CIRCLE

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point (in the same given plane) remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle. The equation of the circle means the equation of its circumference.

1. EQUATION OF CIRCLE IN VARIOUS FORMS :

1.1 Central Form :

If (h, k) is the centre and r is the radius of the circle then its equation is

$\mathbf{(x-h)^2 + (y-k)^2 = r^2}$

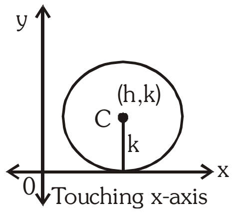
Special Cases :

(i) If centre is origin (0,0) and radius is 'r' then equation of circle is $\mathbf{x^2 + y^2 = r^2}$ and this is called the standard form.

(ii) If radius of circle is zero then equation of circle is $\mathbf{(x - h)^2 + (y - k)^2 = 0}$. Such circle is called zero circle or **point circle.**

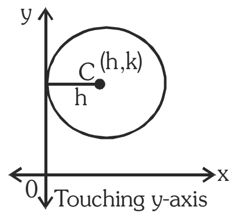
(iii) When circle touches x-axis then equation of the circle is

$\mathbf{(x-h)^2 + (y-k)^2 = k^2}$.

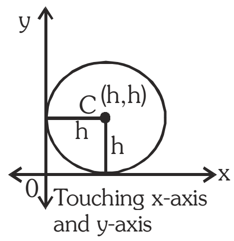


(iv) When circle touches y-axis then equation of circle is

$\mathbf{(x-h)^2 + (y-k)^2 = h^2}$

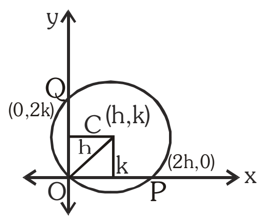


(v) When circle touches both the axes (x-axis and y-axis) then equation of circle $\mathbf{(x-h)^2 + (y-h)^2 = h^2}$.



(vi) When circle passes through the origin and centre of the circle is (h,k) then radius $\sqrt{h^2+k^2} = r$ and intercept cut on x-axis OP =2h, and intercept cut on y-axis is OQ = 2k and equation of circle is

$\mathbf{(x-h)^2 + (y-k)^2 = h^2 + k^2}$ or $\mathbf{x^2 + y^2 - 2hx - 2ky = 0}$



Note : Centre of the circle may exist in any quadrant hence for general cases use ± sign before h & k.

1.2 General form :

$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$. where g,f,c are constants and centre is (–g,–f)

i.e. $\left(-\dfrac{\text{coefficient of } x}{2}, -\dfrac{\text{coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2+f^2-c}$

Note :

(i) If $(g^2 + f^2 - c) > 0$, then circle is a real circle.

(ii) If $(g^2 + f^2 - c) = 0$, then circle is a point circle.

(iii) If $(g^2 + f^2 - c) < 0$, then circle is an imaginary circle with real centre.

(iv) $x^2 + y^2 + 2gx + 2fy + c = 0$, has three constants and to get the equation of the circle at least three conditions should be known $\Rightarrow$ A unique circle passes through three non collinear points.

(v) **The general second degree in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if :**

- coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
- coefficient of xy = 0 or $h = 0$
- $(g^2 + f^2 - c) \geq 0$ (for a real circle)

(vi) Intercepts cut by the circle on axes :

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on :

(a) x-axis = $2\sqrt{g^2 - c}$ (b) y-axis $= 2\sqrt{f^2 - c}$

SOLVED EXAMPLE

Example 1 : Find the centre & radius of the circle whose equation is $x^2 + y^2 - 4x + 6y + 12 = 0$

Solution : Comparing it with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$2g = -4 \Rightarrow g = -2$

$2f = 6 \Rightarrow f = 3$

& $c = 12$

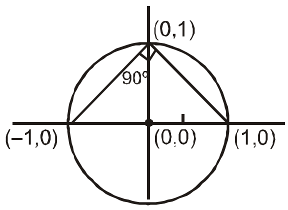
$\therefore$ centre is $(-g, -f)$ i.e. $(2, -3)$

and radius = $\sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (3)^2 - 12} = 1$

Example 2 : Find the equation of the circle that passes through the points (1, 0), (– 1, 0) and (0, 1).

Solution : Centre (0, 0), radius 1

$x^2 + y^2 = 1$



Example 3 : A circle has radius equal to 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).

Solution : Let the centre of the circle be (α, β). It lies on the line $y = x - 1$

$\Rightarrow \beta = \alpha - 1$. Hence the centre is $(\alpha, \alpha - 1)$.

$\Rightarrow$ The equation of the circle is $(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$

It passes through (7, 3) $\Rightarrow (7 - \alpha)^2 + (4 - \alpha)^2 = 9$

$\Rightarrow 2\alpha^2 - 22\alpha + 56 = 0 \Rightarrow \alpha^2 - 11\alpha + 28 = 0$

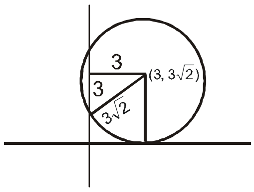
$\Rightarrow (\alpha - 4)(\alpha - 7) = 0 \Rightarrow \alpha = 4, 7$

Hence the required equations are

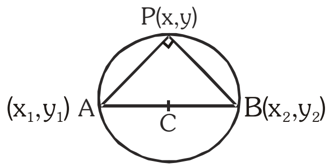
$x^2 + y^2 - 8x - 6y + 16 = 0$ and $x^2 + y^2 - 14x - 12y + 76 = 0$. **Ans.**

Example 4 : Find the equation to the circle which touches the axis of x at a distance 3 from the origin and intercepts a distance 6 on the axis of y.

Solution : $(x \pm 3)^2 + (y \pm 3\sqrt{2})^2 = (3\sqrt{2})^2$



1.3 Diametric form :



If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of the diameter of the circle, then its equation is

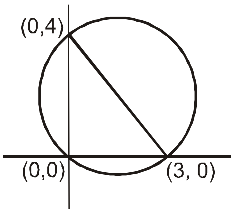
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$

Note : This will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2)

SOLVED EXAMPLE

Example 5 : Find the equation to the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the positive axes.

Solution : $(x - 3)(x - 0) + (y - 0)(y - 4) = 0$

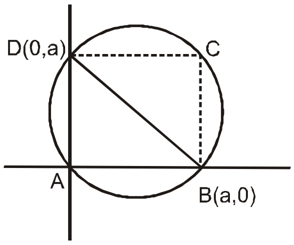


Example 6 : ABCD is a square in first quadrant whose side is a, taking AB and AD as axes, prove that the equation to the circle circumscribing the square is $x^2 + y^2 = a(x + y)$.

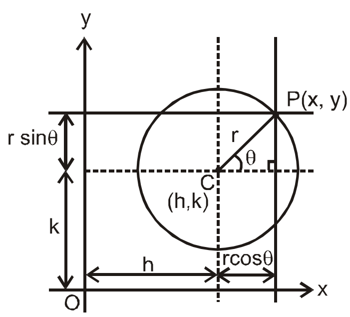
Solution : Since BD is diameter of circle

Hence $(x - a)(x - 0) + (y - 0)(y - a) = 0$

$\Rightarrow x^2 + y^2 = a(x + y)$



1.4 Parametric form :



The parametric equations of $(x - h)^2 + (y - k)^2 = r^2$ are:

$x = h + r\cos\theta$; $y = k + r\sin\theta$; $-\pi < \theta \le \pi$

where (h, k) is the centre, r is the radius & θ is a parameter.

Note : Equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

SOLVED EXAMPLE

Example 7 : Find equation of circle whose cartesian equation are $x = -3 + 2 \sin \theta$, $y = 4 + 2 \cos \theta$

Solution : $x = -3 + 2\sin \theta \Rightarrow x + 3 = 2 \sin \theta$

$y = 4 + 2\cos \theta \Rightarrow y - 4 = 2 \cos \theta$

Squarring and add $(x + 3)^2 + (y - 4)^2 = 4$

Problems for Self Practice-1:

(1) Find the centre and radius of the circle $2x^2 + 2y^2 = 3x - 5y + 7$

(2) Find the equation of the circle whose centre is (1, 2) and which passes through the point (4, 6)

(3) Find the equation of a circle which touches positive y-axis at a distance of 2 units from the origin and cuts an intercept of 3 units with the positive direction of x-axis.

(4) Find the equation of the circle the end points of whose diameter are the centres of the circles $x^2 + y^2 + 16x - 14y = 1$ & $x^2 + y^2 - 4x + 10y = 2$

(5) Find the parametric equations of the circle $x^2 + y^2 - 4x - 2y + 1 = 0$

Answers : (1) Centre $\left(\frac{3}{4}, -\frac{5}{4}\right)$, Radius $\frac{3\sqrt{10}}{4}$ (2) $x^2 + y^2 - 2x - 4y - 20 = 0$

(3) $x^2 + y^2 - 5x - 4y + 4 = 0$ (4) $x^2 + y^2 + 6x - 2y - 51 = 0$

(5) $x = 2 + 2 \cos \theta$, $y = 1 + 2 \sin \theta$.

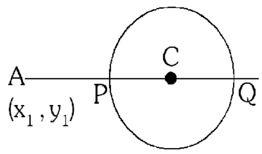
2. POSITION OF A POINT W.R.T CIRCLE :

Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1,y_1) then point (x_1,y_1) lies out side the circle or on the circle or inside the circle according as

$\Rightarrow$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$

Note :

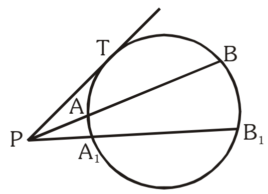
(i) The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & |AC – r| respectively.



(ii) **Power of a point w.r.t. circle :**

The power of point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_1, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$. If P outside, inside or on the circle then power of point is positive, negative or zero respectively.

If from a point $P(x_1, y_1)$, inside or outside the circle, a secant be drawn intersecting the circle in two points A & B, then PA . PB = constant. The product PA . PB is called power of point $P(x_1, y_1)$ w.r.t. the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$.



SOLVED EXAMPLE

Example 8 : Find the position of the points (1, 2) & (6, 0) w.r.t. the circle $x^2 + y^2 - 4x + 2y - 11 = 0$

Solution : Power of point (1, 2) and (6, 0) is negative & positive resepctively.

Hence (1, 2) lie inside the circle and the point (6, 0) lies outside the circle

Example 9 : Find the greatest and least distance of a point P(7, 3) from circle $x^2 + y^2 - 8x - 6y + 16 = 0$. Also find the power of point P w.r.t. circle.

Solution : Centre C $\equiv$ (4, 3) & radius r = $\sqrt{16+9-16}$ = 3

Greatest distance of point P = PC + r = 3 + 3 = 6

Least distance of point P = |PC – r| = 3 – 3 = 0

Power of point P = S_1 = 0

3. POSITION OF A LINE W.R.T CIRCLE :

Let L = 0 be a line & S = 0 be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then:

(i) $p > r \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.

(ii) $p = r \Leftrightarrow$ the line touches the circle. (It is tangent to the circle)

(iii) $p < r \Leftrightarrow$ the line is a secant of the circle.

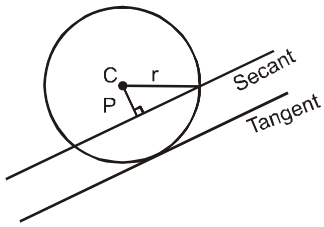
(iv) $p = 0 \Rightarrow$ the line is a diameter of the circle.

Also, if $y = mx + c$ is line and $x^2 + y^2 = a^2$ is circle then

(i) $c^2 < a^2(1 + m^2) \Leftrightarrow$ the line is a secant of the circle.

(ii) $c^2 = a^2(1 + m^2) \Leftrightarrow$ the line touches the circle. (It is tangent to the circle)

(iii) $c^2 > a^2(1 + m^2) \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.



These conditions can also be obtained by solving y = mx + c with $x^2 + y^2 = a^2$ and making the discriminant of the quadratic greater than zero for secant, equal to zero for tangent and less the zero for the last case.

Note : Length of chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $2\sqrt{a^2 - P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.

SOLVED EXAMPLE

Example 10 : For what value of c will the line y = 2x + c be a tangent to the circle $x^2 + y^2 = 5$?

Solution : We have : y = 2x + c or 2x – y + c = 0 ......(i) and $x^2 + y^2 = 5$ ....(ii)

If the line (i) touches the circle (ii), then

length of the $\perp$ from the centre (0, 0) = radius of circle (ii)

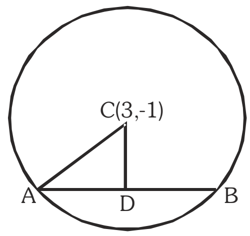
$$\Rightarrow \left|\frac{2\times 0 - 0 + c}{\sqrt{2^2+(-1)^2}}\right| = \sqrt{5} \quad \Rightarrow \quad \left|\frac{c}{\sqrt{5}}\right| = \sqrt{5}$$

$$\Rightarrow \frac{c}{\sqrt{5}} = \pm\sqrt{5} \quad \Rightarrow \quad c = \pm 5$$

Hence, the line (i) touches the circle (ii) for c = ± 5

Example 11 : Find the equation of a circle whose centre is (3, –1) and which cuts off a chord of length 6 on the line 2x– 5y+ 18 = 0.

Solution : Let AB(= 6) be the chord intercepted by the line 2x – 5y + 18 = 0 from the circle and let CD be the perpendicular drawn from centre (3, –1) to the chord AB.



