

22.

INDEFINITE INTEGRATION

1. INTRODUCTION

Integration is a reverse process of differentiation. The integral or primitive of a function $f(x)$ with respect to x is a differential function $\phi(x)$ such that the derivative of $\phi(x)$ with respect to x is the given function $f(x)$. It is expressed symbolically as $\int f(x)dx = \phi(x)$

Thus, $\int f(x)dx = \phi(x) \Leftrightarrow \frac{d}{dx}[\phi(x)] = f(x)$.

The process of finding the integral of a function is called Integration and the given function is called Integrand. Now, it is obvious that the operation of integration is the inverse operation of differentiation. Hence the integral of a function is also named as the anti-derivative of that function.

Further we observe that

$$\left. \begin{aligned} \frac{d}{dx}(x^2) &= 2x \\ \frac{d}{dx}(x^2 + 2) &= 2x \\ \frac{d}{dx}(x^2 + k) &= 2x \end{aligned} \right\} \Rightarrow \int 2x dx = x^2 + \text{constant}$$

So we always add a constant to the integral of function, which is called the constant of Integration. It is generally denoted by c . Due to the presence of this arbitrary constant such an integral is called an Indefinite Integral.

2. ELEMENTARY INTEGRATION

The following integrals are directly obtained from the derivatives of standard functions.

- (a) $\int 0 \cdot dx = c$
- (b) $\int 1 \cdot dx = x + c$
- (c) $\int k \cdot dx = kx + c (k \in \mathbb{R})$
- (d) $\int x^n dx = \frac{x^{n+1}}{n+1} + c (n \neq -1)$
- (e) $\int \frac{1}{x} dx = \log_e x + c$
- (f) $\int e^x dx = e^x + c$

$$(g) \int a^x dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$$

$$(h) \int \sin x dx = -\cos x + c$$

$$(i) \int \cos x dx = \sin x + c$$

3. BASIC THEOREMS OF INTEGRATION

If $f(x)$, $g(x)$ are two functions of a variable x and k is a constant, then

$$(a) \int k f(x) dx = k \int f(x) dx$$

$$(b) \int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

$$(c) \frac{d}{dx} \left(\int f(x) dx \right) = f(x)$$

$$(d) \int \left(\frac{d}{dx} f(x) \right) dx = f(x) + c$$

PLANCESS CONCEPTS

The results of integration are very different from differentiation. There is no standard formula for integration.

Always make sure to write the constant of integration. NEVER assume it as zero from your side.

Vaibhav Gupta (JEE 2009, AIR54)

Illustration 1: Evaluate: $\int \frac{1 - \sin x}{\cos^2 x} dx$

(JEE MAIN)

Sol: As we know, $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$ therefore we can split

$\int \frac{1 - \sin x}{\cos^2 x} dx$ as $\int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx$ and then by solving we can get result.

$$\int \frac{1 - \sin x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx = \int \sec^2 x dx - \int \tan x \sec x dx = \tan x - \sec x + c$$

Illustration 2: Evaluate: $\int \sqrt{1 + \sin 2x} dx$

(JEE MAIN)

Sol: Here $\sin^2 x + \cos^2 x = 1$ and $\sin 2x = 2 \sin x \cos x$, therefore by using these formulae and solving we will get the result.

$$\int \sqrt{1 + \sin 2x} dx = \int \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x} dx = \int \sqrt{(\sin x + \cos x)^2} dx$$

$$= \int (\sin x + \cos x) dx = \int \sin x dx + \int \cos x dx$$

$$= -\cos x + \sin x + c$$

Illustration 3: Evaluate: $\int \sin^4 x \, dx$

(JEE MAIN)

Sol: Here as we know, $\sin^2 x = \frac{1 - \cos 2x}{2}$, Now by putting this in the above integration and solving we will get the term $\frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx$, After that by using the formula

$\cos^2 2x = \frac{1 + \cos 4x}{2}$ we can solve the problem given above.

$$\begin{aligned} \int \sin^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx = \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx = \frac{1}{8} \int (3 - 4\cos 2x + \cos 4x) dx = \frac{1}{8} \left[3x - 2\sin 2x + \frac{\sin 4x}{4} \right] + C \end{aligned}$$

Illustration 4: If $f'(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2)=0$, then, find $f(x)$

(JEE ADVANCED)

Sol: Here $f'(x) = 4x^3 - \frac{3}{x^4}$ therefore $f(x) = \int \left(4x^3 - \frac{3}{x^4} \right) dx$ hence by splitting this integration and solving we will get the result.

$$\text{We have, } \frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4} \Rightarrow f(x) = \int \left(4x^3 - \frac{3}{x^4} \right) dx = \int 4x^3 dx - \int \frac{3}{x^4} dx = 4 \int x^3 dx - 3 \int x^{-4} dx$$

$$= 4 \frac{x^{3+1}}{3+1} - 3 \frac{x^{-4+1}}{-4+1} + C = x^4 + \frac{1}{x^3} + C \quad \dots(i)$$

$$\text{Given } f(2) = 2^4 + \frac{1}{2^3} + C = 0 \Rightarrow 0 = 16 + \frac{1}{8} + C \Rightarrow C = -\frac{129}{8}$$

$$\text{Putting the value of } C \text{ in (i), we get } f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$$

4. METHODS OF INTEGRATION

When the integrand can't be reduced into some standard form then integration is performed using following methods

4.1 Integration by Substitution

4.1.1 Integrand is a Function of Another Function

If the integral is of the form $\int f[\phi(x)] \phi'(x) dx$, then we put $\phi(x) = t$ so that $\phi'(x) dx = dt$. Now integral is reduced $\int f(t) dt$.

PLANCESS CONCEPTS

In this method the function is broken into two factors so that one factor can be expressed in terms of the function whose differential coefficient is the second factor.

In case of objective questions in which direct indefinite integration is asked, function being very complicated to integrate, then try differentiating the options.

If $I = \int \frac{dx}{\sin(x-a)\cos(x-b)}$, then I is Equal to

(a) $\frac{1}{\sin(a-b)} \log \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$

(b) $\frac{1}{\cos(a-b)} \log \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$

(c) $\frac{1}{\sin(a+b)} \log \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$

(d) $\frac{1}{\cos(a+b)} \log \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$

Vaibhav Krishnan JEE 2009, AIR 22

Illustration 5: Evaluate: $\int x \tan x^2 \sec x^2 dx$

(JEE MAIN)

Sol: This problem is based on integration using substitution method. In this we can put $x^2 = t$ and therefore $2x dx = dt$ and then solving we will get the result.

Let $x^2 = t$

$$\Rightarrow 2x dx = dt \Rightarrow x dx = \frac{1}{2} dt \quad \therefore \int x \tan x^2 \sec x^2 dx = \frac{1}{2} \int \tan t \sec t dt = \frac{1}{2} \sec t + c = \frac{1}{2} \sec x^2 + c$$

4.1.2 Integrand is the Product of Function and its Derivative

If the integral is of the form $I = \int f'(x) f(x) dx$ we put $f(x) = t$ and convert it into a standard integral.

Illustration 6: Evaluate: $\int \tan x \sec^2 x dx$

(JEE MAIN)

Sol: Here $\sec^2 x$ is a derivatives of $\tan x$ hence we can put $\tan x = t$ and $\sec^2 x \cdot dx = dt$ thereafter we can solve the given problem.

Let $\tan x = t \Rightarrow \sec^2 x \cdot dx = dt$

$$\therefore I = \int \tan x \sec^2 x dx = \int t dt = \frac{t^2}{2} + c = \frac{\tan^2 x}{2} + c$$

4.1.3 Integrand is a Function of the Form $f(ax+b)$

Here we put $ax+b = t$ and convert it into a standard integral. Now if,

$$\int f(x) dx = \phi(x), \text{ then } \int f(ax+b) dx = \frac{1}{a} \phi(ax+b)$$

Illustration 7: Evaluate: $\int \cos 3x \cos 5x dx$

(JEE MAIN)

Sol: By multiplying and dividing by 2 in the given integration and using the formula

$2 \cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$ we can solve it.

$$I = \int \cos 3x \cos 5x dx = \frac{1}{2} \int (\cos 8x + \cos 2x) dx = \frac{1}{2} \left[\frac{1}{8} \sin 8x + \frac{1}{2} \sin 2x \right] + c$$

Illustration 8: Evaluate: $I = \int \frac{x dx}{x^4 + x^2 + 1}$

(JEE ADVANCED)

Sol: Here by putting $x^2 = t \Rightarrow dt = 2x dx$ we will get the term $\frac{1}{2} \int \frac{dt}{t^2 + t + 1} = \frac{1}{2} \int \frac{dt}{(t + (1/2))^2 + (\sqrt{3}/2)^2}$ and then by putting $t + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta$, we can solve it.

$$\text{Let } x^2 = t \Rightarrow dt = 2x dx \quad \therefore I = \frac{1}{2} \int \frac{dt}{t^2 + t + 1} = \frac{1}{2} \int \frac{dt}{(t + (1/2))^2 + (\sqrt{3}/2)^2}$$

$$\int \frac{1}{[f(x)]^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left[\frac{f(x)}{a} \right] \times \frac{1}{f'(x)} + c \quad \therefore I = \frac{1}{2} \left[\frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \right] + c$$

$$= \frac{1}{\sqrt{3}} \left[\tan^{-1} \left(\frac{2t+1}{\sqrt{3}} \right) \right] + c = \frac{1}{\sqrt{3}} \left[\tan^{-1} \left(\frac{2x^2+1}{\sqrt{3}} \right) \right] + c$$

$$\text{Now put } t + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta \Rightarrow dt = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta$$

$$\therefore \frac{1}{2} \int \frac{(\sqrt{3}/2) \sec^2 \theta d\theta}{(3/4)(\tan^2 \theta + 1)} = \frac{1}{\sqrt{3}} \int d\theta = \frac{1}{\sqrt{3}} \theta + c = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2t+1}{\sqrt{3}} \right) + c = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x^2+1}{\sqrt{3}} \right) + c$$

Standard integration results

- (a) $\int \frac{f'(x)}{f(x)} dx = \log_e [f(x)] + c$
- (b) $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$ (provided $n \neq -1$)
- (c) $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

Illustration 9: Evaluate: $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

(JEE MAIN)

Sol: Here simply substituting $t = \tan x \Rightarrow dt = \sec^2 x dx$ we can solve it.

$$\text{Let } t = \tan x \Rightarrow dt = \sec^2 x dx$$

$$\therefore I = \int \frac{dt}{\sqrt{t}} = 2t^{\frac{1}{2}} + c = 2\sqrt{\tan x} + c$$

4.1.4 Integral of the Form

$\int \frac{dx}{a \sin x + b \cos x}$ then substitute $a = r \cos \theta$ and $b = r \sin \theta$, $\tan \theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1} \left(\frac{b}{a} \right)$, we get

$$I = \int \frac{dx}{r \sin(x + \theta)} = \frac{1}{r} \int \operatorname{cosec}(x + \theta) dx = \frac{1}{r} \log \tan \left(\frac{x + \theta}{2} \right) + c = \frac{\log \tan \left((x/2) + (1/2) \tan^{-1}(b/a) \right)}{\sqrt{a^2 + b^2}} + c$$

PLANCESS CONCEPTS

$$\int \sin^m x \cos^n x \, dx, \text{ where } m, n \in \mathbb{N}$$

$\Rightarrow$ If m is odd put $\cos x = t$

If n is odd put $\sin x = t$

If both m and n are odd, put $\sin x = t$ if $m \geq n$ and $\cos x = t$ otherwise.

If both m and n are even, use power reducing formulae

$$\sin^2 x = \frac{1 - \cos 2x}{2} \text{ or } \cos^2 x = \frac{1 + \cos 2x}{2}$$

If $m+n$ is a negative even integer, put $\tan x = t$

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 10: Evaluate: $\int \frac{1}{\sin x + \cos x} dx$

(JEE ADVANCED)

Sol: As we know, if integration is in the form of $\int \frac{dx}{a \sin x + b \cos x}$ then we can put

$a = r \cos \theta$ and $b = r \sin \theta$ hence the integration will be $\frac{1}{r} \log \tan \left(\frac{x + \theta}{2} \right) + c$.

Here $a=1$ & $b=1$

$$\text{So } \int \frac{1}{\sin x + \cos x} dx = \frac{1}{\sqrt{1+1}} \log \tan \left(\frac{x}{2} + \frac{1}{2} \tan^{-1} 1 \right) + c = \frac{1}{\sqrt{2}} \log \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) + c$$

4.1.5 Standard Substitutions

The following standard substitutions will be useful

Integrand form	Substitutions
$\sqrt{a^2 - x^2}$ or $\frac{1}{\sqrt{a^2 - x^2}}$	$x = a \sin \theta$ or $x = a \cos \theta$
$\sqrt{x^2 + a^2}$ or $\frac{1}{\sqrt{x^2 + a^2}}$	$x = a \tan \theta$ or $x = a \cot \theta$ or $x = a \sinh \theta$
$\sqrt{x^2 - a^2}$ or $\frac{1}{\sqrt{x^2 - a^2}}$	$x = a \sec \theta$ or $x = a \operatorname{cosec} \theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$ or $\sqrt{x(a+x)}$ or $\frac{1}{\sqrt{x(a+x)}}$	$x = a \tan^2 \theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$ or $\sqrt{x(a-x)}$ or $\frac{1}{\sqrt{x(a-x)}}$	$x = a \sin^2 \theta$ or $x = a \cos^2 \theta$

$\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}$ or $\sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$	$x = a \sec^2 \theta$ or $x = a \operatorname{cosec}^2 \theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ ($\beta > \alpha$)	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

Some Standard Integrals

- (a) $\int \tan x dx = \log \sec x + c = -\log \cos x + c$
- (b) $\int \cot x dx = \log \sin x + c = -\log \operatorname{cosec} x + c$
- (c) $\int \sec x dx = \log(\sec x + \tan x) + c = -\log(\sec x - \tan x) + c = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$
- (d) $\int \operatorname{cosec} x dx = -\log(\operatorname{cosec} x + \cot x) + c = \log(\operatorname{cosec} x - \cot x) + c = \log \tan\left(\frac{x}{2}\right) + c$
- (e) $\int \sec x \tan x dx = \sec x + c$
- (f) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$
- (g) $\int \sec^2 x dx = \tan x + c$
- (h) $\int \operatorname{cosec}^2 x dx = -\cot x + c$
- (i) $\int \log x dx = x \log\left(\frac{x}{e}\right) + c = x(\log x + 1) + c$

PLANCESS CONCEPTS

If the integral is of the form $\int R\left(x^{\frac{1}{p}}, x^{\frac{1}{q}}, x^{\frac{1}{r}}, \dots\right) dx$, where R is a rational function then,

Let $a = \text{lcm of } (p, q, r, \dots)$ and put $x = t^a$

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 11: Prove that: $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2}) + C$

(JEE ADVANCED)

Sol: By putting $x = a \sec \theta \Rightarrow dx = a \sec \theta \tan \theta d\theta$, we can solve the problem given above.

$$\begin{aligned} \text{Let } x &= a \sec \theta \Rightarrow dx = a \sec \theta \tan \theta d\theta \Rightarrow \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta d\theta}{a \tan \theta} = \int \sec \theta d\theta = \log(\sec \theta + \tan \theta) + C \\ &= \log\left(\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right) + C = \log(x + \sqrt{x^2 + a^2}) + C \end{aligned}$$

Illustration 12: Evaluate: $\int \cos 2x \cdot \cos 4x \cdot \cos 6x dx$ **(JEE MAIN)****Sol:** By multiplying and dividing by 2 in the given integration and then by using $2 \cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$ we can solve it.

$$\text{Let } I = \int \cos 2x \cdot \cos 4x \cdot \cos 6x dx = \frac{1}{2} \int (2 \cos 2x \cdot \cos 4x) \cos 6x dx = \frac{1}{2} \int (\cos 6x + \cos 2x) \cos 6x dx$$

$$[\because 2 \cos A \cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{2} \int (\cos^2 6x + \cos 6x \cdot \cos 2x) dx = \frac{1}{4} \int (2 \cos^2 6x + 2 \cos 6x \cdot \cos 2x) dx$$

$$= \frac{1}{4} \left[\int (1 + \cos 12x) + (\cos 8x + \cos 4x) dx \right] = \frac{1}{4} \int dx + \frac{1}{4} \int \cos 12x dx + \frac{1}{4} \int \cos 8x dx + \frac{1}{4} \int \cos 4x dx$$

$$\therefore \int \cos \left[f(x) \right] dx = \frac{\sin[f(x)]}{f'(x)} + C$$

$$I = \frac{1}{4} \left[x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} \right] + C$$

Illustration 13: Evaluate: $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$ **(JEE ADVANCED)****Sol:** Here in this problem by using the formulae

$$\cos C - \cos D = -2 \sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2}, \quad \sin 2A = 2 \sin A \cos A \quad \text{and} \quad 2 \cos C \cos D = \cos(C+D) + \cos(C-D)$$

We can solve the problem above step by step.

We have,

$$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx = \int \frac{-2 \sin(x+\alpha) \sin(x-\alpha)}{-2 \sin((x+\alpha)/2) \sin((x-\alpha)/2)} dx = \int \frac{\sin(x+\alpha) \sin(x-\alpha)}{\sin((x+\alpha)/2) \sin((x-\alpha)/2)} dx$$

$$= \int \frac{2 \sin((x+\alpha)/2) \cos((x+\alpha)/2) \cdot 2 \sin((x-\alpha)/2) \cos((x-\alpha)/2)}{\sin((x+\alpha)/2) \sin((x-\alpha)/2)} dx \quad (\sin 2A = 2 \sin A \cos A)$$

$$= 4 \int \cos\left(\frac{x+\alpha}{2}\right) \cos\left(\frac{x-\alpha}{2}\right) dx \quad \because [2 \cos C \cos D = \cos(C+D) + \cos(C-D)]$$

$$= 2 \int (\cos x + \cos \alpha) dx = 2 \int \cos x dx + 2 \cos \alpha \int dx = 2 \sin x + 2x \cos \alpha + C$$

Illustration 14: Evaluate: $\int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cdot \cos^2 x} dx$ **(JEE ADVANCED)****Sol:** Here by using the formula $a^2 - b^2 = (a+b)(a-b)$ and putting $(\sin^2 x + \cos^2 x)^2$ in place of 1 in the denominator, we can reduce the above integration and then using $\cos 2x = \cos^2 x - \sin^2 x$ we can solve it.

$$\text{We have, } \int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cdot \cos^2 x} dx = \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cdot \cos^2 x} dx$$

$$\begin{aligned}
 &= \int \frac{(\sin^4 x + \cos^4 x)(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{(\sin^4 x + \cos^4 x)} dx = \int 1 \cdot (\sin^2 x - \cos^2 x) dx = -\int \cos 2x dx \left[\because \cos 2x = \cos^2 x - \sin^2 x \right] \\
 &= -\frac{\sin 2x}{2} + C
 \end{aligned}$$

4.2 Integration by Parts

If u and v are two functions of x , then

$$\int (u \cdot v) dx = u \left(\int v dx \right) - \int \left(\frac{du}{dx} \right) \left(\int v dx \right) dx$$

This is also known as uv rule of integration. This method of integrating is called integration by parts.

PLANCESS CONCEPTS

- From the first letter of the words inverse circular, logarithmic, Algebraic, Trigonometric, Exponential functions, we get a word ILATE. Therefore the preference of selecting the u function will be according to the order ILATE.
- In some problems we have to give preference to logarithmic function over inverse trigonometric functions. Hence sometimes the word LIATE is used for reference.
- For the integration of Logarithmic or Inverse trigonometric functions alone, take unity (1) as the v function.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 15: Evaluate: $\int (1+x) \log x dx$

(JEE MAIN)

Sol: Here we can integrate the given problem by using Integration by parts i.e.

$$\int (u \cdot v) dx = u \left(\int v dx \right) - \int \left(\frac{du}{dx} \right) \left(\int v dx \right) dx$$

Here $u = \log x$ and $v = (1+x)$.

$$\text{Let } I = \int (1+x) \log x dx$$

Integrating by parts, taking $\log x$ as 1st function, (by LIATE rule) we get

$$\begin{aligned}
 I &= \log x \int (1+x) dx - \int \left[\frac{d}{dx} (\log x) \cdot \int (1+x) dx \right] dx = \log x \left(x + \frac{x^2}{2} \right) - \int \frac{1}{x} \left(x + \frac{x^2}{2} \right) dx = \left(x + \frac{x^2}{2} \right) \log x - \int \left(1 + \frac{x}{2} \right) dx \\
 &= \left(x + \frac{x^2}{2} \right) \log x - \left(x + \frac{x^2}{4} \right) + C
 \end{aligned}$$

Illustration 16: Evaluate: $\int \sec^3 x dx$

(JEE ADVANCED)

Sol: Here we can solve by integrating by parts, taking $\sec x$ as the first function.

$$I = \int \sec^3 x dx = \int \sec x \cdot \sec^2 x dx \quad \text{Let } u = \sec x \text{ \& } v = \sec^2 x$$

$$I = \sec x \tan x - \int (\sec x \tan x) \cdot \tan x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$I = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \Rightarrow I = \sec x + \tan x - I + \int \sec x dx$$

$$\Rightarrow 2I = \sec x \cdot \tan x + \log(\sec x + \tan x) + C \Rightarrow I = \frac{1}{2} [\sec x \tan x + \log(\sec x \tan x)] + C$$

Illustration 17: Evaluate : $\int (\sin^{-1} x)^2 dx$

(JEE ADVANCED)

Sol: We can write the given integration as $\int (\sin^{-1} x)^2 \cdot 1 dx$ and then taking $u = (\sin^{-1} x)^2$ & $v = 1$ solving by integration by parts.

$$I = (\sin^{-1} x)^2 \cdot x - \int \left\{ \frac{d}{dx} (\sin^{-1} x)^2 \cdot x \right\} dx = (\sin^{-1} x)^2 \cdot x - \left\{ 2(\sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}} \cdot x \right\} dx$$

Now, putting $\sin^{-1} x = t \Rightarrow x = \sin t$ so that $\frac{dx}{\sqrt{1-x^2}} = dt$

$$\Rightarrow I = x(\sin^{-1} x)^2 - 2 \int t \cdot \sin t dt = x(\sin^{-1} x)^2 - 2 \left\{ -t \cos t + \int \cos t dt \right\} \text{ (again Integrating by parts)}$$

$$= x(\sin^{-1} x)^2 - 2 \left\{ -t \cos t + \sin t \right\} + C = x(\sin^{-1} x)^2 + 2t \cos t - 2 \sin t + C = x(\sin^{-1} x)^2 + 2 \sin^{-1} x \cdot \sqrt{1-x^2} - 2x + C$$

Illustration 18: Evaluate: $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$

(JEE ADVANCED)

Sol: By using the formula $\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2}$, we can solve the above problem.

$$\text{Let } I = \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx = \int \frac{\sin^{-1} \sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1} \sqrt{x} \right)}{\frac{\pi}{2}} dx \left[\because \sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2} \right]$$

$$= \frac{2}{\pi} \int \left(2 \sin^{-1} \sqrt{x} - \frac{\pi}{2} \right) dx = \frac{4}{\pi} \int \sin^{-1} \sqrt{x} dx - \int 1 dx$$

$$= \frac{4}{\pi} \int \sin^{-1} \sqrt{x} dx - x \quad \dots (i)$$

Putting $\sin^{-1} \sqrt{x} = \theta \Rightarrow x = \sin^2 \theta$ so that $dx = 2 \sin \theta \cdot \cos \theta d\theta = \sin 2\theta d\theta$.

