14. Integration (Definite & Indefinite)

1. DEFINITION :

If f & g are functions of x such that $g'(x) = f(x)$ then the function g is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)\,dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x) + c\} = f(x)$, where c is called the **constant of integration**.

2. STANDARD RESULTS :

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad n \neq -1$ **(ii)** $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$ **(iv)** $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ell n\, a}\ (a > 0) + c$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$ **(vi)** $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ln\sec(ax+b) + c$ **(viii)** $\int \cot(ax+b)\,dx = \frac{1}{a}\ln\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$ **(x)** $\int \mathrm{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b)\,.\,\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \mathrm{cosec}(ax+b)\,.\,\cot(ax+b)\,dx = -\frac{1}{a}\mathrm{cosec}(ax+b) + c$

(xiii) $\int \sec x\,dx = \ln(\sec x + \tan x) + c$ **OR** $\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xiv) $\int \mathrm{cosec}\,x\,dx = \ln(\mathrm{cosec}\,x - \cot x) + c$ **OR** $\ln\tan\frac{x}{2} + c$ **OR** $-\ln(\mathrm{cosec}\,x + \cot x)$

(xv) $\int \sinh x\,dx = \cosh x + c$ **(xvi)** $\int \cosh x\,dx = \sinh x + c$ **(xvii)** $\int \mathrm{sech}^2 x\,dx = \tanh x + c$

(xviii) $\int \mathrm{cosech}^2 x\,dx = -\coth x + c$ **(xix)** $\int \mathrm{sech}\,x\,.\,\tanh x\,dx = -\mathrm{sech}\,x + c$

(xx) $\int \mathrm{cosech}\,x\,.\,\coth x\,dx = -\mathrm{cosech}\,x + c$ **(xxi)** $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(xxii) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ **(xxiii)** $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xxiv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x + \sqrt{x^2+a^2}\right]$ **OR** $\sinh^{-1}\frac{x}{a} + c$

(xxv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left[x + \sqrt{x^2-a^2}\right]$ **OR** $\cosh^{-1}\frac{x}{a} + c$

(xxvi) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x} + c$

(xxvii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + c$

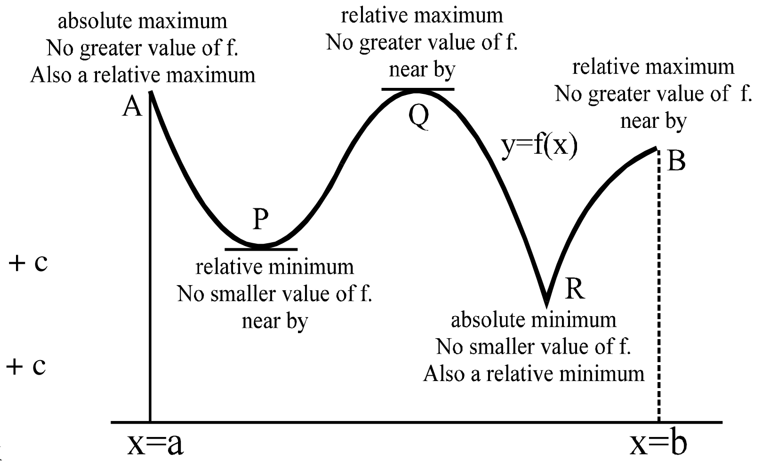
(xxviii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

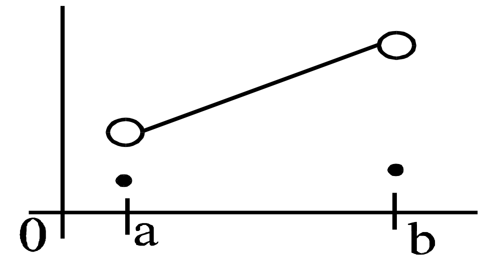
(xxix) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$

(xxx) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c$

(xxxi) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxxii) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$





3. TECHNIQUES OF INTEGRATION :

(i) Substitution or change of independent variable .

Integral $I = \int f(x)\,dx$ is changed to $\int f(\phi(t))\,f'(t)\,dt$, by a suitable substitution $x = \phi(t)$ provided the later integral is easier to integrate .

(ii) Integration by part : $\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$ where u & v are differentiable function . **Note :** While using integration by parts, choose u & v such that

(a) $\int v\,dx$ is simple & **(b)** $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ is simple to integrate.

This is generally obtained, by keeping the order of u & v as per the order of the letters in **ILATE**, where ; I – Inverse function, L – Logarithmic function , A – Algebraic function, T – Trigonometric function & E – Exponential function

(iii) Partial fraction , spiliting a bigger fraction into smaller fraction by known methods .

4. INTEGRALS OF THE TYPE :www.MathsBySuhag.com , www.TekoClasses.com

(i) $\int [f(x)]^n f'(x)\,dx$ **OR** $\int \frac{f'(x)}{[f(x)]^n}\,dx$ put $f(x) = t$ & proceed .

(ii) $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, $\int \sqrt{ax^2+bx+c}\,dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results .

(iii) $\int \frac{px+q}{ax^2+bx+c}\,dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$.