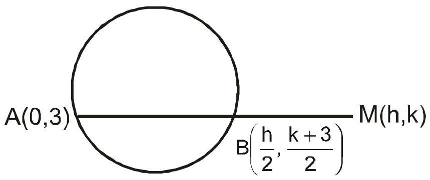
i.e., AD = 3, CD = $\dfrac{2.3-5(-1)+18}{\sqrt{2^2+5^2}} = \sqrt{29}$

Therefore, $CA^2 = 3^2 + (\sqrt{29})^2 = 38$

Hence required equation is $(x-3)^2 + (y+1)^2 = 38$

Example 12 : From the point A (0, 3) on the circle $x^2 + 4x + (y-3)^2 = 0$ a chord AB is drawn & extended to a point M such that AM = 2 AB. Find the equation of the locus of M.

Solution :



B lies on circle $\left(\dfrac{h}{2}\right)^2 + 4\left(\dfrac{h}{2}\right) + \left(\dfrac{k+3}{2}-3\right)^2 = 0 \Rightarrow \dfrac{h^2}{4} + 2h + \dfrac{(k-3)^2}{4} = 0$

Hence locus of (h, k) $\quad x^2 + 8x + (y-3)^2 = 0$.

<u>Problems for Self Practice</u>-2 :

(1) If P(2, 8) is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then find the values of p.

(2) Let x & y be the real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M & m respectively, then find the numerical value of (M + m).

(3) Find the value of λ so that the line $x + y + \lambda = 0$ is secant to the circle $x^2 + y^2 = 9$.

Answers : (1) $(-\infty, -4)$ (2) 10 (3) $\lambda \in \left(-3\sqrt{2}, 3\sqrt{2}\right)$

4. TANGENT OF CIRCLE :

4.1 Equation of the tangent in various forms :

4.1.1 Point form (T = 0) :

(i) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is, $x x_1 + y y_1 = a^2$.

(ii) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is : $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$ which is designated by T = 0.

Note :

In general the equation of tangent to any second degree curve at point (x_1, y_1) on it can be obtained by replacing x^2 by $x x_1$, y^2 by yy_1, x by $\dfrac{x+x_1}{2}$, y by $\dfrac{y+y_1}{2}$, xy by $\dfrac{x_1y+xy_1}{2}$ and c remains as c.

4.1.2 Slope form :

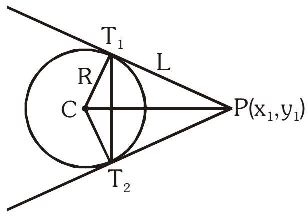
If line y = mx + c is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact **points are** $\left(\mp\frac{am}{\sqrt{1+m^2}}, \pm\frac{a}{\sqrt{1+m^2}}\right)$ **or** $\left(\mp\frac{a^2m}{c}, \pm\frac{a^2}{c}\right)$ and equation of tangent is $\mathbf{y = mx \pm a\sqrt{1+m^2}}$ **.**

4.1.3 Parametric form :

(i) The tangent at the point (acos t, asin t) on the circle $x^2 + y^2 = a^2$ is **xcos t + ysin t = a**

(ii) The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is $\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

4.2 Tangents from an external point :



Let PT_1 & PT_2 are two tangents drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the line joining the points of contact of two tangents, is called chord of contact of the point P.

Note :

(i) Length of tangent $L = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Note : When we use this formula the coefficient of x^2 and y^2 must be unity.

(ii) Length of chord of contact $T_1T_2 = \frac{2LR}{\sqrt{R^2 + L^2}}$, where R is radius of the circle.

(iii) Area of the triangle formed by the pair of the tangents & its chord of contact $= \frac{RL^3}{R^2 + L^2}$.

(vi) Angle between the pair of tangents from $P(x_1, y_1) = \tan^{-1}\left(\frac{2RL}{L^2 - R^2}\right)$

(v) Equation of the circle circumscribing the triangle PT_1T_2 or quadrilateral CT_1PT_2 is :
$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

(vi) The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $T^2 = SS_1$.

Where $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

$S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

SOLVED EXAMPLE

Example 13 : Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0$ at (4, –1).

Solution : Equation of tangent is

$$4x + (-y) - 30\left(\frac{x+4}{2}\right) + 6\left(\frac{y+(-1)}{2}\right) + 109 = 0$$

or $4x - y - 15x - 60 + 3y - 3 + 109 = 0$ or $-11x + 2y + 46 = 0$

or $11x - 2y - 46 = 0$

Hence, the required equation of the tangent is $11x - 2y - 46 = 0$

Example 14 : Show that the line $7y - x = 5$ touches the circle $x^2 + y^2 - 5x + 5y = 0$ and find the equation of the other parallel tangent.

Solution : Other tangent is $-x + 7y + \lambda = 0$ then $\left|\dfrac{-\frac{5}{2} - 7 \times \frac{5}{2} + \lambda}{\sqrt{50}}\right| = \dfrac{5}{\sqrt{2}} \Rightarrow \lambda = 45$ and -5

$\therefore$ other tangent is $x - 7y - 45 = 0$

Example 15 : Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1)

Solution : Given circle is $S = x^2 + y^2 - 2x + 4y = 0$ ....(i)

Let $P \equiv (0, 1)$

For point P, $S_1 = 0^2 + 1^2 - 2.0 + 4.1 = 5$

Clearly P lies outside the circle

and $T \equiv x\,.\,0 + y\,.\,1 - (x + 0) + 2\,(y + 1)$

i.e. $T \equiv -x + 3y + 2$.

Now equation of pair of tangents from P(0, 1) to circle (1) is $SS_1 = T^2$

or $5\,(x^2 + y^2 - 2x + 4y) = (-x + 3y + 2)^2$

or $5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$

or $4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$

or $2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$ ....(ii)

Separate equation of pair of tangents : From (ii), $2x^2 + 3(y - 1)\,x - 2(2y^2 - 4y + 2) = 0$

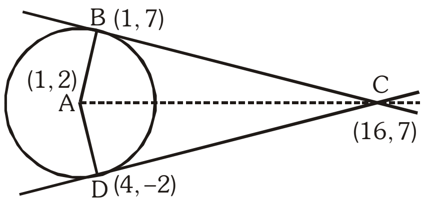
$\therefore$ $x = \dfrac{-3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$

or $4x + 3y - 3 = \pm\sqrt{25y^2 - 50y + 25} = \pm\, 5(y - 1)$

$\therefore$ Separate equations of tangents are $2x - y + 1 = 0$ and $x + 2y - 2 = 0$

Example 16 : Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and B(1, 7) and D(4, –2) are points on the circle then, if tangents be drawn at B and D, which meet at C, then find the area of quadrilateral ABCD.

Solution :



Here centre A(1, 2) and Tangent at (1, 7) is

$x.1 + y.7 - 1(x + 1) - 2(y + 7) - 20 = 0$ or $y = 7$ .... (i)

Tangent at D(4, –2) is $3x - 4y - 20 = 0$ .... (ii)

Solving (i) and (ii), C is (16, 7)

Area ABCD = AB × BC = 5 × 15 = 75 units.

Example 17 : If the length of the tangent from (f, g) to the circle $x^2 + y^2 = 6$ be twice the length of the tangent from (f, g) to the circle $x^2 + y^2 + 3x + 3y = 0$, then will $f^2 + g^2 + 4f + 4g + 2 = 0$?

Solution : given $\sqrt{f^2 + g^2 - 6} = 2\sqrt{f^2 + g^2 + 3g + 3f} \Rightarrow 3g^2 + 3f^2 + 12g + 12f + 6 = 0$

$\Rightarrow g^2 + f^2 + 4g + 4f + 2 = 0$

Problems for Self Practice - 3 :

(1) Find the equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $(a(1 + \cos\alpha), a\sin\alpha)$.

(2) Find the equation of the tangents to the circle $x^2 + y^2 = 4$ which are perpendicular to the line $12x - 5y + 9 = 0$. Also find the points of contact.

(3) Find the joint equation of the tangents through (7, 1) to the circle $x^2 + y^2 = 25$.

(4) If from any point P on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, tangents are drawn to the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$, then find the angle between the tangents.

Answers : (1) $x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$ (2) $5x + 12y = \pm 26$; $\left(\mp\frac{10}{13}, \mp\frac{24}{13}\right)$

(3) $12x^2 - 12y^2 + 7xy - 175x - 25y + 625 = 0$ (4) 2α

4. NORMAL OF CIRCLE :

Normal at a point is the straight line which is perpendicular to the tangent at the point of contact.

If a line is normal / orthogonal to a circle, then it must pass through the centre of the circle. Hence normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is; $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$.

SOLVED EXAMPLE

Example 18 : Find the equation of the normal to the circle $x^2 + y^2 = 2x$, which is parallel to the line $x + 2y = 3$.

Solution : Since normal to the circle passes through its centre, hence its equation is

$$y - 0 = \frac{-1}{2}(x - 1) \Rightarrow x + 2y = 1$$

Example 19 : If the straight line $ax + by = 2$; $a, b \neq 0$ touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then find the values of a and b.

Solution : Given $x^2 + y^2 - 2x = 3$

$\therefore$ centre is (1, 0) and radius is 2

Given $x^2 + y^2 - 4y = 6$

$\therefore$ centre is (0, 2) and radius is $\sqrt{10}$. Since line $ax + by = 2$ touches the first circle

$$\therefore \quad \frac{|a(1) + b(0) - 2|}{\sqrt{a^2 + b^2}} = 2 \quad \text{or} \quad |(a - 2)| = [2\sqrt{a^2 + b^2}] \qquad \text{......... (i)}$$

Also the given line is normal to the second circle. Hence it will pass through the centre of the second circle.

$\therefore$ $a(0) + b(2) = 2$ or $2b = 2$ or $b = 1$

Putting this value in equation (i) we get $|a - 2| = 2\sqrt{a^2 + 1^2}$ or $(a - 2)^2 = 4(a^2 + 1)$

or $a^2 + 4 - 4a = 4a^2 + 4$ or $3a^2 + 4a = 0$ or $a(3a + 4) = 0$ or $a = 0, -\frac{4}{3}$ $(a \neq 0)$

$\therefore$ values of a and b are $\left(-\frac{4}{3}, 1\right)$.

5. DIRECTOR CIRCLE :

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let P(h,k) is the point of intersection of two tangents drawn on the circle $x^2 + y^2 = a^2$, then locus of (h,k) is $x^2 + y^2 = 2a^2$ which is the equation of the director circle.

Note : Director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle. Therefore director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

SOLVED EXAMPLE

Example 20 : Find the equation of director circle of the circle $(x + 4)^2 + y^2 = 8$

Solution : given circle $(x + 4)^2 + y^2 = 8$ centre of director circle $\equiv (-4, 0)$ radius of director circle $\equiv 4$

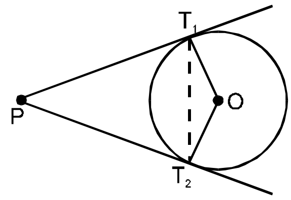
Hence equation of director circle $(x + 4)^2 + (y - 0)^2 = 4^2$

Example 21 : If two tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the circle $x^2 + y^2 = 25$, then find the angle between the tangents.

Solution. Since $x^2 + y^2 = 50$ is director circle of the circle $x^2 + y^2 = 25$,

hence angle between the tangents = 90°

6. EQUATION OF CHORD OF CONTACT (T = 0) :



If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ which is designated by T = 0.

SOLVED EXAMPLE

Example 22 : Find the equation of the chord of contact of the point (1, 2) with respect to the circle $x^2 + y^2 + 2x + 3y + 1 = 0$

Solution : Equation of chord of contact is T = 0 $\Rightarrow$ $4x + 7y + 10 = 0$

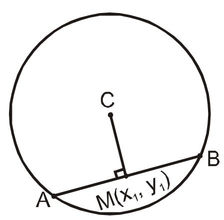
Example 23 : Find the distance between the chords of contact of tangents to the circle; $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin & the point (g, f).

Solution : Equation of chords of contact from (0, 0) & (g, f)

$$gx + fy + c = 0 \Rightarrow gx + fy + g(x + g) + f(y + f) + c = 0 \Rightarrow gx + fy + \frac{(g^2 + f^2 + c)}{2} = 0$$

Distance between these parallel lines = $\left|\dfrac{g^2 + f^2 - c}{2\sqrt{g^2 + f^2}}\right|$

7. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$) :



The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

Notes :

(i) The shortest chord of a circle passing through a point 'M' inside the circle is one chord whose middle point is M.

(ii) The chord passing through a point 'M' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.

SOLVED EXAMPLE

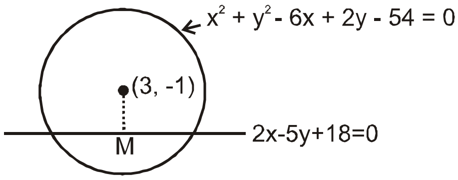
Example 24 : Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y + 9 = 0$ whose middle point is $(-2, -3)$.

Solution : $T = S_1 \Rightarrow -2x - 3y + 3(x-2) + 4(y-3) + 9 = 4 + 9 - 12 - 24 + 9 \Rightarrow x + y + 5 = 0$

Example 25 : Find the co-ordinates of the middle point of the chord cut off on $2x - 5y + 18 = 0$ by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$.

Solution : Required point is foot of $\perp$

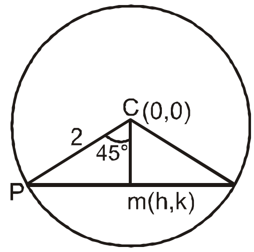
$$\frac{x-3}{2} = \frac{y+1}{-5} = -\left(\frac{6+5+18}{4+25}\right) = -1 \Rightarrow x = 1,\ y = 4$$



Example 26 : Find the locus of the mid point of a chord of the circle $x^2 + y^2 = 4$ which subtends a right angle at the origin.