$\therefore \int \sin^{-1} \sqrt{x} dx = \int \theta \cdot \sin 2\theta d\theta$ Let $u = \theta$ & $V = \sin 2\theta$, then integrating by parts we get

$$= \frac{-\theta}{2} (1 - 2 \sin^2 \theta) + \frac{1}{4} 2 \sin \theta \cdot \cos \theta$$

$$= -\theta \cdot \frac{\cos 2\theta}{2} + \frac{1}{2} \int \cos 2\theta d\theta = -\frac{\theta}{2} \cos 2\theta + \frac{1}{4} \sin 2\theta = -\frac{1}{2} \theta (1 - 2 \sin^2 \theta) + \frac{1}{2} \sin \theta \sqrt{1 - \sin^2 \theta}$$

$$= -\frac{1}{2} (\sin^{-1} \sqrt{x}) (1 - 2x) + \frac{1}{2} \sqrt{x} \cdot \sqrt{1-x} + C \quad \dots (ii)$$

From (i) and (ii), we get

$$I = \frac{4}{\pi} \left\{ -\frac{1}{2} (1 - 2x) \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x-x^2} \right\} - x + C = \frac{2}{\pi} \left\{ \sqrt{x-x^2} - (1 - 2x) \sin^{-1} \sqrt{x} \right\} - x + C$$

4.2.1 Integration by Cancellation

Illustration 19: Evaluate : $\int \left(3x^2 \tan \frac{1}{x} - x \sec^2 \frac{1}{x} \right) dx$

(JEE MAIN)

Sol: Let $\int \left(3x^2 \tan \frac{1}{x} - x \sec^2 \frac{1}{x} \right) dx = \int 3x^2 \tan \frac{1}{x} dx - \int x \sec^2 \frac{1}{x} dx$ and then by using the integration by parts formula

i.e. $\int (u.v) dx = u \left(\int v dx \right) - \int \left(\frac{du}{dx} \right) \left(\int v dx \right) dx$ we can solve the problem above.

$$\int \left(3x^2 \tan \frac{1}{x} - x \sec^2 \frac{1}{x} \right) dx = \int 3x^2 \tan \frac{1}{x} dx - \int x \sec^2 \frac{1}{x} dx = \tan \frac{1}{x} x^3 - \int \left(\sec^2 \frac{1}{x} \right) \left(-\frac{1}{x^2} \right) x^3 dx - \int x \sec^2 \frac{1}{x} dx = x^3 \tan \frac{1}{x} + c$$

4.2.2 Integration of the Form:

If the integral is of the form $\int e^x [f(x) + f'(x)] dx$, then use the formula;

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$$

Illustration 20: Evaluate: $\int e^x (\log x + 1/x) dx$

(JEE MAIN)

Sol: Solution of this problem is based on the method mentioned above, here $f(x) = \log x$ and $f'(x)$

$$= 1/x. \quad = \int e^x \left[\log x + \frac{1}{x} \right] dx$$

$$I = \int \left(e^x \log x + \frac{e^x}{x} \right) dx = e^x \log x + c \quad \left[\begin{array}{l} \text{Here, } f(x) = \log x \\ \& f'(x) = 1/x \end{array} \right]$$

If the integral is of the form $\int [xf'(x) + f(x)] dx$ then use the formula; $\int [xf'(x) + f(x)] dx = xf(x) + c$

Illustration 21: Evaluate : $\int (x \sec^2 x + \tan x) dx$

(JEE MAIN)

Sol: Similar to the problem above.

$$\text{Here } I = \int (x \sec^2 x + \tan x) dx = \int [xf'(x) + f(x)] dx \quad [\text{where } f(x) = \tan x] = x \cdot \tan x + c$$

4.2.3 Special Integrals

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + c$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + c$$

Illustration 22: Evaluate : $\int e^{\sin^{-1} x} dx$

(JEE MAIN)

Sol: By putting $\sin^{-1} x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t dt$ and then integrating by parts we can solve the given problem.

$$I = \int e^{\sin^{-1} x} dx$$

$$\text{Let } \sin^{-1} x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t dt$$

$$\Rightarrow I = \int e^t \cos t dt = \frac{e^t}{2} (\sin t + \cos t) + c = \frac{e^{\sin^{-1} x}}{2} (x + \sqrt{1-x^2}) + c$$

4.3 Integration of Rational Functions

4.3.1 When the Denominator can be Factorized (Using Partial Fraction)

Let the integrand be of the form $\frac{f(x)}{g(x)}$, where both $f(x)$ and $g(x)$ are polynomials. If degree of $f(x)$ is greater than degree of $g(x)$, then first divide $f(x)$ by $g(x)$ till the degree of the remainder becomes less than the degree of $g(x)$. Let $Q(x)$ be the quotient and $R(x)$, the remainder then

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$$

Now in $R(x)/g(x)$, factorize $g(x)$ and then write partial fractions in the following manner:

(a) For every non-repeated linear factor in the denominator. Write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

(b) For every repeated linear factor in the denominator. Write

$$\frac{1}{(x-a)^3(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$$

(c) For every non-repeated quadratic factor in the denominator. Write

$$\frac{1}{(ax^2+bx+c)(x-d)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x-d}$$

(d) For every repeated quadratic factor in the denominator. Write

$$\frac{1}{(ax^2+bx+c)^2(x-d)} = \frac{Ax+B}{(ax^2+bx+c)^2} + \frac{Cx+D}{ax^2+bx+c} + \frac{E}{x-d}$$

PLANCESS CONCEPTS

Consider $f(x)$ as the function we need to factorize

1. For non- repeated linear factor in the denominator.

$$\text{Let } f(x) = \frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

To obtain the value of A remove $(x-a)$ from $f(x)$ and find $f(a)$.

Similarly, to obtain value of B , remove $(x-b)$ from $f(x)$ and find $f(b)$.

2. For repeated linear factor in the denominator.

$$\text{Let } f(x) = \frac{1}{(x-a)^3(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$$

To obtain value of D remove $(x-b)$ from $f(x)$ and find $f(b)$.

To obtain value of c remove $(x-a)^3$ from $f(x)$ and find $f(a)$.

Now that we have reduced the number of unknowns from 4 to 2, we can find A and B easily by equating.

Now let's try this method for $\frac{x^4 + x^3 + 2x^2 - x + 4}{x(x^2 + 2)(x^2 + 1)^3}$

Partial fraction will be of the form

$$\frac{x^4 + x^3 + 2x^2 - x + 4}{x(x^2 + 2)(x^2 + 1)^3} = \frac{A}{x} + \frac{Bx + C}{(x^2 + 2)} + \frac{Dx + E}{(x^2 + 1)} + \frac{Fx + G}{(x^2 + 1)^2} + \frac{Hx + I}{(x^2 + 1)^3}$$

Now remove x and put x=0, we get A=2

Now remove $(x^2 + 1)^3$ and put $x^2 = -1$ i.e. $x = i$ (you can also substitute $x = -i$).

We get $Hi + I = -3i - 2$. Hence $H = -3$ and $I = -2$.

Now remove $(x^2 + 2)$ and put $x = \sqrt{2}i$. We get $B(\sqrt{2}i) + C = 2\sqrt{2}i + 3$. Hence $B = 2$ and $C = 3$

Now the number of unknowns have reduced from 9 to 4 and the remaining unknowns can be solved easily.

This method very useful instead of solving for all the unknowns at the same time.

Also remember that substituting an imaginary number for x is not discussed anywhere in NCERT. So, use this method only for competitive exams.

Ravi Vooda (JEE 2009, AIR 71)

Illustration 23: Evaluate : $\int \frac{x}{x^2 - x - 2} dx$

(JEE MAIN)

Sol: Here the given integration is in the form of $\frac{1}{(x-a)(x-b)}$, hence by using partial fractions we can split it as $\frac{A}{(x-a)} + \frac{B}{(x-b)}$ and then by solving we will get the required result.

$$\text{Here } I = \int \frac{x}{(x-2)(x+1)} dx = \int \frac{1}{3} \left(\frac{2}{x-2} + \frac{1}{x+1} \right) dx = \frac{1}{3} [2 \log(x-2) + \log(x+1)] + c = \frac{1}{3} \log[(x-2)^2(x+1)] + c$$

Illustration 24: Evaluate : $\int \frac{x dx}{3x^4 - 18x^2 + 11}$

(JEE ADVANCED)

Sol: Here simply by putting $t = x^2 \Rightarrow dt = 2x dx$ and then by using partial fractions we can solve the given problem.

$$\begin{aligned} I &= \int \frac{x dx}{3x^4 - 18x^2 + 11} = \int \frac{\frac{1}{2} dt}{3t^2 - 18t + 11} \quad (\text{Put } t = x^2 \Rightarrow dt = 2x dx) \\ &= \frac{1}{6} \int \frac{dt}{t^2 - 6t + (11/3)} = \frac{1}{6} \int \frac{dt}{(t-3)^2 - (16/3)} = \frac{1}{6} \int \frac{dt}{(t-3)^2 - (4/\sqrt{3})^2} \\ &= \frac{1}{6} \frac{1}{2 \times (4/\sqrt{3})} \log \frac{(t-3) - (4/\sqrt{3})}{(t-3) + (4/\sqrt{3})} + C = \frac{\sqrt{3}}{48} \log \frac{\sqrt{3}t - 3\sqrt{3} - 4}{\sqrt{3}t - 3\sqrt{3} + 4} + C = \frac{\sqrt{3}}{48} \log \frac{\sqrt{3}x^2 - 3\sqrt{3} - 4}{\sqrt{3}x^2 - 3\sqrt{3} + 4} + C \end{aligned}$$

4.3.2 When the Denominator cannot be Factorized

In this case the integral may be in the form

$$(i) \int \frac{dx}{ax^2 + bx + c} \quad (ii) \int \frac{(px + q)}{ax^2 + bx + c} dx$$

Method:

(i) Here taking the coefficient of x^2 common from the denominator, write

$$x^2 + (b/a)x + c/a = (x + b/2a)^2 - \frac{b^2 - 4ac}{4a^2}$$

Now the integrand obtained can be evaluated easily by using standard formulae.

$$(ii) \text{ Here suppose that } px + q = A \left[\frac{d}{dx}(ax^2 + bx + c) \right] + B = A(2ax + b) + B \quad \dots(i)$$

Now comparing coefficient of x and constant terms.

$$\text{We get } A = p/2a, B = q - (pb/2a)$$

$$\therefore I = \frac{p}{2a} \int \frac{2ax + b}{ax^2 + bx + c} dx + \left(q - \frac{pb}{2a} \right) \int \frac{dx}{ax^2 + bx + c}$$

Now we can integrate it easily.

4.3.3 Integrand Containing Only Even Powers of x

To find integral of such functions, first we divide numerator and denominator by x^2 , then express the numerator as $d(x \pm 1/x)$ and the denominator as a function of $(x \pm 1/x)$. The following examples illustrate it.

PLANCESS CONCEPTS

$\int R(\sin x, \cos x) dx$ where R is a rational function (universal substitution $\tan(x/2) = t$)

Special cases:

(a) If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

Put $\cos x = t$

(b) If $R(\sin x, \cos x) = -R(\sin x, -\cos x)$

Put $\sin x = t$

(c) If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

Put $\tan x = t$

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 25: Evaluate: $\int \frac{x^2 + 1}{x^4 + 1} dx$

(JEE ADVANCED)

Sol: Here dividing the numerator and denominator by x^2 , we get $\int \frac{1 + (1/x^2)}{x^2 + (1/x^2)} dx = \int \frac{1 + (1/x^2)}{\left[x - (1/x) \right]^2 + 2} dx$ and then by putting $x - 1/x = t \Rightarrow [1 + 1/x^2] dx = dt$, we can solve it.

$$I = \int \frac{1 + (1/x^2)}{x^2 + (1/x^2)} dx = \int \frac{1 + (1/x^2)}{\left\{x - (1/x)\right\}^2 + 2} dx$$

Now taking $x - 1/x = t \Rightarrow [1 + 1/x^2] dx = dt$, we get

$$I = \int \frac{dt}{t^2 + 2} = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) + c = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + c$$

4.4 Integration of Irrational Functions

If any one term in numerator or denominator is irrational then it is made rational by a suitable substitution. Also if the integral is of the form

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} \text{ or } \int \sqrt{ax^2 + bx + c} dx$$

Then we integrate it by expressing $ax^2 + bx + c = (x + \alpha)^2 + \beta^2$

Also for integrals of the form $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$ or $\int (px + q) \sqrt{ax^2 + bx + c} dx$

First we express $px + q$ in the form

$px + q = A \left[\frac{d}{dx} (ax^2 + bx + c) \right] + B$ and then proceed as usual with standard form.

Illustration 26: Evaluate : $\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx$

(JEE MAIN)

Sol: Simply by putting $e^x = t$, then $e^x dx = dt$, we can solve the given problem.

Put $e^x = t$, then $e^x dx = dt$

$$\begin{aligned} \therefore \int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx &= \int \frac{dt}{\sqrt{5 - 4t - t^2}} = \int \frac{dt}{\sqrt{5 - (t^2 + 4t)}} = \int \frac{dt}{\sqrt{5 - (t^2 + 4t + 4) + 4}} = \int \frac{dt}{\sqrt{9 - (t + 2)^2}} \\ &= \int \frac{dt}{\sqrt{(3)^2 - (t + 2)^2}} = \sin^{-1} \left(\frac{t + 2}{3} \right) + C = \sin^{-1} \left(\frac{e^x + 2}{3} \right) + C \end{aligned}$$

Illustration 27: Evaluate : $\int \frac{1}{\sqrt{(x-a)(x-b)}} dx$

(JEE ADVANCED)

Sol: Here first expand $(x-a)(x-b)$ and then adding and subtracting by $\left(\frac{a+b}{2}\right)^2$, we can reduce the above integration. After that by putting $x - \left(\frac{a+b}{2}\right) = u$, we can solve the given problem.

$$\begin{aligned} \text{Let, } I &= \int \frac{1}{\sqrt{(x-a)(x-b)}} dx = \int \frac{1}{\sqrt{x^2 - (a+b)x + ab}} dx = \int \frac{dx}{\sqrt{x^2 - (a+b)x + \left(\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab}} \\ &= \int \frac{dx}{\sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left[\left(\frac{a^2 + b^2 + 2ab}{4}\right) - ab\right]}} = \int \frac{dx}{\sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left[\frac{(a-b)^2}{4}\right]}} \\ &= \int \frac{dx}{\sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2}} \end{aligned}$$

... (i)

On putting $x - \left(\frac{a+b}{2}\right) = u$ so that $dx = du$ in (i), we get

$$I = \int \frac{du}{\sqrt{u^2 - \left(\frac{a-b}{2}\right)^2}} \left[\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log \left| x + \sqrt{x^2 - a^2} \right| \right] = \log \left| u + \sqrt{u^2 - \left(\frac{a-b}{2}\right)^2} \right| + c$$

Putting $u = \left(x - \left(\frac{a+b}{2}\right)\right)$, we get

$$I = \log \left| \left(x - \left(\frac{a+b}{2}\right)\right) + \sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2} \right| + c = \log \left| x - \frac{a+b}{2} + \sqrt{(x-a)(x-b)} \right| + c$$

5. STANDARD INTEGRALS

$$(a) \quad \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$(b) \quad \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$(c) \quad \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$(d) \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c = -\cos^{-1} \left(\frac{x}{a} \right) + c$$

$$(e) \quad \int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1} \left(\frac{x}{a} \right) + c = \log \left(x + \sqrt{x^2 + a^2} \right) + c$$

$$(f) \quad \int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \left(\frac{x}{a} \right) + c = \log \left(x + \sqrt{x^2 - a^2} \right) + c$$

$$(g) \quad \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c \quad (\text{Substitute } x = a \cos \theta \text{ or } x = a \sin \theta \text{ and proceed})$$

$$(h) \quad \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + c \quad (\text{Substitute } x = a \tan \theta \text{ or } x = a \cot \theta \text{ and proceed})$$

$$(i) \quad \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + c \quad (\text{Substitute } x = a \sec \theta \text{ or } x = a \operatorname{cosec} \theta \text{ and proceed})$$

$$(j) \quad \int \frac{1}{x \sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c \quad (\text{Valid for } x > a > 0)$$

$$(k) \quad \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin \left\{ bx - \tan^{-1} \left(\frac{b}{a} \right) \right\} + c$$

$$(I) \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos \left\{ bx - \tan^{-1} \frac{b}{a} \right\} + c$$

Integration of irrational algebraic functions:

Type 1: (a) $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$ (Put : $x = a \cos^2 q + b \sin^2 q$)

(b) $\int \frac{dx}{(x-\alpha)\sqrt{(x-\beta)}}$ (Put : $x = a \sec^2 q - b \tan^2 q$)

Type 2: $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ (Put: $px+q = t^2$)

Type 3: $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ (Put: $ax+b = \frac{1}{t}$)

Type 4: $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ (Put: $px+q = t^2$)

Type 5: $\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$

Case I: When (ax^2+bx+c) breaks up into two linear factors, e.g.

$$I = \int \frac{dx}{(x^2-x-2)\sqrt{x^2+x+1}} \text{ then } = \int \left(\frac{A}{x-2} + \frac{B}{x+1} \right) \frac{1}{\sqrt{x^2+x+1}} dx = A \int \frac{dx}{(x-2)\sqrt{x^2+x+1}} + B \int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

$$\text{Put } x-2 = \frac{1}{t} \quad \text{Put } x+1 = \frac{1}{t}$$

Case II: If ax^2+bx+c is a perfect square say $(lx+m)^2$, then put $lx+m = \frac{1}{t}$

Case III: If $b = 2, q = 0$

e.g. $\int \frac{dx}{(ax^2+b)\sqrt{px^2+r}}$ then, put $x = \frac{1}{t}$ or trigonometric substitutions are also helpful.

Integral of the form $\int \frac{dx}{P\sqrt{Q}}$, where P, Q are linear or quadratic functions of x.

Integral

$$\int \frac{1}{(ax+b)\sqrt{cx+d}} dx$$

$$\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$$

$$\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$$

$$\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$$

Substitutions

$$cx+d = z^2$$

$$px+q = z^2$$

$$px+q = \frac{1}{z}$$

$$x = \frac{1}{z}$$

$$\int \frac{dx}{(ax+b)^m \sqrt{ax^2+bx+c}}$$

$$ax+b=1/t$$

Illustration 28: Evaluate : $\int \frac{dx}{(x+1)\sqrt{x-2}}$

(JEE MAIN)

Sol: Simply by putting $x-2=t^2$, $\therefore dx=2t dt$ we can solve the given problem by using the appropriate formula.

$$\int \frac{dx}{(x+1)\sqrt{x-2}}$$

Put $x-2=t^2$

$\therefore dx=2t dt$

$$\therefore I = \int \frac{2t dt}{(t^2+3)t} = 2 \int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + c = \frac{2}{\sqrt{3}} \tan^{-1} \left(\sqrt{\frac{x-2}{3}} \right) + c \quad (\because t = \sqrt{x-2})$$

Illustration 29: Evaluate : $\int \frac{dx}{(x^2-4)\sqrt{x}}$

(JEE MAIN)

Sol: Here first put $x=t^2$ therefore $dx=2t dt$ and then using partial fractions we reduce the given integration in standard form. After that by solving we will get the result.

Let $I = \int \frac{dx}{(x^2-4)\sqrt{x}}$

Put $x=t^2 \therefore dx=2t dt$ then $I = \int \frac{2t}{(t^4-4)t} dt = 2 \int \frac{dt}{(t^2+2)(t^2-2)}$

Put $t^2=z \therefore \frac{1}{(t^2+2)(t^2-2)} = \frac{1}{(z+2)(z-2)} = \frac{A}{z+2} + \frac{B}{z-2}$

$A = -\frac{1}{4}$ and $B = \frac{1}{4} \Rightarrow \frac{1}{(t^2+2)(t^2-2)} = \frac{1}{4(t^2+2)} + \frac{1}{4(t^2-2)}$

$$\therefore I = 2 \int \frac{1}{(t^2+2)(t^2-2)} = -\frac{1}{2} \int \frac{dt}{t^2+2} + \frac{1}{2} \int \frac{dt}{t^2-2} = -\frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) + \frac{1}{4\sqrt{2}} \log \left| \frac{t-\sqrt{2}}{t+\sqrt{2}} \right| + c$$

$$= -\frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x}{2} \right) + \frac{1}{4\sqrt{2}} \log \left| \frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}} \right| + c \quad (\because t = \sqrt{x})$$

6. SPECIAL TRIGONOMETRIC FUNCTIONS

Here we shall study the methods for evaluation of the following types of integrals.

Type 1

(i) $\int \frac{dx}{a+b\sin^2 x}$ (ii) $\int \frac{dx}{a+b\cos^2 x}$ (iii) $\int \frac{dx}{a\cos^2 x + b\sin x \cos x + c\sin^2 x}$ (iv) $\int \frac{dx}{(a\sin x + b\cos x)^2}$

Method: Divide the numerator and denominator by $\cos^2 x$ in all such types of integrals and then put $\tan x=t$

Illustration 30: Evaluate : $\int \frac{dx}{1+3\sin^2 x}$

(JEE MAIN)

Sol: Here dividing the numerator and denominator by $\cos^2 x$ we can solve it.

$$I = \int \frac{\sec^2 x dx}{\sec^2 x + 3 \tan^2 x} = \int \frac{\sec^2 x}{1 + 4 \tan^2 x} dx = \frac{1}{2} \tan^{-1}(2 \tan x) + c$$

Type 2

$$(i) \int \frac{dx}{a+b\cos x} \quad (ii) \int \frac{dx}{a+b\sin x} \quad (iii) \int \frac{dx}{a\cos x+b\sin x} \quad (iv) \int \frac{dx}{a\sin x+b\cos x+c}$$

Method: In such types of integrals we use the following substitutions

$$\sin x = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)} = \frac{2t}{1+t^2}, \quad \cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \frac{1-t^2}{1+t^2}; \quad dx = \frac{2dt}{1+t^2}$$

and integrate another method for the evaluation of the integral.

Illustration 31: Evaluate: $\int \frac{dx}{5+4\cos x}$

(JEE MAIN)

Sol: Here by putting $\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$ and then by taking $\tan(x/2) = t$ we can solve the given problem

$$I = \int \frac{dx}{5 + 4 \left[\frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} \right]} = \int \frac{\sec^2(x/2) dx}{9 + \tan^2(x/2)} = 2 \int \frac{dt}{3^2 + t^2} \quad \text{where } \tan(x/2) = t$$

$$= 2 \left(\frac{1}{3} \tan^{-1} \left(\frac{t}{3} \right) \right) + C = 2 \left[\frac{1}{3} \tan^{-1} \left(\frac{\tan x/2}{3} \right) \right] + C$$

Type 3

$$(i) \int \frac{p\sin x + q\cos x}{a\sin x + b\cos x} dx \quad (ii) \int \frac{p\sin x}{a\sin x + b\cos x} dx \quad (iii) \int \frac{q\cos x}{a\sin x + b\cos x} dx$$

For their integration, we first express numerator as follows-

Numerator = A (denominator) + B (derivative of denominator)

Then integral = Ax + B log (denominator) + C

Illustration 32: Evaluate : $\int \frac{6+3\sin x+14\cos x}{3+4\sin x+5\cos x} dx$

(JEE ADVANCED)

Sol: By using partial fractions, we can reduce the given integration to the standard form.

$$\int \frac{6+3\sin x+14\cos x}{3+4\sin x+5\cos x} dx$$

$$\Rightarrow 6+3\sin x+14\cos x = A(3+4\sin x+5\cos x) + B(4\cos x-5\sin x) + c$$

Solving R.H.S. & comparing both sides, we get $4A - 5B = 3$ $5A + 4B = 14$

Also, $3A+C=6$ $\therefore \int \frac{A(3+4\sin x+5\cos x)+B(4\cos x-5\sin x)+c}{3+4\sin x+5\cos x}$

$$\Rightarrow Ax + \log(3 + 4\sin x + 5\cos x) + \int \frac{C \, dx}{\underbrace{3 + 4\sin x + 5\cos x}_{\text{this is of type 2}}}$$

Illustration 33: Evaluate : $\int \frac{2\sin 2\phi - \cos \phi}{6 - \cos^2 \phi - 4\sin \phi} d\phi$

(JEE ADVANCED)

Sol: Here we can write the given integration as $\int \frac{2(\sin 2\phi - 4\cos \phi) + 7\cos \phi}{6 - \cos^2 \phi - 4\sin \phi} d\phi$ and as we know $2(\sin 2\phi - 4\cos \phi)$ is the derivative of $6 - \cos^2 \phi - 4\sin \phi$ hence by putting $6 - \cos^2 \phi - 4\sin \phi = t$, we can solve the given problem.

$$\begin{aligned} I &= \int \frac{2\sin 2\phi - \cos \phi}{6 - \cos^2 \phi - 4\sin \phi} d\phi = \frac{d}{d\phi}(6 - \cos^2 \phi - 4\sin \phi) \\ &= 2\cos \phi \sin \phi - 4\cos \phi = \sin 2\phi - 4\cos \phi = 2\sin 2\phi - \cos \phi = 2(\sin 2\phi - 4\cos \phi) + 7\cos \phi \\ I &= \int \frac{2(\sin 2\phi - 4\cos \phi) + 7\cos \phi}{6 - \cos^2 \phi - 4\sin \phi} d\phi = \int \frac{2(\sin 2\phi - 4\cos \phi)d\phi}{6 - \cos^2 \phi - 4\sin \phi} + \int \frac{7\cos \phi d\phi}{6 - \cos^2 \phi - 4\sin \phi} = 2\int \frac{dt}{t} + \int \frac{7\cos \phi d\phi}{6 - (1 - \sin^2 \phi) - 4\sin \phi} \\ &= 2\log t + C_1 + \int \frac{7\cos \phi d\phi}{5 + \sin^2 \phi - 4\sin \phi} = 2\log(6 - \cos^2 \phi - 4\sin \phi) + C_1 + \int \frac{7dx}{x^2 - 4x + 5} \quad (\sin \phi = x) \\ &= 2\log(6 - \cos^2 \phi - 4\sin \phi) + C_1 + \int \frac{7dx}{(x-2)^2 + 1} = 2\log(6 - \cos^2 \phi - 4\sin \phi) + C_1 + 7\tan^{-1} \frac{x-2}{1} + C_2 \\ &= 2\log(6 - \cos^2 \phi - 4\sin \phi) + 7\tan^{-1}(\sin \phi - 2) + C \end{aligned}$$

7. SPECIAL EXPONENTIAL FUNCTIONS

- (a) $\int \frac{ae^x}{b+ce^x} dx$ [put $e^x = t$]
- (b) $\int \frac{1}{1+e^x} dx$ [Multiply and divide by e^{-x} and $e^{-x} = t$]
- (c) $\int \frac{1}{1-e^x} dx$ [Multiply and divide by e^{-x} and $e^{-x} = t$]
- (d) $\int \frac{1}{e^x - e^{-x}} dx$ [Multiply and divide by e^x]
- (e) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$ $\left[\frac{f'(x)}{f(x)} \text{ form} \right]$
- (f) $\int \frac{e^x + 1}{e^x - 1} dx$ [Multiply and divide by $e^{-x/2}$]
- (g) $\int \frac{1}{(1+e^x)(1-e^{-x})} dx$ [Multiply and divide by e^x and put $e^x = t$]
- (h) $\int \frac{1}{\sqrt{1-e^x}} dx$ [Multiply and divide by $e^{-x/2}$]

- (i) $\int \frac{1}{\sqrt{1+e^x}} dx$ [Multiply and divide by $e^{-x/2}$]
- (j) $\int \frac{1}{\sqrt{e^x-1}} dx$ [Multiply and divide by $e^{-x/2}$]
- (k) $\int \frac{1}{\sqrt{2e^x-1}} dx$ [Multiply and divide by $\sqrt{2}e^{-x/2}$]
- (l) $\int \sqrt{1-e^x} dx$ [Integrand = $(1-e^x)/\sqrt{1-e^x}$]
- (m) $\int \sqrt{1+e^x} dx$ [Integrand = $(1+e^x)/\sqrt{1+e^x}$]
- (n) $\int \sqrt{e^x-1} dx$ [Integrand = $(e^x-1)/\sqrt{e^x-1}$]
- (o) $\int \sqrt{\frac{e^x+a}{e^x-a}} dx$ [Integrand = $(e^x+a)/\sqrt{e^{2x}-a^2}$]

Illustration 34: Evaluate : $\int \sqrt{e^x-1} dx$

(JEE MAIN)

Sol: Here by multiplying and dividing by $\sqrt{e^x-1}$ in the given integration and then by putting $e^x-1 = t^2$ we can evaluate the given integration.

$$\text{Here } I = \int \sqrt{e^x-1} dx = \int \frac{e^x-1}{\sqrt{e^x-1}} dx = \int \frac{e^x}{\sqrt{e^x-1}} dx - \int \frac{1}{\sqrt{e^x-1}} dx$$

Let $e^x-1 = t^2$, then $e^x = dx = 2t \quad dt$

$$\therefore I = 2 \int dt - \int \frac{2}{t^2+1} dt = 2t - 2 \tan^{-1}(t) + c = 2 \left[\sqrt{e^x-1} - \tan^{-1} \sqrt{e^x-1} \right] + c$$

Illustration 35: Evaluate : $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$

(JEE MAIN)

Sol: We have, $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$

Put $e^x = t$, then $e^x dx = dt$

$$\begin{aligned} \therefore \int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx &= \int \frac{dt}{\sqrt{5-4t-t^2}} \\ &= \int \frac{dt}{\sqrt{5-(t^2+4t)}} = \int \frac{dt}{\sqrt{5-(t^2+4t+4)+4}} \\ &= \int \frac{dt}{\sqrt{9-(t+2)^2}} = \int \frac{dt}{\sqrt{(3)^2-(t+2)^2}} = \sin^{-1} \left(\frac{t+2}{3} \right) + C = \sin^{-1} \left(\frac{e^x+2}{3} \right) + C \end{aligned}$$

PROBLEM-SOLVING TACTICS

Integration by Parts

- (a) Integration by parts is useful for dealing with integrals of the products of the following functions
 $u \ll \tan^{-1} x, \sin^{-1} x, \cos^{-1} x, (\log x)^k, \sin x, \cos x, e^x \gg dv$
 Priority for choosing u and dv : ILATE
- (b) Integration by parts is sometimes useful for finding integrals of functions involving inverse functions such as $\ln x$ and $\sin^{-1} x$.
- (c) Sometimes when dealing with integrals, the integrand involves inverse functions (like $\sin^{-1} x$), it is useful to substitute $x =$ the inverse of that inverse function (like $x = \sin u$), then do integration by parts.
- (d) Sometimes you will have to do integration by parts more than once (for example, $\int x^2 e^x dx$ and $\int x^3 \sin x dx$).
 Sometimes you need to do it twice by parts, then manipulate the equation (for example, $\int e^x \sin x dx$).
- (e) Try u – substitution first before integration by parts.

