)$
The n^{th} term is given by $u_n = \lambda r^{n-1} + a_0 n^{k-1} + a_1 n^{k-2} + \ldots + a_{k-1}$

SOLVED EXAMPLE

Example # 41 : Find the sum of n terms of the series 3 + 7 + 14 + 24 + 37 + ...

Solution : Clearly here the differences between the successive terms are
7 – 3, 14 – 7, 24 – 14, ... i.e. 4, 7, 10, 13,........., which are in A.P.
Let $S = 3 + 7 + 14 + 24 + \ldots + T_n$
$S = 3 + 7 + 14 + \ldots + T_{n-1} + T_n$
Subtracting, we get
$0 = 3 + [4 + 7 + 10 + 13 + \ldots (n-1) \text{ terms}] - T_n$
$\therefore$ $T_n = 3 + S_{n-1}$ of an A.P. whose a = 4 and d = 3.

$$\therefore \quad T_n = 3 + \left(\frac{n-1}{2}\right)(2.4 + (n-2)3) = \frac{6 + (n-1)(3n+2)}{4} \text{ or, } T_n = \frac{1}{2}\left(3n^2 - n + 4\right)$$

Now putting n = 1, 2, 3,........, n and adding

$\therefore \quad S_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2 + n + 4)$ **Ans.**

Aliter Method :

Let $T_n = an^2 + bn + c$

Now, $T_1 = 3 = a + b + c$ .....(i)

$T_2 = 7 = 4a + 2b + c$ .....(ii)

$T_3 = 14 = 8a + 3b + c$ .....(iii)

Solving (i), (ii) & (iii) we get

$a = \frac{3}{2}, b = -\frac{1}{2}$ & c = 2

$\therefore \quad T_n = \frac{1}{2}(3n^2 - n + 4)$

$\Rightarrow \quad s_n = \Sigma T_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2 + n + 4)$ **Ans.**

Example # 42 : Find the sum of n-terms of the series 1 + 4 + 10 + 22 + ...

Solution : Let $S = 1 + 4 + 10 + 22 + ... + T_n$ .... (i)

$S = 1 + 4 + 10 + ... + T_{n-1} + T_n$ .... (ii)

(i) – (ii) $\Rightarrow T_n = 1 + (3 + 6 + 12 + ... + T_n - T_{n-1})$

$T_n = 1 + 3\left(\frac{2^{n-1} - 1}{2 - 1}\right)$

$T_n = 3 . 2^{n-1} - 2$

So $S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$= 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n = 3.2^n - 2n - 3$ **Ans.**

Aliter Method :

Let $T_n = ar^n + b$, where r = 2

Now $T_1 = 1 = ar + b$ ....(i)

$T_2 = 4 = ar^2 + b$ ....(ii)

Solving (i) & (ii), we get

$a = \frac{3}{2}, b = -2$

$\therefore \quad T_n = 3.2^{n-1} - 2$

$\Rightarrow \quad S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$= 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n = 3.2^n - 2n - 3$ **Ans.**

9. METHOD OF DIFFERENCE FOR FINDING SUM OF n TERMS (V(n) METHOD)

If possible express r^{th} term as difference of two terms as $t_r = \pm (f(r) - f(r \pm 1))$. This can be explained with the help of examples given below.

$t_1 = f(1) - f(0)$,

$t_2 = f(2) - f(1)$,

$\vdots \quad \vdots \quad \vdots$

$t_n = f(n) - f(n-1)$

$\Rightarrow \quad S_n = f(n) - f(0)$

SOLVED EXAMPLE

Example # 43 : Find the sum of n terms of the series 1 . 3 . 5 + 3 . 5 . 7 + 5 . 7 . 9 + ...

Solution : The n^{th} term is $(2n - 1)(2n + 1)(2n + 3)$

$T_n = (2n - 1)(2n + 1)(2n + 3)$

$T_n = \frac{1}{8}(2n-1)(2n + 1)(2n + 3)\{(2n + 5) - (2n - 3)\}$

$= \frac{1}{8}(V_n - V_{n-1})$ [Let $V_n = (2n - 1)(2n + 1)(2n + 3)(2n + 5)$]

$S_n = \sum T_n = \frac{1}{8}[V_n - V_0]$

$\therefore \ S_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{8} + \frac{15}{8} = n(2n^3 + 8n^2 + 7n - 2)$

Example # 44 : Sum to n terms of the series $\frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \frac{1}{(1+3x)(1+4x)} + \ldots\ldots\ldots$

Solution : Let T_r be the general term of the series

$T_r = \frac{1}{(1+rx)(1+(r+1)x)}$

So $T_r = \frac{1}{x}\left[\frac{(1+(r+1)x)-(1+rx)}{(1+rx)(1+(r+1)x)}\right] = \frac{1}{x}\left[\frac{1}{1+rx} - \frac{1}{1+(r+1)x}\right]$

$T_r = f(r) - f(r + 1)$

$\therefore \quad S = \sum T_r = T_1 + T_2 + T_3 + \ldots\ldots\ldots + T_n$

$= \frac{1}{x}\left[\frac{1}{1+x} - \frac{1}{1+(n+1)x}\right] = \frac{n}{(1+x)[1+(n+1)x]}$

Example # 45 : Sum to n terms of the series $\frac{4}{1.2.3} + \frac{5}{2.3.4} + \frac{6}{3.4.5} + ...$

Solution : Let $T_r = \frac{r+3}{r(r+1)(r+2)} = \frac{1}{(r+1)(r+2)} + \frac{3}{r(r+1)(r+2)}$

$$= \left[\frac{1}{r+1} - \frac{1}{r+2}\right] + \frac{3}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$$

$$\therefore \quad S = \left[\frac{1}{2} - \frac{1}{n+2}\right] + \frac{3}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{5}{4} - \frac{1}{n+2}\left[1 + \frac{3}{2(n+1)}\right] = \frac{5}{4} - \frac{1}{2(n+1)(n+2)}[2n+5]$$

Example # 46 : If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

Solution : $\because \quad T_n = S_n - S_{n-1}$

$$= \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r = \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8} = \frac{n(n+1)(n+2)}{8}[(n+3)-(n-1)]$$

$$T_n = \frac{n(n+1)(n+2)}{8}(4) = \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \quad \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} = \frac{(n+2)-n}{n(n+1)(n+2)} = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \quad .... (i)$$

Let $V_n = \frac{1}{n(n+1)}$

$$\therefore \quad \frac{1}{T_n} = V_n - V_{n+1}$$

Putting n = 1, 2, 3, ... n

$$\Rightarrow \quad \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} + ... + \frac{1}{T_n} = (V_1 - V_{n+1})$$

$$\Rightarrow \quad \sum_{r=1}^{n} \frac{1}{T_r} = \frac{n^2+3n}{2(n+1)(n+2)}$$

Problems for Self Practice -10:

(1) Sum to n terms the following series

(i) $4 + 14 + 30 + 52 + 80 + 114 + \ldots\ldots\ldots\ldots$

(ii) $2 + 5 + 12 + 31 + 86 + \ldots\ldots\ldots\ldots\ldots$

(iii) $\frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \frac{1+2+3}{1^3+2^3+3^3} + \ldots\ldots\ldots$

(iv) $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \ldots\ldots\ldots$

(v) $1 . 5 . 9 + 2 . 6 . 10 + 3 . 7. 11 + \ldots\ldots\ldots$

Answers : (1) (i) $n(n + 1)^2$

(ii) $\frac{3^n + n^2 + n - 1}{2}$

(iii) $\frac{2n}{n+1}$

(iv) $\frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right]$

(v) $\frac{n}{4}(n + 1)(n + 8)(n + 9)$

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Arithmetic Progression

A-1. Find the sum of first 48 terms of the series whose k^{th} term is $\frac{k}{7} + 9$.

A-2. Find the sum of all those integers between 100 and 800 each of which on division by 16 leaves the remainder 7.

A-3. Find the number of integers between 100 & 1000 that are not divisible by 13

A-4. If the sum of first 10 terms of an A.P. is 140 and the sum of first 16 terms is 320, find the sum of n terms

A-5. Find the sum of the series $(a + b)^2 + (a^2 + b^2) + (a - b)^2 + \ldots + $ to n terms.

A-6. If $x > 0$, and $\log_2 x + \log_2 \left(\sqrt{x}\right) + \log_2 \left(\sqrt[4]{x}\right) + \log_2 \left(\sqrt[8]{x}\right) + \log_2 \left(\sqrt[16]{x}\right) + \ldots\ldots = 4$, then find x.

A-7. If pth, qth and rth terms of an A.P are a,b,c respectively than prove that $a(q - r) + b(r - p) + c(p - q) = 0$

A-8. The sum of three consecutive numbers in A.P. is 27, and their product is 504, find them.