Solution : $\cos 45° = \frac{cm}{cp} = \frac{\sqrt{h^2+k^2}}{2}$

Hence locus $x^2 + y^2 = 2$



Problems for Self Practice -4:

(1) Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

(2) Find the equation of the director circle of the circle $(x - h)^2 + (y - k)^2 = a^2$.

(3) Tangents are drawn from the point P(4, 6) to the circle $x^2 + y^2 = 25$. Find the area of the triangle formed by them and their chord of contact.

(4) Find the co-ordinates of the point of intersection of tangents at the points where the line $2x + y + 12 = 0$ meets the circle $x^2 + y^2 - 4x + 3y - 1 = 0$

(5) Find the equation of the chord of $x^2 + y^2 - 6x + 10 - a = 0$ which is bisected at $(-2, 4)$.

(6) Find the locus of mid point of chord of $x^2 + y^2 + 2gx + 2fy + c = 0$ that pass through the origin.

Answers :

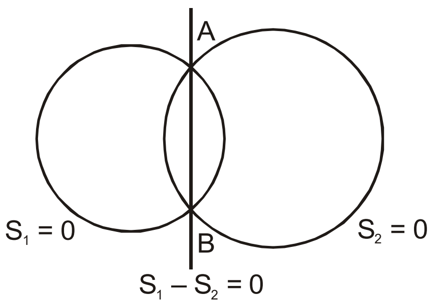
(1) $14x - 5y - 40 = 0$ (2) $(x - h)^2 + (y - k)^2 = 2a^2$

(3) $\frac{405\sqrt{3}}{52}$ sq. units (4) $(1, -2)$

(5) $5x - 4y + 26 = 0$ (6) $x^2 + y^2 + gx + fy = 0$

8. COMMON CHORD :

If the circle $S_1 = 0$ & $S_2 = 0$ intersect at points A and B, then equation of their common chord AB is $S_1 - S_2 = 0$ (provided coefficient of x^2 and y^2 are same in both equation of circle).



Note :

(i) If the circle $S_1 = 0$, bisects the circumference of the circle $S_2 = 0$, then their common chord will be the diameter of the circle $S_2 = 0$.

(ii) If the length of common chord is maximum, then it becomes diameter of the smaller circle.

SOLVED EXAMPLE

Example 27 : Find the equation of common chord of two circles $x^2 + y^2 - 2x - 3 = 0$ & $x^2 + y^2 + 4x + 6y + 4 = 0$

Solution : Equation of common chord is $S_1 - S_2 = 0 \Rightarrow 6x + 6y + 7 = 0$

Example 28 : The circumference of the circle $x^2 + y^2 - 2x + 8y - q = 0$ is bisected by the circle $x^2 + y^2 + 4x + 12y + p = 0$, then find the value of p + q

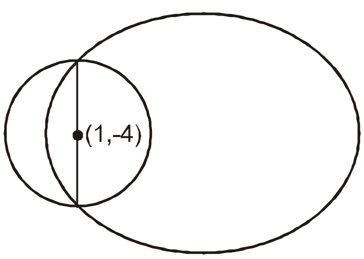
Solution : Common chord of given circle

$6x + 4y + (p + q) = 0$

This is diameter of $x^2 + y^2 - 2x + 8y - q = 0$

centre (1, –4)

$6 - 16 + (p + q) = 0 \Rightarrow p + q = 10$

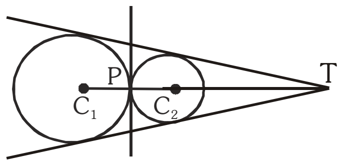


9. COMMON TANGENTS :

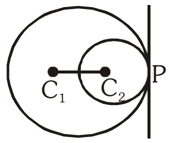
Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

(i) Both circles will touch :

(a) Externally if $C_1C_2 = r_1 + r_2$ i.e. the distance between their centres is equal to sum of their radii and point P & T divides C_1C_2 in the ratio $r_1 : r_2$ (internally & externally respectively). In this case there are **three common tangents.**



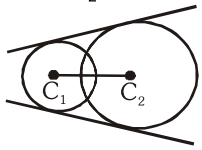
(b) Internally if $C_1C_2 = |r_1 - r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent.**



Note : Equation of common tangent at common point of contact is $\mathbf{S_1 - S_2 = 0}$ (provided coefficient of x^2 and y^2 are same in both equation of circle $S_1 = 0$ and $S_2 = 0$).

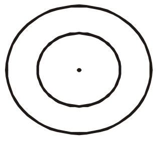
(ii) The circles will intersect :

when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**

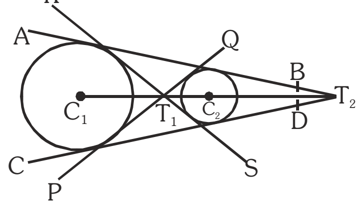


(iii) The circles will not intersect :

(a) One circle will lie inside the other circle if $C_1C_2 < | r_1 - r_2|$ In this case there will be no common tangent.



(b) When circle are apart from each other then $C_1C_2 > r_1 + r_2$ and in this case there will be **four common tangents.** Lines PQ and RS are called **transverse** or **indirect** or **internal** common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct** or **external** common tangents. These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.



Note : • Length of direct common tangent = $\sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

• Length of transverse common tangent = $\sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

SOLVED EXAMPLE

Example 29 : If the two circles, $x^2 + y^2 + 2g_1x + 2f_1y = 0$ & $x^2 + y^2 + 2g_2x + 2f_2y = 0$ touch each other then prove that $\frac{f_1}{g_1} = \frac{f_2}{g_2}$.

Solution : If two circles touch each other, then $C_1C_2 = r_1 + r_2$

$\sqrt{(-g_1+g_2)^2+(-f_1+f_2)^2} = \sqrt{g_1^2+f_1^2} + \sqrt{g_2^2+f_2^2}$ squaring both sides

$-2g_1g_2 - 2f_1f_2 = 2\sqrt{(g_1^2+f_1^2)(g_2^2+f_2^2)} \Rightarrow (g_1f_2)^2 + (g_2f_1)^2 - 2g_1g_2f_1f_2 = 0 \Rightarrow \frac{g_1}{g_2} = \frac{f_1}{f_2}$

Example 30 : Find the equation of common tangents of the circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$.

Solution : Given circles are $x^2 + y^2 - 2x - 4y = 0$ ....(i)

and $x^2 + y^2 - 8y - 4 = 0$ ....(ii)

Let A and B be the centres and r_1 and r_2 the radii of circles (i) and (ii) respectively, then

$A \equiv (1, 2)$, $B \equiv (0, 4)$, $r_1 = \sqrt{5}$, $r_2 = 2\sqrt{5}$

Now $AB = \sqrt{(1-0)^2+(2-4)^2} = \sqrt{5}$ and $r_1 + r_2 = 3\sqrt{5}$, $|r_1 - r_2| = \sqrt{5}$

Thus $AB = |r_1 - r_2| \Rightarrow$ two circles touch each other internally,

hence there exist only one common tangent whose equation is $S_1 - S_2 = 0$

$\Rightarrow \quad x - 2y - 2 = 0$

Example 31 : Find the number of common tangents to the circles $x^2 + y^2 = 1$ and $x^2 + y^2 - 2x - 6y + 6 = 0$.

Solution : Given circles are $x^2 + y^2 = 1$ ....(i)

and $x^2 + y^2 - 2x - 6y + 6 = 0$ ....(ii)

Let A and B be the centres and r_1 and r_2 the radii of circles (i) and (ii) respectively, then

$A \equiv (0, 0)$, $B \equiv (1, 3)$, $r_1 = 1$, $r_2 = 2$

Now $AB = \sqrt{(1-0)^2+(3-0)^2} = \sqrt{10}$ and $r_1 + r_2 = 3$

Thus $AB > r_1 + r_2$, hence there exist four common tangents.

Problems for Self Practice - 5 :

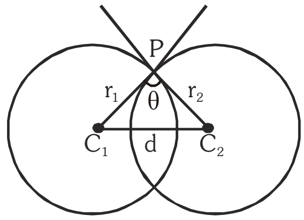
(1) Find the angle subtended by the common chord of the two circles $S_1 \equiv x^2 + y^2 - 4x - 1 = 0$ and $S_2 \equiv x^2 + y^2 - 7x + y = 0$ in the minor arc of $S_1 = 0$.

(2) Two circles with radius 5 touches at the point (1, 2). If the equation of common tangent is $4x + 3y = 10$ and one of the circle is $x^2 + y^2 + 6x + 2y - 15 = 0$. Find the equation of other circle.

(3) Find the length of transverse common tangent to the two circles $x^2 + y^2 + 2x + 6y + 1 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$.

Answers : (1) 135° (2) $(x - 5)^2 + (y - 5)^2 = 25$ (3) 9

10. THE ANGLE OF INTERSECTION OF TWO CIRCLES :



The angle between the tangents or normals of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If two circles are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the acute angle between them

then $\cos\theta = \left|\dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}\right|$ or $\cos\theta = \left|\left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)\right|$

Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.

If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and conditions for the circles to be orthogonal is $\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$

SOLVED EXAMPLE

Example 32 : Show that the circles $x^2 + y^2 - 2x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 6 = 0$ cut each other orthogonally.

Solution : $C_1 \equiv (1, 3)$ C_2 $(-3, -2) \Rightarrow r_1 = \sqrt{22}$ $\Rightarrow$ $r_2 = \sqrt{19} \Rightarrow (C_1C_2)^2 = r_1^2 + r_2^2$

Example 33 : Find the equation of the circle passing through the origin and cutting the circles $x^2 + y^2 - 4x + 6y + 10 = 0$ and $x^2 + y^2 + 12y + 6 = 0$ at right angles.

Solution : Equation of circle passing through origin is $x^2 + y^2 + 2gx + 2fy = 0$

This circle cuts the circle $x^2 + y^2 - 4x + 6y + 10 = 0$ orthogonally

$2g(-2) + 2f(3) = 0 + 10 \Rightarrow -2g + 3f - 5 = 0$ ...(1)

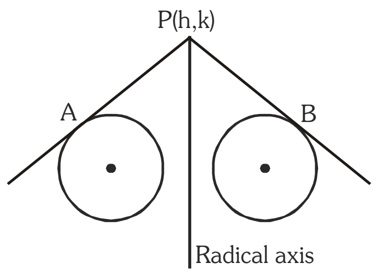
& $x^2 + y^2 + 12y + 6 = 0$ also

$2g(0) + 2f(6) = 6 + 0 \Rightarrow f = \dfrac{1}{2} \Rightarrow -2g + \dfrac{3}{2} - 5 = 0 \Rightarrow 2g = -\dfrac{7}{2} \Rightarrow g = -\dfrac{7}{4}$

Hence circle $x^2 + y^2 + 2\left(-\dfrac{7}{4}\right)x + 2\left(\dfrac{1}{2}\right)y = 0 \Rightarrow 2x^2 + 2y^2 - 7x + 2y = 0$

11. RADICAL AXIS AND RADICAL CENTRE :

11.1 Radical axis :



The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. If two circles are

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1\, y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

then equation of radical axis is $2x(g_1 - g_2) + 2y(f_1 - f_2)k + c_1 - c_2 = 0$ which is designated by $\mathbf{S_1 - S_2 = 0}$

Note :

(i) To get the equation of the radical axis first of all make the coefficient of x^2 and y^2 same in both equation of circle.

(ii) If two circles touch each other, then the radical axis is the common tangent of the two circles at the common point of contact.

(iii) When the two circles intersect on real points then common chord is the radical axis of the two circles.

(iv) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

(v) Radical axis (if exist) bisects common tangent to two circles.

(vi) If circles are concentric then the radical axis does not always exist but if circles are not concentric then radical axis always exists.

(vii) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

(viii) A system of circle, every pair of which have the same radical axis, is called a **coaxial system of circles.**

11.2 Radical centre :

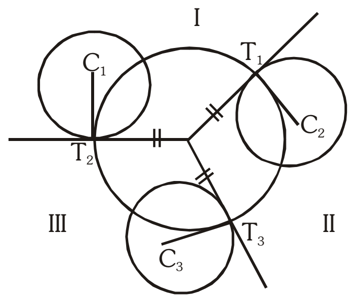
The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles.

Note :

(i) The length of tangents from radical centre to the three circles are equal.

(ii) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.

(iii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.



SOLVED EXAMPLE

Example 34 : Find the equation of the radical axis of the circle $x^2 + y^2 - 3x - 4y + 5 = 0$ and $3x^2 + 3y^2 - 7x - 8y + 11 = 0$

Solution : Equation of the radical axis is $S_1 - S_2 = 0$

$\Rightarrow \quad 3(x^2 + y^2 - 3x - 4y + 5) - (3x^2 + 3y^2 - 7x - 8y + 11) = 0$

$\Rightarrow \quad x + 2y = 2$

Example 35 : Given the three circles $x^2 + y^2 - 16x + 60 = 0$, $3x^2 + 3y^2 - 36x + 81 = 0$ and $x^2 + y^2 - 16x - 12y + 84 = 0$, find (1) the point from which the tangents to them are equal in length and (2) this length.

Solution : (1) $S_1 - S_2 = 0 \Rightarrow \quad 4x - 33 = 0$

$S_3 - S_1 = 0 \Rightarrow \quad -12y + 24 = 0 \qquad \Rightarrow \left(\frac{33}{4}, 2\right)$

(2) Length of tangent from $\left(\frac{33}{4}, 2\right)$ to any circle is $= \frac{1}{4}$.