Trigonometric Integral

- (a) Integral Type : $\int \sin^m x \cos^n x dx$

Case 1: One of m or n are even, and the other odd

Use u – substitution by setting $u = \sin$ or $\cos$ that with an even power. Use the identity $\sin^2 x + \cos^2 x = 1$.

Case 2: Both m and n are odd

Use u – substitution by setting $u = \sin$ or $\cos$ that with a higher power. Use the identity $\sin^2 x + \cos^2 x = 1$.

Case 3: Both m and n are even (hard case)

Do not use u – substitution. Use the half double angle formula to reduce the integrand into case 1 or 2:

$$\sin x \cos x = \frac{1}{2} \sin 2x; \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x); \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

(Note: 0 is also an even number. For example, $\sin^3 x = \sin^3 x \cos^0 x$, so it is in case 1)

Just remember that when both are even, you can't use u -substitution, but you can use the half – double angle formula. When it is not that case, let $u = \sin x$ or $\cos x$, and one will work (at the end there is no square root term after substitution).

- (b) Integral type : $\int \tan^m x \sec^n x dx$

Case 1: $\sec$ is odd power, $\tan$ is even power.

Hard to do, we omit (most likely won't pop out in the exam).

Case 2: Else

Set $u = \sec x$ or $\tan x$, and use $1 + \tan^2 x = \sec^2 x$. One will work at the end (there is no square root term after substitution).

- (c) Integral type : $\int \sin(Ax) \cos(Bx) dx, \int \cos(Ax) \cos(Bx) dx, \int \sin(Ax) \sin(Bx) dx$

Use the product to sum formula:

$$\cos \theta \cos \phi = \frac{1}{2}(\cos(\theta - \phi) + \cos(\theta + \phi)); \quad \sin \theta \cos \phi = \frac{1}{2}(\sin(\theta - \phi) + \sin(\theta + \phi))$$

$$\sin \theta \sin \phi = \frac{1}{2}(\sin(\theta - \phi) - \sin(\theta + \phi))$$

Reduce product into sum and then integrate.

Trigonometric Substitution

(a) Trigonometric substitution is useful for quadratic form with square root:

$$\sqrt{a^2 - x^2} : \text{Let } x = a \sin \theta$$

$$\sqrt{x^2 + a^2} : \text{Let } x = a \tan \theta$$

$$\sqrt{x^2 - a^2} : \text{Let } x = a \sec \theta$$

(b) General procedure for doing trig sub:

Step 1: Draw the right triangle, and decide what trigonometric function to substitute for x .

Step 2: Find dx , then substitute the integrand using triangle, convert integral into trigonometric integral.

Step 3: Solve the trigonometric integral.

Step 4: Substitute back using triangle.

(i) If the quadratic form is not in the Pythagoras form (for example, $\sqrt{2 + 2x + x^2}$, then use the perfecting the square method to transform it into Pythagoras form).

(ii) Try u – substitution before trigonometric substitution.

(iii) Integrals involving $(1 - x^2)$ and $(x^2 - 1)$ without square roots can be solved easily with partial fractions. So don't use trigonometric substitution.

Rational Integral and Partial Fraction

(a) General step for solving rational integral:

Step 1: Do long division for the rational function if the degree of the numerator is higher than the denominator.

Step 2: Do partial fraction decomposition.

Step 3: Evaluate the integral of each simple fraction.

(b) General step for partial fraction:

Step 1: Factorize the denominator.

Step 2: Set the partial fraction according to "rule".

Step 3: Solve the unknown of the numerator of the partial fraction.

Improper Integral

(a) General steps for evaluating improper integral:

Step 1: Change the improper integral into the appropriate limit. [Change $\pm\infty$ or singular point (where) to appropriate limit.]

Step 2: Evaluate the integral.

Step 3: Find the limit.

(b) The very first step to test improper integral involving ∞ is to check its limit. If its limit is not zero, then the integral diverges.

(c) Whenever you see improper integrals involving the quotient of a rational or irrational function, such as

$$\int_a^\infty \frac{x^3 + \sqrt{3x}}{(8x^3 + 7x)^{3/2}} dx$$

Use **limit comparison test**. The appropriate comparing function can be found by looking at the Integrand (quotient of rational irrational). "Discard" the lower degree terms.

- (d) Sometimes, using u – substitution before using any test will be easier.
- (e) Sometimes, to determine if an improper integral converges or diverges, directly evaluating the improper integral is easier.
- (f) When doing a **comparison test**, beware of the comparing function that you choose. It might not give an appropriate conclusion if the comparing function is not correct.
- (g) Try the **limit comparison test** before the **comparison test**.
- (h) Useful comparing function, which is good to know their convergence or divergence

$$\int_a^{\infty} x^k e^{-\beta x} dx < \infty \quad \text{For } k \geq 0, \beta > 0$$

$$\int_a^{\infty} \frac{1}{x^p} dx \begin{cases} < \infty & \text{if } p > 1; \\ = \infty & \text{if } p \leq 1; \end{cases}$$

$$\int_a^{\infty} \frac{1}{x^p} dx \begin{cases} < \infty & \text{if } p < 1; \\ = \infty & \text{if } p \geq 1; \end{cases}$$

FORMULAE SHEET

Basic theorems of Integration:

1. $\int k f(x) dx = k \int f(x) dx$	2. $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$
3. $\frac{d}{dx} \left(\int f(x) dx \right) = f(x)$	4. $\int \left(\frac{d}{dx} f(x) \right) dx = f(x)$

Elementary Integration:

1. $\int 0 dx = c$	2. $\int 1 dx = x + c$
3. $\int k dx = kx + c (k \in \mathbb{R})$	4. $\int x^n dx = \frac{x^{n+1}}{n+1} + c (n \neq -1)$
5. $\int \frac{1}{x} dx = \log_e x + c$	6. $\int e^x dx = e^x + c$
7. $\int a^x dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$	8. $\int \sin x dx = -\cos x + c$
9. $\int \cos x dx = \sin x + c$	10. $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c$
11. $\int \frac{c}{ax + b} dx = \frac{c}{a} \log ax + b + c$	12. $\int f'(x) e^{f(x)} dx = e^{f(x)} + c$
13. $\int \log x dx = x \log x - x + c$	14. $\int \log_a x dx = x \log_a x - \frac{x}{\log a} + c$

Standard substitution:

1. $\sqrt{a^2 - x^2}$ or $\frac{1}{\sqrt{a^2 - x^2}}$	$x = a \sin \theta$ or $x = a \cos \theta$
2. $\sqrt{x^2 + a^2}$ or $\frac{1}{\sqrt{x^2 + a^2}}$	$x = a \tan \theta$ or $x = a \cot \theta$
3. $\sqrt{x^2 - a^2}$ or $\frac{1}{\sqrt{x^2 - a^2}}$	$x = a \sec \theta$ or $x = a \operatorname{cosec} \theta$
4. $\sqrt{\frac{x}{a+x}}, \sqrt{\frac{a+x}{x}}, \sqrt{x(a+x)}$ and $\frac{1}{\sqrt{x(a+x)}}$	$x = a \tan^2 \theta$
5. $\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}, \sqrt{x(a-x)}$ and $\frac{1}{\sqrt{x(a-x)}}$	$x = a \sin^2 \theta$ or $x = a \cos^2 \theta$
6. $\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}, \sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$	$x = a \sec^2 \theta$ or $x = a \operatorname{cosec}^2 \theta$
7. $\sqrt{\frac{a-x}{a+x}}$ and $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
8. $\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ ($\beta > \alpha$)	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

Some standard Integrals:

1. $\int \tan x dx = \log \sec x + c = -\log \cos x + c$	2. $\int \cot x dx = \log \sin x + c$
3. $\int \sec x dx = \log(\sec x + \tan x) + c$ $= -\log(\sec x - \tan x) + c = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$	4. $\int \operatorname{cosec} x dx = -\log(\operatorname{cosec} x + \cot x) + c$ $= \log(\operatorname{cosec} x - \cot x) + c = \log \tan\left(\frac{x}{2}\right) + c$
5. $\int \sec x \tan x dx = \sec x + c$	6. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$
7. $\int \sec^2 x dx = \tan x + c$	8. $\int \operatorname{cosec}^2 x dx = -\cot x + c$
9. $\int \log x dx = x \log\left(\frac{x}{e}\right) + c = x(\log x + 1) + c$	10. $\int \sin^2 x dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C$ $= \frac{1}{2}(x - \sin x \cos x) + C$

11. $\int \cos^2 x dx = \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) + C =$ $\frac{1}{2}(x + \sin x \cos x) + C$	12. $\int \sec^3 x dx = \frac{1}{2} \sec x \tan x$ $+ \frac{1}{2} \log \sec x + \tan x + c$
13. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} +$ $\frac{n-1}{n} \int \sin^{n-2} x dx$	14. $\int \cos^n x dx = -\frac{\cos^{n-1} x \sin x}{n} +$ $\frac{n-1}{n} \int \cos^{n-2} x dx$

Integration by Parts:

1. $\int (u \cdot v) dx = u \left(\int v dx \right) - \int \left(\frac{du}{dx} \right) \left(\int v dx \right) dx$	2. $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$
--	---

Standard Integrals:

1. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$
2. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left \frac{x-a}{x+a} \right + c$
3. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left \frac{a+x}{a-x} \right + c$
4. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c = -\cos^{-1} \left(\frac{x}{a} \right) + c$
5. $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1} \left(\frac{x}{a} \right) + c = \log \left(x + \sqrt{x^2 + a^2} \right) + c$
6. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \left(\frac{x}{a} \right) + c = \log \left(x + \sqrt{x^2 - a^2} \right) + c$
7. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$
8. $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left x + \sqrt{x^2 + a^2} \right + c$
9. $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left x + \sqrt{x^2 - a^2} \right + c$

$$10. \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c \quad (\text{Valid for } x > a > 0)$$

$$11. \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin \left\{ bx - \tan^{-1} \left(\frac{b}{a} \right) \right\} + c$$

$$12. \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos \left\{ bx - \tan^{-1} \frac{b}{a} \right\} + c$$

Solved Examples

JEE Main/Boards

Example 1: Evaluate : $\int \frac{x + \sin x}{1 + \cos x} dx$

Sol: Here by using the formula

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \text{ and } 1 + \cos x = 2 \cos^2 \frac{x}{2}$$

we can solve the given problem.

$$\begin{aligned} \int \frac{x + \sin x}{1 + \cos x} dx &= \int \frac{x + 2 \sin x / 2 \cos x / 2}{2 \cos^2 x / 2} dx \\ &= \int \frac{x}{2} \sec^2 x / 2 + \tan \frac{x}{2} dx = x \tan x / 2 + c \end{aligned}$$

Example 2: Evaluate : $\int \sqrt{x^2 + a^2} dx$

Sol: By applying integration by parts and taking $\sqrt{x^2 + a^2}$ as the first function we can solve the given problem.

$$\begin{aligned} \int \sqrt{x^2 + a^2} dx &= \\ &= \sqrt{x^2 + a^2} x - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}} dx = \sqrt{x^2 + a^2} x - \int \frac{x^2}{\sqrt{x^2 + a^2}} dx \\ &= \frac{x\sqrt{x^2 + a^2} + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}}{2} \end{aligned}$$

Put $x = a \tan \theta$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \sec \theta d\theta = \log (\sec \theta + \tan \theta)$$

$$= \log \left(\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right)$$

$$\therefore I = \frac{1}{2} \left(x\sqrt{x^2 + a^2} + \log \left(\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right) \right) + c$$

Example 3: Evaluate : $\int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}} dx$

Sol: Here first write $\cos((\pi/2) - x)$ at the place of $\sin x$ then by using the formula $1 - \cos x = 2 \sin^2 \frac{x}{2}$

And $1 + \cos x = 2 \cos^2 \frac{x}{2}$ we can solve it.

$$\begin{aligned} I &= \int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}} dx = \int \tan^{-1} \sqrt{\frac{1 - \cos((\pi/2) - x)}{1 + \cos((\pi/2) - x)}} dx \\ &= \int \tan^{-1} \sqrt{\frac{2 \sin^2((\pi/4) - (x/2))}{2 \cos^2((\pi/4) - (x/2))}} dx \\ &= \int \tan^{-1} \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) dx = \int \left(\frac{\pi}{4} - \frac{x}{2} \right) dx = \frac{\pi}{4} x - \frac{x^2}{4} + C \end{aligned}$$

Example 4: Evaluate : $\int \log(2 + x^2) dx$

Sol: Here integrating by parts by taking $\log(2 + x^2)$ as the first function we can solve the given problem.

$$I = \int \log(2+x^2) dx = \int \log(2+x^2) \cdot 1 dx$$

Taking $\log(2+x^2)$ as first function and integrating by parts, we get

$$\begin{aligned} I &= \left[\log(2+x^2) \right] x - \int x \cdot \frac{2x}{2+x^2} dx \\ &= x \log(2+x^2) - 2 \int \frac{(x^2+2)-2}{2+x^2} dx \\ &= x \log(2+x^2) - 2 \int \left[1 - \frac{2}{x^2+2} \right] dx \\ &= x \log(x^2+2) - 2 \left[x - \sqrt{2} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) \right] + c \end{aligned}$$

Example 5: Evaluate : $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$

Sol: Simply put $e^{2x} + e^{-2x} = t \Rightarrow (e^{2x} - e^{-2x}) dx = \frac{dt}{2}$ and then solving we will get the result.

$$\begin{aligned} I &= \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx \\ \text{Put } e^{2x} + e^{-2x} &= t \Rightarrow (e^{2x} - e^{-2x}) dx = \frac{dt}{2} \\ \therefore I &= \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log|t| + C = \frac{1}{2} \log|e^{2x} + e^{-2x}| + c \end{aligned}$$

Example 6: Evaluate : $\int \frac{x^3 - 1}{x^3 + x} dx$

Sol: By splitting the given integration as

$$\int \frac{x^3}{x(x^2+1)} dx - \int \frac{1}{x(x^2+1)} dx$$

We can solve the given problem.

$$\begin{aligned} \int \frac{x^3 - 1}{x^3 + x} dx &= \int \frac{x^3}{x(x^2+1)} dx - \int \frac{1}{x(x^2+1)} dx \\ &= \int \frac{x^2}{x^2+1} dx - \int \frac{1}{x} - \frac{x}{(x^2+1)} dx \\ &= \int \left(1 - \frac{1}{x^2+1} \right) dx - \int \frac{1}{x} dx + \int \frac{x}{x^2+1} dx \\ &= x - \tan^{-1} x - \log x + \log \sqrt{x^2+1} + c \end{aligned}$$

Example 7: Evaluate: $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Sol: By putting $x = \sin \theta \Rightarrow dx = \cos \theta d\theta$, we will reduce the given integration as

$$\int \frac{\sec^2 \theta}{1+2\tan^2 \theta} d\theta \text{ and then put } t = \tan \theta \Rightarrow dt = \sec^2 \theta d\theta$$

and solve it.

$$\text{Put } x = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

$$\begin{aligned} \Rightarrow I &= \int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{1}{1+\sin^2 \theta} d\theta \\ &= \int \frac{\sec^2 \theta}{1+2\tan^2 \theta} d\theta \end{aligned}$$

$$\text{Again put } t = \tan \theta \Rightarrow dt = \sec^2 \theta d\theta$$

$$\begin{aligned} I &= \int \frac{1}{1+t^2} dt = \frac{1}{\sqrt{2}} \int \frac{1}{t^2 + (1/\sqrt{2})^2} dt \\ &= \frac{1}{2} \left(\frac{1}{1/\sqrt{2}} \right) \tan^{-1} \left(\frac{t}{1/\sqrt{2}} \right) + c \\ &= \frac{1}{2} \tan^{-1}(\sqrt{2} \tan \theta) + c = \frac{1}{2} \tan^{-1} \left(\frac{x\sqrt{2}}{\sqrt{1-x^2}} \right) + c \end{aligned}$$

Example 8: Evaluate : $\int \frac{xdx}{(x-1)(x^2+4)}$

Sol: By partial fractions, we can reduce the given fraction as a sum of two fractions which will be easier to integrate.

$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$x = 1 \Rightarrow A = 1/5$$

$$x = 2i \Rightarrow B = -1/5, C = 4/5$$

$$\begin{aligned} \therefore I &= \int \left(\frac{1}{5(x-1)} + \frac{4-2x}{5(x^2+4)} \right) dx \\ &= \frac{1}{5} \log(x-1) - \frac{1}{10} \left[\frac{2x}{x^2+4} - \frac{8}{x^2+4} \right] \\ &= \frac{1}{5} \log(x-1) - \frac{\log(x^2+4)}{10} + \frac{2}{5} \tan^{-1} \frac{x}{2} \end{aligned}$$

Example 9: Evaluate: $\int \frac{\sin x}{\sin 4x} dx$

Sol: By using the formula $\sin 2x = 2 \sin x \cos x$, we can reduce the given fraction and then by putting $\sin x = t$ we can solve it.

$$\begin{aligned}\int \frac{\sin x}{\sin 4x} dx &= \int \frac{\sin x dx}{2 \cos 2x \sin 2x} = \int \frac{dx}{4 \cos x \cos 2x} \\ &= \int \frac{\cos x dx}{4(1 - \sin^2 x)(1 - 2\sin^2 x)}\end{aligned}$$

Put $\sin x = t$

$$\begin{aligned}\Rightarrow I &= \frac{1}{4} \int \frac{dt}{(1-t^2)(1-2t^2)} \\ &= \frac{1}{4} \int \left(\frac{1}{(t^2-1)} - \frac{2}{(2t^2-1)} \right) dt \\ &= \frac{1}{4} \left(\frac{1}{2} \log \left| \frac{t-1}{t+1} \right| - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2}t-1}{\sqrt{2}t+1} \right| \right) + c \\ &= \frac{1}{8} \log \left| \frac{\sin x - 1}{\sin x + 1} \right| + \frac{1}{4\sqrt{2}} \log \left| \frac{\sqrt{2} \sin x + 1}{\sqrt{2} \sin x - 1} \right| + c\end{aligned}$$

Example 10: Evaluate: $\int \frac{dx}{(x)(x^4-1)}$

Sol: Here we can write the given integration as

$$\int \frac{x^{-5}}{(1-x^{-4})} dx \text{ and then by putting } 1-x^{-4} = t$$

$\Rightarrow 4x^{-5} = dt$ we can solve it.

$$\int \frac{dx}{(x)(x^4-1)} = \int \frac{x^{-5}}{(1-x^{-4})} dx$$

Put $1-x^{-4} = t \Rightarrow 4x^{-5} = dt$

$$\Rightarrow \int \frac{dt}{4t} = \frac{1}{4} \log |t| + c = \frac{1}{4} \log |1-x^{-4}| + c$$

JEE Advanced/Boards

Example 1: Evaluate: $\int x \sin^{-1} x dx$

Sol: Integrating by parts taking $\sin^{-1} x$ as the first term we can solve the given integration.

$$\int x \sin^{-1} x dx$$

Let $u = \sin^{-1} x$, $v = x$

$$\begin{aligned}&= \frac{x^2}{2} \sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{(1-x^2)-1}{\sqrt{1-x^2}} dx \\ &= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx \\ &= \frac{x^2}{2} \sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} - \frac{1}{4} \sin^{-1} x + c \\ &= \frac{2x^2-1}{4} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + c\end{aligned}$$

Example 2: Evaluate: $\int \frac{e^x(2-x^2)dx}{(1-x)\sqrt{1-x^2}}$

Sol: We can split the given fraction as

$\int e^x \left\{ \frac{1+x}{\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}} \right\} dx$ and this integration is in the form of $e^x(f(x) + f'(x))$

$$I = \int \frac{e^x(2-x^2)dx}{(1-x)\sqrt{1-x^2}} = \int e^x \frac{(1-x^2)+1}{(1-x)\sqrt{1-x^2}} dx$$

$$= \int e^x \left\{ \frac{1+x}{\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}} \right\} dx$$

$$\text{But } \frac{d}{dx} \left(\frac{1+x}{\sqrt{1-x^2}} \right) = \frac{1}{\sqrt{1-x^2}} + (1+x) \frac{x}{(1-x^2)^{3/2}}$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{(1+x)x}{(1-x)(1+x)\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{x}{(1-x)\sqrt{1-x^2}} = \frac{1-x+x}{(1-x)\sqrt{1-x^2}}$$

$$= \frac{1}{(1-x)\sqrt{1-x^2}}$$

Hence, the integrand is of type $e^x(f(x) + f'(x))$

$$\therefore I = e^x \frac{1+x}{\sqrt{1-x^2}} + C$$

Example 3: Evaluate: $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$

Sol: Here by taking $\cos^3 x$ and $\sin^2 x$ common from the numerator and denominator respectively and then by putting $\sin x = t$ we can solve the given problem.

$$I = \int \frac{\cos^3 x(1+\cos^2 x)}{\sin^2 x(1+\sin^2 x)} dx$$

Since power of $\cos x$ is odd, put $\sin x = t$;

Then, $\cos x \, dx = dt$

$$\begin{aligned} I &= \int \frac{(1-t^2)(1+1-t^2)}{t^2(1+t^2)} dt = \int \frac{(1-t^2)(2-t^2)}{t^2(1+t^2)} dt \\ &= \int \left(1 + \frac{2}{t^2} - \frac{6}{1+t^2} \right) dt = t - \frac{2}{t} - 6 \tan^{-1} t + c \\ &= \sin x - 2 \operatorname{cosec} x - 6 \tan^{-1}(\sin x) + c \end{aligned}$$

Example 4: Evaluate : $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx$

Sol: By partial fractions we can reduce the given fraction as a sum of two fractions and then by integrating them we will get the result.

$$\begin{aligned} I &= \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx \\ \text{Let } 4e^x + 6e^{-x} &= A(9e^x - 4e^{-x}) + B \left(\frac{d}{dx}(9e^x - 4e^{-x}) \right) \\ \text{By comparing the coefficients of } e^x \text{ and } e^{-x}, \text{ we get} \\ A &= \frac{-19}{36}, B = \frac{35}{36} \\ \therefore I &= \int \frac{A(9e^x - 4e^{-x}) + B(9e^x + 4e^{-x})}{9e^x - 4e^{-x}} dx \\ &= A \int dx + B \int \frac{9e^x + 4e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \log |9e^x - 4e^{-x}| + c \\ &= -\frac{19}{36}x + \frac{35}{36} \log |9e^x - 4e^{-x}| + c \end{aligned}$$

Example 5: Evaluate : $\int \frac{1+x^2}{1-x^2} \frac{dx}{\sqrt{1+3x^2+x^4}}$ for $x > 0$

Sol: Dividing by x^2 in the numerator and denominator and then putting $x - 1/x = t$ we can solve the given problem.

$$I = \int \frac{1+x^2}{1-x^2} \frac{dx}{\sqrt{1+3x^2+x^4}} = \int \frac{(1/x^2 + 1)dx}{((1/x) - x)\sqrt{(x - (1/x))^2 + 5}}$$

$$\text{Put } x - 1/x = t; \left(1 + \frac{1}{x^2} \right) dx = dt$$

$$I = -\int \frac{dt}{t\sqrt{t^2+5}}$$

$$\text{Put } t^2 + 5 = z^2 \Rightarrow 2t \, dt = 2z \, dz$$

$$\begin{aligned} \Rightarrow I &= -\int \frac{dt}{z^2 - 5} = -\frac{1}{2\sqrt{5}} \log \left| \frac{z - \sqrt{5}}{z + \sqrt{5}} \right| + c \\ &= -\frac{1}{2\sqrt{5}} \log \left| \frac{\sqrt{(x^2 + (1/x^2) + 3)} - \sqrt{5}}{\sqrt{(x^2 + (1/x^2) + 3)} + \sqrt{5}} \right| + c \end{aligned}$$

Example 6: Evaluate :

$$\int \operatorname{cosec}^2 x \cdot \log(\cos x + \sqrt{\cos 2x}) dx \text{ for } \sin x > 0$$

Sol: By substituting $\cos^2 x - \sin^2 x$ in place of $\cos 2x$, we can reduce the given integration as the sum of two integrations and then by integrating them separately we will obtain the result.

$$\begin{aligned} &\int \operatorname{cosec}^2 x \cdot \log(\cos x + \sqrt{\cos 2x}) dx \\ &= \int \operatorname{cosec}^2 x \log(\cos x + \sqrt{\cos^2 x - \sin^2 x}) dx \\ &= \int \operatorname{cosec}^2 x \log(\sin x (\cot x + \sqrt{\cot^2 x - 1})) dx \\ &= \int \operatorname{cosec}^2 x \cdot \log \sin x \, dx + \int \operatorname{cosec}^2 x \cdot \log[\cot x + \sqrt{\cot^2 x - 1}] dx \\ &= I_1 + I_2 \end{aligned}$$

$$\begin{aligned} I_1 &= \int \operatorname{cosec}^2 x \cdot \log \sin x \, dx \\ &= (-\cot x) \cdot \log \sin x - \int (-\cot x) \cdot \cot x \, dx \\ &= -\cot x \cdot \log \sin x + \int (\operatorname{cosec}^2 x - 1) dx \\ &= -\cot x \cdot \log \sin x - \cot x - x + C_1 \end{aligned}$$

$$I_2 = \int \operatorname{cosec}^2 x \cdot \log[\cot x + \sqrt{\cot^2 x - 1}] dx$$

Put $\cot x = t$; $-\operatorname{cosec}^2 x \, dx = dt$

$$I_2 = -\int \log[t + \sqrt{t^2 - 1}] dt$$

(Integrate by parts)

$$\begin{aligned} &= -t \cdot \log(t + \sqrt{t^2 - 1}) + \int t \cdot \frac{1 + (t/\sqrt{t^2 - 1})}{t + \sqrt{t^2 - 1}} dt \\ &= -t \cdot \log(t + \sqrt{t^2 - 1}) + \int \frac{t}{\sqrt{t^2 - 1}} dt \\ &= -t \cdot \log(t + \sqrt{t^2 - 1}) + \sqrt{t^2 - 1} + C_2 \\ &= -\cot x \log(\cot x + \sqrt{\cot^2 x - 1}) + \sqrt{\cot^2 x - 1} + C_2 \end{aligned}$$

Example 7: Evaluate : $\int \frac{\sin x}{\sin^3 x + \cos^3 x} dx$

Sol: By taking $\cos^3 x$ common from the denominator and then by putting $\tan x = t$ we can solve it.