A-9. If a, b, c are in A.P., then show that:

(i) $b + c - a,\ c + a - b,\ a + b - c$ are in A.P.

(ii) $a + \frac{1}{bc},\ b + \frac{1}{ca},\ c + \frac{1}{ab}$ are in AP

(iii) $\frac{1}{\sqrt{b}+\sqrt{c}},\ \frac{1}{\sqrt{c}+\sqrt{a}},\ \frac{1}{\sqrt{a}+\sqrt{b}}$ are in AP

(iv) $a^2 (b + c),\ b^2 (c + a),\ c^2 (a + b)$ are also in A.P.

A-10. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer.

A-11. If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between a and b (where $a \neq b$) then find the value of n

A-12. There are 11 AMs between 28 and 10. Find the number of integral AMs

A-13. If a, b, c, d, e, f are AMs between 2 and 12 then find thesum $a + b + c + d + e + f$

Section (B) : Geometric Progression

B-1. A boy agrees to work at the rate of one rupee the first day, two rupees the second day, four rupees the third day, eight rupees the fourth day and so on in the month of April. How much would he get on the 20th of April.

B-2. The seventh term of a G.P. is 8 times the fourth term. Find the G.P. when its 5th term is 48

B-3. The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 & the third is increased by 1, we obtain three consecutive terms of a G.P., find the numbers.

B-4. If the p^{th}, q^{th} & r^{th} terms of an AP are in GP. Find the common ratio of the GP.

B-5. The sum of infinite number of terms of a G.P. is 4 and the sum of their cubes is 192. Find the series.

B-6. The third term of a G.P. is the square of the first term. If the second term is 8, find its sixth term.

B-7. The continued product of three numbers in G.P. is 216, and the sum of the products of them in pairs is 156; find the numbers

B-8. If a, b, c, d are in G.P., prove that :

(i) $(a^2 - b^2)$, $(b^2 - c^2)$, $(c^2 - d^2)$ are in G.P.

(ii) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.

B-9. In a set of four numbers, the first three are in G.P. and the last three are in A.P. with a common difference of 6. If the first number is same as the fourth find the four number.

B-10. If G is the geometric mean between two distinct positive numbers a and b, then show that $\frac{1}{G-a} + \frac{1}{G-b} = \frac{1}{G}$

.

B-11. If a is the A.M. of b and c and GMs inserted between b and c are G_1, G_2 then prove that $G_1^3 + G_2^3 = 2abc$

Section (C) : Harmonic and Arithmetic Geometric Progression

C-1. If mth term of a H.P. be n and nth term be m then find the $(mn)^{th}$ term

C-2 If a, b, c are in H.P. and a > c > 0, then prove that $\frac{1}{b-c} - \frac{1}{a-b}$ is positive

C-3. If b is the harmonic mean between a and c, then prove that $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$.

C-4. If a, b, c, d are in H.P. then find the value of $\frac{a^{-2} - d^{-2}}{b^{-2} - c^{-2}}$

C-6. If the first and $(2n + 1)^{th}$ term of a A.P, G.P and H.P (consisting of positive terms) are equal and their $(n + 1)^{th}$ terms are a, b, c respectively then prove that a, b, c are in G.P.

C-7. If a be a positive real number and A.M of a and 2a exceed their HM by 2, then find a

C-8. The H.M. between two numbers is $\frac{16}{5}$, their A.M. is A and G.M. is G. If $2A + G^2 = 26$, then find the numbers.

C-9. If 9 AMs A_1, ...A_9 and 9HMs H_1, H_2, ... H_9 are inserted between 2 and 3 alternatively then find the value of

$A_i + \frac{6}{H_i}$

C-10. Sum the following series $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \dots$ to n terms.

C-11. Find the sum of n terms of the series the r^{th} term of which is $(2r + 1)2^r$.

C-12. Find the sum : $\frac{3}{2} - \frac{5}{6} + \frac{7}{18} - \frac{9}{54} + \dots \infty$

Section (D) : Means, Inequalities A.M. ≥ G.M. ≥ H.M

D-1. If the product of three positive real numbers say a, b, c be 27, then find the minimum value of ab + bc + ca

D-2. Using the relation A.M. ≥ G.M. prove that
$(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq 9x^2y^2z^2$. (x, y, z are positive real number)

D-3 If a, b, c are three positive real numbers, then the minimum value of the expression $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$

D-4. If x > 0, then find greatest value of the expression $\frac{x^{100}}{1+x+x^2+x^3+.....+x^{200}}$.

D-5. If $x_i > 0$, i = 1, 2, ..., 50 and $x_1 + x_2 + .. + x_{50} = 50$, then find the minimum value of $\frac{1}{x_1} + \frac{1}{x_2} + ...+ \frac{1}{x_{50}}$.

D-6. If a, b, c are positive real numbers and sides of the triangle, then prove that
$(a + b + c)^3 \geq 27 (a + b - c)(c + a - b)(b + c - a)$

D-7. If a, b, c be three unequal positive quantities in H.P. then prove that $a^n + c^n > 2b^n$

D-8 If abc = 8 and a, b, c > 0, then find the minimum value of (2 + a) (2 + b) (2 + c)

D-9 Using the relation A.M. ≥ G.M. prove that (a + b) . (b + c) . (c + a) > abc ; if a, b, c are positive real numbers

Section (E) : Summation of series

E-1. (i) If $t_n = 3^n - 2^n$ then find $\sum_{n=1}^{k} t_n$. (ii) If $t_n = n(n + 2)$ then find $\sum_{n=1}^{k} t_n$.

E-2. Find the sum of the series $31^3 + 32^3 + ... + 50^3$

E-3. If $S_n = \sum_{r=1}^{n} t_r = \frac{1}{6}n(2n^2+9n+13)$, then $\sum_{r=1}^{\infty} \frac{1}{r\cdot\sqrt{t_r}}$ equals

E-4. Find the sum of n terms : $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$

Section (F) : Method of difference and V_n Method

F-1. Sum to n terms : 3 + 15 + 35 + 63 + ...

F-2. Find the sum to n-terms of the sequence : 1 + 5 + 13 + 29 + 61 + ...

F-3. Find the sum to n-terms of the sequence : $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + ...$

F-4. Find the sum $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$

F-5 If Sn = $\sum_{r=1}^{n} T_r = n(n+1)(n+2)(n+3)$ then find $\sum_{r=1}^{10} \frac{1}{T_r}$

F-6. If $t_r = \frac{r+2}{r(r+1)}\cdot\frac{1}{2^{r+1}}$, then find $\sum_{r=1}^{n} t_r$

PART-II : OBJECTIVE QUESTIONS

Section (A) : Arithmetic Progression

A-1. If $a_1, a_2, \ldots\ldots, a_n$ are distinct terms of an A.P., then

(A) $a_1 + 2a_2 + a_3 = 0$ (B) $a_1 - 2a_2 + a_4 = 0$
(C) $a_1 + 3a_2 - 3a_3 - a_4 = 0$ (D) $a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 = 0$

A-2. If the sum of the first 2n terms of the A.P. 2, 5, 8, ..., is equal to the sum of the first n terms of the A.P. 57, 59, 61, ..., then n equals

(A) 10 (B) 12 (C) 11 (D) 13

A-3. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

(A) 2550 (B) 1050 (C) 3050 (D) none of these

A-4. Let $a_1, a_2, a_3, \ldots\ldots$ be terms of an AP. If $\dfrac{a_1 + a_2 + \ldots\ldots + a_p}{a_1 + a_2 + \ldots\ldots + a_q} = \dfrac{p^2}{q^2}$, $p \neq q$, then $\dfrac{a_6}{a_{21}}$ equals :

(A) $\dfrac{7}{2}$ (B) $\dfrac{2}{7}$ (C) $\dfrac{11}{41}$ (D) $\dfrac{41}{11}$

A-5. The sum of four integers in A.P. is 24 and their product is 945. The common difference of A.P. is

(A) ± 1 (B) ± 2 (C) ± 3 (D) ± 4

A-6. If $a_1, a_2, a_3, \ldots\ldots\ldots$ are in A.P. such that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, then $a_1 + a_2 + a_3 + \ldots\ldots + a_{23} + a_{24}$ is equal to

(A) 909 (B) 75 (C) 750 (D) 900

A-7. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = \ldots\ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots\ldots$are in an AP with common difference –2, then the time taken by him to count all notes is

(A) 34 minutes (B) 125 minutes (C) 135 minutes (D) 24 minutes

A-8. There are n A.M's between 3 and 54, such that the 8th mean: $(n-2)^{th}$ mean:: 3: 5. The value of n is.