<u>**Problems for Self Practice**</u> **-6 :**

(1) Find the angle of intersection of two circles

$S : x^2 + y^2 - 4x + 6y + 11 = 0$ & $S' : x^2 + y^2 - 2x + 8y + 13 = 0$

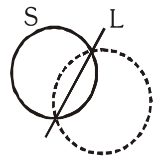
(2) When the circles $x^2 + y^2 + 4x + 6y + 3 = 0$ and $2(x^2 + y^2) + 6x + 4y + c = 0$ intersect orthogonally, then find the value of c is

(3) Find the radical centre of the following set of circles

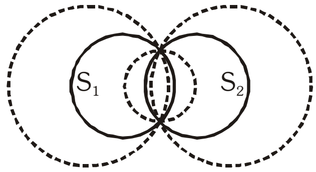
$x^2 + y^2 - 3x - 6y + 14 = 0$; $x^2 + y^2 - x - 4y + 8 = 0$; $x^2 + y^2 + 2x - 6y + 9 = 0$

Answers : (1) 135° (2) 18 (3) (1,2)

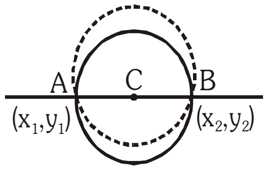
12. FAMILY OF CIRCLES :



(i) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + \lambda L = 0$.



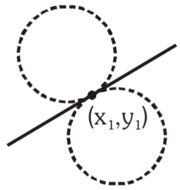
(ii) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + \lambda S_2 = 0$ where $\lambda \neq -1$, provided the co–efficient of x^2 & y^2 in S_1 & S_2 are same



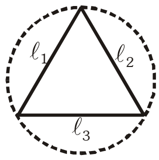
(iii) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

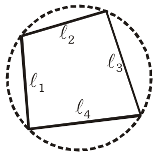
where λ is a parameter.



(iv) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + \lambda [y - y_1 - m(x - x_1)] = 0$, where λ is a parameter.



(v) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ & coefficient of x^2 = coefficient of y^2.



(vi) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

SOLVED EXAMPLE

Example 36 : Find the equation of the circle through the point (0, 1) and which passes through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$.

Solution : The equation of the circle through the intersection of the given circles is

$$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0 \qquad \text{.......... (i)}$$

circle passes through the point (0, 1)

$$\therefore \quad \lambda = \frac{-17}{10}$$

Hence equation of required circle is $x^2 + y^2 + 13x + 11y - 12 = 0$

Example 37 : Find the equation of a circle which passes through the point (1 , –2) and (4 , –3) and whose centre lies on the line 3x + 4y = 7.

Solution : Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ..... (i)

Hence, substituting the points, (1, –2) and (4 , –3) in equation (i)

$5 + 2g - 4f + c = 0$ ..... (ii)

$25 + 8g - 6f + c = 0$ ..... (iii)

centre (–g , –f) lies on line 3x + 4y = 7

Hence $-3g - 4f = 7$

solving for g, f,c, we get

Here $g = \frac{-47}{15}$, $f = \frac{9}{15}$, $c = \frac{55}{15}$

Hence the equation is $15(x^2 + y^2) - 94x + 18y + 55 = 0$

Example 38 : Find the equation of circle touching the line 2x + 3y + 1 = 0 at (1, – 1) and cutting orthogonally the circle having line segment joining (0, 3) and (– 2, – 1) as diameter.

Solution : The equation of circle having tangent 2x + 3y + 1 = 0 at (1, – 1)

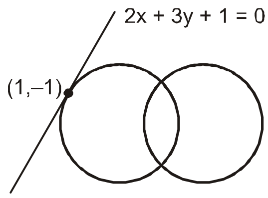
$\Rightarrow$ $(x - 1)^2 + (y + 1)^2 + \lambda(2x + 3y + 1) = 0$

$x^2 + y^2 + 2x(\lambda - 1) + y(3\lambda + 2) + (\lambda + 2) = 0$ ... (i)

equation of circle having end points of diameter (0, – 1) and (–2, 3) is

$x(x + 2) + (y + 1)(y - 3) = 0$ or $x^2 + y^2 + 2x - 2y - 3 = 0$ ... (ii)

since (i) & (ii) cut orthogonally



$\therefore$ $\frac{2(2\lambda - 2)}{2}.1 + \frac{2(3\lambda + 2)}{2}(-1) = \lambda + 2 - 3$

$\Rightarrow$ $2\lambda - 2 - 3\lambda - 2 = \lambda - 1$ $\Rightarrow 2\lambda = -3 \Rightarrow \lambda = -3/2$

$\therefore$ from equation (i), equation of required circle is

$2x^2 + 2y^2 - 10x - 5y + 1 = 0$

Problems for Self Practice -7:

(1) Find the equation of the circle passing through the points of intersection of the circle $x^2 + y^2 - 6x + 2y + 4 = 0$ & $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line y = x.

(2) Find the equation of the circle which passes through the point (0, 0) & which touches the circle $x^2 + y^2 = 25$ at the point (3, 4) on it.

Answers : (1) $x^2 + y^2 - \frac{10x}{7} - \frac{10y}{7} - \frac{12}{7} = 0$

(2) $x^2 + y^2 - 3x - 4y = 0$

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Equation of Circle in various forms

A-1. Find the centre and the radius of the circles

(i) $3x^2 + 3y^2 - 8x - 10y + 3 = 0$

(ii) $x^2 + y^2 + 2x \sin\theta + 2y \cos\theta - 8 = 0$

(iii) $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$, for some λ.

A-2. Find the equation of the circle whose centre is the point of intersection of the lines $2x - 3y + 4 = 0$ & $3x + 4y - 5 = 0$ and passes through the origin.

A-3. Find the equation to the circle which touches the axis of :

(i) x at a distance +3 from the origin and intercepts a distance 6 on the axis of y.

(ii) x, pass through the point (1, 1) and have line $x + y = 3$ as diameter.

A-4. Find equation of circle which touches x & y axis & perpendicular distance of centre of circle from $3x + 4y + 11 = 0$ is 5. Given that circle lies in Ist quadrant.

A-5. Find the equation to the circles which pass through the points :

(i) (0, 0), (a, 0) and (0, b) **(ii)** (1, 2), (3, –4) and (5, –6)

A-6. Find the equation to the circle which goes through the origin and cuts off intercepts equal to h and k from the positive parts of the axes.

A-7. Find the parametric form of the equation of the circle $x^2 + y^2 + px + py = 0$

Section (B) : Position of Point and Line w.r.t. Circle

B-1. If the points $(\lambda, -\lambda)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then find the range of λ.

B-2. (i) Find the shortest distance from the point M(–7, 2) to the circle $x^2 + y^2 - 10x - 14y - 151 = 0$.

(ii) Find the co-ordinate of the point on the circle $x^2 + y^2 - 12x - 4y + 30 = 0$, which is farthest from the origin.

B-3. For what value of λ, does the line $3x + 4y = \lambda$ touch the circle $x^2 + y^2 = 10x$.

B-4. Find the points of intersection of the line $x - y + 2 = 0$ and the circle $3x^2 + 3y^2 - 29x - 19y + 56 = 0$. Also determine the length of the chord intercepted.

B-5. Find the range of parameter 'a' for which the variable line $y = 2x + a$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.

Section (C) : Tangent, Length of tangent and Pair of tangents

C-1. Find the equation of the tangent to the circle

(i) $x^2 + y^2 - 6x + 4y = 12$, which are parallel to the straight line $4x + 3y + 5 = 0$.

(ii) $x^2 + y^2 - 22x - 4y + 25 = 0$, which are perpendicular to the straight line $5x + 12y + 9 = 0$

(iii) $x^2 + y^2 = 25$, which are inclined at 30° to the axis of x.

C-2 A line with gradient 2 is passing through the point P(1, 7) and touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point Q. If (a, b) are the coordinates of the point Q, then find the value of (7a + 7b + c).

C-3. Show that two tangents can be drawn from the point (9, 0) to the circle $x^2 + y^2 = 16$. Also find the equation of tangents and the angle between them.

C-4. Find the area of the quadrilateral formed by a pair of tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii.

C-5. If the length of the tangent from a point (f, g) to the circle $x^2 + y^2 = 4$ be four times the length of the tangent from it to the circle $x^2 + y^2 = 4x$, show that $15f^2 + 15g^2 - 64f + 4 = 0$.

Section (D) : Normal, Director Circle, Chord of contact, Chord with mid point

D-1. Find the equation of the normal to the circle

(i) $x^2 + y^2 = 5$ at the point (1, 2)

(ii) $x^2 + y^2 = 2x$, which is parallel to the line x + 2y = 3.

D-2. If the angle between the tangents drawn to $x^2 + y^2 + 4x + 8y + c = 0$ from (0, 0) is $\frac{\pi}{2}$, then find value of 'c'

D-3. The straight line x – 2y + 1 = 0 intersects the circle $x^2 + y^2 = 25$ in points T and T', find the co-ordinates of a point of intersection of tangents drawn at T and T' to the circle.

D-4. Tangents are drawn from the point (h, k) to the circle $x^2 + y^2 = a^2$; prove that the area of the triangle formed by them and chord of contact is $\dfrac{a(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$.

D-5. Find the co-ordinates of the middle point of the chord which the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ cuts off on the line y = x – 1.

Find also the equation of the locus of the middle point of all chords of the circle which are parallel to the line y = x – 1.

Section (E) : Common chord and Common tangents

E-1. Find the length of common chord of two circles S : $x^2 + y^2 - 4x + 6y + 11 = 0$ and S' : $x^2 + y^2 - 2x + 8y + 13 = 0$

E-2. Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$; find the point of intersection of these tangents.

E-3. A circle S = 0 is drawn with its centre at (–1, 1) so as to touch the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ externally. Find the intercept made by the circle S = 0 on the coordinate axes.

E-4. Find the equations to the common tangents of the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and

$x^2 + y^2 + 6x - 2y + 1 = 0$

E-5. Let C_1 and C_2 are circles defined by $x^2 + y^2 - 20x + 64 = 0$ and $x^2 + y^2 + 30x + 144 = 0$. Find the length of the shortest line segment PQ that is tangent to C_1 at P and to C_2 at Q.

Section (F) : Orthogonality, Radical axis and Radical centre

F-1. For what value of k the circles $x^2 + y^2 + 5x + 3y + 7 = 0$ and $x^2 + y^2 - 8x + 6y + k = 0$ cut orthogonally.

F-2. Find the equation to the circle which passes through the origin and has its centre on the line x + y + 4 = 0 and cuts the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

F-3. Find the equation to the circle orthogonal to the two circles $x^2 + y^2 - 4x - 6y + 11 = 0$; $x^2 + y^2 - 10x - 4y + 21 = 0$ and has 2x + 3y = 7 as diameter.

F-4. Find the radical centre of circles $x^2 + y^2 + 3x + 2y + 1 = 0$, $x^2 + y^2 - x + 6y + 5 = 0$ and $x^2 + y^2 + 5x - 8y + 15 = 0$. Also find the equation of the circle cutting them orthogonally.

Section (G) : Family of Circles

G-1. Find the equation of the circle through the points of intersection of circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ & cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

G-2. If $y = 2x$ is a chord of the circle $x^2 + y^2 - 10x = 0$, find the equation of a circle with this chord as diameter.

G-3. The line $2x - 3y + 1 = 0$ is tangent to a circle $S = 0$ at (1, 1). If the radius of the circle is $\sqrt{13}$. Find the equation of the circle S.

