3.Sequence & Progression(AP, GP, HP, AGP, Spl. Series)

DEFINITION : A sequence is a set of terms in a definite order with a rule for obtaining the terms. e.g. 1 , 1/2 , 1/3 , ... , 1/n , ... is a sequence.

AN ARITHMETIC PROGRESSION (AP) : AP is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then AP can be written as a, a + d, a + 2 d, ... a + (n – 1)d, ... n^{th} term of this AP $t_n = a + (n-1)d$, where $d = a_n - a_{n-1}$. The sum of the first n terms of the AP is given by ; $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$. where l is the last term.

NOTES :(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(ii) Three numbers in AP can be taken as a – d , a , a + d ; four numbers in AP can be taken as a – 3d, a – d, a + d, a + 3d ; five numbers in AP are a – 2d , a – d , a, a + d, a + 2d & six terms in AP are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(iii) The common difference can be zero, positive or negative.

(iv) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it.

(vi) $t_r = S_r - S_{r-1}$ **(vii)** If a , b , c are in AP $\Rightarrow 2b = a + c$.

GEOMETRIC PROGRESSION (GP) :

GP is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by a constant . Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by that which immediately proceeds it. Therefore a, ar, ar^2, ar^3, ar^4, ... is a GP with a as the first term & r as common ratio.www.MathsBySuhag.com , www.TekoClasses.com

(i) n^{th} term = $a\, r^{n-1}$ **(ii)** Sum of the I^{st} n terms i.e. $S_n = \frac{a(r^n - 1)}{r-1}$, if $r \neq 1$.

(iii) Sum of an infinite GP when $|r| < 1$ when $n \to \infty$ $r^n \to 0$ if $|r| < 1$ therefore,

$$S_\infty = \frac{a}{1-r} (|r| < 1) .$$

(iv) If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP.

(v) Any 3 consecutive terms of a GP can be taken as a/r, a, ar ; any 4 consecutive terms of a GP can be taken as a/r^3, a/r, ar, ar^3 & so on.

(vi) If a, b, c are in GP $\Rightarrow b^2 = ac$.www.MathsBySuhag.com , www.TekoClasses.com

HARMONIC PROGRESSION (HP) :A sequence is said to HP if the reciprocals of its terms are in AP.If the sequence $a_1, a_2, a_3, ... , a_n$ is an HP then $1/a_1, 1/a_2, ... , 1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. For HP whose first term is a & second term is b, the n^{th} term is $t_n = \frac{ab}{b + (n-1)(a-b)}$.

If a, b, c are in HP $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

MEANS

ARITHMETIC MEAN :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c . AM for any n positive number $a_1, a_2, ... , a_n$ is ;

$A = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}$.www.MathsBySuhag.com , www.TekoClasses.com

n - ARITHMETIC MEANS BETWEEN TWO NUMBERS :

If a, b are any two given numbers & a, $A_1, A_2, ... , A_n$, b are in AP then $A_1, A_2, ... A_n$ are the n AM's between a & b .

$$A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1},\ ......,\ A_n = a + \frac{n(b-a)}{n+1} \quad = a + d,\ \ = a + 2d,\ ......,\ A_n = a + nd,$$

where $d = \frac{b-a}{n+1}$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e.

$\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

GEOMETRIC MEANS :

If a, b, c are in GP, b is the GM between a & c.

$b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0$, $c > 0$.

n-GEOMETRIC MEANS BETWEEN a, b :

If a, b are two given numbers & a, $G_1, G_2, ... , G_n$, b are in GP. Then $G_1, G_2, G_3, ..., G_n$ are n GMs between a & b .

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$, ... , $G_n = a(b/a)^{n/n+1}$
$= ar$, $= ar^2$, ... $= ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to the n^{th} power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b.

HARMONIC MEAN :

If a, b, c are in HP, b is the HM between a & c, then $b = 2ac/[a + c]$.

THEOREM :

If A, G, H are respectively AM, GM, HM between a & b both being unequal & positive then,

(i) $G^2 = AH$

(ii) $A > G > H$ $(G > 0)$. Note that A, G, H constitute a GP.

ARITHMETICO-GEOMETRIC SERIES :

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the **Arithmetico-Geometric Series**. e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5, ... are in AP & 1, x, x^2, x^3 ..... are in GP.

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Standart appearance of an Arithmetico-Geometric Series is

Let $S_n = a + (a + d)\, r + (a + 2d)\, r^2 + ... + [a + (n-1)d]\, r^{n-1}$

SUM TO INFINITY :

$$\text{If } |r| < 1 \ \& \ n \to \infty \text{ then } \lim_{n \to \infty} r^n = 0 .\ S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} .$$

SIGMA NOTATIONS **THEOREMS :**

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$.**(ii)** $\sum_{r=1}^{n} k\, a_r = k \sum_{r=1}^{n} a_r$.**(iii)** $\sum_{r=1}^{n} k = nk$; where k is a constant.

RESULTS **(i)** $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural nos.)

(ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(iii) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(iv) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2 + 3n - 1)$www.MathsBySuhag.com , www.TekoClasses.com

METHOD OF DIFFERENCE : If $T_1, T_2, T_3, ... , T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2$, ... constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.

Remember that to find the sum of n terms of a series each term of which is composed of r factors in AP, the first factors of several terms being in the same AP, we "write down the nth term, affix the next factor at the end, divide by the number of factors thus increased and by the common difference and add a constant. Determine the value of the constant by applying the initial conditions".