(A) 12 (B) 16 (C) 18 (D) 20

Section (B) : Geometric Progression

B-1. The third term of a G.P is 4. The product of the first five terms is

(A) 4^3 (B) 4^5 (C) 4^4 (D) 4

B-2. For a sequence $\{a_n\}$, $a_1 = 2$ and $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{3}$. Then $\sum_{r=1}^{20} a_r$ is

(A) $\dfrac{20}{2}$ [4 + 19 × 3] (B) $3\left(1-\dfrac{1}{3^{20}}\right)$ (C) $2(1-3^{20})$ (D) $\left(1-\dfrac{1}{3^{20}}\right)$

B-3. One side of an equilateral triangle is 24 cm. The mid–points of its sides are joined to form another triangle whose mid – points are in turn joined to form still another triangle. This process continues indefinitely. Then the sum of the perimeters of all the triangles is

(A) 144 cm (B) 212 cm (C) 288 cm (D) 172 cm

B-4. If S is the sum to infinity of a G.P. whose first term is 'a', then the sum of the first n terms is

(A) $S\left(1-\dfrac{a}{S}\right)^n$ (B) $S\left[1-\left(1-\dfrac{a}{S}\right)^n\right]$ (C) $a\left[1-\left(1-\dfrac{a}{S}\right)^n\right]$ (D) $S\left[1-\left(1-\dfrac{S}{a}\right)^n\right]$

B-5 If y, x, z are in A.P., then

(A) x + y, y + z, z + x are in G.P.
(B) 2^{x+y}, 2^{y+z}, 2^{z+x} are in G.P.
(C) 3^x, 3^y, 3^z are in G.P.
(D) none of these

B-6 The real values of x for which y = 1 + $\frac{2}{x}+\frac{4}{x^2}+\frac{8}{x^3}+\frac{16}{x^4}+\ldots\infty$ is finite is

(A) (–2, 2)
(B) $(-\infty, -2) \cup (2, \infty)$
(C) $(2, \infty)$
(D) none of these

B-7 If $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P. then 9^{ax+1}, 9^{bx+1}, 9^{cx+1} (where $x \in R$) are in

(A) G.P. (B) G.P. only if x < 0 (C) G.P. only if x > 0 (D) none of these

B-8 A G.P. consists of even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying odd places, then its common ratio is

(A) 2 (B) 3 (C) 4 (D) 5

B-9 If k_1, k_2, k_3 and k_4 are GM's between a and b, then roots of the equation $k_2k_3x^2 - \frac{k_2}{k_1+k_3}x - k_1k_4 = 0$ are

(A) One positive, one negative
(B) Both negative
(C) Both positive
(D) Imaginary

Section (C) : Harmonic and Arithmetic Geometric Progression

C-1. $\log_4 5$, $\log_{20} 5$, $\log_{100} 5$ are in

(A) A.P. (B) G.P. (C) H.P. (D) none of these

C-2. Let the positive numbers a, b, c, d be in A.P. Then abc, abd, acd, bcd are:

(A) not in A.P./G.P./H.P.
(B) in A.P.
(C) in G.P.
(D) in H.P.

C-3. **Statement 1 :** 3,6,12 are in G.P., then 9,12,18 are in H.P.

Statement 2 : If three consecutive terms of a G.P. are positive and if middle term is added in these terms, then resultant will be in H.P.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true

C-4 If $a_i \in R$ and a_1, a_2, a_3 are in A.P., a_2, a_3, a_4 are in G.P. and a_3, a_4, a_5 are in H.P. then $\frac{a_1-a_3}{a_3-a_5}$ is equal to

(A) $\frac{a_1}{a_3}$ (B) $\frac{a_3}{a_1}$ (C) $\frac{a_5}{a_1}$ (D) 1

C-5 The harmonic mean of two numbers is 4 and their arithmetic mean and geometric mean satisfy the relation $2A + G^2 = 27$. The numbers are

(A) 6, 3 (B) 5, 4 (C) 5, -5/2 (D) -3, 1

C-6. If $3 + \frac{1}{4}(3 + d) + \frac{1}{4^2}(3 + 2d) + ... +$ upto $\infty = 8$, then the value of d is

(A) 9 (B) 5 (C) 1 (D) 4

C-7. The sum of the series $1 + \frac{3}{4} + \frac{7}{16} + \frac{15}{64} + \frac{31}{256} + ...$ to infinity is

(A) $\frac{8}{3}$ (B) $\frac{4}{3}$ (C) 2 (D) 3

C-8. If A, G & H are respectively the A.M., G.M. & H.M. of three positive numbers a, b, & c, then the equation whose roots are a, b, & c is given by:

(A) $x^3 - 3Ax^2 + 3G^3x - G^3 = 0$ (B) $x^3 - 3Ax^2 + 3(G^3/H)x - G^3 = 0$
(C) $x^3 + 3Ax^2 + 3(G^3/H)x - G^3 = 0$ (D) $x^3 - 3Ax^2 - 3(G^3/H)x + G^3 = 0$

Section (D) : Inequality A.M. ≥ G.M. ≥ H.M

D-1. If a, b, c, d are positive real numbers such that a + b + c + d = 2, then M = (a + b) (c + d) satisfies the relation:

(A) $0 \le M \le 1$ (B) $1 \le M \le 2$ (C) $2 \le M \le 3$ (D) $3 \le M \le 4$

D-2. If $x \in R$, the numbers $5^{1+x} + 5^{1-x}$, a/2, $25^x + 25^{-x}$ form an A.P. then 'a' must lie in the interval:

(A) [1, 5] (B) [2, 5] (C) [5, 12] (D) [12, ∞)

D-3. If a + b + c = 3 and a > 0, b > 0, c > 0, the greatest value of $a^2b^3c^2$.

(A) $\frac{3^{10}.2^4}{7^7}$ (B) $\frac{3^9.2^4}{7^7}$ (C) $\frac{3^9.2^5}{7^7}$ (D) $\frac{3^{10}.2^5}{7^7}$

D- 4. If x, y, z are positive numbers then minimum value of $x^{\ln y - \ln z} + y^{\ln z - \ln x} + z^{\ln x - \ln y}$ is

(A) 1 (B) 2 (C) 3 (D) 4

D-5 If 2x + 3y + 5z = 10 and 81xyz = 100, where x, y, z $\in R^+$, then number of ordered triplet (x, y, z) is

(A) 0 (B) 1 (C) infinite (D) none of these

D-6 Let p, q, r $\in R^+$ and $27pqr \ge (p + q + r)^3$ and 3p + 4q + 5r = 12 then $p^3 + q^4 + r^5$ is equal to

(A) 3 (B) 6 (C) 2 (D) none of these

Section (E) : Summation of series

E-1. Sum of the series $S = 1^2 - 2^2 + 3^2 - 4^2 + ... - 2002^2 + 2003^2$ is

(A) 2007006 (B) 1005004 (C) 2000506 (D) none of these

E-2. If $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$, then value of $1 + \frac{3}{2} + \frac{5}{3} + ... + \frac{2n-1}{n}$ is

(A) $2n - H_n$ (B) $2n + H_n$ (C) $H_n - 2n$ (D) $H_n + n$

E-3. Given that $a_1, a_2, a_3, ...a_n$ form an A.P. find then following sum $\sum_{i=1}^{10} \frac{a_i a_{i+1} a_{i+2}}{a_i + a_{i+2}}$

Given that $a_1 = 1$; $a_2 = 2$

(A) $\frac{495}{2}$ (B) $\frac{415}{2}$ (C) 112 (D) 115

E-4. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is

(A) $\frac{n(4n^2-1)c^2}{6}$ (B) $\frac{n(4n^2+1)c^2}{3}$

(C) $\frac{n(4n^2-1)c^2}{3}$ (D) $\frac{n(4n^2+1)c^2}{6}$

Section (F) : Method of difference and V_n Method

F-1. **Statement 1 :** The sum of the first 30 terms of the sequence 1,2,4,7,11,16, 22,...... is 4520.

Statement 2 : If the successive differences of the terms of a sequence form an A.P., then general term of sequence is of the form $an^2 + bn + c$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

F-2. $\sum_{r=1}^{\infty} \frac{r^3 + (r^2+1)^2}{(r^4+r^2+1)(r^2+r)}$ is equal to

(A) $\frac{3}{2}$ (B) 1 (C) 2 (D) infinite

F-3. If $H_1, H_2, H_3, \ldots\ldots, H_{2n+1}$ are in H.P., then $\sum_{i=1}^{2n} (-1)^i \left(\frac{H_i + H_{i+1}}{H_i - H_{i+1}}\right)$ is equal to

(A) 2n - 1 (B) 2n + 1 (C) 2n (D) 2n + 2

F-4. If $a_1, a_2, ..., a_n$ are in HP, then the expression $a_1a_2 + a_2a_3 + ... + a_{n-1}a_n$ is equal to :

(A) $(n-1)(a_1 - a_n)$ (B) na_1a_n (C) $(n-1)a_1a_n$ (D) $n(a_1 - a_n)$

F-5. Find the sum to ten terms of the sequence : $1 . 3 . 2^2 + 2 . 4 . 3^2 + 3 . 5 . 4^2 + ...$

(A) 11.12 .13 .23 (B) 10.11 .12 .23

(C) 10.12 .13 .23 (D) None of these

PART - III : MATCH THE COLUMN

1.