G-4. Consider a family of circles passing through two fixed points A(3, 7) & B(6, 5). Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

PART-II : OBJECTIVE QUESTIONS

Section (A) : Equation of Circle in various forms

A-1. Centres of the three circles $x^2 + y^2 - 4x - 6y - 14 = 0$, $x^2 + y^2 + 2x + 4y - 5 = 0$ and $x^2 + y^2 - 10x - 16y + 7 = 0$

(A) are the vertices of a right triangle

(B) the vertices of an isosceles triangle which is not regular

(C) vertices of a regular triangle

(D) are collinear

A-2. The equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$ is

(A) $x^2 + y^2 + 17x - 19y + 50 = 0$ (B) $x^2 + y^2 - 17x - 19y + 50 = 0$

(C) $x^2 + y^2 - 17x + 19y + 50 = 0$ (D) $x^2 + y^2 + 17x + 19y + 50 = 0$

A-3. The intercepts made by the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ on the x-axis and y-axis are respectively

(A) 9, 13 (B) 5, 13 (C) 9, 15 (D) none

A-4. Equation of line passing through mid point of intercepts made by circle $x^2 + y^2 - 4x - 6y = 0$ on co-ordinate axes is

(A) $3x + 2y - 12 = 0$ (B) $3x + y - 6 = 0$ (C) $3x + 4y - 12 = 0$ (D) $3x + 2y - 6 = 0$

A-5. Two thin rods AB & CD of lengths 2a & 2b move along OX & OY respectively, when 'O' is the origin. The equation of the locus of the centre of the circle passing through the extremities of the two rods is:

(A) $x^2 + y^2 = a^2 + b^2$ (B) $x^2 - y^2 = a^2 - b^2$ (C) $x^2 + y^2 = a^2 - b^2$ (D) $x^2 - y^2 = a^2 + b^2$

A-6. The circle described on the line joining the points (0, 1), (a, b) as diameter cuts the x–axis in points whose abscissa are roots of the equation:

(A) $x^2 + ax + b = 0$ (B) $x^2 - ax + b = 0$ (C) $x^2 + ax - b = 0$ (D) $x^2 - ax - b = 0$

A-7. Let A and B be two fixed points then the locus of a point C which moves so that $(\tan\angle BAC)(\tan \angle ABC)=1$, $0 < \angle BAC < \frac{\pi}{2}$, $0 < \angle ABC < \frac{\pi}{2}$ is

(A) Circle (B) pair of straight line (C) A point (D) Straight line

A-8. Let x and y be the real numbers satisfying $x^2 - 4x + y^2 + 3 = 0$, then maximum value of $x + y$ is

(A) $2-\sqrt{2}$ (B) $2+\sqrt{2}$ (C) $2\sqrt{2}$ (D) None of these

Section (B) : Position of Point and Line w.r.t. Circle

B-1. Consider the points P(2, 1); Q(0, 0); P(4, –3) and the circle S : $x^2 + y^2 - 5x + 2y - 5 = 0$

(A) exactly one point lies outside S (B) exactly two points lie outside S

(C) all the three points lie outside S (D) none of the point lies outside S

B-2. Find the co-ordinates of a point P on line $x + y = -13$, nearest to the circle $x^2 + y^2 + 4x + 6y - 5 = 0$

(A) (– 6, – 7) (B) (– 15, 2) (C) (– 5, – 6) (D) (– 7, – 6)

B-3. The condition so that the line $(x + g)\cos\theta + (y + f)\sin\theta = k$ is a tangent to $x^2 + y^2 + 2gx + 2fy + c = 0$ is

(A) $g^2 + f^2 = c + k^2$ (B) $g^2 + f^2 = c^2 + k$ (C) $g^2 + f^2 = c^2 + k^2$ (D) $g^2 + f^2 = c + k$

B-4. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre (2, 1), then the radius of the circle is

(A) 3 (B) 2 (C) 3/2 (D) 1

B-5. Two lines through (2, 3) from which the circle $x^2 + y^2 = 25$ intercepts chords of length 8 units have equations

(A) $2x + 3y = 13$, $x + 5y = 17$ (B) $y = 3$, $12x + 5y = 39$

(C) $x = 2$, $9x - 11y = 51$ (D) $y = 0$, $12x + 5y = 39$

B-6. Chord AB of the circle $x^2 + y^2 = 100$ passes through the point (7, 1) and subtends an angle of 60° at the circumference of the circle. If m_1 and m_2 are the slopes of two such chords then the value of m_1m_2, is

(A) –1 (B) 1 (C) 7/12 (D) –3

Section (C) : Tangent, Length of tangent and Pair of tangents

C-1. The tangent lines to the circle $x^2 + y^2 - 6x + 4y = 12$ which are parallel to the line $4x + 3y + 5 = 0$ are given by:

(A) $4x + 3y - 7 = 0$, $4x + 3y + 15 = 0$ (B) $4x + 3y - 31 = 0$, $4x + 3y + 19 = 0$

(C) $4x + 3y - 17 = 0$, $4x + 3y + 13 = 0$ (D) $4x + 3y - 31 = 0$, $4x + 3y - 19 = 0$

C-2. The tangent to the circle $x^2 + y^2 = 5$ at the point (1, –2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ at

(A) (–2, 1) (B) (–3, 0) (C) (–1, –1) (D) (3, –1)

C-3. Tangents PA and PB are drawn to the circle $x^2 + y^2 = 4$, then the locus of the point P if the triangle PAB is equilateral, is equal to-

(A) $x^2 + y^2 = 16$ (B) $x^2 + y^2 = 8$ (C) $x^2 + y^2 = 64$ (D) $x^2 + y^2 = 32$

C-4. Combined equation to the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$ is

(A) $3(x^2 + y^2) = (x + 2y)^2$ (B) $2(x^2 + y^2) = (3x + y)^2$ (C) $9(x^2 + y^2) = (2x + 3y)^2$ (D) $x^2 + y^2 = (2x + 3y)^2$

C-5. A line segment through a point P cuts a given circle in 2 points A & B, such that PA = 16 & PB = 9, find the length of tangent from points to the circle

(A) 7 (B) 25 (C) 12 (D) 8

C-6. The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + p = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + q = 0$ is:

(A) $\sqrt{q-p}$ (B) $\sqrt{p-q}$ (C) $\sqrt{q+p}$ (D) $\sqrt{2q + p}$

C-7. A point A(2, 1) is outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ & AP, AQ are tangents to the circle. The equation of the circle circumscribing the triangle APQ is :

(A) $(x + g)(x - 2) + (y + f)(y - 1) = 0$
(B) $(x + g)(x - 2) - (y + f)(y - 1) = 0$
(C) $(x - g)(x + 2) + (y - f)(y + 1) = 0$
(D) $(x - g)(x - 2) + (y - f)(y - 1) = 0$

C-8. The locus of the point of intersection of the tangents to the circle $x^2 + y^2 = a^2$ at points whose parametric angles differ by $\frac{\pi}{3}$ is

(A) $x^2 + y^2 = \frac{4a^2}{3}$
(B) $x^2 + y^2 = \frac{2a^2}{3}$
(C) $x^2 + y^2 = \frac{a^2}{3}$
(D) $x^2 + y^2 = \frac{a^2}{9}$

Section (D) : Normal, Director Circle, Chord of contact, Chord with mid point

D-1. The normal at the point (3, 4) on a circle cuts the circle at the point (–1, –2). Then the equation of the circle is

(A) $x^2 + y^2 + 2x - 2y - 13 = 0$
(B) $x^2 + y^2 - 2x - 2y - 11 = 0$
(C) $x^2 + y^2 - 2x + 2y + 12 = 0$
(D) $x^2 + y^2 - 2x - 2y + 14 = 0$

D-2. The angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ equals

(A) $\frac{\pi}{4}$
(B) $\frac{\pi}{3}$
(C) $\frac{\pi}{2}$
(D) $\frac{\pi}{6}$

D-3. The length of chord of contact of point (4, 4) with respect to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ is

(A) $2\sqrt{3}$
(B) $3\sqrt{2}$
(C) $2\sqrt{6}$
(D) $6\sqrt{2}$

D-4. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through the point

(A) (1, 2)
(B) $\left(\frac{1}{2}, \frac{1}{4}\right)$
(C) (2, 4)
(D) (4, 4)

D-5. The locus of the centers of the circles such that the point (2, 3) is the mid point of the chord $5x + 2y = 16$ is:

(A) $2x - 5y + 11 = 0$
(B) $2x + 5y - 11 = 0$
(C) $2x + 5y + 11 = 0$
(D) $2x - 5y - 11 = 0$

D-6. Find the locus of the mid point of the chord of a circle $x^2 + y^2 = 4$ such that the segment intercepted by the chord on the curve $x^2 - 2x - 2y = 0$ subtends a right angle at the origin.

(A) $x^2 + y^2 - 2x - 2y = 0$
(B) $x^2 + y^2 + 2x - 2y = 0$
(C) $x^2 + y^2 + 2x + 2y = 0$
(D) $x^2 + y^2 - 2x + 2y = 0$

D-7. From (3, 4) chords are drawn to the circle $x^2 + y^2 - 4x = 0$. The locus of the mid points of the chords is :

(A) $x^2 + y^2 - 5x - 4y + 6 = 0$
(B) $x^2 + y^2 + 5x - 4y + 6 = 0$
(C) $x^2 + y^2 - 5x + 4y + 6 = 0$
(D) $x^2 + y^2 - 5x - 4y - 6 = 0$

Section (E) : Common chord and Common tangents

E-1. If the circle $x^2 + y^2 + 6x - 2y + k = 0$ bisects the circumference of the circle $x^2 + y^2 + 2x - 6y - 15 = 0$, then k =

(A) 21
(B) –21
(C) 23
(D) –23

E-2. Number of common tangents of the circles $(x + 2)^2 + (y - 2)^2 = 49$ and $(x - 2)^2 + (y + 1)^2 = 4$ is:

(A) 0
(B) 1
(C) 2
(D) 3

E-3. The equation of the common tangent to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ at their point of contact is

(A) $12x + 5y + 19 = 0$ (B) $5x + 12y + 19 = 0$

(C) $5x - 12y + 19 = 0$ (D) $12x - 5y + 19 = 0$

E-4. If the length of a common internal tangent to two circles is 7, and that of a common external tangent is 11, then the product of the radii of the two circles is:

(A) 18 (B) 20 (C) 16 (D) 12

E-5. The equation of the circle whose radius is 3 and which touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ internally at the point $(-1, -1)$ is

(A) $5x^2 + 5y^2 + 8x + 14y - 32 = 0$ (B) $x^2 + y^2 - 8x - 14y - 32 = 0$

(C) $5x^2 + 5y^2 - 14x - 8y - 32 = 0$ (D) $5x^2 + 5y^2 - 8x - 14y - 32 = 0$

Section (F) : Orthogonality, Radical axis and Radical centre

F-1. The angle at which the circle $(x-1)^2 + y^2 = 10$ and $x^2 + (y - 2)^2 = 5$ intersect is -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

F-2. Two congruent circles with centres at (2,3) and (5,6) which intersect at right angles has radius equal to-

(A) $2\sqrt{2}$ (B) 3 (C) 4 (D) none

F-3. The locus of the centre of the circle which cuts the circles $x^2 + y^2 + 2x + 6y + 9 = 0$ and $x^2 + y^2 = 1$ orthogonally is

(A) $x^2 + y^2 + 2x + 6y + 10 = 0$ (B) $2x + 6y - 1 = 0$

(C) $2x + 6y + 10 = 0$ (D) $2x + 6y + 9 = 0$

F-4. Equation of the circle cutting orthogonally the three circles $x^2 + y^2 - 2x + 3y - 7 = 0$, $x^2 + y^2 + 5x - 5y + 9 = 0$ and $x^2 + y^2 + 7x - 9y + 29 = 0$ is

(A) $x^2 + y^2 - 16x - 18y - 4 = 0$ (B) $x^2 + y^2 - 7x + 11y + 6 = 0$

(C) $x^2 + y^2 + 2x - 8y + 9 = 0$ (D) $x^2 + y^2 + 16x - 18y - 4 = 0$

Section (G) : Family of Circles

G-1. The equation of the circle through the points of intersection of $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is -

(A) $x^2 + y^2 + x + 2y = 0$ (B) $x^2 + y^2 - x + 20 = 0$ (C) $x^2 + y^2 - x - 2y = 0$ (D) $2(x^2 + y^2) - x - 2y = 0$

G-2. Find the equation of the circle which passes through the point (1, 1) & which touches the circle $x^2 + y^2 + 4x - 6y - 3 = 0$ at the point (2, 3) on it.

(A) $x^2 + y^2 + x - 6y + 3 = 0$ (B) $x^2 + y^2 + x - 6y - 3 = 0$

(C) $x^2 + y^2 + x + 6y + 3 = 0$ (D) $x^2 + y^2 + x - 3y + 3 = 0$

G-3. The equation of a circle which touches the line $x + y = 5$ at $N(-2,7)$ and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally, is -

(A) $x^2 + y^2 + 7x - 11y + 38 = 0$ (B) $x^2 + y^2 = 53$

(C) $x^2 + y^2 + x - y - 44 = 0$ (D) $x^2 + y^2 - x + y - 62 = 0$

G-4. The system of circles $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$; $\lambda \in R$ passes through two fixed points whose abscissa are roots of equation

(A) $x^2 + 2x - 8 = 0$ (B) $x^2 + 2x + 8 = 0$ (C) $x^2 - 2x - 8 = 0$ (D) $x^2 - 8x - 2 = 0$

PART - III : MATCH THE COLUMN

1.

	Column – I		**Column – II**
(A)	Number of values of a for which the common chord of the circles $x^2 + y^2 = 8$ and $(x - a)^2 + y^2 = 8$ subtends a right angle at the origin is	(p)	4
(B)	A chord of the circle $(x - 1)^2 + y^2 = 4$ lies along the line $y = 22\sqrt{3}\,(x - 1)$. The length of the chord is equal to	(q)	2
(C)	The number of circles touching all the three lines $3x + 7y = 2$, $21x + 49y = 5$ and $9x + 21y = 0$ are	(r)	0
(D)	If radii of the smallest and largest circle passing through the point $\left(\sqrt{3}, \sqrt{2}\right)$ and touching the circle $x^2 + y^2 - 2\sqrt{2}\,y - 2 = 0$ are r_1 and r_2 respectively, then the mean of r_1 and r_2 is	(s)	1

2.