Integrand contains odd powers in $\sin x$ and $\cos x$. So, put $\tan x = t$.

$$\begin{aligned}\Rightarrow I &= \int \frac{1}{\cos^3 x} \frac{\sin x}{1 + \tan^3 x} dx \\&= \int \frac{\tan x \cdot \sec^2 x}{1 + \tan^3 x} dx \text{ (put } \tan x = t) \\&= \int \frac{t}{1+t^3} dt = -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{3} \int \frac{t+1}{t^2-t+1} dt \\&= -\frac{1}{3} \log|t+1| + \frac{1}{6} \int \frac{(2t-1)+3}{t^2-t+1} dt \\&= -\frac{1}{3} \log|t+1| + \frac{1}{6} \log|t^2-t+1| + \\&\quad \frac{1}{2} \frac{dt}{(t-(1/2))^2 + (3/4)} \\&= -\frac{1}{3} \log|t+1| + \frac{1}{6} \log|t^2-t+1| + \\&\quad \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{t-(1/2)}{\sqrt{3}/2} \right) + C \\&= \frac{1}{6} \log \left| \frac{1 - \tan x + \tan^2 x}{(1 + \tan x)^2} \right| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x - 1}{\sqrt{3}} \right) + C\end{aligned}$$

Example 8: If $I_{m,n} = \int \cos^m x \cdot \cos nx \, dx$, show that $(m+n)I_{m,n} = \cos^m x \sin nx + mI_{m-1,n-1}$

Sol: By using integration by parts and by taking $\cos^m x$ as the first term we can prove the given equation.

Integrating by parts,

$$I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin x \sin nx \, dx \quad \dots (i)$$

But $\cos(n-1)x = \cos(nx-x)$

$$\begin{aligned}&= \cos nx \cos x + \sin nx \sin x \\&\Rightarrow \sin x \sin nx = \cos(n-1)x - \cos nx \cos x\end{aligned} \quad \dots (ii)$$

From (i) and (ii):

$$\begin{aligned}I_{m,n} &= \frac{1}{n} \cos^m x \sin nx + \\&\quad \frac{m}{n} \int \cos^{m-1} x [\cos(n-1)x - \cos nx \cos x] dx\end{aligned}$$

$$\begin{aligned}&= \frac{1}{n} \cos^m x \sin nx + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n} \\&\Rightarrow (m+n)I_{m,n} = \cos^m x \sin nx + mI_{m-1,n-1}\end{aligned}$$

Example 9: Evaluate : $I = \int \left(x + \sqrt{1+x^2} \right)^n dx$

Sol: Simply putting $x + \sqrt{1+x^2} = t$ and integrating we can solve the given problem.

Let $x + \sqrt{1+x^2} = t$, then

$$\left(1 + \frac{x}{\sqrt{1+x^2}} \right) dx = dt \Rightarrow \frac{t}{\sqrt{1+x^2}} dx = dt$$

As $\sqrt{1+x^2} + x = t$

$$\frac{1}{t} = \frac{1}{\sqrt{1+x^2} + x} = \frac{\sqrt{1+x^2} - x}{1}$$

$$\Rightarrow 2\sqrt{1+x^2} = t + \frac{1}{t} = \frac{t^2+1}{t}$$

$$\begin{aligned}\text{Thus } I &= \int t^n \left(\frac{t^2+1}{2t} \right) \frac{dt}{t} \\&= \frac{1}{2} \int t^{n-2} (t^2+1) dt = \frac{1}{2} \int (t^n + t^{n-2}) dt = \frac{1}{2} \left(\frac{t^{n+1}}{n+1} + \frac{t^{n-1}}{n-1} \right) + C\end{aligned}$$

Where $t = x + \sqrt{1+x^2}$

Example 10: Evaluate:

$$I = \int \frac{2 \sin^3(x/2) dx}{(\cos(x/2)) \sqrt{\cos^3 x + 3 \cos^2 x + \cos x}} \text{ for } \cos x > 0$$

Sol: Here we can reduce the given fraction by using the formula $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$ and then by putting $\cos x = t$ we can solve it.

$$\begin{aligned}I &= \int \frac{(2 \sin(x/2) \cos(x/2)) (2 \sin^2(x/2)) dx}{(2 \cos^2(x/2)) \sqrt{\cos^3 x + 3 \cos^2 x + \cos x}} \\&= \int \frac{(1 - \cos x) \sin x \, dx}{(1 + \cos x) \sqrt{\cos^3 x + 3 \cos^2 x + \cos x}}\end{aligned}$$

[Put $\cos x = t$]

$$\Rightarrow I = \int \frac{(t-1) dt}{(1+t) \sqrt{t^3 + 3t^2 + t}}$$

$$\begin{aligned}
&= \int \frac{(t^2 - 1)dt}{(1+t)^2 t \sqrt{t+3+(1/t)}} \\
&= \int \frac{t^2(1 - (1/t^2))dt}{t(t^2 + 2t + 1)\sqrt{t+(1/t)+3}} \\
&= \int \frac{(1 - (1/t^2))dt}{(t + (1/t) + 2)\sqrt{(t + (1/t)) + 3}}
\end{aligned}$$

Put $t + \frac{1}{t} + 3 = z^2 : z > 0 ;$

Then $\left(1 - \frac{1}{t^2}\right)dt = 2zdz$

$$\Rightarrow I = \int \frac{2zdz}{(z^2 - 1)z} = 2 \int \frac{dz}{z^2 - 1} = \log \left| \frac{z-1}{z+1} \right| + c$$

$$\Rightarrow I = \log \left| \frac{\sqrt{\cos x + \sec x + 3} - 1}{\sqrt{\cos x + \sec x + 3} + 1} \right| + c$$

Example 11: Evaluate:

$$I = \int \frac{(\sin x - \cos x) dx}{(\sin x + \cos x) \sqrt{\sin x \cos x + \sin^2 x \cos^2 x}}$$

Sol: Here by putting $\sin x + \cos x = t$ we can integrate the given fraction using the appropriate formula.

Let $\sin x + \cos x = t$

$$\Rightarrow (\cos x - \sin x) dx = dt$$

Also, $t^2 = (\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$

$$\therefore \sin x \cos x = \frac{t^2 - 1}{2}$$

$$\Rightarrow I = - \int \frac{dt}{t \sqrt{\left(\frac{t^2 - 1}{2}\right) \left(1 + \left(\frac{t^2 - 1}{2}\right)\right)}}$$

$$= - \int \frac{dt}{t \sqrt{((t^2 - 1)(t^2 + 1))/4}} = -2 \int \frac{t^3 dt}{t^4 \sqrt{t^4 - 1}}$$

Put $t^4 - 1 = z^2 : z > 0$

$$\Rightarrow I = -2 \int \frac{1}{4} \frac{2z dz}{(z^2 + 1)z} = - \int \frac{dz}{1 + z^2}$$

$$= -\tan^{-1} z + c = -\tan^{-1} \sqrt{t^4 - 1} + c$$

$$= -\tan^{-1} \sqrt{(1 + \sin 2x)^2 - 1} + c$$

$$= -\tan^{-1} \sqrt{\sin^2 2x + 2 \sin 2x} + c$$

Example 12: Evaluate : $I = \int \frac{dx}{1 + \sqrt{x^2 + x + 2}}$

Sol: We can reduce the given fraction as

$$\int \frac{dx}{1 + \sqrt{(x + (1/2))^2 + (7/4)}} \text{ and then by putting}$$

$x + \frac{1}{2} = \frac{\sqrt{7}}{2} \tan \theta : -\frac{\pi}{2} < \theta < \frac{\pi}{2}$; and using appropriate integration formula we can integrate the given fraction.

$$I = \int \frac{dx}{1 + \sqrt{(x + (1/2))^2 + (7/4)}}$$

Put $x + \frac{1}{2} = \frac{\sqrt{7}}{2} \tan \theta : -\frac{\pi}{2} < \theta < \frac{\pi}{2}$; then

$$dx = \frac{\sqrt{7}}{2} \sec^2 \theta d\theta$$

$$\Rightarrow I = \int \frac{\sqrt{7}}{2} \frac{\sec^2 \theta d\theta}{1 + (\sqrt{7}/2) \sec \theta}$$

$$= \frac{\sqrt{7}}{2} \int \frac{d\theta}{\cos \theta (\cos \theta + (\sqrt{7}/2))}$$

$$= \int \left(\frac{1}{\cos \theta} - \frac{1}{\cos \theta + (\sqrt{7}/2)} \right) d\theta$$

$$= \log |\sec \theta + \tan \theta| - \int \frac{d\theta}{a + \cos \theta}; a = \frac{\sqrt{7}}{2}$$

$$I = \log |\sec \theta + \tan \theta| - I_1 \quad \dots(i)$$

Where, $I_1 = \int \frac{d\theta}{a + \cos \theta}$

Put $\tan \frac{\theta}{2} = t; \cos \theta = \frac{1 - t^2}{1 + t^2}$

$$I_1 = \int \frac{2dt}{1 + t^2} \frac{1}{a + ((1 - t^2)/(1 + t^2))}$$

$$= 2 \int \frac{dt}{a(1 + t^2) + 1 - t^2}$$

$$= \frac{2}{a-1} \int \frac{dt}{((a+1)/(a-1)) + t^2}$$

$$= \frac{2}{a-1} \sqrt{\frac{a-1}{a+1}} \tan^{-1} \left(\sqrt{\frac{a-1}{a+1}} t \right) + c$$

$$= \frac{2}{\sqrt{a^2 - 1}} \tan^{-1} \left(\frac{\sqrt{a-1}}{a+1} \tan \frac{\theta}{2} \right) + c \quad \dots(ii)$$

From (i) and (ii), we get I.

JEE Main/Boards

Exercise 1

Q.1 $\int \frac{\sec x}{\sec x + \tan x} dx$

Q.2 $\int \left(1 + \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + 5 \frac{1}{|x|\sqrt{x^2-1}} + a^x \right) dx$

Q.3 $\int \tan^{-1} \left(\frac{\sin 2x}{1 + \cos 2x} \right) dx : x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

Q.4 $\int \frac{1 + \tan x}{x + \log \sec x} dx$

Q.5 $\int \frac{2 \cos x - 3 \sin x}{3 \cos x + 2 \sin x} dx$

Q.6 $\int \frac{2x-1}{\sqrt{x^2-x-1}} dx$

Q.7 $\int \frac{dx}{\sqrt{1-3x} - \sqrt{5-3x}}$

Q.8 $\int x^2 e^{x^3} \cos(e^{x^3}) dx$

Q.9 $\int \frac{\sec^2(2 \tan^{-1} x)}{1+x^2} dx$

Q.10 $\int \frac{dx}{(2 \sin x + 3 \cos x)^2}$

Q.11 $\int \cos^{3/5} x \sin^3 x dx$

Q.12 $\int \frac{\log x}{x^2} dx$

Q.13 $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx : a > 0$

Q.14 $\int e^x \frac{2 + \sin 2x}{1 + \cos 2x} dx$

Q.15 $\int \frac{dx}{x \left[6(\log x)^2 + 7 \log x + 2 \right]}$

Q.16 $\int \frac{x^2+1}{(x+3)(x-1)^2} dx$

Q.17 $\int \frac{1}{1 - \tan x} dx$

Q.18 If $f'(x) = x - \frac{1}{x^2}$ and $f(1) = \frac{1}{2}$, find $f(x)$.

Q.19 For any natural number evaluate m

$\int (x^{3m} + x^{2m} + x^m) (2x^{2m} + 3x^m + 6)^{1/m} dx, x > 0$

Q.20 $\int \frac{x^3 + 3x + 2}{(x^2 + 1)^2 (x + 1)} dx$

Q.21 $\int \frac{dx}{\sin x + \sec x}$

Q.22 $\int \frac{\sqrt{\cos 2x}}{\sin x} dx : \cos x > 0$

Q.23 $\int \frac{\sqrt{x^2+1} (\log(x^2+1) - 2 \log x)}{x^4} dx$

Q.24 $\int \frac{\sin x}{\sin x - \cos x} dx$

Q.25 $\int x \sin^{-1} \left(\frac{1}{2} \frac{\sqrt{2a-x}}{a} \right) dx$

Q.26 $\int \sec^4 x \operatorname{cosec}^2 x dx$

Q.27 $\int \frac{d\theta}{(a + b \cos \theta)^2} : a > b > 0$

Q.28 Evaluate $\int \frac{dx}{x \sqrt{x^4 - 1}}$

Q.29 $\int \frac{dx}{\sqrt{\sin^3 x \sin(x + \alpha)}}$

Q.30 $\int \frac{dx}{(1 + \sqrt{x}) \sqrt{x - x^2}}$

Q.31 $\int \frac{\cos 8x - \cos 7x}{1 + 2 \cos 5x} dx$

Q.32 $\int \frac{x^3 + 1}{x(x-1)^3} dx$

Q.33 $\int \frac{dx}{\sqrt[3]{\sin^{11} x \cdot \cos x}}$

Q.34 Evaluate $\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx$

Q.35 Evaluate $\int \frac{\log x}{(1+x)^3} dx$

Q.36 Evaluate $\int \frac{f(x)}{x^3 - 1} dx$, where $f(x)$ is polynomial of the second degree in x such that $f(0) = f(1) = 3f(2) = -3$

Exercise 2

Single Correct Choice Type

Q.1 $\int \left(\frac{x}{1+x^5} \right)^{3/2} dx$ equals-

- (A) $\frac{2}{5} \sqrt{\frac{x^5}{1+x^5}} + c$ (B) $\frac{2}{5} \sqrt{\frac{x}{1+x^5}} + c$
 (C) $\frac{2}{5} \frac{1}{\sqrt{1+x^5}} + c$ (D) None of these

Q.2 $\frac{\cos^8 x - \sin^8 x}{1 - 2\sin^2 x \cos^2 x} dx$ equals-

- (A) $-\frac{\sin 2x}{2} + c$ (B) $\frac{\sin 2x}{2} + c$
 (C) $\frac{\cos 2x}{3} + c$ (D) $-\frac{\cos 2x}{1} + c$

Q.3 Identify the correct expression

- (A) $x \int \log x dx = x^2 \log -x^2 + c$
 (B) $x \int \log |x| dx = xe^x + c$
 (C) $x \int e^x dx = xe^x + cx$
 (D) $\int \frac{dx}{\sqrt{a^2 + x^2}} = \frac{1}{a} \tan^{-1} \left[\frac{x}{a} \right] + c$

Q.4 Primitive of $\sqrt{1 + 2 \tan x (\sec x + \tan x)}$ w.r.t. x is

- (A) $\log(\sec x + \tan x) + \log \cos x + c$
 (B) $\log(\sec x + \tan x) + \log \sec x + c$

(C) $2 \log \left(\sec \frac{x}{2} + \tan \frac{x}{2} \right) + c$

(D) $\log \sqrt{1 + \sin x} + c$

Q.5 $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$ is equal to

- (A) $\cot^{-1}(\cot^2 x) + c$ (B) $-\cot^{-1}(\tan^2 x) + c$
 (C) $\tan^{-1}(\tan^2 x) + c$ (D) $-\tan^{-1}(\cos 2x) + c$

Q.6 The value of integral $\int \frac{d\theta}{\cos^3 \theta \sqrt{\sin 2\theta}}$ can be expressed as irrational function of $\tan \theta$ as

- (A) $\frac{\sqrt{2}}{5} \left(\sqrt{\tan^2 \theta + 5} \right) \tan \theta + c$
 (B) $\frac{2}{5} (\tan^2 \theta + 5) \sqrt{\tan \theta} + c$
 (C) $\frac{\sqrt{2}}{5} (\tan^2 \theta + 5) \sqrt{\tan \theta} + c$
 (D) $\frac{\sqrt{2}}{5} (\tan^2 \theta + 5)^{\sqrt{\tan \theta}} + c$

Q.7 $\int \frac{dx}{a+bx^2}$ $a, b \neq 0$ and $a/b > 0$

- (A) $\frac{1}{\sqrt{ab}} \tan^{-1} x \sqrt{\frac{b}{a}} + c$ (B) $\sqrt{\frac{b}{a}} \tan^{-1} x \sqrt{\frac{b}{a}} + c$
 (C) $\sqrt{\frac{a}{b}} \tan^{-1} x \sqrt{\frac{a}{b}} + c$ (D) $\sqrt{ab} \tan^{-1} x \sqrt{\frac{b}{a}} + c$

Q.8 $\int \frac{1-x^7}{x(1+x^7)} dx$ equals

- (A) $\log x + \frac{2}{7} \log(1+x^7) + c$ (B) $\log x - \frac{2}{4} \log(1-x^7) + c$
 (C) $\log x - \frac{2}{7} \log(1+x^7) + c$ (D) $\log x + \frac{2}{4} \log(1-x^7) + c$

Q.9 $\int \frac{\log |x|}{x \sqrt{1 + \log |x|}} dx$ equal

- (A) $\frac{2}{3} \sqrt{1 + \log |x|} (\log |x| - 2) + c$
 (B) $\frac{2}{3} \sqrt{1 + \log |x|} (\log |x| + 2) + c$

(C) $\frac{1}{3}\sqrt{1+\log|x|}(\log|x|-2)+c$

(D) $2\sqrt{1+\log|x|}(3\log|x|-2)+c$

Q.10 If $\int \frac{x^4+1}{x(x^2+1)^2} dx = A\log|x| + \frac{B}{1+x^2} + c$,

Where c is the constant of integration then:

(A) A=1; B=-1 (B) A=-1; B=1

(C) A=1; B=1 (D) A=-1; B=-1

Q.11 $\int 4\sin x \cos \frac{x}{2} \cos \frac{3x}{2} dx$ equals

(A) $\cos x - \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + c$

(B) $\cos x - \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + c$

(C) $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + c$

(D) $\cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x + c$

Q.12 $\int \sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x dx$ equals

(A) $\frac{\sin 16x}{1024} + c$ (B) $-\frac{\cos 32x}{1024} + c$

(C) $\frac{\cos 32x}{1096} + c$ (D) $-\frac{\cos 32x}{1096} + c$

Q.13 If $\int f(x) dx = F(x)$, then $\int x^3 f(x^2) dx$ is equal to

(A) $\frac{1}{2} \left((F(x))^2 - \int (f(x))^2 dx \right)$ (B) $\frac{1}{2} \left(x^2 F(x^2) - \int (f(x^2) dx^2) \right)$

(C) $\frac{1}{2} \left(F(x) - \frac{1}{2} \int (F(x^2) dx \right)$ (D) None of these

Q.14 If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \log (9e^{2x} - 4) + C$ then

A, B and C are

(A) $A = \frac{3}{2}, B = \frac{36}{35}, C = \frac{3}{2} \log 3 + \text{constant}$

(B) $A = \frac{3}{2}, B = \frac{35}{36}, C = -\frac{3}{2} \log 3 + \text{constant}$

(C) $A = -\frac{3}{2}, B = \frac{35}{36}, C = \frac{3}{2} \log 3 + \text{constant}$

(D) None of these

Q.15 $\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{\sqrt{2}(\cos x + \sin x)} dx$ equals

(A) $\sec^{-1}(\sin x + \cos x) + c$

(B) $\sec^{-1}(\sin x - \cos x) + c$

(C) $\log [(\sin x + \cos x) + \sqrt{\sin 2x}] + c$

(D) $\log [(\sin x - \cos x) + \sqrt{\sin 2x}] + c$

Previous Years' Questions

Q.1 The value of $\int \frac{(x^2-1)dx}{x^3\sqrt{2x^4-2x^2+1}}$ (2006)

(A) $2\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+c$ (B) $2\sqrt{2+\frac{2}{x^2}+\frac{1}{x^4}}+c$

(C) $\frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}+c$ (D) None of these

Q.2 If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \log (9e^{2x} - 4) + C$, then

A=....., B=..... and C=..... (1989)

Q.3 Integrate $\frac{1}{1-\cot x}$ or $\frac{\sin x}{\sin x - \cos x}$ (1978)

Q.4 Integrate the curve $\frac{x}{1+x^4}$ (1978)

Q.5 Integrate $\sin x \cdot \sin 2x \cdot \sin 3x + \sec^2 x \cdot \cos^2 2x + \sin^4 x \cdot \cos^4 x$. (1979)

Q.6 Integrate $\frac{x^2}{(a+bx)^2}$ (1979)

Q.7 Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x}) dx$. (1988)

Q.8 Evaluate $\int \frac{(x+1)}{x(1+xe^x)^2} dx$. (1996)

Q.9 Integrate the following (1997)

$\int \left(\frac{1-\sqrt{x}}{1+\sqrt{x}} \right)^{1/2} \frac{dx}{x}$

Q.10 The value of $\sqrt{2} \frac{\sin x dx}{\sin\left(x - \frac{\pi}{4}\right)}$ is **(2008)**

- (A) $x + \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + c$ (B) $x - \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c$
 (C) $x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c$ (D) $x - \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + c$

Q.11 If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\log 2)$ is equal to **(2011)**

- (A) 5 (B) 13 (C) -2 (D) 7

Q.12 If the integral

$$\int \frac{5 \tan x}{\tan x - 2} dx = x + a \log|\sin x - 2 \cos x| + k$$

then a is equal to

- (2012)**
 (A) -1 (B) -2 (C) 1 (D) 2

Q.13 If $\int f(x) dx = \psi(x)$, then $\int x^5 f(x^3) dx$ is equal to **(2013)**

- (A) $\frac{1}{3} \left[x^3 \psi(x^3) - \int x^2 \psi(x^3) dx \right] + c$
 (B) $\frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c$
 (C) $\frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + c$
 (D) $\frac{1}{3} \left[x^3 \psi(x^3) - \int x^2 \psi(x^3) dx \right] + c$

Q.14 The integral $\int \left(1 + x - \frac{1}{x}\right)^{x + \frac{1}{x}} dx$ is equal to **(2014)**

- (A) $(x+1)^{x + \frac{1}{x}} + c$ (B) $-x e^{x + \frac{1}{x}} + c$
 (C) $(x-1)^{x + \frac{1}{x}} + c$ (D) $x e^{x + \frac{1}{x}} + c$

Q.15 The integral $\int \frac{dx}{x^2 (x^4 + 1)^{3/4}}$ equals **(2015)**

- (A) $\left(\frac{x^4 + 1}{x^4}\right)^{1/4} + c$ (B) $(x^4 + 1)^{1/4} + c$
 (C) $-(x^4 + 1)^{1/4} + c$ (D) $-\left(\frac{x^4 + 1}{x^4}\right)^{1/4} + c$

Q.16 The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$ is equal to **(2016)**

- (A) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + c$ (B) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + c$
 (C) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + c$ (D) $\frac{-x^5}{2(x^5 + x^3 + 1)^2} + c$