	Column-I		Column-II
(A)	The cofficient of x^{49} in the product $(x-1)(x-3)(x-5)(x-7)\ ....\ (x-99)$	(p)	–2500
(B)	Let S_n denote sum of first n terms of an A.P. If $S_{2n} = 3S_n$, then $\frac{S_{3n}}{S_n}$ is	(q)	9
(C)	Sum of infinite series $4 + \frac{8}{3} + \frac{12}{3^2} + \frac{16}{3^3} + - - -$ is	(r)	3
(D)	The length,breadth, height of a rectangular box are in G.P. (length > breadth > height) The volume is 27, the total surface area is 78. Then the length is	(s)	6

2.

	Column-I		Column-II
(A)	The value of xyz is 15/2 or 18/5 according as the series a, x, y, z, b are in an A.P. or H.P. then 'a + b' equals where a, b are positive integers.	(p)	2
(B)	The value of $2^{\frac{1}{4}} . 4^{\frac{1}{8}} . 8^{\frac{1}{16}} - - - \infty$ is equal to	(q)	1
(C)	If x, y, z are in A.P., then $(x + 2y - z)(2y + z - x)(z + x - y) = kxyz$, where $k \in N$, then k is equal to	(r)	3
(D)	There are m A.M. between 1 and 31. If the ratio of the 7^{th} and $(m-1)^{th}$ means is 5 : 9, then $\frac{m}{7}$ is equal to	(s)	4

Exercise # 2

PART-I : OBJECTIVE

1. If 1, 2, 3 ... are first terms; 1, 3, 5 .... are common differences and S_1, S_2, S_3 .... are sums of n terms of given p AP's; then $S_1 + S_2 + S_3 + ... + S_p$ is equal to

(A) $\frac{np(np+1)}{2}$ (B) $\frac{n(np+1)}{2}$ (C) $\frac{np(p+1)}{2}$ (D) $\frac{np(np-1)}{2}$

2. If a, b and c are three terms of an A.P. such that $a \neq b$, then $\frac{b-c}{a-b}$ may be equal to

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 1 (D) None of these

3. Sum of the first n terms of an A.P (having positive terms) is given by $S_n = (1 + 2T_n)(1 - T_n)$, where T_n is the nth term of the series, then the value of T_2^2 is

(A) $\frac{\sqrt{2}+1}{2\sqrt{2}}$ (B) $\frac{\sqrt{2}-1}{2\sqrt{2}}$ (C) $\frac{1}{2\sqrt{2}}$ (D) none of these

4. If the sum of n terms of a G.P. (with common ratio r) beginning with the p^{th} term is k times the sum of an equal number of terms of the same series beginning with the q^{th} term, then the value of k is:

(A) $r^{p/q}$ (B) $r^{q/p}$ (C) r^{p-q} (D) r^{p+q}

5. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. if $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{2\sqrt{3}}$ (C) $\frac{1}{2}-\frac{1}{\sqrt{3}}$ (D) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

6. Area of the triangle with vertices (a, b), (x_1, y_1) and (x_2, y_2) where a, x_1, x_2 are in G.P. with common ratio r and b, y_1, y_2 are in G.P. with common ratio s is

(A) $ab(r-1)(s-1)(s-r)$ (B) $ab(r+1)(s+1)(s-r)$

(C) $ab(r-1)(s-1)(s-r)$ (D) $ab(r+1)(s+1)(r-s)$

7. Let $S = \frac{4}{19}+\frac{44}{19^2}+\frac{444}{19^3}+.......\infty$. Then S is equal to

(A) $\frac{38}{81}$ (B) $\frac{4}{19}$ (C) $\frac{36}{171}$ (D) none of these

8. If three distinct real numbers a, b, c are in G.P and $a + b + c = ax$, then

(A) $x \in \left[\frac{3}{4},\infty\right) - \{1, 3\}$ (B) $x \in R^+$ (C) $x \in (-1, \infty)$ (D) none of these

9. If a, b, c are in A.P a, x, b are in G.P and b, y, c are in G.P., then x^2, b^2, y^2 are in

(A) A.P. (B) G.P. (C) H.P. (D) None of these

10. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$ where a,b,c are in AP and $|a| < 1$, $|b| < 1$, $|c| < 1$, then x,y,z are in :

(A) HP (B) Arithmetico–Geometric Progression
(C) AP (D) GP

11. Given the sequence of numbers x_1, x_2, x_3, ... x_{2013} which satisfy $\frac{x_1}{x_1+1} = \frac{x_2}{x_2+3} = \frac{x_3}{x_3+5} = ...$ $= \frac{x_{2013}}{x_{2013}+4025}$, nature of the sequence is

(A) A.P. (B) G.P. (C) H.P. (D) A.G.P.

12. If $\frac{1}{a}+\frac{1}{a-2b}+\frac{1}{c}+\frac{1}{c-2b} = 0$ and a, b, c are not in A.P, then

(A) a, b, c are in G.P (B) a, $\frac{b}{2}$, c are in A.P
(C) a, $\frac{b}{2}$, c are in H.P (D) a, 2b, c are in H.P

13. If $3x^2 - 2(a-d)x + (a^2 + 2(b^2 + c^2) + d^2) = 2(ab + bc + cd)$, then
(A) a, b, c, d are in G .P. (B) a, b, c, d are in H .P.
(C) a, b, c, d are in A .P. (D) none of these

14. If x, y, z are real numbers such that $x^2 + 18y^2 + 81z^2 = 6xy + 54yz$, then x, y, z are in
(A) A.P. (B) G.P. (C) H.P. (D) A.G.P.

15. The sum of the first n-terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ...$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is

(A) $\frac{n(n+1)^2}{4}$ (B) $\frac{n^2(n+2)}{4}$ (C) $\frac{n^2(n+1)}{2}$ (D) $\frac{n(n+2)^2}{4}$

16. Let T_r and S_r be the r^{th} term and sum up to r^{th} term of a series respectively. If for an odd number n, $S_n = n$ and $T_n = \frac{T_{n-1}}{n^2}$ then T_m (m being even) is

(A) $\frac{2}{1+m^2}$ (B) $\frac{2m^2}{1+m^2}$ (C) $\frac{(m+1)^2}{2+(m+1)^2}$ (D) $\frac{2(m+1)^2}{1+(m+1)^2}$

17. Consider the sequence 2, 3, 5, 6, 7, 8, 10, 11, ... of all positive integer, then 2011^{th} term of this sequence is
(A) 2056 (B) 2011 (C) 2013 (D) 2060

18. If $a_1, a_2, a_3, ..., a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + a_3 + ... + a_{n-1} + 2a_n$ is
(A) $n(2c)^{1/n}$ (B) $(n + 1) c^{1/n}$ (C) $2nc^{1/n}$ (D) $(n + 1)(2c)^{1/n}$

19. If $1^2 + 2^2 + 3^2 + ... + 2003^2 = (2003)(4007)(334)$ and
$(1)(2003) + (2)(2002) + (3)(2001) + ... + (2003)(1) = (2003)(334)(x)$., then x equals
(A) 2005 (B) 2004 (C) 2003 (D) 2001

20. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$ upto $\infty = \frac{\pi^2}{6}$, then $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... =$
(A) $\pi^2/12$ (B) $\pi^2/24$ (C) $\pi^2/8$ (D) $\pi^2/4$

21. The sum of $\frac{7}{2\times3}\left(\frac{1}{3}\right) + \frac{9}{3\times4}\left(\frac{1}{3}\right)^2 + \frac{11}{4\times5}\left(\frac{1}{3}\right)^3 + ...$ upto 10 terms is equal to

(A) $\frac{1}{2} - \frac{1}{12\times3^{10}}$ (B) $\frac{1}{3} - \frac{1}{12\times3^{10}}$ (C) $\frac{1}{2} - \frac{1}{10\times3^{10}}$ (D) none of these

22. The value of $\sum_{r=0}^{n}(a + r + ar)(-a)^r$ is equal to

(A) $(-1)^n[(n+1)a^{n+1} - a]$ (B) $(-1)^n(n+1)a^{n+1}$

(C) $(-1)^n\frac{(n+2)a^{n+1}}{2}$ (D) $(-1)^n\frac{na^n}{2}$

PART-II : NUMERICAL QUESTIONS

1. A man arranges to pay off a debt of Rs. 3600 by 40 annual installments which form an arithmetic series. When 30 of the installments are paid he dies leaving a third of the debt unpaid. Find the value of the first installment.