	Column – I		**Column – II**
(A)	Number of common tangents of the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ is	(p)	1
(B)	Number of indirect common tangents of the circles $x^2 + y^2 - 4x - 10y + 4 = 0$ & $x^2 + y^2 - 6x - 12y - 55 = 0$ is	(q)	2
(C)	Number of common tangents of the circles $x^2 + y^2 - 2x - 4y = 0$ & $x^2 + y^2 - 8y - 4 = 0$ is	(r)	3
(D)	Number of direct common tangents of the circles $x^2 + y^2 + 2x - 8y + 13 = 0$ & $x^2 + y^2 - 6x - 2y + 6 = 0$ is	(s)	0

Exercise # 2

PART-I : OBJECTIVE

1. The equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is
(A) $(x + 16)^2 + (y + 2)^2 = 5^2$
(B) $(x - 16)^2 + (y - 2)^2 = 5^2$
(C) $(x + 2)^2 + (y + 16)^2 = 5^2$
(D) $(x + 16)^2 - (y + 2)^2 = 5^2$

2. If $\left(a, \frac{2}{a}\right), \left(b, \frac{2}{b}\right), \left(c, \frac{2}{c}\right)$ & $\left(d, \frac{2}{d}\right)$ are four distinct points on a circle of radius 1 unit, then abcd is equal to:
(A) 4
(B) 16
(C) 1
(D) 2

3. $y - 1 = m_1(x - 3)$ and $y - 3 = m_2(x - 1)$ are two family of straight lines, at right angled to each other. The locus of their point of intersection is
(A) $x^2 + y^2 - 2x - 6y + 10 = 0$
(B) $x^2 + y^2 - 4x - 4y + 6 = 0$
(C) $x^2 + y^2 - 2x - 6y + 6 = 0$
(D) $x^2 + y^2 - 4x - 4y - 6 = 0$

4. The area of the triangle formed by line joining the origin to the points of intersection(s) of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is
(A) 3
(B) 4
(C) 5
(D) 6

5. If tangent at (1, 2) to the circle c_1: $x^2 + y^2 = 5$ intersects the circle c_2: $x^2 + y^2 = 9$ at A & B and tangents at A & B to the second circle meet at point C, then the co–ordinates of C is
(A) (4, 5)
(B) $\left(\frac{9}{15}, \frac{18}{5}\right)$
(C) (4, –5)
(D) $\left(\frac{9}{5}, \frac{18}{5}\right)$

6. A circle passes through point $\left(3, \sqrt{\frac{7}{2}}\right)$ touches the line pair $x^2 - y^2 - 2x + 1 = 0$. Centre of circle lies inside the circle $x^2 + y^2 - 8x + 10y + 15 = 0$. Co-ordinate of centre of circle is
(A) (4, 0)
(B) (5, 0)
(C) (6, 0)
(D) (0, 4)

7. The length of the tangents from any point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$ to the two circles $5x^2 + 5y^2 - 24x + 32y + 75 = 0$ and $5x^2 + 5y^2 - 48x + 64y + 300 = 0$ are in the ratio
(A) 1 : 2
(B) 2 : 3
(C) 3 : 4
(D) 2 : 1

8. Let the chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$, then a, b, c are in
(A) A.P.
(B) G.P.
(C) H.P.
(D) None of these

9. The circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if
(A) $a^2 + b^2 = c^2$
(B) $a + b = c$
(C) $\frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{c^2}$
(D) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

10. Two circles whose radii are equal to 4 and 8 intersect at right angles. The length of their common chord is-
(A) $\frac{16}{\sqrt{5}}$
(B) 8
(C) $4\sqrt{6}$
(D) $\frac{8\sqrt{5}}{5}$

11. A circle touches a straight line $\ell x + my + n = 0$ & cuts the circle $x^2 + y^2 = 9$ orthogonally. The locus of centres of such circles is:
(A) $(\ell x + my + n)^2 = (\ell^2 + m^2)(x^2 + y^2 - 9)$
(B) $(\ell x + my - n)^2 = (\ell^2 + m^2)(x^2 + y^2 - 9)$
(C) $(\ell x + my + n)^2 = (\ell^2 + m^2)(x^2 + y^2 + 9)$
(D) $(\ell x + my - n)^2 = (\ell^2 + m^2)(x^2 + y^2 - 9)$

12. The locus of the point at which two given unequal circles subtend equal angles is:
(A) a straight line (B) a circle (C) a parabola (D) an ellipse

13. A circle is given by $x^2 + (y - 1)^2 = 1$. Another circle C touches it externally and also the x-axis, then the locus of its centre is
(A) $\{(x, y) : x^2 = 4y\} \cup \{(x, y) : y \le 0\}$ (B) $\{(x, y) : x^2 + (y - 1)^2 = 4\} \cup \{(x, y) : y \le 0\}$
(C) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y \le 0\}$ (D) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$

14. The equation of a circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normal and having size just sufficient to contain the circle $x(x - 4) + y(y - 3) = 0$ is
(A) $x^2 + y^2 + 6x + 3y - 45 = 0$ (B) $x^2 + y^2 + 6x - 3y - 45 = 0$
(C) $x^2 + y^2 + 3x - 6y - 45 = 0$ (D) $x^2 + y^2 + 6x - 3y + 45 = 0$

15. **STATEMENT-1 :** If three circles which are such that their centres are non-collinear, then exactly one circle exists which cuts the three circles orthogonally.
STATEMENT-2 : Radical axis for two intersecting circles is the common chord.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true

16. The centre of family of circles cutting the family of
circles $x^2 + y^2 + 4x\left(\lambda - \frac{3}{2}\right) + 3y\left(\lambda - \frac{4}{3}\right) - 6(\lambda + 2) = 0$ orthogonally, lies on
(A) $x - y - 1 = 0$ (B) $4x + 3y - 6 = 0$
(C) $4x + 3y + 7 = 0$ (D) $3x - 4y - 1 = 0$

17. The circle $x^2 + y^2 = 4$ cuts the circle $x^2 + y^2 + 2x + 3y - 5 = 0$ in A & B. Then the equation of the circle on AB as a diameter is:
(A) $13(x^2 + y^2) - 4x - 6y - 50 = 0$ (B) $9(x^2 + y^2) + 8x - 4y + 25 = 0$
(C) $x^2 + y^2 - 5x + 2y + 72 = 0$ (D) $13(x^2 + y^2) - 4x - 6y + 50 = 0$

PART-II : NUMERICAL QUESTIONS

1. Find maximum number of points having integer coordinates (both x, y integer) which can lie on a circle with centre at $\left(\sqrt{2}, \sqrt{3}\right)$ is (are)

2. A circle is inscribed (i.e. touches all four sides) into a rhombous ABCD with one angle 60º. The distance from the centre of the circle to the nearest vertex is equal to 1. If P is any point of the circle, then $|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$ is equal to:

3. The number of integral values of k so that circle $x^2 + y^2 - 6x - 10y + k = 0$ does not touch or intersect the coordinate axes and the point (1, 4) is inside the circle is

4. A variable circle passes through the point A (a, b) & touches the x–axis and the locus of the other end of the diameter through A is $\lambda(x - a)^2 = by$, then the value of λ is

5. The exhaustive set of values of 'a' for which the point (2a, a + 1) is an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$ made by the chord $x - y + 1 = 0$ is (α, β). The value of $\alpha + \beta$ is

6. The number of integral points which lie on or inside the circle $x^2 + y^2 = 4$ is

7. A circle with center in the first quadrant is tangent to $y = x + 10$, $y = x - 6$, and the y-axis. Let (h, k) be the center of the circle, then the value of h + k is (take $\sqrt{2} = 1.41$)

8. If M and m are the maximum and minimum values of $\frac{y}{x}$ for pair of real number (x,y) which satisfy the equation $(x-3)^2 + (y-3)^2 = 6$, then the value of (M + m) is

9. Let P be any moving point on the circle $x^2 + y^2 - 2x = 1$, from this point chord of contact is drawn w.r.t. the circle $x^2 + y^2 - 2x = 0$. The locus of the circumcentre of the triangle CAB, C being centre of the circle and A, B are the points of contact is $(x - a)^2 + y^2 = b$, then the value of $|a - b|$ is

10. The circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q, then the number of real values of 'a' for which the line $5x + by - a = 0$ passes through P and Q.

11. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles and the area of the rhombus is $a\sqrt{3}$ sq. units, then find the value of a.

12. The co-ordinates of the centre of the smallest circle touching the circles $x^2 + y^2 - 2y - 3 = 0$ and $x^2 + y^2 - 8x - 18y + 93 = 0$ is (a, b), then the value of $\frac{a}{b}$ is

13. Circles C_1 and C_2 are externally tangent and they are both internally tangent to the circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively and the centres of the three circles are collinear. A chord of C_3 is also a common internal tangent of C_1 and C_2. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$ where *m*, *n* and *p* are positive integers, *m* and *p* are relatively prime and *n* is not divisible by the square of any prime, then value of (*m* + *n* + *p*) is

14. If (α, β) is a point on the circle whose centre is on the x-axis and which touches the line $x + y = 0$ at (2, –2), then the least integral value of 'α' is

15. Two circles drawn through the point (a, 5a) and (4a, a) to touch the axis of y. They intersect at an angle of θ, then the value of $\tan\theta$ is

16. The centre of the circle S = 0 lie on the line $2x - 2y + 9 = 0$ & S = 0 cuts orthogonally the circle $x^2 + y^2 = 4$. The circle S = 0 passes through two fixed points whose sum of ordinate is

PART - III : ONE OR MORE THAN ONE CORRECT

1. $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$, represents :

(A) equation of a straight line, if θ is constant and r is variable

(B) equation of a circle, if r is constant and θ is a variable

(C) a straight line passing through a fixed point and having a known slope

(D) a circle with a known centre and a given radius.

2. Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 & L_2 are equal, then which of the following equations can represent L_1?

(A) $x + y = 0$ (B) $x - y = 0$ (C) $x + 7y = 0$ (D) $x - 7y = 0$

3. The equation of the circle which touches both the axes and the line $\frac{x}{3}+\frac{y}{4}=1$ and lies in the first quadrant is $(x-c)^2+(y-c)^2=c^2$ where c is

(A) 1 (B) 2 (C) 4 (D) 6

4. Find the equations of straight lines which pass through the intersection of the lines $x-2y-5=0$, $7x+y=50$ & divide the circumference of the circle $x^2+y^2=100$ into two arcs whose lengths are in the ratio 2 : 1.

(A) $3x-4y-25=0$ (B) $4x+3y-25=0$

(C) $4x-3y-25=0$ (D) $3x+4y-25=0$

5. One of the diameter of the circle circumscribing the rectangle ABCD is $x-3y+1=0$. If two verticles of rectangle are the points $(-2, 5)$ and $(6, 5)$ respectively, then which of the following hold(s) good?

(A) Area of rectangle ABCD is 64 square units.

(B) Centre of circle is (2, 1)

(C) The other two vertices of the rectangle are $(-2,-3)$ and $(6,-3)$

(D) Equation of sides are $x=-2$, $y=-3$, $x=5$ and $y=6$.

6. Consider two circles $C_1: x^2+y^2-1=0$ and $C_2: x^2+y^2-2=0$. Let A(1,0) be a fixed point on the circle C_1 and B be any variable point on the circle C_2. The line BA meets the curve C_2 again at C. Which of the following alternative(s) is/are correct ?

(A) $OA^2+OB^2+BC^2 \in [7, 11]$, where O is the origin. (B) $OA^2+OB^2+BC^2 \in [4, 7]$, where O is the origin.

(C) Locus of midpoint of AB is a circle of radius $\frac{1}{\sqrt{2}}$. (D) Locus of midpoint of AB is a circle of area $\frac{\pi}{2}$.

7. The equation (s) of the tangent at the point (0, 0) to the circle where circle makes intercepts of length 2a and 2b units on the coordinate axes, is (are) -

(A) $ax+by=0$ (B) $ax-by=0$ (C) $x=y$ (D) $bx+ay=ab$

8. Tangents PA and PB are drawn to the circle $S \equiv x^2+y^2-2y-3=0$ from the point P(3,4). Which of the following alternative(s) is/are correct ?

(A) The power of point P(3,4) with respect to circle S = 0 is 14.

(B) The angle between tangents from P(3,4) to the circle S = 0 is $\frac{\pi}{3}$

(C) The equation of circumcircle of ΔPAB is $x^2+y^2-3x-5y+4=0$

(D) The area of quadrilateral PACB is $3\sqrt{7}$ square units where C is the centre of circle S = 0.

9. Tangents are drawn to the circle $x^2+y^2=50$ from a point 'P' lying on the x-axis. These tangents meet the y-axis at points 'P_1' and 'P_2'. Possible coordinates of 'P' so that area of triangle PP_1P_2 is minimum, is/are

(A) (10, 0) (B) $(10\sqrt{2}, 0)$ (C) (–10, 0) (D) $(-10\sqrt{2}, 0)$

10. If $a\ell^2-bm^2+2d\ell+1=0$, where a, b, d are fixed real numbers such that $a+b=d^2$, then the line $\ell x+my+1=0$ touches a fixed circle

(A) which cuts the x–axis orthogonally

(B) with radius equal to b

(C) on which the length of the tangent from the origin is $\sqrt{d^2-b}$

(D) none of these.

11. The tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular if

(A) $h = r$ (B) $h = -r$ (C) $r^2 + h^2 = 1$ (D) $r^2 = h^2$

12. If the circle C_1: $x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to 3/4, then the co–ordinates of the centre of C_2 are:

(A) $\left(\frac{9}{5}, \frac{12}{5}\right)$ (B) $\left(\frac{9}{5}, \frac{-12}{5}\right)$ (C) $\left(\frac{-9}{5}, \frac{-12}{5}\right)$ (D) $\left(\frac{-9}{5}, \frac{+12}{5}\right)$

13. For the circles $x^2 + y^2 - 10x + 16y + 89 - r^2 = 0$ and $x^2 + y^2 + 6x - 14y + 42 = 0$ which of the following is/are true.

(A) Number of integral values of r are 14 for which circles are intersecting.

(B) Number of integral values of r are 18 for which circles are intersecting.

(C) For r equal to 13 number of common tangents are 3.

(D) For r equal to 21 number of common tangents are 2.

14. $x^2 + y^2 = a^2$ and $(x - 2a)^2 + y^2 = a^2$ are two equal circles touching each other. Find the equation of circle (or circles) of the same radius touching both the circles.