JEE Advanced/Boards

Exercise 1

Q.1 (i) $\int \frac{dx}{\cot(x/2) \cdot \cot(9x/3) \cdot \cot(x/6)}$;

(ii) $\int \frac{\tan(\log x) \tan(\log(x/2)) \tan(\log 2)}{x} dx$

Q.2 $\int \frac{dx}{(x - \alpha)\sqrt{(x - \alpha)(x - \beta)}}$

Q.3 $\int \frac{\log(\log((1+x)/(1-x)))}{1-x^2} dx$

Q.4 $\int \left[\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x \right] \log x dx$

Q.5 $\int \sqrt{\frac{\cos(x-a)}{\sin(x+a)}} dx$

Q.6 $\int \frac{x^5 + 3x^4 - x^3 + 8x^2 - x + 8}{x^2 + 1} dx$

Q.7 $\int \frac{(\sqrt{x} + 1) dx}{\sqrt{x}(\sqrt[3]{x} + 1)}$

Q.8 $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$

Q.9 $\int \frac{x \ln x}{(x^2 - 1)^{3/2}} dx$

Q.10 $\int \frac{\log 6 \sqrt[6]{((\sin x) 6^{\cos x})} \cos x}{\sin x} dx$

Q.11 $\int \left[\frac{\sqrt{x^2 + 1} [\log(x^2 + 1) - 2 \log x]}{x^4} \right] dx$

Q.12 If $f(x) = \frac{\sin x}{\sin^2 x + 4 \cos^2 x}$ the antiderivative of $\left(\frac{1}{\sqrt{3}} \right) \tan^{-1} \left(\left(\frac{1}{\sqrt{3}} \right) g(x) \right) + c$ is then $g(x)$ is equal to

Q.13 $\int \frac{\cot x dx}{(1 - \sin x)(\sec x + 1)}$

Q.14 $\int \frac{3x^2 + 1}{(x^2 - 1)^3} dx$

Q.15 $\int \frac{(ax^2 - b) dx}{x \sqrt{c^2 x^2 - (ax^2 + b)^2}}$

Q.16 Evaluate $\int \frac{e^{m \tan^{-1} x}}{(1 + x^2)^{3/2}} dx$

Q.17 Evaluate $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$

Q.18

$\int \frac{(e^{\sqrt{x}} - e^{-\sqrt{x}}) \cos \left(e^{\sqrt{x}} + e^{-\sqrt{x}} + \frac{\pi}{4} \right) + (e^{\sqrt{x}} + e^{-\sqrt{x}}) \cos \left(e^{\sqrt{x}} - e^{-\sqrt{x}} + \frac{\pi}{4} \right)}{\sqrt{x}} dx$

Q.19 $\int \frac{x^2 + x}{(e^x + x + 1)^2} dx$

Q.20 $\int \frac{e^{\cos x} (x \sin^3 x + \cos x)}{\sin^2 x} dx$

Q.21 $\int \frac{5x^4 + 4x^5}{(x^5 + x + 1)^2} dx$

Q.22 $\int \frac{a^2 \sin^2 x + b^2 \cos^2 x}{a^4 \sin^2 x + b^4 \cos^2 x} dx$

Q.23 $\int \frac{\cos^2 x}{1 + \tan x} dx$

Q.24 $\int \log x \cdot \sin^{-1} x dx$

Q.25 Evaluate $\int \frac{(x^2 + 1)e^x}{(x + 1)^2} dx$

Q.26 Evaluate $\int \frac{e^{\sin z}}{\cos^2 x} (x \cos^3 x - \sin x) dx$

Q.27 Evaluate $\int \frac{dx}{\sqrt{1 - 2x - x^2}}$

Q.28 $\int \frac{dx}{\sec x + \operatorname{cosec} x}$

Q.29 Evaluate $\int \sqrt{2x^2 + 3x + 4} dx$

Q.30 Evaluate $\int \frac{dx}{x(x^n + 1)}$

Q.31 $\int \frac{\cos x - \sin x}{7 - 9 \sin 2x} dx$

Q.32 $\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{1 + 3 \sin 2x} dx$

Q.33 $\int \frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2(x^2 + 1)^2} dx$

Q.34 $\int \frac{dx}{\cos^3 x - \sin^3 x}$

Q.35 $\int \frac{x^2}{(x \cos x - \sin x)(x \sin x + \cos x)} dx$

Q.36 $\int \cos 2\theta \cdot \ln \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} d\theta$

Q.37 Match the columns

Column I	Column II
(A) $\int \frac{x^4 - 1}{x^2 \sqrt{x^4 + x^2 + 1}} dx$	(p) $\log \left(\frac{(x^2 + 1) + \sqrt{x^4 + 1}}{x} \right) + c$
(B) $\int \frac{x^2 - 1}{x \sqrt{1 + x^4}} dx$	(q) $C - \frac{1}{2} \log \left(\frac{\sqrt{x^4 + 1} - 2\sqrt{x}}{(x^2 - 1)} \right)$
(C) $\int \frac{1 + x^2}{(1 - x^2) \sqrt{1 + x^4}} dx$	(r) $C - \tan^{-1} \left(\sqrt{1 + \frac{1}{x^4}} - 1 \right)$

Column I	Column II
(D) $\int \frac{1}{(1+x^4)\sqrt{1+x^4-x^2}} dx$	(s) $\frac{\sqrt{x^4+x^2+1}}{x} + c$

Exercise 2

Single Correct Choice Type

Q.1 If $\int \frac{\tan^{-1} x - \cot^{-1} x}{\tan^{-1} x + \cot^{-1} x} dx$ is equal to :

- (A) $\frac{4}{\pi} x \tan^{-1} x + \frac{2}{\pi} \log(1+x^2) - x + c$
 (B) $\frac{4}{\pi} x \tan^{-1} x - \frac{2}{\pi} \log(1+x^2) + x + c$
 (C) $\frac{4}{\pi} x \tan^{-1} x + \frac{2}{\pi} \log(1+x^2) + x + c$
 (D) $\frac{4}{\pi} x \tan^{-1} x - \frac{2}{\pi} \log(1+x^2) - x + c$

Q.2 $\int \frac{x^2 - 4}{x^4 + 24x^2 + 16} dx$ equals

- (A) $\frac{1}{4} \tan^{-1} \left(\frac{x^2 + 4}{4x} \right) + c$ (B) $-\frac{1}{4} \cot^{-1} \left(\frac{x^2 + 4}{x} \right) + c$
 (C) $-\frac{1}{4} \cot^{-1} \left(\frac{4x^2 + 4}{x} \right) + c$ (D) $\frac{1}{4} \cot^{-1} \left(\frac{x^2 + 4}{x} \right) + c$

Q.3 $\int \frac{(x-1)^2}{x^4 + 2x^2 + 1} dx$ equals

- (A) $\frac{x^3}{3} + x + \frac{x}{x^2 + 1} + c$ (B) $\frac{x^5 + x^3 + x + 3}{3(x^2 + 1)} + c$
 (C) $\frac{x^5 + 4x^3 + 3x + 3}{3(x^2 + 1)} + c$ (D) None of these

Q.4 $\int \frac{x^4 - 4}{x^2 \sqrt{4 + x^2 + x^4}} dx$ equals

- (A) $\frac{\sqrt{4 + x^2 + x^4}}{x} + c$ (B) $\sqrt{4 + x^2 + x^4} + c$
 (C) $\frac{\sqrt{4 + x^2 + x^4}}{2} + c$ (D) $\frac{\sqrt{4 + x^2 + x^4}}{2x} + c$

Q.5 $\int \frac{\sec x + \tan x - 1}{\tan x - \sec x + 1} dx$ equals

- (A) $\log [(\sec x + \tan x)] + \log \sec x + c$
 (B) $\log [\sec x - \tan x] - \log \cos x + c$
 (C) $\log (\sec x + \tan x) - \log \sec x + c$
 (D) $-\log (\sec x + \tan x) + \log \cos x + c$

Q.6 $\int e^x \frac{(x^2 - 3)}{(x + 3)^2} dx$

- (A) $e^x \cdot \frac{x}{x+3} + c$ (B) $e^x \left(2 - \frac{6}{x+3} \right) + c$
 (C) $e^x \left(1 - \frac{6}{x+3} \right) + c$ (D) $e^x \frac{3}{x+3} + c$

Q.7 $\int \frac{\sqrt{1 - \cos x}}{\cos \alpha - \cos x} dx$ where $0 < \alpha < x < \pi$, equals

- (A) $2 \log \left(\cos \frac{\alpha}{2} - \cos \frac{x}{2} \right) + c$ (B) $2 \cos^{-1} \left(\frac{\cos(x/2)}{\cos(\alpha/2)} \right) + c$
 (C) $2\sqrt{2} \log \left(\cos \frac{\alpha}{2} - \cos \frac{x}{2} \right) + c$ (D) $-2 \sin^{-1} \left(\frac{\cos(x/2)}{\cos(\alpha/2)} \right) + c$

Q.8 Primitive of $\frac{3x^4 - 1}{(x^4 + x + 1)^2}$ w.r.t. x is :

- (A) $\frac{x}{x^4 + x + 1} + c$ (B) $-\frac{x}{x^4 + x + 1} + c$
 (C) $\frac{x+1}{x^4 + x + 1} + c$ (D) $-\frac{x+1}{x^4 + x + 1} + c$

Q.9 If $\int e^{3x} \cos 4x dx = e^{3x} (A \sin 4x + B \cos 4x) + c$

Then

- (A) $4A = 3B$ (B) $2A = 3B$
 (C) $3A = 4B$ (D) $4B + 3A = 1$

Q.10 The evaluation of $\int \frac{px^{p+2q-1} - qx^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1} dx$ is

- (A) $-\frac{x^p}{x^{p+q} + 1} + c$ (B) $\frac{x^q}{x^{p+q} + 1} + c$
 (C) $-\frac{x^q}{x^{p+q} + 1} + c$ (D) $\frac{x^p}{x^{p+q} + 1} + c$

Q.11 If $\int f(x)dx = g(x)$, then $\int f^{-1}(x)dx$ is equal to –

- (A) $g^{-1}(x)$ (B) $xf^{-1}(x) - g(f^{-1}(x))$
 (C) $xf^{-1}(x)g^{-1}(x)$ (D) $f^{-1}(x)$

Q.12 Primitive of $\sqrt[3]{\frac{x}{(x^4-1)^4}}$ w.r.t x is –

- (A) $\frac{3}{4}\left(1 + \frac{1}{x^4-1}\right)^{\frac{1}{3}} + c$ (B) $-\frac{3}{4}\left(1 + \frac{1}{x^4-1}\right)^{\frac{1}{3}} + c$
 (C) $\frac{4}{3}\left(1 + \frac{1}{x^4-1}\right)^{\frac{1}{3}} + c$ (D) $-\frac{4}{3}\left(1 + \frac{1}{x^4-1}\right)^{\frac{1}{3}} + c$

Q.13 If $\int e^u \cdot \sin 2x dx$ can be found in terms of known functions of x then u can be :

- (A) x (B) $\sin x$ (C) $\cos x$ (D) $\cos 2x$

Previous Years' Questions

Q.1 The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$ is
 (1995)

- (A) $\sin x - 6 \tan^{-1}(\sin x) + c$
 (B) $\sin x - 2(\sin x)^{-1} + c$
 (C) $\sin x - 2(\sin x)^{-1} - 6 \tan^{-1}(\sin x) + c$
 (D) $\sin x - 2(\sin x)^{-1} + 5 \tan^{-1}(\sin x) + c$

Q.2 Let $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and

$g(x) = \underbrace{\text{fofo...of}}_{f \text{ occurs } n \text{ times}}(x)$. Then, $\int x^{n-2} g(x) dx$ equals (2007)

- (A) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + c$
 (B) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}} + c$
 (C) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}} + c$
 (D) $\frac{1}{n+1}(1+nx^n)^{1-\frac{1}{n}} + c$

Q.3 Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx$

Then, for an arbitrary constant c, the value of J-I equals
 (2008)

- (A) $\frac{1}{2} \log \left| \frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1} \right| + c$ (B) $\frac{1}{2} \log \left| \frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1} \right| + c$
 (C) $\frac{1}{2} \log \left| \frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right| + c$ (D) $\frac{1}{2} \log \left| \frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1} \right| + c$

Q.4 $f(x)$ is the integral of

$\frac{2 \sin x - \sin 2x}{x^3}$, $x \neq 0$ find $\lim_{x \rightarrow 0} f'(x)$. (1979)

Q.5 Evaluate the following integrals (1980)

- (i) $\int \sqrt{1 + \sin\left(\frac{1}{2}x\right)} dx$ (ii) $\int \frac{x^2}{\sqrt{1-x}} dx$

Q.6 Evaluate $\int (e^{\log x} + \sin x) \cos x dx$. (1981)

Q.7 Evaluate $\int \frac{(x-1)e^x}{(x+1)^3} dx$. (1983)

Q.8 Evaluate $\int \frac{dx}{x^2(x^4+1)^{3/4}}$. (1984)

Q.9 $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$. (1985)

Q.10 Evaluate $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$. (1986)

Q.11 Evaluate $\int \frac{(\cos 2x)^{1/2}}{\sin x} dx$. (1987)

Q.12 Find the indefinite integral

$\int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\log(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}} \right) dx$ (1992)

Q.13 Integrate $\int \frac{x^3 + 3x + 2}{(x^2 + 1)^2 (x + 1)} dx$ (1999)

Q.14 Evaluate $\int \sin^{-1} \left(\frac{2x + 2}{\sqrt{4x^2 + 8x + 13}} \right) dx$. (2000)

Q.15 For any natural number m , evaluate $\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6x^m)^{1/m} dx, x > 0$. (2002)

Q.16 Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx, J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx$.

Then, for an arbitrary constant C , the value of $J - I$ equals (2008)

(A) $\frac{1}{2} \log \left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1} \right) + c$ (B) $\frac{1}{2} \log \left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1} \right) + c$

(C) $\frac{1}{2} \log \left(\frac{e^x - e^{-x} - 1}{e^x + e^{-x} + 1} \right) + c$ (D) $\frac{1}{2} \log \left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^x + 1} \right) + c$

Q.17 The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K) (2012)

(A) $-\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$

(B) $\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$

(C) $-\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$

(D) $\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.14 Q.19 Q.23 Q.25
Q.29 Q.32 Q.35 Q.38

Exercise 2

Q.4 Q.10 Q.13 Q.15

Previous Years' Questions

Q.5 Q.7 Q.9

JEE Advanced/Boards

Exercise 1

Q.3 Q.11 Q.18 Q.20
Q.30 Q.37 Q.38

Exercise 2

Q.1 Q.5 Q.7 Q.10
Q.12

Previous Years' Questions

Q.1 Q.2 Q.4 Q.10
Q.12

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $\tan x - \sec x + A$

Q.2 $x + \tan^{-1} x - 2 \sin^{-1} x + 5 \sec^{-1} x + \frac{a^x}{\log a} + A$

Q.3 $\frac{x^2}{2} + A$

Q.4 $\log |x + \log(\sec x)| + A$

Q.5 $\log |3 \cos x + 2 \sin x| + A$

Q.6 $2\sqrt{x^2 - x - 1} + A$

Q.7 $\frac{1}{18} \left[(1 - 3x)^{3/2} + (5 - 3x)^{3/2} \right] + A$

Q.8 $\frac{1}{3} \sin(e^{x^3}) + A$

Q.9 $\frac{x}{1+x^2} + c$

Q.10 $-\frac{1}{2(2 \tan x + 3)} + A$

Q.11 $-\frac{5}{8} \cos^{8/5} x + \frac{5}{18} \cos^{18/5} x + A$

Q.12 $\frac{-\log x}{x} - \frac{1}{x} + A$

Q.13 $\left(\tan^{-1} \sqrt{\frac{x}{a}} \right) (a+x) - a \sqrt{\frac{x}{a}} + c$

Q.14 $e^x \tan x + A$

Q.15 $\log |2 \log x + 1| - \log |3 \log x + 2| + A$

Q.16 $\frac{5}{8} \log |x+3| + \frac{3}{8} \log |x-1| - \frac{1}{2(x-1)} + A$

Q.17 $\frac{1}{2} x - \frac{1}{2} \log |\cos x - \sin x| + A$

Q.18 $f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$

Q.19 $\frac{1}{6(m+1)} (2x^{3m} + 3x^{2m} + 6x^m)^{m+1/m}$

Q.20 $\frac{1}{4} \log(x^2 + 1) - \frac{1}{2} \log(x+1) + \frac{3}{2} \tan^{-1} x + \frac{x}{x^2 + 1}$

Q.21 $\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+p}{\sqrt{3}-p} \right| + \tan^{-1}(q) + c$

Q.22 $-\log |\cot x + \sqrt{\cot^2 x - 1}| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + c$

Q.23 $-\frac{1}{3} \left(1 + \frac{1}{x^2} \right)^{3/2} \left[\log \left(1 + \frac{1}{x^2} \right) - \frac{2}{3} \right] + c$

Q.24 $\frac{1}{2} x + \frac{1}{2} \log |\sin x - \cos x| + c$

Q.25 $\frac{x^2}{2} \sin^{-1} \left(\frac{1}{2} \sqrt{\frac{2a-x}{a}} \right) + \frac{a^2}{2} \left(\sin^{-1} \frac{x}{2a} - \frac{x}{2a} \sqrt{1 - \frac{x^2}{4a^2}} \right) + c$

Q.26 $\frac{1}{3} \tan^3 x + 2 \tan x - \cot x + c$

Q.27 $\frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \tan(\theta/2) \right] + c_1$

Q.28 $\int \frac{1}{2} d\theta = \frac{1}{2} \theta + c = \frac{1}{2} \sec^{-1}(x^2) + c$

Q.29 $\frac{-2}{\sin \alpha} \sqrt{\cos \alpha + \cot x \sin \alpha} + c$

Q.30 $2 \left(\sqrt{\frac{x}{1-x}} - \frac{1}{\sqrt{1-x}} \right) + c$

Q.31 $\frac{1}{2} \int (\cos 3x - \cos 2x) dx = \frac{1}{3} \sin 3x - \frac{1}{2} \sin 2x + c$

Q.32 $-\log |x| + 2 \log |x-1| + \frac{1}{x-1} - \frac{1}{(x-1)^2} + c$

Q.33 $2\log t - \frac{3}{2}\log(2t-1) - \frac{3}{2}\frac{1}{(2t-1)} + c$, where $t = x + \sqrt{x^2 - x + 1}$

Q.34 $\log(e^x + \sqrt{e^{2x} - 4}) - \log 2 + c = \log(e^x + \sqrt{e^{2x} - 4}) + c$

Q.35 $I = -\frac{1}{2}\frac{\log x}{(1+x)^2} + \frac{1}{2}\log \frac{x}{1+x} + \frac{1}{2(1+x)} + c$

Q.36 $\log \frac{x^2 + x + 1}{|x-1|} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$

Exercise 2

Single Correct Choice Type

Q.1 A

Q.2 B

Q.3 C

Q.4 B

Q.5 C

Q.6 C

Q.7 A

Q.8 C

Q.9 A

Q.10 C

Q.11 B

Q.12 B

Q.13 B

Q.14 D

Q.15 A

Previous Years' Questions

Q.1 C **Q.2** $A = -\frac{3}{2}$, $B = \frac{35}{36}$ and $C \in \mathbb{R}$ **Q.3** $\frac{1}{2}\log(\sin x - \cos x) + \frac{x}{2} + c$

Q.4 $\frac{1}{2}\tan^{-1}(x^2) + c$ **Q.5** $-\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24} + \sin 2x + \tan x - 2x + \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$

Q.6 $\frac{1}{b^3} \left(a + bx - 2a \log(a + bx) - \frac{a^2}{a + bx} + c \right)$ **Q.7** $\sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right) + c$

Q.8 $\int \frac{dt}{t^2(t-1)} = \log \left| \frac{xe^x}{1+xe^x} \right| + \frac{1}{1+xe^x} + c$ **Q.9** $2[\cos^{-1} \sqrt{x} - \log|1 + \sqrt{1-x}| - \frac{1}{2}\log|x|] + c$

Q.10 C

Q.11 D

Q.12 D

Q.13 C

Q.14 D

Q.15 D

Q.16 A

JEE Advanced/Boards

Exercise 1

Q.1 (i) $2\log\left(\sec \frac{x}{2}\right) - 3\log\left(\sec \frac{x}{3}\right) - 6\log\left(\sec \frac{x}{6}\right) + c$ (ii) $\log \left(\frac{\sec(\log x)}{\sec\left(\log \frac{x}{2}\right)} \right) - x \tan \log |2| + c$

$$\text{Q.2 } \frac{-2}{\alpha - \beta} \cdot \frac{\sqrt{x - \beta}}{x - \alpha} + c$$

$$\text{Q.3 } \log\left(\frac{1+x}{1-x}\right) \log\left(\frac{1+x}{1-x}\right) \log\left(\frac{1+x}{1-x}\right) + c$$

$$\text{Q.4 } \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + c$$

$$\text{Q.5 } \cos a \cdot \arccos\left(\frac{\cos x}{\cos a}\right) - \sin a \cdot \log\left(\sin x + \sqrt{\sin^2 x - \sin^2 a}\right) + c$$

$$\text{Q.6 } \frac{x^4}{4} + x^3 - x^2 + 5x + \frac{1}{2} \log(x^2 + 1) + 3 \tan^{-1} x + c$$

+ c where $t = x^{1/6}$

$$\text{Q.7 } 6 \left[\frac{t^4}{4} - \frac{t^2}{2} + t + \frac{1}{2} \log(1 + t^2) - \tan^{-1} t \right] + C \quad \text{where } t = x^{1/6}$$

$$\text{Q.8 } (a+x) \tan^{-1} \left[\frac{\sqrt{x}}{a} - \sqrt{ax} \right] + c$$

$$\text{Q.9 } \sec^{-1} x - \frac{\log x}{\sqrt{x^2 - 1}} + c$$

$$\text{Q.10 } \frac{1}{6} \left[\frac{\log^2(\sin x)}{\log 36} + \log \left[\tan \frac{x}{2} + \cos x \right] \right] + c$$

$$\text{Q.11 } -\frac{1}{3} \left[1 + \frac{1}{x^2} \right]^{3/2} \left[\log \left(1 + \frac{1}{x^2} \right) - \frac{2}{3} \right] + c$$

$$\text{Q.12 } \int f(x) dx = \int \frac{dt}{t^2 + 3} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{\sec x}{\sqrt{3}} + c$$

$$\text{Q.13 } \frac{1}{2} \log \left| \tan \frac{x}{2} \right| + \frac{1}{4} \tan^2 \frac{x}{2} + \tan \frac{x}{2} + c$$

$$\text{Q.14 } -\frac{x}{(x^2 - 1)^2} + C$$

$$\text{Q.15 } \sin^{-1} \left(\frac{ax^2 + b}{cx} \right) + k$$

$$\text{Q.16 } 1 = \frac{e^{m \tan^{-1} x}}{m^2 + 1} \left(m \cdot \frac{1}{\sqrt{x^2 + 1}} + \frac{x}{\sqrt{x^2 + 1}} \right) + c = \frac{e^{m \tan^{-1} x} (m + x)}{(m^2 + 1) \sqrt{x^2 + 1}} + c$$

$$\text{Q.17 } a \left[\tan^{-1} \sqrt{\frac{x}{a}} \left(\frac{a+x}{a} \right) - \sqrt{\frac{x}{a}} \right] + c$$

$$\text{Q.18 } \sqrt{2} \left(\cos(e^{-\sqrt{x}}) \right) \left(\sin(e\sqrt{x}) + \cos(e\sqrt{x}) \right) + c$$

$$\text{Q.19 } -\log \left(\frac{x+1}{e^x} + 1 \right) - \frac{1}{\left(\frac{x+1}{e^x} + 1 \right)} + c$$

$$\text{Q.20 } -e^{\cos x} (x + \operatorname{cosec} x) + c$$

$$\text{Q.21 } C - \frac{x+1}{x^5 + x + 1} \text{ or } c + \frac{x^5}{x^5 + x + 1}$$

$$\text{Q.22 } \frac{1}{a^2 + b^2} \left(x + \tan^{-1} \left(\frac{a^2 \tan x}{b^2} \right) \right) + c$$

$$\text{Q.23 } \frac{1}{4} \log(\cos x + \sin x) + \frac{x}{2} + \frac{1}{8} (\sin 2x + \cos 2x) + c$$

$$\text{Q.24 } (x \log x - x) \sin^{-1} x + \sqrt{1 - x^2} \log x - \log \left(\frac{1 - \sqrt{1 - x^2}}{x} \right) + c$$

$$\text{Q.25 } e^x f(x) + c = e^x \left(\frac{x-1}{x+1} \right) + c$$

$$\text{Q.26 } e^{zf(z)} + c = e^z \left(\sin^{-1} z - \frac{1}{\sqrt{1 - z^2}} \right) + c = e^{\sin x} (x - \sec x) + c$$

$$\text{Q.27 } \sin^{-1} \left(\frac{1+x}{\sqrt{2}} \right) + c$$

$$\text{Q.28 } \frac{1}{2} \left[\sin x - \cos x - \frac{1}{\sqrt{2}} \log \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right] + C$$

$$\text{Q.29 } I = \frac{4x+3}{8} \sqrt{2x^2 + 3x + 4} + \frac{23}{16\sqrt{2}} \log \left[\frac{4x+3}{4} + \frac{\sqrt{2x^2 + 3x + 4}}{\sqrt{2}} \right] + c$$

$$\text{Q.30 } \frac{1}{n} \log \left| \frac{z-1}{z} \right| + c = \frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + c$$

$$\text{Q.31 } \frac{1}{24} \log \frac{(4+3\sin x+3\cos x)}{(4-3\sin x-3\cos x)} + c$$

$$\text{Q.32 } \tan^{-1} \left(\frac{\sqrt{2} \sin 2x}{\sin x + \cos x} \right) + c$$

$$\text{Q.33 } 4 \log x + \frac{7}{x} + 6 \tan^{-1}(x) + \frac{6x}{1+x^2} + c$$

$$\text{Q.34 } \frac{2}{3} \tan^{-1}(\sin x + \cos x) + \frac{1}{3\sqrt{2}} \log \left| \frac{\sqrt{2} + \sin x + \cos x}{\sqrt{2} - \sin x - \cos x} \right| + c$$

$$\text{Q.35 } \log \left| \frac{x \sin x + \cos x}{x \cos x - \sin x} \right| + c$$

$$\text{Q.36 } \frac{1}{2}(\sin 2\theta) \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) - \frac{1}{2} \log(\sec 2\theta) + c = \frac{2}{\cos 2\theta} + c$$

$$\text{Q.37 } A \rightarrow s; B \rightarrow p; C \rightarrow q; D \rightarrow r$$