2. If the common difference of the A.P. in which $T_7 = 9$ and $T_1T_2T_7$ is least, is 'd' then 20d is–

3. The number of terms in an A.P. is even ; the sum of the odd terms is 24, sum of the even terms is 30, and the last term exceeds the first by 10½; find the number of terms.

4. If $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A.P. and α, β be the roots of equation $2acx^2 + 2abcx + (a + c) = 0$ then $(1 + \alpha)(1 + \beta)$ is equal to $(a, b, c \neq 0)$

5. If S denote the sum to infinity and S_n the sum of n terms of the series $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...$ such that $S - S_n < \frac{1}{300}$, then find the least value of n

6. In a circle of radius R a square is inscribed, then a circle is inscribed in the square, a new square in the circle and so on for n times. If the ratio of the limit of the sum of areas of all the circles to the limit of the sum of areas of all the squares as $n \to \infty$ is k, then find the value of $\frac{4k}{\pi}$.

7. Find the sum of the infinitely decreasing G.P. whose third term, three times the product of the first and fourth term and second term form an A.P. in the indicated order, with common difference equal to 1/8.

8. If a, b, c are in GP, a – b, c – a, b – c are in HP, then the value of a + 4b + c is

9. Given that α, γ are roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ the roots of the equation $Bx^2 - 6x + 1 = 0$, then find value of (A + B), such that α, β, γ & δ are in H.P.

10. $a, a_1, a_2, a_3, \ldots, a_{2n}$, b are in A.P. and $a, g_1, g_2, g_3, \ldots\ldots g_{2n}$, b are in G.P. and h is the harmonic mean of a and b, if $\frac{a_1 + a_{2n}}{g_1 g_{2n}} + \frac{a_2 + a_{2n-1}}{g_2 g_{2n-1}} + \ldots + \frac{a_n + a_{n+1}}{g_n g_{n+1}}$ is equal to $\frac{Kn}{20h}$, then find value of K.

11. If the arithmetic mean of two numbers a & b (0 < a < b) is 6 and their geometric mean G and harmonic mean H satisfy the relation $G^2 + 3H = 48$. Then find the value of (2a – b)

12. If a_1, a_2, a_3, a_4 are positive real numbers such that $a_1 + a_2 + a_3 + a_4 = 16$ then find maximum value of $(a_1 + a_2)(a_3 + a_4)$.

13. If a, b, c are the sides of a triangle, then find the minimum value of $\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c}$

14. Find the number of solutions of equation $x\,a^{1/x} + \frac{1}{x}a^x = 2a$, $a \geq 1$

15. If S_1, S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, then $\frac{S_3(1+8S_1)}{S_2^2}$ is equal to

16. If $\frac{25}{k} = 1^2 - \frac{2^2}{5} + \frac{3^2}{5^2} - \frac{4^2}{5^3} + \frac{5^2}{5^4} - \frac{6^2}{5^5} + \ldots\ldots\ \infty$, then find the value of k

17. If $S = \frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots\ldots\infty$, then find the value of 14S.

18. If $\sum_{r=1}^{n} t_r = \frac{n(n+1)(n+2)(n+3)}{8}$, then $\sum_{r=1}^{\infty} \frac{4}{t_r}$ equals

PART - III : ONE OR MORE THAN ONE CORRECT

1. The interior angles of a polygon are in A.P. If the smallest angle is 120º & the common difference is 5º, then the number of sides in the polygon is:

(A) 7 (B) 9 (C) 16 (D) 5

2. If 1, $\log_y x$, $\log_z y$, $-15\log_x z$ are in A.P., then

(A) $z^3 = x$ (B) $x = y^{-1}$ (C) $z^{-3} = y$ (D) $x = y^{-1} = z^3$

3. Consider an infinite geometric series with first term 'a' and common ratio r. If the sum is 4 and the second term is 3/4, then:

(A) $a = \frac{7}{4}$, $r = \frac{3}{7}$ (B) $a = 2$, $r = \frac{3}{8}$ (C) $a = \frac{3}{2}$, $r = \frac{1}{2}$ (D) $a = 3$, $r = \frac{1}{4}$

4. Three numbers a, b, c between 2 and 18 are such that

(i) their sum is 25

(ii) the numbers 2, a, b, are in A.P.

(iii) the number b, c, 18 are in G.P.

then which of the following options are correct.

(A) a = 5 (B) b = 8 (C) b + c = 20 (D) a + b + c = 25

5. If $\frac{a_{k+1}}{a_k}$ is constant for every $k \geq 1$. If $n > m \Rightarrow a_n > a_m$ and $a_1 + a_n = 66$, $a_2 a_{n-1} = 128$ and $\sum_{i=1}^{n} a_i = 126$ then

(A) n = 6 (B) n = 5 (C) $\frac{a_{k+1}}{a_k} = 2$ (D) $\frac{a_{k+1}}{a_k} = 4$

6. The sides of a right triangle form a G.P. The tangent of the smallest angle is

(A) $\sqrt{\frac{\sqrt{5}+1}{2}}$ (B) $\sqrt{\frac{\sqrt{5}-1}{2}}$ (C) $\sqrt{\frac{2}{\sqrt{5}+1}}$ (D) $\sqrt{\frac{2}{\sqrt{5}-1}}$

7. If b_1, b_2, b_3 ($b_i > 0$) are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(A) r > 3 (B) 0 < r < 1 (C) r = 3.5 (D) r = 5.2

8. If a satisfies the equation $a^{2017} - 2a + 1 = 0$ and $S = 1 + a + a^2 + ... + a^{2016}$. then posible value(s) of S is/are

(A) 2016 (B) 2018 (C) 2017 (D) 2

9. Which of the following numbers is/are composite

(A) 1111......1 (91 digits) (B) 1111......1 (81 digits)

(C) 1111......1 (75 digits) (D) 1111......1 (105 digits)

10. If the arithmetic mean of two positive numbers a & b (a > b) is twice their geometric mean, then a: b is:

(A) $2+\sqrt{3} : 2-\sqrt{3}$ (B) $7+4\sqrt{3} : 1$ (C) $1 : 7-4\sqrt{3}$ (D) $2 : \sqrt{3}$

11. Which of the following is/are TRUE

(A) Equal numbers are always in A.P. , G.P. and H.P.

(B) If a, b, c be in H.P., then $a-\frac{b}{2}, \frac{b}{2}, c-\frac{b}{2}$ will be in AP

(C) If G_1 and G_2 are two geometric means and A is the arithmetic mean inserted between two positive numbers, then the value of $\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1}$ is 2A.

(D) Let general term of a G.P. (with positive terms) with common ratio r be T_{k+1} and general term of another G.P. (with positive terms) with common ratio r be T'_{k+1}, then the series whose general term $T''_{k+1} = T_{k+1} + T'_{k+1}$ is also a G.P. with common ratio r.

12. Let a, x, b be in A.P; a, y, b be in G.P and a, z, b be in H.P. If x = y + 2 and a = 5z, then

(A) $y^2 = xz$ (B) x > y > z (C) a = 9, b = 1 (D) a = 1/4, b = 9/4

13. If $a_k a_{k-1} + a_{k-1} a_{k-2} = 2a_k a_{k-2}$, $k \geq 3$ and $a_1 = 1$, here $S_p = \sum_{k=1}^{p} \frac{1}{a_k}$ and given that $\frac{S_{2p}}{S_p}$ does not depend on p then $\frac{1}{a_{2016}}$ may be

(A) 4031 (B) 1 (C) 2016 (D) 2017/2

14. First three terms of the sequence 1/16, a, b, 1/6 are in geometric series and last three terms are in harmonic series if

(A) $a = \frac{1}{9}, b = \frac{1}{12}$ (B) $a = \frac{1}{12}, b = \frac{1}{9}$

(C) $a = 1, b = -\frac{1}{4}$ (D) $a = -\frac{1}{4}, b = 1$

15. For the series $2 + \left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) + \left((2\sqrt{2} - 1) + \frac{1}{2}\right) + \left(\left(3\sqrt{2} - 2\right) + \frac{1}{2\sqrt{2}}\right) + ...$

(A) $S_n = \sqrt{2}\left(\sqrt{2} + n - 1\right) - n + \left(\frac{\left(2^{n/2} - 1\right)}{\left(\sqrt{2} - 1\right)2^{\frac{n-1}{2}}}\right)$ (B) $T_n = \sqrt{2}\left(\sqrt{2} + n - 1\right) - n + \left(\frac{1}{2}\right)^{\frac{n-1}{2}}$