Express px + q = A (differential co-efficient of denominator) + B .

(iv) $\int e^x[f(x) + f'(x)]\,dx = e^x.f(x) + c$ **(v)** $\int [f(x) + xf'(x)]\,dx = x\,f(x) + c$

(vi) $\int \frac{dx}{x(x^n+1)}$ $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

(vii) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$ $n \in N$, take x^n common & put $1+x^{-n} = t^n$

(viii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$ take x^n common as x and put $1 + x^{-n} = t$.

(ix) $\int \frac{dx}{a+b\sin^2 x}$ **OR** $\int \frac{dx}{a+b\cos^2 x}$ **OR** $\int \frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$

Multiply $N^{r.}$ & $D^{r.}$ by $\sec^2 x$ & put $\tan x = t$.

(x) $\int \frac{dx}{a+b\sin x}$ **OR** $\int \frac{dx}{a+b\cos x}$ **OR** $\int \frac{dx}{a+b\sin x+c\cos x}$

Hint Convert sines & cosines into their respective tangents of half the angles , put $\tan\frac{x}{2} = t$

(xi) $\int \frac{a.\cos x+b.\sin x+c}{\ell.\cos x+m.\sin x+n}\,dx$. Express $Nr \equiv A(Dr) + B\frac{d}{dx}(Dr) + c$ & proceed .

(xii) $\int \frac{x^2+1}{x^4+Kx^2+1}\,dx$ **OR** $\int \frac{x^2-1}{x^4+Kx^2+1}\,dx$ where K is any constant .

Hint : Divide Nr & Dr by x^2 & proceed .

(xiii) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px + q = t^2$.

(xiv) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$,put $ax + b = \frac{1}{t}$; $\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$, put $x = \frac{1}{t}$

(xv) $\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

DEFINITE INTEGRAL

1. $\int_a^b f(x)\,dx = F(b) - F(a)$ where $\int f(x)\,dx = F(x) + c$

Very Important Note : If $\int_a^b f(x)\,dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a , b) provided f is a continuous function in (a , b) .

2. Properties Of Definite Integral :www.MathsBySuhag.com , www.TekoClasses.com

P–1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ provided f is same **P – 2** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P–3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a, b] . This property to be used when f is piecewise continuous in (a, b) .

P–4 $\int_{-a}^a f(x)\,dx = 0$ if f(x) is an odd function i.e. $f(x) = -f(-x)$.

$= 2\int_0^a f(x)\,dx$ if f(x) is an even function i.e. $f(x) = f(-x)$.

P–5 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, In particular $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

P–6 $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a - x) = f(x)$

$= 0$ if $f(2a - x) = - f(x)$

P–7 $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$; where 'a' is the period of the function i.e. $f(a + x) = f(x)$

P–8 $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$ where f(x) is periodic with period T & $n \in I$.

P–9 $\int_{ma}^{na} f(x)\,dx = (n-m)\int_0^a f(x)\,dx$ if f(x) is periodic with period 'a' .

P–10 If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

P–11 $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$.**P–12** If $f(x) \ge 0$ on the interval [a, b] , then $\int_a^b f(x)\,dx \ge 0$.

3. WALLI'S FORMULA :www.MathsBySuhag.com , www.TekoClasses.com

$$\int_0^{\pi/2} \sin^n x\,.\,\cos^m x\,dx = \frac{[(n-1)(n-3)(n-5)....1or2][(m-1)(m-3)....1or2]}{(m+n)(m+n-2)(m+n-4)....1or2}\,K$$

Where $K = \frac{\pi}{2}$ if both m and n are even $(m, n \in N)$; = 1 otherwise

4. DERIVATIVE OF ANTIDERIVATIVE FUNCTION :

If h(x) & g(x) are differentiable functions of x then ,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f[h(x)]\,.\,h'(x) - f[g(x)]\,.\,g'(x)$$

5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Limit}}\ h\left[f(a) + f(a+h) + f(a+2h) + ... + f\left(a + \overline{n-1}\,h\right)\right]$$

$$= \underset{h\to 0}{\text{Limit}}\ h\sum_{r=0}^{n-1} f(a+rh) \quad \text{where } b - a = nh$$

If a = 0 & b = 1 then , $\underset{n\to\infty}{\text{Limit}}\ h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)\,dx$; where nh = 1 **OR**

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx \quad .$$

6. ESTIMATION OF DEFINITE INTEGRAL :

(i) For a monotonic decreasing function in (a , b) ; $f(b).(b-a) < \int_a^b f(x)\,dx < f(a).(b-a)$

(ii) For a monotonic increasing function in (a , b) ; $f(a).(b-a) < \int_a^b f(x)\,dx < f(b).(b-a)$

7. SOME IMPORTANT EXPANSIONS :www.MathsBySuhag.com , www.TekoClasses.com

(i) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\ldots.\infty=\ln 2$ **(ii)** $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.\infty=\frac{\pi^2}{6}$

(iii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots.\infty=\frac{\pi^2}{12}$ **(iv)** $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots.\infty=\frac{\pi^2}{8}$

(v) $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots.\infty=\frac{\pi^2}{24}$