Exercise 2

Q.1 D

Q.2 A

Q.3 D

Q.4 A

Q.5 A

Q.6 C

Q.7 B

Q.8 B

Q.9 C

Q.10 C

Q.11 B

Q.12 B

Q.13 A

Previous Years' Questions

Q.1 C

Q.2 A

Q.3 C

Q.4 1

$$\text{Q.5 (i) } 4 \sin \frac{x}{4} - 4 \cos \frac{x}{4} + c \quad \text{(ii) } -2 \left\{ \sqrt{1-x} - \frac{2}{2}(1-x)^{3/2} + \frac{1}{5}(1-x)^{5/2} \right\} + c$$

$$\text{Q.6 } x \sin x + \cos x - \frac{\cos 2x}{4} + c$$

$$\text{Q.7 } \frac{e^x}{(x+1)^2} + c$$

$$\text{Q.8 } -\frac{(x^4+1)^{1/4}}{x} + c$$

$$\text{Q.9 } -2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x(1-x)} + c$$

$$\text{Q.10 } \frac{2}{\pi} [\sqrt{x-x^2} - (1-2x) \sin^{-1} \sqrt{x}] - x + c$$

$$\text{Q.11 } -\log |\cot x + \sqrt{\cot^2 x - 1}| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + c$$

$$\text{Q.12 } \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} - \frac{4}{3}x^{1/2} - \frac{12}{5}x^{5/12} + \frac{1}{2}x^{1/3} - 4x^{1/4} - 7x^{1/6} - 12x^{1/12} + (2x^{1/2} - 3x^{1/3} - 6x^{1/6} + 11) \log(1+x^{1/6}) + 12 \log(1+x^{1/2}) - 3[\log(1+x^{1/6})]^2 + c$$

$$\text{Q.13 } -\frac{1}{2} \log |x+1| + \frac{1}{4} \log |x^2+1| + \frac{3}{2} \tan^{-1} x + \frac{x}{x^2+1} + c$$

$$\text{Q.14 } (x+1) \tan^{-1} \left(\frac{2x+2}{3} \right) - \frac{3}{4} \log(4x^2+8x+13) + c$$

$$\text{Q.15 } \frac{1}{6(m+1)} \cdot (2x^{3m} + 3x^{2m} + 6x^m)^{(m+1)/m} + c$$

Q.16 C

Q.17 C

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\int \frac{\sec x}{\sec x + \tan x} dx = \int \frac{\sec x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx$

$= \int (\sec^2 x - \sec x \tan x) dx = \tan x - \sec x + C$

Sol 2: $\int \left(1 + \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + 5 \frac{1}{|x|\sqrt{x^2-1}} + a^x \right) dx$

$= \int dx + \int \frac{dx}{1+x^2} - \int \frac{2dx}{\sqrt{1-x^2}} + 5 \int \frac{dx}{|x|\sqrt{x^2-1}} + \int a^x dx$

$= x + \tan^{-1}x - 2\sin^{-1}x + 5\sec^{-1}x + \frac{a^x}{\log a} + C$

Sol 3: $\frac{\sin 2x}{1 + \cos 2x} = \tan x$

$\int \tan^{-1} \left(\frac{\sin 2x}{1 + \cos 2x} \right) dx = \int \tan^{-1} \tan x dx$

$= \int x dx = \frac{x^2}{2} + C$

Sol 4: $\int \frac{1 + \tan x}{x + \log \sec x} dx$

Put $x + \log \sec x = t$

$\Rightarrow \left(1 + \frac{1}{\sec x} \sec x \tan x \right) dx = dt$

$\therefore \int \frac{dt}{t} = \log t + C = \log(x + \log \sec x) + C$

Sol 5: Put $3\cos x + 2\sin x = t$

$-3\sin x + 2\cos x = \frac{dt}{dx}$

$\therefore I = \int \frac{dt}{t} = \log t = \log(3\cos x + 2\sin x)$

Sol 6: $\int \frac{2x-1}{\sqrt{x^2-x-1}} dx$

Put $x^2 - x - 1 = t$

$\Rightarrow (2x-1)dx = dt$

$\int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} dt = 2\sqrt{t} + C = 2\sqrt{x^2-x-1} + C$

Sol 7: $\int \frac{dx}{\sqrt{1-3x} - \sqrt{5-3x}}$

$= \int \frac{\sqrt{1-3x} + \sqrt{5-3x}}{(-4)} dx$

$= -\frac{1}{4} \int \sqrt{1-3x} dx - \frac{1}{4} \int \sqrt{5-3x} dx$

$= -\frac{1}{12} \times \frac{2}{3} (\sqrt{1-3x})^{3/2} - \frac{1}{12} \times \frac{2}{3} (\sqrt{5-3x})^{3/2}$

$= -\frac{1}{18} \left[(1-3x)^{3/2} + (5-3x)^{3/2} \right] + C$

Sol 8: $\int x^2 e^{x^3} \cos(e^{x^3}) dx$

$e^{x^3} = t \Rightarrow 3x^2 e^{x^3} dx = dt$

$\therefore \frac{1}{3} \int \cos t dt = \frac{1}{3} \sin t + C = \frac{1}{3} \sin e^{x^3} + C$

Sol 9: $\int \frac{\sec^2(2\tan^{-1}x)}{1+x^2} dx$

$2\tan^{-1}x = t \Rightarrow \frac{2}{1+x^2} dx = dt$

$\frac{1}{2} \int \sec^2 t dt = \frac{1}{2} \tan t + C = \frac{1}{2} \tan(2\tan^{-1}x) + C$

$= \frac{1}{2} \frac{2\tan(\tan^{-1}x)}{1+\tan^2(\tan^{-1}x)} + C = \frac{x}{1+x^2} + C$

Sol 10: $\int \frac{dx}{(2\sin x + 3\cos x)^2} = \int \frac{\sec^2 x}{(2\tan x + 3)} dx$

Put $2\tan x + 3 = t \Rightarrow 2\sec^2 x dx = dt$

$\therefore \frac{1}{2} \int \frac{dt}{t^2} = -\frac{1}{2t} + C = -\frac{1}{2} \times \frac{1}{(2\tan x + 3)} + C$

Sol 11: $\int \cos^{3/5} x \sin^3 x dx$

$= \int \cos^{3/5} x (1 - \cos^2 x) \sin x dx$

$$= \int (\cos^{3/5} x - \cos^{13/5} x) \sin x dx$$

Put $\cos x = t$; $-\sin x dx = dt$

$$= \int (t^{13/5} - t^{3/5}) dt = \frac{5}{18} t^{18/5} - \frac{5}{8} t^{8/5} + c$$

$$= \frac{5}{18} (\cos x)^{18/5} - \frac{5}{8} (\cos x)^{8/5} + c$$

Sol 12: $\int \frac{\log x}{x^2} dx$

$$\log x \int \frac{1}{x^2} dx - \int \left(\frac{d \log x}{dx} \int \frac{1}{x^2} dx \right) dx$$

$$= (\log x) \left(-\frac{1}{x} \right) - \int \frac{1}{x} \times \left(-\frac{1}{x} \right) dx + c$$

$$= -\frac{1}{x} \log x + \int \frac{1}{x^2} dx + c = -\frac{1}{x} (\log x + 1) + c$$

Sol 13: $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$, $a > 0$

$$x = a \tan^2 \theta \Rightarrow dx = 2a \tan \theta \sec^2 \theta d\theta$$

$$\int \sin^{-1} \sqrt{\frac{a \tan^2 \theta}{a \sec^2 \theta}} \cdot 2a \tan \theta \sec^2 \theta d\theta$$

$$= 2a \int (\sin^{-1} \sin \theta) \tan \theta \sec^2 \theta d\theta$$

$$\tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt$$

$$= 2a \int t \tan^{-1} t dt$$

$$= 2a \left[\tan^{-1} t \int t dt - \frac{1}{2} \int \frac{1}{1+t^2} \times t^2 dt \right]$$

$$= 2a \left[(\tan^{-1} t) \frac{t^2}{2} - \frac{1}{2} \left[t - \tan^{-1} t \right] \right] + c$$

$$= at^2 \tan^{-1} t + \left[\frac{1}{2} \tan^{-1} t - \frac{1}{2} t \right] 2a + c$$

$$\therefore t = \tan \theta = \frac{\sqrt{x}}{\sqrt{a}}$$

$$\therefore I = a \frac{x}{a} \tan^{-1} \sqrt{\frac{x}{a}} + \left[\frac{1}{2} \tan^{-1} \sqrt{\frac{x}{a}} - \frac{1}{2} \sqrt{\frac{x}{a}} \right] 2a + c$$

$$= x \tan^{-1} \sqrt{\frac{x}{a}} + \left[\frac{1}{2} \tan^{-1} \sqrt{\frac{x}{a}} - \frac{1}{2} \sqrt{\frac{x}{a}} \right] 2a + c$$

$$= x \tan^{-1} \sqrt{\frac{x}{a}} + a \tan^{-1} \sqrt{\frac{x}{a}} - a \sqrt{\frac{x}{a}} + c$$

$$= \left(\tan^{-1} \sqrt{\frac{x}{a}} \right) (a+x) - a \sqrt{\frac{x}{a}} + c$$

Sol 14: $\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) dx$

$$1 + \cos 2x = 2 \cos^2 x$$

$$2 + \sin 2x = 2 + 2 \sin x \cos x$$

$$\Rightarrow \int e^x \left(\frac{2 + 2 \sin x \cos x}{2 \cos^2 x} \right) = \int e^x (\sec^2 x + \tan x) dx$$

$$\tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\therefore \text{This is of form } \int e^x (f(x) + f'(x)) dx = e^x f(x)$$

$$\therefore I = e^x \tan x + c$$

Sol 15: $\int \frac{dx}{x[6(\log x)^2 + 7 \log x + 2]}$

$$\log x = t; \frac{1}{x} dx = dt$$

$$\int \frac{dt}{(6t^2 + 7t + 2)} = \int \frac{dt}{(6t^2 + 3t + 4t + 2)}$$

$$= \int \frac{dt}{(3t+2)(2t+1)} = - \int \left[\frac{3}{(3t+2)} - \frac{2}{(2t+1)} \right] dt$$

$$= -3 \int \frac{1}{3t+2} dt + 2 \int \frac{1}{2t+1} dt$$

$$= -3 \frac{1}{3} \log(3t+2) + \frac{2}{2} \log(2t+1) + c$$

$$= \frac{2}{2} \left(\frac{2t+1}{3t+2} \right) + c = \frac{2}{2} \log \left(\frac{2 \log x + 1}{3 \log x + 2} \right) + c$$

$$= \log | 2 \log x + 1 | - \log | 3 \log x + 2 | + c$$

Sol 16: $\int \frac{x^2 + 1}{(x+3)(x-1)^2} dx$

$$I = \int \frac{5}{8} \times \frac{1}{(x+3)} dx + \int \frac{3}{8(x-1)} dx + \frac{1}{2} \int \frac{1}{(x-1)^2} dx$$

$$= \frac{5}{8} \log(x+3) + \frac{3}{8} \log(x-1) - \frac{1}{2(x-1)} + c$$

Sol 17: $\int \frac{1}{(1 - \tan x)} dx$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\text{or } dx = \frac{dt}{1 + \tan^2 x} = \frac{dt}{1 + t^2}$$

$$\begin{aligned}
\therefore I &= \int \frac{1}{(1-t)(1+t^2)} dt \\
&= \frac{1}{2} \int \frac{1}{(1-t)} dt + \frac{1}{2} \int \frac{t+1}{t^2+1} dt \\
&= -\frac{1}{2} \log(1-t) + \frac{1}{4} \log(1+t^2) + \frac{1}{2} \tan^{-1} t + c \\
&= \frac{1}{2} \log\left(\frac{1}{1-\tan x}\right) + \frac{1}{2} \log \sqrt{1+\tan^2 x} + \frac{x}{2} + c \\
&= \frac{1}{2} \log\left(\frac{\sec x}{1-\tan x}\right) + \frac{x}{2} + c \\
&= \frac{x}{2} - \frac{1}{2} \log |\cos x - \sin x| + c
\end{aligned}$$

Sol 18: $f'(x) = x - \frac{1}{x^2}$

$$f(x) = \int f'(x) dx = \int \left(x - \frac{1}{x^2}\right) dx = \frac{x^2}{2} + \frac{1}{x} + c$$

$$f(1) = \frac{1}{2} + 1 + c = \frac{1}{2} \Rightarrow c = -1$$

$$\therefore f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$$

Sol 19: $\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx$

Put $x^m = t$ and integrate.

Sol 20: $\int \frac{x^3 + 3x + 2}{(x^2 + 1)^2(x + 1)} dx$

$$x^3 + 3x + 2 = x^3 + x + 2x + 2$$

$$= x(x^2 + 1) + 2(x + 1)$$

$$I = \int \frac{x(x^2 + 1) + 2(x + 1)}{(x^2 + 1)^2(x + 1)} dx$$

$$= \int \frac{x}{(x^2 + 1)(x + 1)} + 2 \int \frac{1}{(1 + x^2)^2} dx$$

$$= \int \frac{(x + 1) - 1}{(x^2 + 1)(x + 1)} dx + 2 \int \left(\frac{1}{1 + x^2}\right)^2 dx$$

$$= 2 \int \frac{1}{(1 + x^2)^2} dx - \int \frac{dx}{(x^2 + 1)(x + 1)} + \int \frac{1}{(1 + x^2)} dx$$

Put $x = \tan \theta$

$$dx = \sec^2 \theta d\theta$$

$$= 2 \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta + \tan^{-1} x + \frac{1}{2} \int \left(\frac{(x-1)}{(1+x^2)} - \frac{1}{x+1}\right) dx$$

$$= \frac{2}{2} \int (\cos 2\theta + 1) d\theta + \tan^{-1} x$$

$$= \frac{1}{2} \left[\frac{1}{2} \tan^{-1}(1 + x^2) - \tan^{-1} x - \log(x + 1) \right] + c$$

$$= \frac{1}{4} \tan^{-1}(1 + x^2) - \frac{1}{2} \tan^{-1} x - \frac{1}{2} \log(x + 1) + c$$

Sol 21: $I = \int \frac{1}{\sin x + \sec x} dx$

$$\Rightarrow \int \frac{2 \cos x}{2 + 2 \sin x \cos x} dx = \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + \sin 2x} dx$$

$$\Rightarrow \int \frac{\cos x + \sin x}{2 + 2 \sin x} dx + \int \frac{\cos x - \sin x}{2 + 2 \sin 2x} dx$$

$$\Rightarrow \int \frac{1}{2 + [1 - (\sin x + \cos x)^2]} \times d(\sin x - \cos x)$$

$$+ \int \frac{1}{2 + [(\sin x + \cos x)^2 - 1]} \times d(\sin x + \cos x)$$

$$\Rightarrow \int \frac{1}{(\sqrt{3})^2 - (\sin x - \cos x)^2} \times d(\sin x - \cos x)$$

$$+ \int \frac{1}{(1)^2 + (\sin x + \cos x)^2} \times d(\sin x + \cos x)$$

$$\Rightarrow \int \frac{1}{(\sqrt{3})^2 - p^2} dp + \int \frac{1}{1^2 + q^2} dq$$

$$\Rightarrow \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + p}{\sqrt{3} - p} \right| + \tan^{-1}(q) + c$$

Where $p = \sin x - \cos x$ & $q = \sin x + \cos x$

Sol 22:

$$I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx = \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}} dx = \int \sqrt{\cot^2 x - 1} dx$$

On putting

$$\cot x = \sec \theta \quad \& \quad -\operatorname{cosec}^2 x dx = \sec \theta \tan \theta d\theta$$

$$\text{We get, } I = \int \sqrt{\sec^2 \theta - 1} \times \frac{\sec \theta \tan \theta}{-\operatorname{cosec}^2 x} d\theta$$

$$= - \int \frac{\sec \theta \tan^2 \theta}{1 + \sec^2 \theta} d\theta$$

$$\begin{aligned}
&= -\int \frac{\sin^2 \theta}{\cos \theta + \cos^3 \theta} d\theta = -\int \frac{1 - \cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} d\theta \\
&= \int \frac{(1 + \cos^2 \theta) - 2 \cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} = -\int \sec \theta d\theta + 2 \int \frac{\cos \theta}{1 + \cos^2 \theta} d\theta \\
&\Rightarrow -\int \sec \theta d\theta + 2 \int \frac{d(\sin \theta)}{1 + \cos^2 \theta} \\
&\Rightarrow -\int \sec \theta d\theta + 2 \int \frac{d(\sin \theta)}{2 - \sin^2 \theta} \\
&\Rightarrow -\log |\sec \theta + \tan \theta| + 2 \times \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sin \theta}{\sqrt{2} - \sin \theta} \right| + c \\
&\Rightarrow -\log |\sec \theta + \sqrt{\sec^2 \theta - 1}| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \cos^2 \theta}}{\sqrt{2} - \sqrt{1 - \cos^2 \theta}} \right| + c \\
&\Rightarrow -\log |\cot x + \sqrt{\cot^2 x - 1}| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + c
\end{aligned}$$

Sol 23: $\int \frac{\sqrt{x^2 + 1} (\log(x^2 + 1) - \log x^2)}{x^4} dx$

$$= \int \frac{\left(\sqrt{1 + \frac{1}{x^2}} \log \left(1 + \frac{1}{x^2} \right) \right)}{x^3} dx$$

$$1 + \frac{1}{x^2} = t \Rightarrow -\frac{2}{x^3} dx = dt$$

$$-\frac{1}{2} \int \sqrt{t} \log t dt = -\frac{1}{2} \left[(\log t) \frac{t^{3/2}}{3/2} - \int \frac{1}{t} \times \frac{t^{3/2}}{3/2} dt \right]$$

$$= \frac{1}{2} \left[-\frac{2}{3} (\log t) t^{3/2} + \frac{2}{3} \int t^{1/2} dt \right]$$

$$= \frac{1}{2} \left[-\frac{2}{3} \left[\log \left(1 + \frac{1}{x^2} \right) \right] \left[1 + \frac{1}{x^2} \right]^{3/2} + \frac{4}{9} \left(1 + \frac{1}{x^2} \right)^{3/2} + c \right]$$

$$= -\frac{1}{3} \left[1 + \frac{1}{x^2} \right]^{3/2} \left[\log \left(1 + \frac{1}{x^2} \right) - \frac{2}{3} \right] + c$$

Sol 24: $\int \frac{\sin x}{\sin x - \cos x} dx$

$$= \frac{1}{2} \int \frac{\sin x - \cos x + \sin x + \cos x}{(\sin x - \cos x)} dx = \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{dt}{t}$$

Put $\sin x - \cos x = t$

$$(\cos x + \sin x) dx = dt$$

$$\Rightarrow \frac{1}{2} x + \frac{1}{2} \log(\sin x - \cos x) + c$$

$$\therefore \int \frac{1}{t} dt = \log t = \log(\sin x - \cos x)$$

Sol 25: $\int x \sin^{-1} \left(\frac{1}{2} \cdot \frac{\sqrt{2a-x}}{a} \right) dx$

$$\sin^{-1} \left(\frac{1}{2} \cdot \frac{\sqrt{2a-x}}{a} \right) \int x dx - \int \left[\frac{d \sin^{-1} \frac{1}{2} \cdot \frac{\sqrt{2a-x}}{a}}{dx} \right] \int x dx dx$$

$$\Rightarrow \frac{x^2}{2} \sin^{-1} \left(\frac{1}{2} \cdot \frac{\sqrt{2a-x}}{a} \right)$$

$$- \int \left[\frac{x^2}{2} \times \frac{1}{\sqrt{1 - \left(\frac{2a-x}{4a^2} \right)}} \times \frac{1}{2a} \times \frac{-1}{2\sqrt{2a-x}} \right] dx$$

$$\Rightarrow \frac{x^2}{2} \sin^{-1} \left(\frac{1}{2} \cdot \frac{\sqrt{2a-x}}{a} \right)$$

$$+ \frac{2a}{8a} \int \frac{x^2}{\sqrt{4a^2 - 2a + x}} \times \frac{1}{\sqrt{2a-x}} dx$$

Sol 26: $\int \frac{1}{\cos^4 x \sin^2 x} dx$

$$\int \frac{(\sin^2 x + \cos^2 x)^2}{\sin^2 x \cos^4 x} dx$$

$$= \int \left(\frac{\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x}{\sin^2 x \cos^4 x} \right) dx$$

$$= \int \left(\frac{\sin^2 x}{\cos^4 x} + \frac{1}{\sin^2 x} + \frac{2}{\cos^2 x} \right) dx$$

$$= \int \tan^2 x \sec^2 x dx + \int \operatorname{cosec}^2 x dx + 2 \int \sec^2 x dx$$

$$= \frac{\tan^3 x}{3} - \cot x + 2 \tan x + c$$

Sol 27: $I = \int \frac{1}{(a + b \cos \theta)^2} d\theta$

Let $P = \frac{\sin \theta}{(a + b \cos \theta)}$

$$\frac{dP}{d\theta} = \frac{(a+b \cos \theta) \cos \theta - \sin \theta (0 - b \sin \theta)}{(a+b \cos \theta)^2}$$

$$= \frac{a \cos \theta + b(\cos^2 \theta + \sin^2 \theta)}{(a+b \cos \theta)^2} = \frac{a \cos \theta + b}{(a+b \cos \theta)^2}$$

$$\text{Let } a+b \cos \theta = Y \Rightarrow \cos \theta = \frac{Y-a}{b}$$

$$\frac{dP}{d\theta} = \frac{a\left(\frac{Y-a}{b}\right) + b}{\left[a+b\left(\frac{Y-a}{b}\right)\right]^2} = \frac{aY + b^2 - a^2}{bY^2}$$

$$\Rightarrow \frac{dP}{d\theta} = \frac{a}{b}\left(\frac{1}{Y}\right) + \left(\frac{b^2 - a^2}{b}\right) \frac{1}{Y^2}$$

$$\Rightarrow \frac{dP}{d\theta} = \frac{a}{b}\left(\frac{1}{a+b \cos \theta}\right) + \left(\frac{b^2 - a^2}{b}\right)\left(\frac{1}{a+b \cos \theta}\right)^2$$

Integrating,

$$P = \frac{a}{b} \int \frac{1}{a+b \cos \theta} d\theta + \frac{b^2 - a^2}{b} \int \frac{1}{(a+b \cos \theta)^2} d\theta$$

$$\Rightarrow \frac{-(b^2 - a^2)}{b} \int \frac{1}{(a+b \cos \theta)^2} d\theta = \frac{a}{b} \int \frac{1}{a+b \cos \theta} d\theta - P$$

$$\Rightarrow \int \frac{1}{(a+b \cos \theta)^2} d\theta = \frac{b}{(a^2 - b^2)} \left[\frac{a}{b} \int \frac{1}{a+b \cos \theta} d\theta - P \right]$$

$$= \frac{b}{a^2 - b^2} \left[\frac{a}{b} I_1 - \frac{\sin \theta}{(a+b \cos \theta)} \right] + c$$

$$\text{Where } I_1 = \int \frac{1}{a+b \cos \theta} d\theta$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \tan(\theta/2) \right] + c_1$$

Sol 28: [Here, $\sqrt{x^4 - 1} = \sqrt{(x^2)^2 - 1}$, which is of the form, $\sqrt{x^2 - a^2}$ hence substitution $x^2 = \sec \theta$ may be tried]

$$\text{Now, } \int \frac{dx}{x\sqrt{x^4 - 1}} = \int \frac{dx}{x\sqrt{(x^2)^2 - 1}} \quad \dots(i)$$

Let $x^2 = \sec \theta$, then $2x dx = \sec \theta \tan \theta d\theta$

$$dx = \frac{\sec \theta \tan \theta}{2x} d\theta = \frac{\sec \theta \tan \theta}{2\sqrt{\sec \theta}} d\theta$$