(A) $x^2 + y^2 + 2ax + 2\sqrt{3}\,ay + 3a^2 = 0$ (B) $x^2 + y^2 - 2ax + 2\sqrt{3}\,ay + 3a^2 = 0$

(C) $x^2 + y^2 + 2ax - 2\sqrt{3}\,ay + 3a^2 = 0$ (D) $x^2 + y^2 - 2ax - 2\sqrt{3}\,ay + 3a^2 = 0$

15. Consider the circles $S_1 : x^2 + y^2 = 4$ and $S_2 : x^2 + y^2 - 2x - 4y + 4 = 0$ which of the following statements are correct ?

(A) Number of common tangents to these circles is 2.

(B) If the power of a variable point P w.r.t. these two circles is same then P moves on the line $x + 2y - 4 = 0$

(C) Sum of the y-intercepts of both the circles is 6.

(D) The circles S_1 and S_2 are orthogonal.

16. Equations of circles which pass through the points (1, –2) and (3, – 4) and touch the x-axis is

(A) $x^2 + y^2 + 6x + 2y + 9 = 0$ (B) $x^2 + y^2 + 10x + 20y + 25 = 0$

(C) $x^2 + y^2 - 6x + 4y + 9 = 0$ (D) $x^2 + y^2 + 10x + 20y - 25 = 0$

17. The centre of a circle passing through the points (0, 0), (1, 0) & touching the circle $x^2 + y^2 = 9$ is :

(A) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (B) $\left(\frac{1}{2}, \sqrt{2}\right)$ (C) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, -\sqrt{2}\right)$

18. Curves $ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$ and $a'x^2 - 2hxy + (a' + a - b)y^2 - 2g'x - 2f'y + c = 0$ intersect at four concyclic point A, B, C and D. If P is the point $\left(\frac{g'+g}{a'+a}, \frac{f'+f}{a'+a}\right)$,then which of the following is/are true

(A) P is also concyclic with points A,B,C,D (B) PA, PB, PC in G.P.

(C) $PA^2 + PB^2 + PC^2 = 3PD^2$ (D) PA, PB, PC in A.P.

PART - IV : COMPREHENSION

Comprehension # 1 (Q. No. 1 to 3)

Let S_1, S_2, S_3 be the circles $x^2 + y^2 + 3x + 2y + 1 = 0$, $x^2 + y^2 - x + 6y + 5 = 0$ and $x^2 + y^2 + 5x - 8y + 15 = 0$, then

1. Point from which length of tangents to these three circles is same is

(A) (1, 0) (B) (3, 2) (C) (10, 5) (D) (– 2, 1)

2. Equation of circle S_4 which cut orthogonally to all given circle is

(A) $x^2 + y^2 - 6x + 4y - 14 = 0$
(B) $x^2 + y_2 + 6x + 4y - 14 = 0$
(C) $x^2 + y^2 - 6x - 4y + 14 = 0$
(D) $x^2 + y^2 - 6x - 4y - 14 = 0$

3. Radical centre of circles S_1, S_2, & S_4 is

(A) $\left(-\frac{3}{5},-\frac{8}{5}\right)$ (B) (3, 2) (C) (1, 0) (D) $\left(-\frac{4}{5},-\frac{3}{2}\right)$

Comprehension # 2 (Q. No. 4 to 5)

Two circles are

$S_1 \equiv (x + 3)^2 + y^2 = 9$

$S_2 \equiv (x - 5)^2 + y^2 = 16$

with centres C_1 & C_2

4. A direct common tangent is drawn from a point P which touches S_1 & S_2 at Q & R, respectively. Find the ratio of area of ΔPQC_1 & ΔPRC_2.

(A) 3 : 4 (B) 9 : 16 (C) 16 : 9 (D) 4 : 3

5. From point 'A' on S_2 which is nearest to C_1, a variable chord is drawn to S_1. The locus of mid point of the chord.

(A) circle
(B) Diameter of s_1
(C) Arc of a circle
(D) chord of s_1 but not diameter

6. Locus of 5 cuts the circle S_1 at B & C, then line segment BC subtends an angle on the major arc of circle S_1 is

(A) $\cos^{-1}\frac{3}{4}$ (B) $\frac{\pi}{2} - \tan^{-1}\frac{4}{3}$ (C) $\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{3}{4}$ (D) $\frac{\pi}{2}\cot^{-1}\frac{4}{3}$

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where k > 0, then the value of [k] is

[**Note** : [k] denotes the largest integer less than or equal to k]

2. The circle passing through the point (–1, 0) and touching the y-axis at (0, 2) also passes through the point

(A) $\left(-\frac{3}{2}, 0\right)$ (B) $\left(-\frac{5}{2}, 2\right)$ (C) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (D) (–4, 0)

[IIT-JEE 2011, (3, –1), 80]

3. The straight line 2x – 3y = 1 divides the circular region $x^2 + y^2 \leq 6$ into two parts.

If S = $\left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\}$, then the number of point(s) in S lying inside the smaller part is

[IIT-JEE 2011, (4, 0), 80]

4. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line 4x – 5y = 20 to the circle $x^2 + y^2 = 9$ is **[IIT-JEE 2012, (3, –1), 70]**

(A) $20(x^2 + y^2) - 36x + 45y = 0$ (B) $20(x^2 + y^2) + 36x - 45y = 0$

(C) $36(x^2 + y^2) - 20x + 45y = 0$ (D) $36(x^2 + y^2) + 20x - 45y = 0$

Paragraph for Question Nos. 5 to 6

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point P($\sqrt{3}$, 1). A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$. **[IIT-JEE 2012, (3, –1), 66]**

5. A common tangent of the two circles is

(A) x = 4 (B) y = 2 (C) $x + \sqrt{3}y = 4$ (D) $x + 2\sqrt{2}y = 6$

6. A possible equation of L is

(A) $x - \sqrt{3}y = 1$ (B) $x + \sqrt{3}y = 1$ (C) $x - \sqrt{3}y = -1$ (D) $x + \sqrt{3}y = 5$

7.* Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is (are)

[JEE (Advanced) 2013, (3, –1), 60]

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$ (B) $x^2 + y^2 - 6x + 7y + 9 = 0$

(C) $x^2 + y^2 - 6x - 8y + 9 = 0$ (D) $x^2 + y^2 - 6x - 7y + 9 = 0$

8.* A circle S passes through the point (0, 1) and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then **[JEE (Advanced) 2014, (3, 0), 60]**

(A) radius of S is 8 (B) radius of S is 7

(C) centre of S is (–7, 1) (D) centre of S is (–8, 1)

9*. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1,0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. then the locus of E passes through the point(s)- **[JEE(Advanced)-2016, (4,–2), 62]**

(A) $\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4},\frac{1}{2}\right)$ (C) $\left(\frac{1}{3},-\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4},-\frac{1}{2}\right)$

10*. The circle C_1: $x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then

[JEE(Advanced)-2016, (4,–2), 62]

(A) $Q_2Q_3 = 12$

(B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$

(D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

11. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points ? **[JEE(Advanced)-2017, (3, 0),61]**

Paragraph for Question 12 and 13

Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

(*There are two question based on Paragraph "X", the question given below is one of them*)

12. Let E_1E_2 and F_1F_2 be the chord of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slop –1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents of S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the points E_3, F_3 and G_3 lie on the curve **[JEE(Advanced)-2018, (3,–1),60]**

(A) $x + y = 4$

(B) $(x - 4)^2 + (y - 4)^2 = 16$

(C) $(x - 4)(y - 4) = 4$

(D) $xy = 4$

13. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve - **[JEE(Advanced)-2018, (3,–1),60]**

(A) $(x + y)^2 = 3xy$ (B) $x^{2/3} + y^{2/3} = 2^{4/3}$ (C) $x^2 + y^2 = 2xy$ (D) $x^2 + y^2 = x^2y^2$

14*. Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2019, (3,–1),62]**

(A) The point (–2, 7) lies in E_1

(B) The point $\left(\frac{4}{5},\frac{7}{5}\right)$ does **NOT** lie in E_2

(C) The point $\left(\frac{1}{2},1\right)$ lies in E_2

(D) The point $\left(0,\frac{3}{2}\right)$ does **NOT** lie in E_1

15. A line $y = mx + 1$ intersects the circle $(x - 3)^2 + (y + 2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

[JEE(Advanced)-2019, (3,0), 62]

(1) $6 \le m < 8$ (2) $2 \le m < 4$ (3) $4 \le m < 6$ (4) $-3 \le m < -1$

16. Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____

[JEE(Advanced)-2019, 3(0)]

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x- 3)^2 + (y - 4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x - h)^2 + (y - k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below :

	List-I		**List-II**
(I)	$2h + k$	(P)	6
(II)	$\dfrac{\text{Length of ZW}}{\text{Length of XY}}$	(Q)	$\sqrt{6}$
(III)	$\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R)	$\dfrac{5}{4}$
(IV)	α	(S)	$\dfrac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\dfrac{10}{3}$

17. Which of the following is the only INCORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (IV), (S) (2) (IV), (U) (3) (III), (R) (4) (I), (P)

18. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (II), (T) (2) (I), (S) (3) (I), (U) (4) (II), (Q)

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

[AIEEE 2010, (4, –1), 144]

(1) $-35 < m < 15$ (2) $15 < m < 65$ (3) $35 < m < 85$ (4) $-85 < m < -35$

2. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2 (c > 0)$ touch each other if : **[AIEEE-2011, (4, –1), 120]**

(1) $2|a| = c$ (2) $|a| = c$ (3) $a = 2c$ (4) $|a| = 2c$

3. The equation of the circle passing through the point (1, 0) and (0, 1) and having the smallest radius is -

[AIEEE-2011, (4, –1), 120]

(1) $x^2 + y^2 - 2x - 2y + 1 = 0$ (2) $x^2 + y^2 - x - y = 0$

(3) $x^2 + y^2 + 2x + 2y - 7 = 0$ (4) $x^2 + y^2 + x + y - 2 = 0$

4. The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is : **[AIEEE- 2012, (4, –1), 120]**

(1) $\frac{10}{3}$ (2) $\frac{3}{5}$ (3) $\frac{6}{5}$ (4) $\frac{5}{3}$

5. The circle passing through (1, –2) and touching the axis of x at (3, 0) also passes through the point

[AIEEE - 2013, (4, –¼),120]

(1) (–5, 2) (2) (2, – 5) (3) (5, – 2) (4) (–2, 5)

6. Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at (0, y), passing through origin and touching the circle C externally, then the radius of T is equal to : **[JEE(Main)-2014, (4, – ¼), 120]**

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$ (3) $\frac{\sqrt{3}}{\sqrt{2}}$ (4) $\frac{\sqrt{3}}{2}$

7. Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a

[JEE(Main)-2015, (4, – ¼), 120]

(1) straight line parallel to x-axis (2) straight line parallel to y-axis

(3) circle of radius $\sqrt{2}$ (4) circle of radius $\sqrt{3}$

8 . The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is

[JEE(Main) 2015, (4, – 1), 120]

(1) 1 (2) 2 (3) 3 (4) 4

9. If one of the diameters of the circle, given by the euqation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is :- **[JEE(Main) 2016, (4, –1), 120]**

(1) 10 (2) $5\sqrt{2}$ (3) $5\sqrt{3}$ (4) 5

10. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on :- **[JEE(Main) 2016, (4, –1), 120]**

(1) A parabola (2) A circle

(3) An ellipse which is not a circle (4) A hyperbola

11. Let the orthocentre and centroid of a triangle be A(–3, 5) and B(3, 3) respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is:

[JEE(Main) 2018, (4, –1), 120]

(1) $2\sqrt{10}$ (2) $3\sqrt{\frac{5}{2}}$ (3) $\frac{3\sqrt{5}}{2}$ (4) $\sqrt{10}$

12. Three circles of radii a, b, c(a < b < c) touch each other externally. If they have x-axis as a common tangent, then : **[JEE(Main)-2019, Online (9-1-19), P-1 (4, –1), 120]**

(1) $\frac{1}{\sqrt{a}}=\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}$ (2) a, b, c are in A.P.