(C) $S_n = \frac{n}{2}\left(3 + (n-1)\sqrt{2} - n\right) + \left(\frac{\left(2^{n/2} - 1\right)}{\left(\sqrt{2} - 1\right)2^{\frac{n-1}{2}}}\right)$ (D) $S_n = \frac{n}{2}\left(3 + (n-1)\sqrt{2} - n\right) + \left(\frac{1}{2}\right)^{\frac{n-1}{2}}$

16. The roots of the equation $x^4 - 8x^3 + ax^2 - bx + 16 = 0$, are positive, if

(A) a = 24 (B) a = 12 (C) b = 8 (D) b = 32

17. If $\sum_{r=1}^{n} r(r+1)(2r + 3) = an^4 + bn^3 + cn^2 + dn + e$, then

(A) a + c = b + d (B) e = 0

(C) a, b – 2/3, c – 1 are in A.P. (D) c/a is an integer

18. The value of $\sum_{r=1}^{n} \frac{1}{\sqrt{a + rx} + \sqrt{a + (r-1)x}}$ is

(A) $\frac{n}{\sqrt{a} + \sqrt{a + nx}}$ (B) $\frac{n}{\sqrt{a} - \sqrt{a + nx}}$

(C) $\frac{\sqrt{a + nx} - \sqrt{a}}{x}$ (D) $\frac{\sqrt{a} + \sqrt{a + nx}}{x}$

PART - IV : COMPREHENSION

Comprehension - 1

Let V_r denotes the sum of the first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is (2r – 1). Let

$$T_r = V_{r+1} - V_r - 2 \text{ and } Q_r = T_{r+1} - T_r \text{ for } r = 1, 2, ...$$

1. The sum $V_1 + V_2 + ... + V_n$ is

(A) $\frac{1}{12}$ n(n + 1) (3n² – n + 1)   (B) $\frac{1}{12}$ n(n + 1) (3n² + n + 2)

(C) $\frac{1}{2}$ n(2n² – n + 1)   (D) $\frac{1}{3}$ (2n³ – 2n + 3)

2. T_r is always

(A) an odd number   (B) an even number

(C) a prime number   (D) a composite number

3. Which one of the following is a correct statement ?

(A) Q_1, Q_2, Q_3,...... are in A.P. with common difference 5

(B) Q_1, Q_2, Q_3,...... are in A.P. with common difference 6

(C) Q_1, Q_2, Q_3,..... are in A.P. with common difference 11

(D) $Q_1 = Q_2 = Q_3 = ...$

Comprehension - 2

In a sequence of (4n + 1) terms the first (2n + 1) terms are in AP whose common difference is 2, and the last (2n + 1) terms are in GP whose common ratio 0.5. If the middle terms of the AP and GP are equal, then

4. Middle term of the sequence is

(A) $\frac{n \,.\, 2^{n+1}}{2^n - 1}$ (B) $\frac{n \,.\, 2^{n+1}}{2^{2n} - 1}$ (C) $n \,.\, 2^n$ (D) None of these

5. First term of the sequence is

(A) $\frac{4n + 2n \,.\, 2^n}{2^n - 1}$ (B) $\frac{4n - 2n \,.\, 2^n}{2^n - 1}$ (C) $\frac{2n - n \,.\, 2^n}{2^n - 1}$ (D) $\frac{2n + n \,.\, 2^n}{2^n - 1}$

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. Let S_k, k = 1, 2, ..., 100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!}$ + $\sum_{k=1}^{100}\left|(k^2-3k+1)S_k\right|$ is **[IIT-JEE -2010, Paper-1, (3, 0), 84]**

2. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for k = 3, 4, ..., 11.
If $\frac{a_1^2+a_2^2+\ldots+a_{11}^2}{11} = 90$, then the value of $\frac{a_1+a_2+\ldots+a_{11}}{11}$ is equal to

[IIT-JEE - 2010, Paper-2, (3, 0), 79]

3. Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i$, $1 \le p \le 100$.

For any integer n with $1 \le n \le 20$, let m = 5n. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is

[IIT-JEE 2011, Paper-1, (4, 0), 80]

4. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} where a > 0 is

[IIT-JEE 2011, Paper-1, (4, 0), 80]

5. Let $a_1, a_2, a_3, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) 22 (B) 23

(C) 24 (D) 25

6. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s) **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

(A) 1056 (B) 1088

(C) 1120 (D) 1332

7. A pack contains n card numbered from 1 to n. Two consecutive numbered card are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k – 20 = **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

8. Let a,b,c be positive integers such that $\frac{b}{a}$ is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is b + 2, then the value of $\frac{a^2+a-14}{a+1}$ is **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

9. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is **[JEE (Advanced) 2015, P-2 (4, 0) / 80]**

10. The least value of $\alpha \in R$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all x > 0, is

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$

(C) $\frac{1}{27}$ (D) $\frac{1}{25}$

11. Let $b_i > 1$ for i = 1, 2, ..., 101. Suppose $\log_e b_1, \log_e b_2, ...,\log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$ then **[JEE(Advanced)-2016, 3(–1)]**

(A) s > t and $a_{101} > b_{101}$

(B) s > t and $a_{101} < b_{101}$

(C) s < t and $a_{101} > b_{101}$

(D) s < t and $a_{101} < b_{101}$

12. The sides of the right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ? **[JEE(Advanced)-2017, 3]**

13. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ... , and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is ______ **[JEE(Advanced)-2018, 3]**

14. Let AP(a, d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference d > 0. If AP (1, 3) $\cap$ AP(2, 5) $\cap$ AP (3,7) = AP (a, d) then a + d equals ......

[JEE(Advanced)-2019, Paper-1, (3, 0), 62]

PART - II : JEE(MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. A man saves Rs. 200 in each of the first three months of his service. In each of the subsequent months his saving increases by Rs. 40 more than the saving of immediately previous month. His total saving from the start of service will be Rs. 11040 after : **[AIEEE 2011, I, (4, –1), 120]**

(1) 18 months (2) 19 months (3) 20 months (4) 21 months

2. Let a_n be the n^{th} term of an A.P. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the A.P. is :

[AIEEE 2011, II, (4, –1), 120]

(1) $\alpha - \beta$ (2) $\frac{\alpha - \beta}{100}$ (3) $\beta - \alpha$ (4) $\frac{\alpha - \beta}{200}$

3. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777,....., is **[AIEEE - 2013, (4, –1),360]**

(1) $\frac{7}{81}(179 - 10^{-20})$ (2) $\frac{7}{9}(99 - 10^{-20})$

(3) $\frac{7}{81}(179 + 10^{-20})$ (4) $\frac{7}{9}(99 + 10^{-20})$

4. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots\ldots\ldots + 10(11)^9 = k(10)^9$, then k is equal to

[JEE(Main) 2014, (4, – 1), 120]

(1) 100 (2) 110 (3) $\frac{121}{10}$ (4) $\frac{441}{100}$

5. Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Then the common ratio of the G.P. is **[JEE(Main) 2014, (4, – 1), 120]**

(1) $2 - \sqrt{3}$ (2) $2 + \sqrt{3}$ (3) $\sqrt{2} + \sqrt{3}$ (4) $3 + \sqrt{2}$

6. If m is the A. M. of two distinct real numbers l and n(l, n > 1) and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals : **[JEE(Main) 2015, (4, – 1), 120]**

(1) $4l^2mn$ (2) $4lm^2n$ (3) $4lmn^2$ (4) $4l^2m^2n^2$

7. The sum of first 9 terms of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots\ldots\ldots$ is :

[JEE(Main) 2015, (4, – 1), 120]

(1) 71 (2) 96 (3) 142 (4) 192

8. If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is:-

[JEE(Main) 2016, (4, – 1), 120]

(1) $\frac{7}{4}$ (2) $\frac{8}{5}$ (3) $\frac{4}{3}$ (4) 1

9. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2+\left(2\frac{2}{5}\right)^2+\left(3\frac{1}{5}\right)^2+4^2+\left(4\frac{4}{5}\right)^2+...,$ is $\frac{16}{5}m,$ then m is equal to :-

[JEE(Main) 2016, (4, – 1), 120]

(1) 99 (2) 102 (3) 101 (4) 100

10. If, for a positive integer n, the quadratic equation,

x(x + 1) + (x + 1) (x + 2) + ... + (x + n – 1) (x + n) = 10n

has two consecutive integral solutions, then n is equal to : **[JEE(Main) 2017, (4, – 1), 120]**

(1) 11 (2) 12 (3) 9 (4) 10

11. For any three positive real numbers a, b and c, $9(25a^2 + b^2) + 25(c^2 – 3ac) = 15b(3a + c)$.

Then : **[JEE(Main) 2017, (4, – 1), 120]**

(1) a, b and c are in G.P.

(2) b, c and a are in G.P.

(3) b, c and a are in A.P.