Now, from (i)

$$\int \frac{dx}{x\sqrt{x^4 - 1}} = \int \left(\frac{1}{\sqrt{\sec \theta} \sqrt{\sec^2 \theta - 1}} \right) \frac{\sec \theta \cdot \tan \theta}{2\sqrt{\sec \theta}} d\theta$$

$$= \int \left(\frac{1}{\sqrt{\sec \theta} \cdot \tan \theta} \right) \frac{\sec \theta \tan \theta}{2\sqrt{\sec \theta}} d\theta$$

$$= \int \frac{1}{2} d\theta = \frac{1}{2} \theta + c = \frac{1}{2} \sec^{-1}(x^2) + c$$

$$\text{Sol 29: } I = \int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

$$= \int \frac{1}{\sqrt{\sin^3 x [\sin x \cos \alpha + \sin \alpha \cos x]}} dx$$

$$= \int \frac{1}{\sin^4 x (\cos \alpha + \cot x \sin \alpha)} dx$$

$$= -\frac{1}{\sin \alpha} \int \frac{1}{\sqrt{\cos \alpha + \cot x \sin \alpha}} d(\cos \alpha + \cot x \sin \alpha)$$

$$= -\frac{1}{\sin \alpha} \int \frac{1}{\sqrt{t}} dt \quad ; \quad \text{where } t = \cos \alpha + \cot x \sin \alpha$$

$$= \frac{-2}{\sin \alpha} \sqrt{\cos \alpha + \cot x \sin \alpha} + c$$

$$\text{Sol 30: } I = \int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}} dx$$

$$\text{put } x = \sin^2 \theta \quad \& \quad dx = 2 \sin \theta \cos \theta d\theta$$

$$\Rightarrow I = \int \frac{2 \sin \theta \cos \theta d\theta}{(1+\sin \theta) \sqrt{\sin^2 \theta - \sin^4 \theta}} = 2 \int \frac{1 - \sin \theta}{\cos^2 \theta} d\theta$$

$$\Rightarrow = 2(\tan \theta - \sec \theta) + c = 2 \left(\sqrt{\frac{x}{1-x}} - \frac{1}{\sqrt{1-x}} \right) + c$$

$$\text{Sol 31: } I = \int \frac{\cos 8x - \cos 7x}{1 + 2 \cos 5x} dx$$

$$= \frac{1}{2} \int \frac{2 \sin \frac{5x}{2} \cos 8x - 2 \sin \frac{5x}{2} \cos 7x}{\sin \frac{5x}{2} + 2 \cos 5x \sin \frac{5x}{2}} dx$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{\left(\sin \frac{21x}{2} - \sin \frac{11x}{2} \right) - \left(\sin \frac{19x}{2} - \sin \frac{9x}{2} \right)}{\sin \frac{15x}{2}} \\
&= \frac{1}{2} \int \frac{2 \sin \frac{15x}{2} \cos 3x - 2 \sin \frac{15x}{2} \cos 2x}{\sin \frac{15x}{2}} dx \\
&= \frac{1}{2} \int (\cos 3x - \cos 2x) dx = \frac{1}{3} \sin 3x - \frac{1}{2} \sin 2x + c
\end{aligned}$$

Sol 32: $I = \int \frac{x^3 + 1}{x(x-1)^3} dx$

$$\frac{x^3 + 1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

put $x = 1 \Rightarrow D = 2$

put $x = 0 \Rightarrow A = -1$

put $x = -1$ & $x = 2 \Rightarrow B = 2$ & $C = 1$

$$\begin{aligned}
\therefore I &= \int \frac{-1}{x} dx + \int \frac{2}{x-1} dx + \int \frac{1}{(x-1)^2} dx + \int \frac{2}{(x-1)^3} dx \\
&= -\log |x| + 2 \log |x-1| + \frac{1}{x-1} - \frac{1}{(x-1)^2} + c
\end{aligned}$$

Sol 33: $I = \int \frac{1}{x + \sqrt{x^2 - x + 1}} dx$

put $t = x + \sqrt{x^2 - x + 1}$

$$\Rightarrow x = \frac{t^2 - 1}{2t - 1} \text{ and } dx = \frac{2t^2 - 2t + 2}{(2t - 1)^2} dt$$

$$\Rightarrow I = 2 \int \frac{t^2 - t + 1}{t(2t - 1)^2} dt$$

let $\frac{t^2 - t + 1}{t(2t - 1)^2} = \frac{A}{t} + \frac{B}{2t - 1} + \frac{C}{(2t - 1)^2}$

Solving by partial fraction method, we get

$$A = 1, C = \frac{3}{2} \text{ and } B = -\frac{3}{2}$$

$$I = 2 \log t - \frac{3}{2} \log (2t - 1) - \frac{3}{2} \frac{1}{(2t - 1)} + c$$

where $t = x + \sqrt{x^2 - x + 1}$

Sol 34: [Here $\sqrt{e^{2x} - 4} = -\sqrt{(e^x)^2 - 2^2}$, which is of the form $\sqrt{x^2 - a^2}$, hence substitution $e^x = 2 \sec \theta$ may be tried]

$$\text{Now, } \int \frac{e^x}{\sqrt{e^{2x} - 4}} dx = \int \frac{e^x}{\sqrt{(e^x)^2 - 2^2}} dx \quad \dots(i)$$

Let $e^x = 2 \sec \theta$, then $e^x dx = 2 \sec \theta \tan \theta d\theta$

Now from (i),

$$\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx = \int \frac{2 \sec \theta \tan \theta}{\sqrt{4 \sec^2 \theta - 4}} d\theta = \int \frac{2 \sec \theta \tan \theta}{2 \tan \theta} d\theta$$

$$= \int \sec \theta d\theta = \log |\sec \theta + \tan \theta| + c \quad \dots(ii)$$

$$\therefore e^x = 2 \sec \theta \quad \therefore \sec \theta = \frac{e^x}{2}$$

$$\therefore \tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\frac{e^{2x}}{4} - 1} = \sqrt{\frac{e^{2x} - 4}{4}}$$

From (ii), $\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx = \log \left| \frac{e^x}{2} + \frac{\sqrt{e^{2x} - 4}}{2} \right| + c$

$$= \log \left(\frac{e^x + \sqrt{e^{2x} - 4}}{2} \right) + c \quad [\because e^x + \sqrt{e^{2x} - 4} > 0]$$

$$= \log(e^x + \sqrt{e^{2x} - 4}) - \log 2 + c = \log(e^x + \sqrt{e^{2x} - 4}) + c'$$

Sol 35: $I = (\log x) \cdot \left[-\frac{1}{2(1+x)^2} \right] - \int \left(-\frac{1}{2(1+x)^2 x} \right) dx$

[Taking $u = \log x$]

$$= -\frac{1}{2} \cdot \frac{\log x}{(1+x)^2} + \frac{1}{2} \int \frac{dx}{x(1+x)^2} \quad \dots(i)$$

$$\text{Now, } \frac{1}{x(1+x)^2} = \frac{1+x-x}{x(1+x)^2} = \frac{1}{x(1+x)} - \frac{1}{(1+x)^2}$$

$$= \frac{1+x-x}{x(1+x)} - \frac{1}{(1+x)^2} = \frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2}$$

$$\therefore \int \frac{dx}{x(1+x)^2} = \int \left[\frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2} \right] dx$$

$$= \log x - \log(1+x) + \frac{1}{1+x} = \log \frac{x}{1+x} + \frac{1}{1+x}$$

[Here $x > 0$ a $\log x$ occurs in the integrand]

$$\therefore \text{From (i), } I = -\frac{1}{2} \frac{\log x}{(1+x)^2} + \frac{1}{2} \log \frac{x}{1+x} + \frac{1}{2(1+x)} + c$$

Sol 36: Let $f(x) = ax^2 + bx + c$

$$f(0) = c = -3$$

$$f(1) = a + b + c = -3$$

$$\text{or } a + b = 0 \because a = 1, b = -1$$

$$f(2) = 4a + 2b + c = -1$$

$$\text{or } 4a + 2b = 2$$

$$\therefore f(x) = x^2 - x - 3$$

$$\begin{aligned} \int \frac{x^2 - x - 3}{x^3 - 1} dx &= \int \frac{x(x-1)}{(x-1)(x^2+x+1)} dx - \int \frac{3}{(x^3-1)} dx \\ &= \int \frac{x}{(x^2+x+1)} dx - \int \frac{3}{(x^3-1)} dx \\ &= \frac{1}{2} \left[\int \frac{2x+1}{(x^2+x+1)} dx - \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right] - \int \frac{3}{(x^3-1)} dx \\ &= \frac{1}{2} \log(x^2+x+1) - \frac{1}{2} \times \frac{2}{\sqrt{3}} \tan^{-1} \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} \\ &\quad - \int \left(\frac{1}{x-1} - \frac{x+2}{(x^2+x+1)} \right) dx \\ &= \frac{1}{2} \log(x^2+x+1) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} - \log(x-1) \\ &\quad + \frac{1}{2} \int \left(\frac{2x+1}{x^2+x+1} + \frac{3}{(x^2+x+1)} \right) dx \\ &= \log(x^2+x+1) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} - \log(x-1) \\ &\quad + \frac{3}{2 \times \frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) \\ &= \log \frac{(x^2+x+1)}{|x-1|} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c \end{aligned}$$

Exercise 2

Single Correct Choice Type

Sol 1: (A) $\int \left(\frac{x}{1+x^5} \right)^{3/2} dx$

$$\int \frac{x^{3/2}}{x^{15/2} \left(1 + \frac{1}{x^5} \right)^{3/2}} dx = \int \frac{x^{-6}}{\left(1 + \frac{1}{x^5} \right)^{3/2}} dx$$

Put $1 + \frac{1}{x^5} = t$

$$\Rightarrow -5x^{-6} dx = dt \quad \text{or} \quad x^{-6} dx = -\frac{1}{5} dt$$

$$\therefore I = -\frac{1}{5} \int \frac{dt}{t^{3/2}} = -\frac{1}{5} \times (-2) \frac{1}{\sqrt{t}} + c$$

$$\therefore I = \frac{2}{5} \frac{1}{\sqrt{1 + \frac{1}{x^5}}} + c = \frac{2}{5} \sqrt{\frac{x^5}{1+x^5}} + c$$

Sol 2: (B) $-\int \frac{(\sin^8 x - \cos^8 x)}{1 - 2\sin^2 x \cos^2 x} dx$

$$\begin{aligned} &= -\int \frac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} dx \\ &= -\int \frac{(\sin^2 x - \cos^2 x)[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x]}{1 - 2\sin^2 x \cos^2 x} dx \\ &= -\int (\sin^2 x - \cos^2 x) dx = -\int (-\cos 2x) dx = \frac{1}{2} \sin 2x + c \end{aligned}$$

Sol 3: (C)

(A) $x \int \log x dx = x \left[(\log x)x - \int x \times \frac{1}{x} dx \right] = x^2 \log x - x^2 + cx$

(B) $x \int \ell |x| dx = x^2 \log |x| - x^2 + cx$

(C) $x \int e^x dx = x[e^x + c] = xe^x + cx$

(D) $\int \frac{dx}{\sqrt{a^2 + x^2}}$

$$x = a \tan \theta, dx = a \sec^2 \theta d\theta$$

$$\Rightarrow I = \int \frac{\sec^2 \theta d\theta}{a \sec \theta} = \int \sec \theta d\theta$$

$$= \log |\sec \theta + \tan \theta| + c = \log \left| \frac{x}{a} + \frac{\sqrt{x^2 + a^2}}{a} \right| + c$$

Sol 4: (B) $\int \sqrt{1+2\tan x(\sec x + \tan x)} dx$

$$= \int \sqrt{1+2\tan x \sec x + 2\tan^2 x} dx$$

$$= \int \sqrt{\tan^2 x + \sec^2 x + 2\tan x \sec x} dx$$

$$= \int (\tan x + \sec x) dx = \int \tan x dx + \int \sec x dx$$

$$= -\log|\cos x| + \log|\sec x + \tan x| + c$$

$$= \log|\sec x| + \log|\sec x + \tan x| + c$$

Sol 5: (C) $I = \int \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{2\tan x \sec^2 x}{\tan^4 x + 1} dx$

Let $\tan^2 x = t$

$$\therefore I = 2\tan x \sec^2 x dx = dt$$

$$\therefore I = \int \frac{1}{t^2 + 1} dt$$

$$\therefore I = \tan^{-1}(t) + c = \tan^{-1}(\tan^2 x) + c$$

$$\tan^{-1}(\tan^2 x) = \tan^{-1}\left(\frac{1}{\cot^2 x}\right) = \cot^{-1}(\cot^2 x)$$

$$\tan^{-1}(\tan^2 x) = \frac{\pi}{2} - \cot^{-1}(\tan^2 x) + c$$

$$= -\cot^{-1}(\tan^2 x) + c_1$$

Sol 6: (C) $\int \frac{1}{\cos^3 \theta \sqrt{2\sin \theta \cos \theta}} d\theta$

$$= \int \frac{1}{\sqrt{2} \cos^{7/2} \theta \sin^{1/2} \theta} d\theta$$

Dividing and multiplying by $\cos^4 \theta$

$$I = \int \frac{\sec^4 \theta}{\sqrt{2} \tan^{1/2} \theta} d\theta$$

let $\tan^{1/2} \theta = t$

$$\therefore \frac{1}{2\sqrt{\tan \theta}} \times \sec^2 \theta d\theta = dt$$

$$\therefore I = \int \sqrt{2} \sec^2 \theta dt$$

$$= \int \sqrt{2} (1+t^4) dt = \sqrt{2} \times \left(t + \frac{t^5}{5} \right) + c$$

$$= \frac{\sqrt{2}}{5} (5\sqrt{\tan \theta} + \tan^2 \theta \times \sqrt{\tan \theta}) + c$$

$$I = \frac{\sqrt{2}}{5} (\tan^2 \theta + 5) \sqrt{\tan \theta} + c$$

Sol 7: (A) $\int \frac{dx}{b\left(\frac{a}{b} + x^2\right)} = \frac{1}{b \times \sqrt{\frac{a}{b}}} \tan^{-1}\left(\frac{x}{\sqrt{\frac{a}{b}}}\right)$

$$= \frac{1}{\sqrt{ab}} \tan^{-1}\left(x\sqrt{\frac{b}{a}}\right) + c$$

Sol 8: (C) $I = \int \frac{1+x^7-2x^7}{x(1+x^7)} dx$

$$= \int \frac{1}{x} - \frac{2x^6}{1+x^7} dx = \log x - \int \frac{2x^6}{1+x^7} dx$$

let $1+x^7 = t \Rightarrow 7x^6 dx = dt$

$$I = \log x - \frac{2}{7} \int \frac{1}{t} dt = \log x - \frac{2}{7} \log t + c$$

$$= \log x - \frac{2}{7} \log(1+x^7) + c$$

Sol 9: (A) Let $\log|x| = t \Rightarrow \frac{1}{x} dx = dt$

$$\therefore I = \int \frac{t}{\sqrt{1+t}} dt = \int \sqrt{1+t} - \frac{1}{\sqrt{1+t}} dt$$

$$= \frac{(t+1)^{3/2}}{3} \times 2 - (t+1)^{1/2} \times 2 + c$$

$$= 2(t+1)^{1/2} \left(\frac{t+1}{3} - 1 \right) + c = \frac{2}{3} (t+1)^{1/2} (t-2) + c$$

$$= \frac{2}{3} (\log|x| + 1)^{1/2} (\log|x| - 2) + c$$

Sol 10: (C)

$$I = \int \frac{x^4 + 2x^2 + 1 - 2x^2}{x(x^2 + 1)^2} dx = \int \frac{(x^2 + 1)^2 - 2x^2}{x(x^2 + 1)^2} dx$$

$$= \int \frac{1}{x} - \frac{2x}{(x^2 + 1)^2} dx = \log|x| - \int \frac{2x}{(x^2 + 1)^2} dx$$

let $x^2 + 1 = t \Rightarrow 2x dx = dt$

$$\therefore I = \log|x| - \int \frac{1}{t^2} dt$$

$$= \log|x| + \frac{1}{t} + c = \log|x| + \frac{1}{1+x^2} + c$$

$$A = 1, B = 1$$

Sol 11: (B) $I = \int 2\sin x (\cos 2x + \cos x) dx$

$$= \int 2\sin x (2\cos^2 x - 1 + \cos x) dx$$

Let $\cos x = t \Rightarrow -\sin x dx = dt$

$$\therefore I = \int 2(1-t-2t^2)dt = 2t - t^2 - \frac{4t^3}{3} + c$$

$$= 2\cos x - \cos^2 x - \frac{4\cos^3 x}{3} + c$$

$$= \cos x - \left(\frac{4\cos^3 x - 3\cos x}{3} \right) - \cos^2 x + c$$

$$= \cos x - \frac{1}{3}\cos 3x - \cos^2 x - \frac{1}{2} + c$$

$$= \cos x - \frac{1}{3}\cos 3x - \frac{\cos 2x}{2} + c$$

Sol 12: (B) $I = \int \frac{1}{2} \sin 2x \cos 2x \cos 4x \cos 18x \cos 16x dx$

$$= \frac{1}{32} \int \sin 32x dx = -\frac{\cos 32x}{1024} + c$$

Sol 13: (B) $I = \int x^3 f(x^2) dx$

let $x^2 = t$

$$\therefore I = \frac{1}{2} \int t f(t) dt$$

$$= \frac{t}{2} \int f(t) dt - \int \frac{1}{2} \left(\int f(t) dt \right) dt$$

$$= \frac{1}{2} \left[x^2 F(x^2) - \int f(x^2) d(x^2) \right]$$

Sol 14: (D) $I = \int \frac{4e^{2x} + 6}{9e^{2x} - 4} dx$

$$\Rightarrow 4e^{2x} + 6 = a(9e^{2x} - 4) + b \times 18 \times e^{2x}$$

$$\Rightarrow 9a + 18b = 4$$

$$-4a = 6$$

$$\therefore a = -\frac{3}{2}$$

$$18b = 4 + \frac{27}{2} \Rightarrow b = \frac{35}{36}$$

$$\therefore I = \int -\frac{3}{2} dx + \frac{35}{36} \frac{18e^{2x}}{9e^{2x} - 4} dx$$

$$= -\frac{3x}{2} + \frac{35}{36} \log(9e^{2x} - 4) + c$$

$$A = -\frac{3}{2} \text{ and } B = \frac{35}{36}$$

Sol 15: (A) $I = \int \frac{\cos x - \sin x}{\sqrt{2 \sin x \cos x} (\cos x + \sin x)} dx$

$$2 \sin x \cos x = (\cos x + \sin x)^2 - 1$$

$$\text{And } d(\cos x + \sin x) = (-\sin x + \cos x) dx$$

$$\therefore \text{let } \cos x + \sin x = t$$

$$\therefore I = \int \frac{1}{\sqrt{t^2 - 1}} dt$$

$$\therefore I = \sec^{-1}(t) + c = \sec^{-1}(\cos x + \sin x) + c$$

Previous Years' Questions

Sol 1: (C) Let $I = \int \frac{(x^2 - 1) dx}{x^3 \sqrt{2x^4 - 2x^2 + 1}}$, dividing numerator

and denominator by x^5

$$= \int \frac{\left(\frac{1}{x^3} - \frac{1}{x^5} \right) dx}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}$$

$$\text{Put } 2 - \frac{2}{x^2} + \frac{1}{x^4} = t \Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5} \right) dx = dt$$

$$\therefore I = \frac{1}{4} \int \frac{dt}{\sqrt{t}} = \frac{1}{4} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{1}{2} \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + c$$

Sol 2: Given, $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \log(9e^{2x} - 4) + c$

$$\text{LHS} = \int \frac{4e^{2x} + 6}{9e^{2x} - 4} dx$$

$$\text{Let } 4e^{2x} + 6 = A(9e^{2x} - 4) + B(18e^{2x})$$

$$\Rightarrow 9A + 18B = 4$$

$$\text{and } -4A = 6$$

$$\Rightarrow A = -\frac{3}{2} \text{ and } B = \frac{35}{36}$$

$$\therefore \int \frac{A(9e^{2x} - 4) + B(18e^{2x})}{9e^{2x} - 4} dx$$

$$= A \int 1 dx + B \int \frac{1}{t} dt, \text{ where } t = 9e^{2x} - 4$$

$$= Ax + B \log(9e^{2x} - 4) + c$$

$$= -\frac{3}{2}x + \frac{35}{36} \log(9e^{2x} - 4) + c$$

$$\therefore A = -\frac{3}{2}, B = \frac{35}{36}$$

and C = any real number

Sol 3: Let $I = \int \frac{\sin x}{\sin x - \cos x} dx$

Again, let $\sin x = A(\cos x + \sin x) + B(\sin x - \cos x)$,

then $A + B = 1$ and $A - B = 0$

$$\Rightarrow A = \frac{1}{2}, B = \frac{1}{2}$$

$$\therefore I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\sin x - \cos x)}{(\sin x - \cos x)} dx$$

$$= \frac{1}{2} \int \frac{\cos x + \sin x}{\sin x - \cos x} dx + \frac{1}{2} \int 1 dx + c$$

$$= \frac{1}{2} \log(\sin x - \cos x) + \frac{1}{2} x + c$$

Sol 4: Let $I = \int \frac{xdx}{1+x^4}$

$$= \frac{1}{2} \int \frac{2x}{1+(x^2)^2} dx$$

Put $x^2 = u \Rightarrow 2xdx = du$

$$\therefore I = \frac{1}{2} \int \frac{du}{1+u^2} = \frac{1}{2} \tan^{-1}(u) + c = \frac{1}{2} \tan^{-1}(x^2) + c$$

Sol 5: Let $I_1 = \int \sin x \sin 2x \sin 3x dx$

$$= \frac{1}{4} \int (\sin 4x + \sin 2x - \sin 6x) dx$$

$$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24}$$

$$I_2 = \int \sec^2 x \cdot \cos^2 2x dx = \int \sec^2 x (2\cos^2 x - 1)^2 dx$$

$$= \int (4\cos^2 x + \sec^2 x - 4) dx = \int (2\cos 2x + \sec^2 x - 2) dx$$

$$= \sin 2x + \tan x - 2x$$

$$\therefore \int \frac{A(9e^{2x} - 4) + B(18e^{2x})}{9e^{2x} - 4} dx$$

$$= A \int 1 dx + B \int \frac{1}{t} dt, \text{ where } t = 9e^{2x} - 4$$

$$= Ax + B \log(9e^{2x} - 4) + c$$

$$= -\frac{3}{2}x + \frac{35}{36} \log(9e^{2x} - 4) + c$$

$$\therefore A = -\frac{3}{2}, B = \frac{35}{36}$$

and C = any real number

and $I_3 = \int \sin^4 x \cos^4 x dx$

$$= \frac{1}{128} \int (3 - 4\cos 4x + \cos 8x) dx = \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$$

$$\therefore I = I_1 + I_2 + I_3$$

$$= -\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24} + \sin 2x + \tan x - 2x$$

$$+ \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024}$$

Sol 6: Let $I = \frac{x^2}{(a+bx)^2}$

Put $a + bx = t$

$$\Rightarrow bdx = dt$$

$$\therefore I = \int \frac{\left(\frac{t-a}{b}\right)^2}{t^2} \cdot \frac{dt}{b} = \frac{1}{b^3} \int \left(\frac{t^2 - 2at + a^2}{t^2}\right) dt$$

$$= \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right) dt = \frac{1}{b^3} \left(t - 2a \log t - \frac{a^2}{t}\right) + c$$

$$= \frac{1}{b^3} \left(a + bx - 2a \log(a + bx) - \frac{a^2}{a + bx} + c\right)$$

Sol 7: Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = \int \frac{\tan x + 1}{\sqrt{\tan x}} dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x dx = 2t dt$

$$\Rightarrow dx = \frac{2t}{1+t^4} dt$$

$$\therefore I = \int \frac{t^2 + 1}{\sqrt{t^2}} \cdot \frac{2t}{t^4 + 1} dt = 2 \int \frac{t^2 + 1}{t^4 + 1} dt$$

$$= 2 \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} - 2 + 2} dt = 2 \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + (\sqrt{2})^2} dt$$

Put $t - \frac{1}{t} = u \Rightarrow 1 + \frac{1}{t^2} dt = du$

$$\therefore I = 2 \int \frac{du}{u^2 + (\sqrt{2})^2}$$

$$\Rightarrow I = \frac{2}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c$$

$$= \sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right) + c$$

Sol 8: $\int \frac{(x+1)}{x(1+xe^x)^2} dx$

This can be rewritten as $\int \frac{e^x(x+1)}{2e^x(1+xe^x)^2} dx$

let $1+xe^x = t \Rightarrow e^x(1+x)dx = dt$

Now integration becomes $\int \frac{dt}{t^2(t-1)}$

$$\Rightarrow \frac{1}{t^2(t-1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2} \text{ (using partial fraction)}$$

$$\Rightarrow 1 = t^2(A+B) + (C-B)t - C$$

Comparing, we get $C = -1$, $B = -1$ and $A = 1$

Now our integration becomes

$$\int \frac{dt}{t^2(t-1)} = \int \frac{1}{t-1} dt - \int \frac{t+1}{t^2} dt = \int \frac{1}{t-1} dt - \int \frac{1}{t} dt - \int t^2 dt$$

$$= \log(t-1) - \log(t) - \frac{t^{-2+1}}{-2+1} + C = \log \frac{t-1}{t} + \frac{1}{t} + C$$

Putting $t = 1 + xe^x$, we get

$$\int \frac{dt}{t^2(t-1)} = \log \left| \frac{xe^x}{1+xe^x} \right| + \frac{1}{1+xe^x} + c$$

Sol 9: Let $I = \int \left(\frac{1-\sqrt{x}}{1+\sqrt{x}} \right)^{\frac{1}{2}} \cdot \frac{dx}{x}$

Put $x = \cos^2 \theta \Rightarrow dx = -2\cos \theta \sin \theta d\theta$

$$\therefore I = \int \left(\frac{1-\cos \theta}{1+\cos \theta} \right)^{\frac{1}{2}} \cdot \frac{-2\cos \theta \sin \theta}{\cos^2 \theta} d\theta$$

$$= \int \frac{\sin \frac{\theta}{2} \cdot \frac{-2\sin \theta}{\cos \frac{\theta}{2}}}{\cos \frac{\theta}{2}} d\theta = - \int \frac{2\sin \frac{\theta}{2} \cdot 2\sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} \cdot \cos \theta} d\theta$$

$$= -2 \int \frac{2\sin^2 \frac{\theta}{2}}{\cos \theta} d\theta = -2 \int \frac{1-\cos \theta}{\cos \theta} d\theta$$

$$= 2 \int (1 - \sec \theta) d\theta = 2[\theta - \log |\sec \theta + \tan \theta|] + c$$

$$\Rightarrow I = 2 \left[\cos^{-1} \sqrt{x} - \log \left| \frac{1}{\sqrt{x}} + \sqrt{\frac{1}{x} - 1} \right| \right] + c$$

$$I = 2 \left[\cos^{-1} \sqrt{x} - \log |1 + \sqrt{1-x}| - \frac{1}{2} \log |x| \right] + c$$