(3) $\sqrt{a},\sqrt{b},\sqrt{c}$ are in A.P. (4) $\frac{1}{\sqrt{b}}=\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{c}}$

13. If a circle C passing through the point (4,0) touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point (1, –1), then the radius of C is : **[JEE(Main)-2019, Online (10-1-19), P-1 (4, –1), 120]**

(1) $\sqrt{57}$ (2) 4 (3) $2\sqrt{5}$ (4) 5

14. If a variable line, $3x + 4y - \lambda = 0$ is such that the two circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ are on its opposite sides, then the set of all values of λ is the interval :-

[JEE(Main)-2019, Online (12-1-19), P-1 (4, –1), 120]

(1) [12, 21] (2) (2, 17) (3) (23, 31) (4) [13, 23]

15. The tangent and the normal lines at the point $\left(\sqrt{3},1\right)$ to the circle $x^2 + y^2 = 4$ and the x-axis form a triangle. The area of this triangle (in square units) is : **[JEE(Main)-2019, Online (8-4-19), P-2 (4, –1), 120]**

(1) $\frac{1}{3}$ (2) $\frac{4}{\sqrt{3}}$ (3) $\frac{1}{\sqrt{3}}$ (4) $\frac{2}{\sqrt{3}}$

16. A circle touching the x-axis at (3, 0) and making an intercept of length 8 on the y-axis passes through the point : **[JEE(Main)-2019, Online (12-4-19), P-2 (4, –1), 120]**

(1) (3, 10) (2) (2,3) (3) (1,5) (4) (3,5)

17. Let the tangents drawn from the origin to the circle, $x^2 + y^2 - 8x - 4y + 16 = 0$ touch it at the points A and B. The $(AB)^2$ is equal to : **[JEE(Main)-2020, Online (7-1-20), P-2 (4, –1), 100]**

(1) $\frac{52}{5}$ (2) $\frac{32}{5}$ (3) $\frac{56}{5}$ (4) $\frac{64}{5}$

18. If a line, $y = mx + c$ is a tangent to the circle, $(x - 3)^2 + y^2 = 1$ and it is perpendicular to a line L_1, where L_1 is the tangent to the circle, $x^2 + y^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, then

[JEE(Main)-2020, Online (8-1-20), P-2 (4, –1), 100]

(1) $c^2 - 6c + 7 = 0$ (2) $c^2 + 6c + 7 = 0$ (3) $c^2 + 7c + 6 = 0$ (4) $c^2 - 7c + 6 = 0$

Answers

Exercise # 1

PART - I

SECTION-(A)

A-1. (i) $\left(\frac{4}{3}, \frac{5}{3}\right); \frac{4\sqrt{2}}{3}$ (ii) $(-\sin\theta, -\cos\theta)$; 3

(iii) $\left(1, -\frac{3}{2}\right); \frac{\sqrt{23}}{2}$

A-2. $17(x^2 + y^2) + 2x - 44y = 0$

A-3. (i) $x^2 + y^2 - 6x \pm 6\sqrt{2}y + 9 = 0$;
(ii) $x^2 + y^2 + 4x - 10y + 4 = 0$; $x^2 + y^2 - 4x - 2y + 4 = 0$

A-4. $x^2 + y^2 - 4x - 4y + 4 = 0$

A-5. (i) $x^2 + y^2 - ax - by = 0$; (ii) $x^2 + y^2 - 22x - 4y + 25 = 0$

A-6. $x^2 + y^2 - hx - ky = 0$

A-7. $x = \frac{p}{2}(-1 + \sqrt{2}\cos\theta)$; $y = \frac{p}{2}(-1 + \sqrt{2}\sin\theta)$

SECTION-(B)

B-1. $\lambda \in (-1, 4)$ **B-2.** (i) 2; (ii) (9, 3)

B-3. 40, –10 **B-4.** (1, 3), (5, 7), $4\sqrt{2}$

B-5. $(2\sqrt{5} - 15, -\sqrt{5} - 1)$.

SECTION-(C)

C-1. (i) $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$;
(ii) $12x - 5y + 8 = 0$ and $12x - 5y - 252 = 0$
(iii) $x - \sqrt{3}y \pm 10 = 0$

C-2. 4

C-3. $y = \pm\frac{4}{\sqrt{65}}(x - 9)$, $\tan^{-1}\left(\frac{8\sqrt{65}}{49}\right)$

C-4. 8 sq. units

SECTION-(D)

D-1. (i) $2x - y = 0$ (ii) $x + 2y - 1 = 0$

D-2. 10 **D-3.** (–25, 50)

D-5. $\left(\frac{1}{2}, -\frac{1}{2}\right)$, $x + y = 0$

SECTION-(E)

E-1. $2\sqrt{2}$ **E-2.** $\left(6, -\frac{18}{5}\right)$ **E-3.** zero, zero

E-4. $x = 0$, $3x + 4y = 10$, $y = 4$ and $3y = 4x$.

E-5. 20

SECTION-(F)

F-1. – 18

F-2. $3x^2 + 3y^2 + 4x + 20y = 0$

F-3. $x^2 + y^2 - 4x - 2y + 3 = 0$

F-4. (3, 2); $x^2 + y^2 - 6x - 4y - 14 = 0$

SECTION-(G)

G-1. $x^2 + y^2 + 16x + 14y - 12 = 0$

G-2. $x^2 + y^2 - 2x - 4y = 0$.

G-3. $x^2 + y^2 - 6x + 4y = 0$ **OR** $x^2 + y^2 + 2x - 8y + 4 = 0$

G-4. $\left(2, \frac{23}{3}\right)$

PART - II

SECTION-(A)

A-1. D	**A-2.** B	**A-3.** C	**A-4.** D
A-5. B	**A-6.** B	**A-7.** A	**A-8.** B

SECTION-(B)

B-1. D	**B-2.** A	**B-3.** A	**B-4.** A
B-5. B	**B-6.** A		

SECTION-(C)

C-1. B	**C-2.** D	**C-3.** A	**C-4.** C
C-5. C	**C-6.** A	**C-7.** A	**C-8.** A

SECTION-(D)

D-1. B	**D-2.** C	**D-3.** B	**D-4.** B
D-5. A	**D-6.** A	**D-7.** A	

SECTION-(E)

E-1. D	**E-2.** B	**E-3.** B	**E-4.** A
E-5. **D**			

SECTION-(F)

F-1. B	**F-2.** B	**F-3.** C	**F-4.** A

SECTION-(G)

G-1. C	**G-2.** A	**G-3.** A	**G-4.** C

PART - III

1. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (s)

2. (A) $\rightarrow$ (r), (B) $\rightarrow$ (s), (C) $\rightarrow$ (p), (D) $\rightarrow$ (q)

Exercise # 2

PART - I

1. A	**2.** A	**3.** B	**4.** C
5. D	**6.** A	**7.** A	**8.** B
9. D	**10.** A	**11.** A	**12.** B
13. D	**14.** B	**15.** D	**16.** B
17. A			

PART - II

1. 1	**2.** 11	**3.** 3	**4.** 0.25
5. 1.80	**6.** 13	**7.** 13.28	**8.** 6
9. 0.50	**10.** 0	**11.** 8	**12.** 0.40
13. 19	**14.** 2	**15.** 4.44	**16.** 4.50

PART - III

1. A,B,C,D	**2.** B,C	**3.** A,D	**4.** C,D
5. A,B,C	**6.** A,C,D	**7.** A,B	**8.** A,C
9. A,C	**10.** A,C	**11.** A,B,D	**12.** B,D
13. A,C	**14.** B,D	**15.** A,B,D	**16.** B,C
17. B,D	**18.** B,C,D		

PART - IV

1. B	**2.** D	**3.** A	**4.** B
5. C	**6.** A		

Exercise # 3

PART - I

1. 3	**2.** D	**3.** 2	**4.** A
5. D	**6.** A	**7.** A,C	**8.** B,C
9. A,C	**10.** A,B,C	**11.** 2	**12.** A
13. D	**14.** B,D	**15.** 2	**16.** 10.00
17. 1	**18.** 4		

PART - II

1. 1	**2.** 2	**3.** 2	**4.** 1
5. 3	**6.** 2	**7.** 3	**8 .** 3
9. 3	**10.** 1	**11.** 2	**12.** 1
13. 4	**14.** 1	**15.** 4	**16.** 1
17. 4	**18.** 2		

Reliable Ranker Problems

1. Two straight lines rotate about two fixed points. If they start from their position of coincidence such that one rotates at the rate double that of the other. Prove that the locus of their point of intersection is a circle.

2. Let a circle be given by $2x(x-a)+y(2y-b)=0$, $(a \neq 0, b \neq 0)$. Find the condition on a & b if two chords, each bisected by the x–axis, can be drawn to the circle from $\left(a, \frac{b}{2}\right)$.

3. A circle touches the line y = x at a point P such that $OP = 4\sqrt{2}$ where O is the origin. The circle contains the point (–10, 2) in its interior and the length of its chord on the line x + y = 0 is $6\sqrt{2}$. Find the equation of the circle.

4. Through a given point P(5, 2), secants are drawn to cut the circle $x^2+y^2=25$ at points $A_1(B_1)$, $A_2(B_2)$, $A_3(B_3)$, $A_4(B_4)$ and $A_5(B_5)$ such that $PA_1+PB_1=5$, $PA_2+PB_2=6$, $PA_3+PB_3=7$, $PA_4+PB_4=8$ and $PA_5+PB_5=9$. Find the value of $\sum_{i=1}^{5} PA_i^2 + \sum_{i=1}^{5} PB_i^2$.

[**Note :** $A_r(B_r)$ denotes that the line passing through P(5,2) meets the circle $x^2+y^2=25$ at two points A_r and B_r]

5. If (a, α) lies inside the circle $x^2+y^2=9$: $x^2-4x-a^2=0$ has exactly one root in (– 1, 0), then find the area of the region in which (a, α) lies.

6. Let $S \equiv x^2+y^2+2gx+2fy+c=0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends right angle at the origin.

7. A ball moving around the circle $x^2+y^2-2x-4y-20=0$ in anti–clockwise direction leaves it tangentially at the point P(–2, –2). After getting reflected from a straight line it passes through the centre of the circle. Find the equation of this straight line if its perpendicular distance from P is $\frac{5}{2}$. You can assume that the angle of incidence is equal to the angle of reflection.

8. The lines $5x+12y-10=0$ and $5x-12y-40=0$ touch a circle C_1 of diameter 6 unit. If the centre of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts of intercepts of length 8 on these lines.

9. The circle $x^2+y^2-4x-8y+16=0$ rolls up the tangent to it at $(2+\sqrt{3}, 3)$ by 2 units, find the equation of the circle in the new position.

10. Find number of values of 'c' for which the set $\{(x, y) \mid x^2+y^2+2x \leq 1\} \cap \{(x, y) \mid x-y+c \geq 0\}$ contains only one point is common.

11. Find the locus of middle points of chords of the circle $x^2+y^2=a^2$, which subtend right angle at the point (c, 0).

12. A and B are two fixed points and P moves such that PA = nPB where $n \neq 1$. Show that locus of P is a circle and for different values of n all the circles have a common radical axis.

13. Prove that the two circles which pass through the points (0, a) , (0, – a) and touch the straight line y = m x + c will cut orthogonaly if $c^2 = a^2(2+m^2)$.

14. Consider points $A(\sqrt{13}, 0)$ and $B(2\sqrt{13}, 0)$ lying on x-axis. These points are rotated in an-anticlockwise direction about the origin through an angle of $\tan^{-1}\left(\frac{2}{3}\right)$. Let the new position of A and B be A′ and B′ respectively. With A′ as centre and radius $\frac{2\sqrt{13}}{3}$ a circle C_1 is drawn and with B′ as a centre and radius $\frac{\sqrt{13}}{3}$ circle C_2 is drawn. Find radical axis of C_1 and C_2.

15. P(a, b) is a point in the first quadrant. If the two circles which pass through P and touch both the co–ordinate axes cut at right angles, then find condition in a and b.

16. Prove that the square of the tangent that can be drawn from any point on one circle to another circle is equal to twice the product of perpendicular distance of the point from the radical axis of two circles and distance between their centres.

17. Find the equation of the circle which cuts each of the circles, $x^2 + y^2 = 4$, $x^2 + y^2 - 6x - 8y + 10 = 0$ & $x^2 + y^2 + 2x - 4y - 2 = 0$ at the extremities of a diameter.

18. Show that if one of the circle $x^2 + y^2 + 2gx + c = 0$ and $x^2 + y^2 + 2g_1x + c = 0$ lies within the other, then gg_1 and c are both positive.

19. Find the equation of the circle with the shortest radius which touches atleast two of the three circles $x^2 + y^2 + 6x + 8 = 0$, $x^2 + y^2 - 18x + 80 = 0$ and $x^2 + y^2 - 8y + 12 = 0$.

20. Circles are drawn passing through the origin O to intersect the coordinate axes at point P and Q such that m. OP + n . OQ is a constant. Show that the circles pass through a fixed point.

21. A triangle has two of its sides along the axes, its third side touches the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. Find the equation of the locus of the circumcentre of the triangle.

22. Find the radical centre of three circles described on the three sides $4x - 7y + 10 = 0$, $x + y - 5 = 0$ and $7x + 4y - 15 = 0$ of a triangle as diameters.

23. The curves whose equations are

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$S' = a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

intersect in four concyclic points then find relation in a, b, h, a′, b′, h′.

24. A circle of constant radius 'r' passes through origin O and cuts the axes of coordinates in points P and Q, then find the equation of the locus of the foot of perpendicular from O to PQ.

25. The ends A , B of a fixed straight line of length 'a' and ends A′ and B′ of another fixed straight line of length 'b' slide upon the axis of X & the axis of Y (one end on axis of X & the other on axis of Y). Find the locus of the centre of the circle passing through A, B, A′ and B′.

26. A fixed circle is cut by a family of circles all of which, pass through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Prove that the chord of intersection of the fixed circle with any circle of the family passes through a fixed point.

Answers

2. $(a^2 > 2b^2)$ **3.** $x^2 + y^2 + 18x - 2y + 32 = 0$ **4.** 215 **5.** 5π

6. $x^2 + y^2 + gx + fy + \frac{c}{2} = 0$ **7.** $(4\sqrt{3} - 3)x - (4 + 3\sqrt{3})y - (39 - 2\sqrt{3}) = 0$

8. $x^2 + y^2 - 10x - 4y + 4 = 0$ **9.** $(x - 3)^2 + (y - 4 - \sqrt{3})^2 = 2^2$ **10.** 1

11. $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$ **14.** $9x + 6y = 65$ **15.** $a^2 - 4ab + b^2 = 0$

17. $x^2 + y^2 - 4x - 6y - 4 = 0$ **19.** $5(x^2 + y^2) + 18x - 16y + 24 = 0$

21. $2(x + y) - a = \frac{2xy}{a}$ **22.** (1, 2) **23.** $\frac{a - b}{h} = \frac{a' - b'}{h'}$

24. $(x^2 + y^2)^2 (x^{-2} + y^{-2}) = 4r^2$ **25.** $(2ax - 2by)^2 + (2bx - 2ay)^2 = (a^2 - b^2)^2$