(4) a, b and c are in A.P.

12. Let a, b, c $\in$ R. If $f(x) = ax^2 + bx + c$ is such that a + b + c = 3 and f(x + y) = f(x) + f(y) + xy, $\forall$ x, y $\in$ R, then $\sum_{n=1}^{10} f(n)$ is equal to : **[JEE(Main) 2017, (4, – 1), 120]**

(1) 255 (2) 330 (3) 165 (4) 190

13. Let $a_1, a_2, a_3, ..., a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + ... + a_{17}^2 = 140m$, then m is equal to- **[JEE(Main) 2018, (4, – 1), 120]**

(1) 68 (2) 34 (3) 33 (4) 66

14. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series

$1^2 + 2\cdot2^2 + 3^2 + 2\cdot4^2 + 5^2 + 2\cdot6^2 + ...$. If B – 2A = 100λ, then λ is equal to :

[JEE(Main) 2018, (4, – 1), 120]

(1) 248 (2) 464 (3) 496 (4) 232

15. If a, b and c be three distinct real numbers in G. P. and a + b + c = xb, then x cannot be :

[JEE(Main) 2019, 09-01-19, P-1, (4, – 1), 120]

(1) 4 (2) –3 (3) –2 (4) 2

16. Let $a_1, a_2, \ldots\ldots, a_{30}$ be an A. P., $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{(2i-1)}$. If $a_5 = -27$ and $S - 2T = 75$, then a_{10} is equal to : **[JEE(Main) 2019, 09-01-19, P-1, (4, – 1), 120]**

(1) 57 (2) 47 (3) 42 (4) 52

17. The sum of the follwing series

$$1+6+\frac{9(1^2+2^2+3^2)}{7}+\frac{12(1^2+2^2+3^2+4^2)}{9}+\frac{15(1^2+2^2+\ldots+5^2)}{11}+\ldots$$ up to 15 terms, is :

[JEE(Main) 2019, 09-01-19, P-2, (4, – 1), 120]

(1) 7820 (2) 7830 (3) 7520 (4) 7510

18. Let a, b and c be the 7^{th}, 11^{th} and 13^{th} terms respectively of a non-constant A.P. If these are also the three consecutive terms of a G.P., then $\frac{a}{c}$ is equal to: **[JEE(Main) 2019, 09-01-19, P-2, (4, – 1), 120]**

(1) $\frac{7}{13}$ (2) 2 (3) $\frac{1}{2}$ (4) 4

19. Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $-\frac{1}{2}$, then the greatest number amongst them is : **[JEE(Main) 2020, 07-01-20, P-1, (4, – 1), 100]**

(1) $\frac{21}{2}$ (2) 27 (3) 16 (4) 7

20. If the sum of the first 40 terms of the series, 3 + 4 + 8 + 9 + 13 + 14 + 18 +19 +… is (102)m, then m is equal to : **[JEE(Main) 2020, 07-01-20, P-2, (4, – 1), 100]**

(1) 20 (2) 5 (3) 10 (4) 25

21. Let $a_1, a_2, a_3, \ldots$ be a G.P. such that $a_1 < 0$, $a_1 + a_2 = 4$ and $a_3 + a_4 = 16$. If $\sum_{i=1}^{9} a_i = 4\lambda$, then λ is equal to :

[JEE(Main) 2020, 07-01-20, P-2, (4, – 1), 100]

(1) –171 (2) 171 (3) $\frac{511}{3}$ (4) –513

Answers

Exercise # 1

PART - I

SECTION-(A)

A-1. 1020 **A-2.** 19668 **A-3.** 830
A-4. $n^2 + 4n$ **A-5.** $n(a^2 + b^2) + nab(3 - n)$
A-6. 4
A-8. 4, 9, 14 **A-11.** 1
A-12. 5 **A-13.** 42

SECTION-(B)

B-2. 3, 6, 12, 24, ...
B-3. 3, 7, 11 or 12, 7, 2 **B-4.** $\frac{q-r}{p-q}$
B-5. 6, –3, 3/2, ... **B-6.** 128
B-7. 2, 6, 18 or 18, 6, 2 **B-9.** 8

SECTION-(C)

C-1. 1 **C-4.** 3
C-7. 12 **C-8.** 2, 8
C-9. 5 **C-10.** $4 - \frac{2+n}{2^{n-1}}$
C-11. $n.2^{n+2} - 2^{n+1} + 2$. **C-12.** $\frac{15}{16}$

SECTION-(D)

D-1. 27 **D-3** 6
D-4. $\frac{1}{201}$ **D-5.** 50
D-8 64

SECTION-(E)

E-1. (i) $\frac{1}{2}(3^{k+1} + 1) - 2^{k+1}$ (ii) $\frac{1}{6}k(k + 1)(2k + 7)$
E-3. 1
E-4. $\frac{n(2n^2+9n+13)}{24}$

SECTION-(F)

F-1. $\frac{n}{3}(4n^2 + 6n - 1)$ **F-2.** $2^{n+2} - 3n - 4$
F-3. $\frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$ **F-4.** $\frac{3}{4}$
F-5 $\frac{65}{1056}$ **F-6.** $\frac{(n+1)2^n - 1}{2^{n+1}(n+1)}$

PART - II

SECTION-(A)

A-1. (D) **A-2.** (C)
A-3. (C) **A-4.** (C)
A-5. (A) **A-6.** (D)
A-7. (A) **A-8.** (B)

SECTION-(B)

B-1. (B) **B-2.** (B)
B-3. (A) **B-4.** (B)
B-5 (B) **B-6** (B)
B-7 (A) **B-8** (C)
B-9 (A)

SECTION-(C)

C-1. (C) **C-2.** (D)
C-3. (A) **C-4** (A)
C-5 (A) **C-6.** (A)
C-7. (A) **C-8.** (B)

SECTION-(D)

D-1. (A) **D-2.** (D)
D-3. (A) **D-4.** (C)
D-5. (B) **D-6.** (A)

SECTION-(E)

E-1. (A) **E-2.** (A)
E-3 (A) **E-4.** (C)

SECTION-(F)

F-1.	(D)	**F-2**	(A)
F-3	(C)	**F-4**	(C)
F-5	(A)		

PART - III

1. (A) → (p), (B) → (s), (C) → (q), (D) → (q)

2. (A) → (s), (B) → (p), (C) → (s), (D) → (p)

Exercise # 2

PART - I

1.	(A)	**2.**	(C)
3.	(B)	**4.**	(C)
5.	(D)	**6.**	(C)
7.	(A)	**8.**	(A)
9.	(A)	**10.**	(A)
11.	(A)	**12.**	(D)
13.	(C)	**14.**	(B)
15.	(C)	**16.**	(D)
17.	(A)	**18.**	(A)
19.	(A)	**20.**	(C)
21.	(A)	**22.**	(B)

PART - II

1.	51	**2.**	33
3.	8	**4.**	1
5.	6	**6.**	2
7.	2	**8.**	0
9.	11	**10.**	40
11.	0	**12.**	64
13.	3	**14.**	1
15.	9	**16.**	54
17.	7	**18.**	2

PART - III

1.	(B)	**2.**	(A,B,C,D)
3.	(D)	**4.**	(A,B,C,D)
5.	(A, C)	**6.**	(B,C)
7.	(A,B,C,D)	**8.**	(C,D)
9.	(A,B,C,D)	**10.**	(A,B,C)
11.	(C,D)	**12.**	(A,B,C)
13.	(A,B)	**14.**	(B,D)
15.	(B,C)	**16.**	(A,D)
17.	(A,B,C,D)	**18.**	(A,C)

PART - IV

1.	(B)	**2.**	(D)
3.	(B)	**4.**	(A)
5.	(B)		

Exercise # 3

PART - I

1. 3 **2.** 0

3. 3 or 9, both 3 and 9 (The common difference of the arithmatic progression can be either 0 or 6, and accordingly the second term can be either 3, or 9 ; thus the answers 3, or 9, or both 3 and 9 are acceptable.)

4.	8	**5.**	(D)
6.	(A,D)	**7.**	5
8.	4	**9.**	9
10.	(C)	**11.**	(B)
12.	6	**13.**	3748
14.	157		

PART - II

1.	(4)	**2.**	(2)
3.	(3)	**4.**	(1)
5.	(2)	**6.**	(2)
7.	(2)	**8.**	(3)
9.	(3)	**10.**	(1)
11.	(3)	**12.**	(2)
13.	(2)	**14.**	(1)
15.	(4)	**16.**	(4)
17.	(1)	**18.**	(4)
19.	(3)	**20.**	(1)
21.	(1)		

Reliable Ranker Problems

1. If a, b are two distinct numbers such that a, $A_1, A_2, \ldots, A_n$, b are in A.P. and a, $H_1, H_2, \ldots, H_n$, b are in H.P. then prove that $A_r > H_r$ where r = 1, 2, ….,n.

2. If x, y, z are distinct positive real numbers satisfying x + y + z = 1, then prove that $\left(\frac{1}{x}+1\right)\left(\frac{1}{y}+1\right)\left(\frac{1}{z}+1\right)$ is always greater than $24\sqrt{3}$.

3. Three positive distinct numbers x, y, z are three terms of geometric progression in an order, and the numbers x + y, y + z, z + x are three terms of arithmetic progression in that order. Prove that $x^x y^y z^z = x^y y^z z^x$.

4. Find the value of n, such that the coefficient of x^{n-2} in the expression
$(x - 1)(x - 2)(x - 2^2) \ldots\ldots (x - 2^{n-1})$ for all $n \in I$, is 290.

5. If $a_1, a_2, a_3, \ldots\ldots, a_{2n}$ are 2n positive real numbers in G.P, prove that

$$\sqrt{a_1a_2}+\sqrt{a_3a_4}+\sqrt{a_5a_6}+\ldots+\sqrt{a_{2n-1}a_{2n}}=\sqrt{a_1+a_3+a_5+\ldots+a_{2n-1}}\sqrt{a_2+a_4+a_6+\ldots+a_{2n}}$$

6. Find the sum in the n^{th} group of sequence,

(i) (1), (2, 3); (4, 5, 6, 7); (8, 9,........, 15); ....

(ii) (1), (2, 3, 4), (5, 6, 7, 8, 9),........

7. If a, b, c are positive real numbers, then prove that

(i) $b^2c^2 + c^2a^2 + a^2b^2 \ge abc\,(a + b + c)$.

(ii) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$

(iii) $\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \ge \frac{9}{a+b+c}$

8. If the sum of the first m terms of an A.P. is equal to the sum of either the next n terms or the next p terms, then prove that $(m + n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m + p)\left(\frac{1}{m}-\frac{1}{n}\right)$.

9. a, b, c, d are four distinct real numbers and they are in A.P. If
$2(a - b) + x(b - c)^2 + (c - a)^3 = 2(a - d) + (b - d)^2 + (c - d)^3$ then find the permissible values of x.

10. Let a, b, c, d, e are positive numbers such that $abcd = e^4$. Show that $(1 + a)(1 + b)(1 + c)(1 + d) \ge (1 + e)^4$.

11. If positive and distinct numbers a, b, c are in H.P., prove that $\frac{a+b}{2a-b}+\frac{c+b}{2c-b}>4$.

12. Find sum of the series $\frac{n}{1.2.3}+\frac{n-1}{2.3.4}+\frac{n-2}{3.4.5}+\cdots$ up to n terms..

13. Let $a_1, a_2, \ldots\ldots$, be positive real numbers in geometric progression. For each n, let A_n, G_n, H_n be respectively the arithmetic mean, geometric mean & harmonic mean of $a_1, a_2, \ldots\ldots, a_n$. Find an expression for the geometric mean of $G_1, G_2, \ldots\ldots, G_n$ in terms of $A_1, A_2, \ldots\ldots, A_n, H_1, H_2, \ldots\ldots, H_n$.

14. Let $a_1, a_2, \ldots\ldots a_n$, be real numbers such that

$$\sqrt{a_1}+\sqrt{a_2-1}+\sqrt{a_3-2}+\ldots\ldots+\sqrt{a_n-(n-1)} = \frac{1}{2}(a_1 + a_2 + \ldots\ldots + a_n) - \frac{n(n-3)}{4}$$

then find the value of $\sum_{i=1}^{100} a_i$

15. If a, b, c are in A.P., p, q, r are in H.P. and ap, bq, cr are in G.P., then find $\frac{p}{r}+\frac{r}{p}$.

16. Let $\{a_n\}$ and $\{b_n\}$ are two sequences given by $a_n = (x)^{1/2^n} + (y)^{1/2^n}$ and $b_n = (x)^{1/2^n} - (y)^{1/2^n}$ for all $n \in N$. Then find $a_1a_2a_3\ldots\ldots a_n$.

17. In an A.P. of which 'a' is the Ist term, if the sum of the Ist 'p' terms is equal to zero, show that the sum of the next 'q' terms is $-\frac{a\,(p+q)\,q}{p-1}$.

18. If $a_i \in R$, i = 1, 2, 3, ...n and all a_i's are distinct such that $\left(\sum_{i=1}^{n-1} a_i^2\right)+6\left(\sum_{i=1}^{n-1} a_i\,a_{i+1}\right)+9\sum_{i=2}^{n} a_i^2 \le 0$

and $a_1 = 8$ then find the sum of first five terms.

19. If a, b, c are in H.P.; b, c, d are in G.P.; and c, d, e are in A.P. such that $(ka - b)^2 e = ab^2$ then value of k.

20. Prove that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ cannot be terms of a single A.P.

21. The value of x + y + z is 15 if a, x, y, z, b are in AP while the value of (1/x) + (1/y) + (1/z) is 5/3 if a, x, y, z, b are in HP. Find a and b.

22. Find the value of $S_n = \sum_{n=1}^{n} \frac{3^n \,.\, 5^n}{(5^n - 3^n)(5^{n+1} - 3^{n+1})}$ and hence S_∞.

23. If n is a root of the equation $x^2(1 - ac) - x(a^2 + c^2) - (1 + ac) = 0$ and if n HM's are inserted between a and c, show that the difference between the first and the last mean is equal to $ac(a - c)$.

24. Circles are inscribed in the acute angle α so that every neighbouring circles touch each other. If the radius of the first circle is R, then find the sum of the radii of the first n circles in terms of R and α.

25. Solve the equation $(2 + x_1 + x_2 + x_3 + x_4)^5 = 6250\, x_1 x_2 x_3 x_4$ .where $x_1, x_2, x_3, x_4 > 0$.

26. Let A, G, H be A.M., G.M. and H.M. of three positive real numbers a, b, c respectively such that $G^2 = AH$, then prove that a, b, c are terms of a GP.

27. If sum of first n terms of an A.P. (having positive terms) is given by $S_n = (1 + 2T_n)(1 - T_n)$ where T_n is the n^{th} term of series, then $T_2^2 = \frac{\sqrt{a}-\sqrt{b}}{4}$, ($a \in N$, $b \in N$), then find the value of (a + b)

28. Sum the series upto infinite terms : $\frac{3}{2^3}+\frac{4}{2^4\cdot 3}+\frac{5}{2^6\cdot 3}+\frac{6}{2^7\cdot 5}+......$

29. If 1.0! + 3.1! + 7.2! + 13.3! + 21.4! + …. upto (n+1) terms = 4000. (4000!), then find the value of n.

30. Show that $\left[(n+1)(2n+1)\right]^n > (n!)^2$

31. Find $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{a^i\cdot a^j}$, where $i \neq j$ and a > 1.

32. Find the sum of n terms of the series $\frac{\sin x}{\cos x+\cos 2x}+\frac{\sin 2x}{\cos x+\cos 4x}+\frac{\sin 3x}{\cos x+\cos 6x}+...........$

33. If a, b, c are three distinct positive real numbers in G.P., then prove that $c^2 + 2ab > 3ac$.

34. Evaluate the sum to n terms of the series $\frac{8}{5}+\frac{16}{65}+\frac{24}{325}$ + ……..

35. If the equation $x^2 - 3x - a_i = 0$ has integral roots for all $a_i \in N$ and $a_i \leq 300$, then find $\sum a_i$.

Answers

4.	No value	**6.**	(i) $2^{n-2}(2^n + 2^{n-1} - 1)$; (ii) $(n-1)^3 + n^3$				
9.	$x \in (-\infty, -8] \cup [16, \infty)$.	**12.**	$\frac{n(n+1)}{4(n+2)}$	**13.**	$G = \prod_{K=1}^{n}(A_k H_k)^{\frac{1}{2n}}$		
14.	5050	**15.**	$\frac{a}{c}+\frac{c}{a}$	**16.**	$\frac{x-y}{b_n}$	**18.**	$\frac{488}{81}$
19.	2	**21.**	a = 1, b = 9 OR b = 1, a = 9			**22.**	$\frac{3}{4}$
24.	$\frac{R\left(1-\sin\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}}\left[\left(\frac{1+\sin\frac{\alpha}{2}}{1-\sin\frac{\alpha}{2}}\right)^n-1\right]$	**27.**	6	**28.**	1/2	**29.**	3999.
32.	$\frac{1}{4}\text{cosec}\frac{x}{2}\left[\sec(2n+1)\frac{x}{2}-\sec\frac{x}{2}\right]$	**34.**	$2\left[1-\frac{1}{2n^2+2n+1}\right]$			**35.**	1600