Sol 10: (C) $\sqrt{2} \frac{\sin x dx}{\sin \left(x - \frac{\pi}{4} \right)} = \sqrt{2} \int \frac{\sin \left(x - \frac{\pi}{4} + \frac{\pi}{4} \right) dx}{\sin \left(x - \frac{\pi}{4} \right)}$

$$= \sqrt{2} \int \left(\cos \frac{\pi}{4} + \cot \left(x - \frac{\pi}{4} \right) \sin \frac{\pi}{4} \right) dx$$

$$= \int dx + \int \cot \left(x - \frac{\pi}{4} \right) dx = x + \ln \left| \sin \left(x - \frac{\pi}{4} \right) \right| + c$$

Sol 11: (D) $\frac{dy}{dx} = y + 3$

$$\Rightarrow \frac{dy}{y+3} = dx$$

$$\log(y+3) = x + c$$

$$x = 0 \Rightarrow y = 2$$

$$\Rightarrow \log 5 = 0 + c$$

$$c = \log 5$$

$$\log(y+3) = x + \log 5$$

$$y + 3 = e^{x - \log 5} \Rightarrow y + 3 = e^{\log 2 + \log 5}$$

$$y + 3 = 10 \Rightarrow y = 7$$

Sol 12: (D)

$$\int \frac{5 \sin x}{\sin x - 2 \cos x} dx$$

$$\Rightarrow \int \left[\frac{2(\cos x + 2 \sin x) + (\sin x - 2 \cos x)}{\sin x - 2 \cos x} \right] dx$$

$$= \int \left(\frac{\cos x + 2 \sin x}{\sin x - 2 \cos x} \right) dx + \int dx + k$$

$$= 2 \log |\sin x - 2 \cos x| + x + k$$

$$\therefore a = 2$$

Sol 13: (C) $\int f(x) dx = \psi(x)$

$$I = \int x^5 f(x^3) dx$$

$$\text{Put } x^3 = t \Rightarrow x^2 dx = \frac{dt}{3} = \frac{1}{3} t f(t) dt$$

$$= \frac{1}{3} [t \psi(t) dt]$$

$$= \frac{1}{3} \left[x^3 \psi(x^3) - 3 \int x^2 \psi(x^3) dx \right] + c = \frac{1}{3} x^3 \psi(x^3) dx + c$$

Sol 14: (D) $I = \int \left\{ e^{\left(x + \frac{1}{x}\right)} + x \left(1 - \frac{1}{x^2}\right) e^{x + \frac{1}{x}} \right\} dx$

$$= x \cdot e^{x + \frac{1}{x}} + c$$

As $\int (xf'(x) + f(x)) dx = xf(x) + c$

Sol 15: (D) $\int \frac{dx}{x^2(x^4 + 1)^{3/4}}$

$$\int \frac{dx}{x^3 \left(1 + \frac{1}{x^4}\right)^{3/4}} \Rightarrow 1 + \frac{1}{x^4} = t^4$$

$$-4 \frac{1}{x^5} = dx = 4t^3 dt$$

$$\frac{dx}{x^3} = t^3 dt$$

$$\int \frac{-t^3 dt}{t^3} = -t + c = -\left(1 + \frac{1}{x^4}\right)^{1/4} + c$$

Sol 16: (A) $\int \frac{2x^{12} + 5x^3}{(x^5 + x^3 + 1)^3} dx$

$$\int \frac{\left(\frac{2}{x^3} + \frac{5}{x^6}\right)}{\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3} dx$$

Let $1 + \frac{1}{x^2} + \frac{1}{x^5} = t$

$$\frac{dt}{dx} = \frac{-2}{x^3} - \frac{5}{x^6}$$

$$\int \frac{-dt}{t^3} = \frac{1}{2t^2} + c = \frac{1}{2\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^2} + c = \frac{x^{10}}{2(x^5 + x^3 + 1)^2} + c$$

JEE Advanced/Boards

Exercise 1

Sol 1: (i) $\int \tan \frac{x}{2} \tan \frac{x}{3} \tan \frac{x}{6} dx$

$$= \int \tan \frac{x}{2} \left[1 - \frac{\left(\tan \frac{x}{3} + \tan \frac{x}{6}\right)}{\tan \frac{x}{2}} \right] dx$$

$$= \int \left(\tan \frac{x}{2} - \tan \frac{x}{3} - \tan \frac{x}{6} \right) dx$$

$$= \int \tan \frac{x}{2} dx - \int \tan \frac{x}{3} dx - \int \tan \frac{x}{6} dx$$

$$= 2 \log \sec \frac{x}{2} - 3 \log \sec \frac{x}{3} - 6 \log \sec \frac{x}{6} + c$$

(ii) $\int \frac{\tan(\log x) \tan\left(\log \frac{x}{2}\right) \tan(\log 2)}{x} dx$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\int \tan t \tan(t - \log 2) \tan(\log 2) dt$$

$$= \int [\tan t - \tan(t - \log 2) - \tan(\log 2)] dt$$

$$= \log \sec t - \log \sec(t - \log 2) - x \tan(\log 2) + c$$

$$= \log \frac{\sec(\log x)}{\sec\left(\log \frac{x}{2}\right)} - x \tan \log |2| + c$$

Sol 2: Put $x = \alpha \sec 2\theta - \beta \tan 2\theta$

$$\int \frac{2(\alpha - \beta) \sec^2 \theta \tan \theta d\theta}{(\alpha - \beta) \tan^2 \theta \sqrt{(\alpha - \beta) \tan^2 \theta (\alpha - \beta) \sec^2 \theta}}$$

$$= \int \frac{2 \sec^2 \theta \tan \theta}{\tan^2 \theta \times (\alpha + \beta) \tan \theta \sec \theta} d\theta$$

$$= \frac{2}{\alpha - \beta} \int \frac{\sec \theta}{\tan^2 \theta} d\theta$$

Put $\tan \theta = t \sec^2 \theta d\theta = dt$

Or $\sec \theta d\theta = \frac{dt}{\sqrt{1+t^2}}$

$$\therefore I = \frac{2}{(\alpha - \beta)} \int \frac{dt}{t^2 \sqrt{1+t^2}} = \frac{2}{(\alpha - \beta)} \int \frac{t^{-3} dt}{\sqrt{t^{-2} + 1}}$$

$$\therefore 1 + t^{-2} = u \Rightarrow -2t^{-3} dt = du$$

$$\text{Or } t^{-3} dt = -\frac{1}{2} du$$

$$= \frac{1}{2} \times \frac{2}{(\alpha - \beta)} \int \frac{-du}{\sqrt{u}} = \frac{-1}{(\alpha - \beta)} \times 2\sqrt{u}$$

$$= \frac{-2}{(\alpha - \beta)} \sqrt{1 + \frac{1}{t^2}}$$

$$= \frac{-2}{(\alpha - \beta)} \sqrt{1 + \frac{1}{\tan^2 \theta}} = \frac{-2}{(\alpha - \beta)} \sqrt{\frac{\sec^2 \theta}{\tan^2 \theta}}$$

$$= \frac{-2}{\alpha - \beta} \sqrt{\frac{(\alpha - \beta) \sec^2 \theta}{(\alpha - \beta) \tan^2 \theta}} = \frac{-2}{\alpha - \beta} \sqrt{\frac{(x - \beta)}{(x - \alpha)}}$$

$$\text{Sol 3: } \int \frac{\ell n \left(\ell n \left(\frac{1+x}{1-x} \right) \right) dx}{1-x^2}$$

$$\text{Put } \log \left(\frac{1+x}{1-x} \right) = t$$

$$\Rightarrow \left(\left(\frac{1-x}{1+x} \right) \times \frac{1-x+1+x}{(1-x)^2} \right) dx = dt$$

$$\text{Or } \left(\frac{2}{1-x^2} \right) dx = dt$$

$$\therefore I = \int \log(t) \frac{dt}{2}$$

$$= \frac{1}{2} \int 1 \cdot \log(t) \frac{dt}{2}$$

Integration by parts,

$$I = \frac{1}{2} \left[\log(t) \int 1 dt - \left[\int \frac{d}{dt} (\log(t)) \int 1 dt \right] dt \right]$$

$$= \frac{1}{2} [t(\log t - 1)] + c$$

$$= \frac{1}{2} \left[\log \left(\frac{1+x}{1-x} \right) \left\{ \log \left(\log \left(\frac{1+x}{1-x} \right) \right) - 1 \right\} \right] + c$$

$$\text{Sol 4: } d \left(\frac{x}{e} \right)^x = \left(\frac{x}{e} \right)^x [\log x] \text{ and } d \left(\frac{e}{x} \right)^x = \left(\frac{e}{x} \right)^x [-\log x]$$

$$\therefore \int \left(\frac{x}{e} \right)^x \log x dx + \int \left(\frac{e}{x} \right)^x \log x dx = \left(\frac{x}{e} \right)^x - \left(\frac{e}{x} \right)^x + c$$

$$\text{Sol 5: } \int \sqrt{\frac{\cos(x-a)}{\sin(x+a)}} dx$$

$$\int \sqrt{\frac{\cos x \cos a - \sin x \sin a}{\sin x \cos a + \cos x \sin a}} dx$$

$$= \int \sqrt{\frac{1 - \tan x \tan a}{\tan x + \tan a}} dx = \int \sqrt{\cot(a+x)} dx$$

$$\text{Sol 6: } \int \frac{x^5 + 3x^4 - x^3 + 8x^2 - x + 8}{(x^2 + 1)} dx$$

$$= \int \frac{(x^3 + 3x^2 - 2x + 5)(x^2 + 1)}{(x^2 + 1)} dx + \int \frac{x + 3}{x^2 + 1} dx$$

$$= \frac{x^4}{4} + x^3 - x^2 + 5x + \frac{1}{2} \log(x^2 + 4) + 3 \tan^{-1} x + c$$

$$\text{Sol 7: } \int \frac{(\sqrt{x} + 1)}{\sqrt{x}(\sqrt[3]{x} + 1)} = \int \frac{(x)^{1/2} + 1}{x^{1/2}(x^{1/3} + 1)} dx$$

$$\text{Put } x^{1/6} = t \Rightarrow dx = 6t^5 dt$$

$$6 \int \frac{(t^3 + 1)t^5}{t^3(t^2 + 1)} dt = 6 \int \frac{(t^3 + 1)t^2}{(t^2 + 1)} dt$$

$$= 6 \int \left[\frac{(t^3 - t + 1)(t^2 + 1)}{(t^2 + 1)} + \frac{t - 1}{(t^2 + 1)} \right] dt$$

$$= 6 \left[\frac{t^4}{4} - \frac{t^2}{2} + t \right] + 3 \log(1 + t^2) - 6 \tan^{-1} t + c$$

$$\text{Where } t = x^{1/6}$$

$$\text{Sol 8: } \int \sin^{-1} \sqrt{\frac{x}{a+x}} dx, a > 0$$

$$x = a \tan^2 \theta$$

$$dx = 2a \tan \theta \sec^2 \theta d\theta$$

$$\int \sin^{-1} \sqrt{\frac{a \tan^2 \theta}{a \sec^2 \theta}} 2a \tan \theta \sec^2 \theta d\theta$$

$$2a \int (\sin^{-1} \sin \theta) \tan \theta \sec^2 \theta d\theta$$

$$\tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt$$

$$2a \int t \tan^{-1} t dt$$

$$2a \left[\tan^{-1} t \int t dt - \frac{1}{2} \int \frac{1}{1+t^2} \times t^2 dt \right]$$

$$= 2a \left[(\tan^{-1} t) \frac{t^2}{2} - \frac{1}{2} [t - \tan^{-1} t] \right] + c$$

$$= at^2 \tan^{-1} t + \left[\frac{1}{2} \tan^{-1} t - \frac{1}{2} t \right] 2a + c$$

$$\therefore t = \tan \theta = \frac{\sqrt{x}}{\sqrt{a}}$$

$$\therefore I = a \frac{x}{a} \tan^{-1} \sqrt{\frac{x}{a}} + \left[\frac{1}{2} \tan^{-1} \sqrt{\frac{x}{a}} - \frac{1}{2} \sqrt{\frac{x}{a}} \right] 2a + c$$

$$= x \tan^{-1} \sqrt{\frac{x}{a}} + \left[\frac{1}{2} \tan^{-1} \sqrt{\frac{x}{a}} - \frac{1}{2} \sqrt{\frac{x}{a}} \right] 2a + c$$

$$= x \tan^{-1} \sqrt{\frac{x}{a}} + a \tan^{-1} \sqrt{\frac{x}{a}} - a \sqrt{\frac{x}{a}} + c$$

$$= \left(\tan^{-1} \sqrt{\frac{x}{a}} \right) (a+x) - a \sqrt{\frac{x}{a}} + c$$

$$= \sqrt{x} \left[\sqrt{x} \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{a} \right] + a \tan^{-1} \sqrt{\frac{x}{a}} + c$$

$$\text{Sol 9: } \int \frac{x \log x}{(x^2-1)^{3/2}} dx$$

$$\therefore \int \left(\frac{x \log x}{(x^2-1)^{3/2}} - \frac{1}{x \sqrt{x^2-1}} \right) dx + \int \frac{1}{x \sqrt{x^2-1}} dx$$

$$\int \left(-d \frac{\log x}{\sqrt{x^2-1}} \right) + \sec^{-1} x + c = -\frac{\log x}{\sqrt{x^2-1}} + \sec^{-1} x + c$$

$$\text{Sol 10: } \int \frac{\log_6[(\sin x) 6^{\cos x}]^{1/6} \cos x}{\sin x} dx$$

$$\int \frac{\frac{1}{6} [\log_6(\sin x) + \log_6 6^{\cos x}] \cos x}{\sin x} dx$$

$$\int \left(\left(\frac{\frac{1}{6} \log(\sin x)}{(\log 6) \sin x} + \frac{1}{6} \frac{\cos x}{\sin x} \right) \cos x \right) dx$$

$$= \int \left[\frac{1}{6} \frac{1}{\log 6} \frac{\cos x}{\sin x} \log(\sin x) + \frac{1}{6} \frac{\cos^2 x}{\sin x} \right] dx$$

$$= \frac{1}{6} \frac{\log^2(\sin x)}{\log 36} + \frac{1}{6} \int (\operatorname{cosec} x - \sin x) dx$$

$$= \frac{1}{6} \left[\frac{\log^2(\sin x)}{\log 36} + \ell \tan \frac{x}{2} + \cos x \right] + c$$

$$\text{Sol 11: } \int \frac{\sqrt{x^2+1} (\log(x^2+1) - \log x^2)}{x^4} dx$$

$$\int \frac{\left(\sqrt{1 + \frac{1}{x^2}} \log \left(1 + \frac{1}{x^2} \right) \right)}{x^3} dx$$

$$1 + \frac{1}{x^2} = t \Rightarrow -\frac{2}{x^3} dx = dt$$

$$-\frac{1}{2} \int \sqrt{t} \log t dt = -\frac{1}{2} \left[(\log t) \frac{t^{3/2}}{3/2} - \int \frac{1}{t} \times \frac{t^{3/2}}{3/2} dt \right]$$

$$= \frac{1}{2} \left[-\frac{2}{3} (\log t) t^{3/2} + \frac{2}{3} \int t^{1/2} dt \right]$$

$$= \frac{1}{2} \left[-\frac{2}{3} \left[\log \left(1 + \frac{1}{x^2} \right) \right] \left[1 + \frac{1}{x^2} \right]^{3/2} + \frac{4}{9} \left(1 + \frac{1}{x^2} \right)^{3/2} + c \right]$$

$$= -\frac{1}{3} \left[1 + \frac{1}{x^2} \right]^{3/2} \left[\log \left(1 + \frac{1}{x^2} \right) - \frac{2}{3} \right] + c$$

$$\text{Sol 12: } f(x) = \frac{\sin x}{\sin^2 x + 4 \cos^2 x} = \frac{\tan x \sec x}{\tan^2 x + 4}$$

$$= \frac{\tan x \sec x}{\sec^2 x + 3}$$

Putting $\sec x = t$, $dx \sec x \tan x = dt$ so

$$\int f(x) dx = \int \frac{dt}{t^2 + 3} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{\sec x}{\sqrt{3}} + c$$

$$\text{Sol 13: } \int \frac{\cos^2 x}{\sin x(1-\sin x)(1+\cos x)} dx = \int \frac{(1+\sin x)}{\sin x(1+\cos x)} dx$$

$$= \int \frac{\left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2 \sin \frac{x}{2} \cos \frac{x}{2} \times 2 \cos^2 \frac{x}{2}} dx = \frac{1}{4} \int \frac{\left(\tan \frac{x}{2} + 1 \right)^2}{\tan \frac{x}{2} \cos^2 \frac{x}{2}} dx$$

$$\text{Let } \tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} dx = 2dt$$

$$= \frac{1}{2} \int \frac{(t+1)^2}{t} dt = \frac{1}{2} \int \left(t + 2 + \frac{1}{t} \right) dt$$

$$= \frac{t^2}{4} + t + \frac{1}{2} \log |t| + c$$

Sol 14: $\int \frac{3x^2 + 1}{(x^2 - 1)^3} dx$

$$\begin{aligned} & \int \left(\frac{3(x^2 - 1) + 4}{(x^2 - 1)^3} \right) dx \text{ or } \int \frac{(3x^2 + 1)(x^2 - 1)}{(x^2 - 1)^4} dx \\ &= \int \left[\frac{3x^2(x^2 - 1) + (x^2 - 1)}{(x^2 - 1)^4} \right] dx = \int \frac{3x^4 - 2x^2 - 1}{(x^2 - 1)^4} dx \\ &= \int - \left[\frac{x^4 + 1 - 2x^2 - 4x^4 + 4x^2}{(x^2 - 1)^4} \right] dx \\ &= \int - \left[\frac{(x^2 - 1)^2 - 2.2x^2(x^2 - 1)}{(x^2 - 1)^4} \right] dx \\ &= \int -d \left(\frac{x}{(x^2 - 1)^2} \right) = - \frac{x}{(x^2 - 1)^2} + c \end{aligned}$$

Sol 15: $\int \frac{(ax^2 - b)dx}{x\sqrt{c^2x^2 - (ax^2 + b)^2}}$

$$\begin{aligned} & \int \frac{(ax^2 - b)}{cx^2 \sqrt{1 - \frac{(ax^2 + b)^2}{c^2x^2}}} \Rightarrow \int \frac{acx^2 - bc}{c^2x^2 \sqrt{1 - \frac{(ax^2 + b)^2}{c^2x^2}}} dx \\ &= \int \frac{2acx^2 - (acx^2 + bc)}{(cx^2) \sqrt{1 - \frac{(ax^2 + b)^2}{c^2x^2}}} dx \\ &= \int \frac{2acx^2 - (ax^2 + b)c}{(cx)^2} \times \frac{1}{\sqrt{1 - \left(\frac{ax^2 + b}{cx} \right)^2}} dx \\ \text{Put } \frac{ax^2 + b}{cx} &= t = \left(\frac{(2ax)cx - c(ax^2 + b)}{(cx)^2} \right) dx \\ \therefore \int \frac{dt}{\sqrt{1 - t^2}} &= \sin^{-1}t + c = \sin^{-1} \left(\frac{ax^2 + b}{cx} \right) + c \end{aligned}$$

Sol 16: Put $z = \tan^{-1}x$, then $dz = \frac{1}{1+x^2} dx$ and $x = \tan z$

$$\begin{aligned} \text{Now, } I &= \int \frac{e^{mz}}{\sqrt{1 + \tan^2 z}} dz = \int e^{mz} \cos z dz \\ &= \frac{e^{mz}}{m} \cos z - \int \frac{e^{mz}}{m} (-\sin z) dz \end{aligned}$$

$$= \frac{e^{mz}}{m} \cos z + \frac{1}{m} \int e^{mz} \sin z dz$$

$$= \frac{e^{mz}}{m} \cos z + \frac{1}{m} \left[\frac{e^{mz}}{m} \sin z - \int \frac{e^{mz}}{m} \cos z dz \right]$$

$$= \frac{e^{mz}}{m} \cos z + \frac{e^{mz}}{m^2} \sin z - \frac{1}{m^2} 1$$

$$\text{Or, } \left(1 + \frac{1}{m^2} \right) 1 = \frac{e^{mz}}{m^2} (m \cos z + \sin z)$$

$$\therefore 1 = \frac{e^{mz} (m \cos z + \sin z)}{m^2 + 1} + c$$

Or,

$$1 = \frac{e^{m \tan^{-1}x}}{m^2 + 1} \left(m \cdot \frac{1}{\sqrt{x^2 + 1}} + \frac{x}{\sqrt{x^2 + 1}} \right) + c = \frac{e^{m \tan^{-1}x} (m + x)}{(m^2 + 1)\sqrt{x^2 + 1}} + c$$

Sol 17: [Here $\sqrt{\frac{x}{a+x}}$ occurs, $\therefore$ put $x = a \tan^2 \theta$]

Put $x = a \tan^2 \theta$, then $dx = 2a \tan \theta \sec^2 \theta d\theta$

Now,

$$I = \int \sin^{-1}(\sin \theta) 2a \tan \theta \sec^2 \theta d\theta = 2a \int \theta \cdot (\tan \theta \sec^2 \theta) d\theta$$

$$= 2a \left[\theta \frac{\sec^2 \theta}{2} - \int 1 \frac{\sec^2 \theta}{2} d\theta \right] + c$$

$$[\int \tan \theta \sec^2 \theta d\theta = \int z dz, \text{ where } z = \sec \theta]$$

$$= a[\theta \sec^2 \theta - \tan \theta] + c$$

$$= a \left[\tan^{-1} \sqrt{\frac{x}{a+x}} \left(\frac{a+x}{a} \right) - \sqrt{\frac{x}{a}} \right] + c$$

Sol 18: $\frac{d \sin \left(e^{\sqrt{x}} + e^{-\sqrt{x}} + \frac{\pi}{4} \right)}{dx}$

$$= \cos \left(e^{\sqrt{x}} + e^{-\sqrt{x}} + \frac{\pi}{4} \right) \left[\frac{e^{\sqrt{x}}}{2\sqrt{x}} - \frac{e^{-\sqrt{x}}}{2\sqrt{x}} \right]$$

$$\text{Also } \frac{d}{dx} \sin \left(e^{\sqrt{x}} - e^{-\sqrt{x}} + \frac{\pi}{4} \right)$$

$$= \cos \left(e^{\sqrt{x}} - e^{-\sqrt{x}} + \frac{\pi}{4} \right) \left[\frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{2e^{-\sqrt{x}}}{2\sqrt{x}} \right]$$

$$\begin{aligned}
 \therefore I &= 2 \int \left[d\sin\left(e^{\sqrt{x}} + e^{-\sqrt{x}} + \frac{\pi}{4}\right) + d\sin\left(e^{\sqrt{x}} - e^{-\sqrt{x}} + \frac{\pi}{4}\right) \right] \\
 &= 2 \left[\sin\left(e^{\sqrt{x}} + e^{-\sqrt{x}} + \frac{\pi}{4}\right) + \sin\left(e^{\sqrt{x}} - e^{-\sqrt{x}} + \frac{\pi}{4}\right) \right] \\
 &= 2 \sin\left(e^{\sqrt{x}} + \frac{\pi}{4}\right) \cos\left(e^{-\sqrt{x}}\right) \\
 &= \sqrt{2} \left(\sin\left(e^{\sqrt{x}}\right) + \cos\left(e^{\sqrt{x}}\right) \right) \cos\left(e^{-\sqrt{x}}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Sol 19: } \int \frac{(x^2 + x)}{(e^x + x + 1)^2} dx &= \int \left(\frac{x(e^x + x + 1) - xe^x}{(e^x + x + 1)^2} \right) dx \\
 &= \int \left(\frac{x}{e^x + x + 1} - \frac{xe^x}{(e^x + x + 1)^2} \right) dx \\
 &= \int \left[\left(\frac{1}{1 + \left(\frac{x+1}{e^x}\right)} \right) \times \frac{x}{e^x} - \frac{1}{\left(1 + \frac{x+1}{e^x}\right)^2} \left(\frac{x}{e^x}\right) \right] dx
 \end{aligned}$$

$$\text{Put } \frac{x+1}{e^x} + 1 = t \Rightarrow \frac{-x}{e^x} dx = dt$$

$$\begin{aligned}
 \int \left(-\frac{dt}{t} + \frac{1}{t^2} \right) dt &= -\log t - \frac{1}{t} + c \\
 &= -\log \left(\frac{x+1}{e^x} + 1 \right) - \frac{1}{\left(\frac{x+1}{e^x} + 1 \right)} + c
 \end{aligned}$$

$$\text{Sol 20: } \int \frac{e^{\cos x} (x \sin^3 x + \cos x)}{\sin^2 x} dx$$

$$\int e^{\cos x} \left(\frac{\cos x}{\sin^2 x} + x \sin x \right) dx$$

$$\int e^{\cos x} (\cot x \operatorname{cosec} x + x \sin x) dx$$

$$\Rightarrow - \int e^{\cos x} (1 - \cot x \operatorname{cosec} x - x \sin x - 1) dx$$

$$= - \int \left[e^{\cos x} (1 - \operatorname{cosec} x \cot x) + (x + \operatorname{cosec} x) e^{\cos x} (-\sin x) \right] dx$$

$$= - \int de^{\cos x} (x + \operatorname{cosec} x) = -e^{\cos x} (x + \operatorname{cosec} x) + c$$

$$\begin{aligned}
 \text{Sol 21: } \int \frac{5x^4 + 4x^5}{(x^5 + x + 1)^2} dx &= \int \frac{5x^5 + 5x^4 + x + 1 - x^5 - x - 1}{(x^5 + x + 1)^2} dx \\
 &= \int -\frac{(x^5 + x + 1) + (5x^4 + 1)(x + 1)}{(x^5 + x + 1)^2} dx \\
 &= \int -d\left(\frac{x+1}{x^5 + x + 1}\right) = -\frac{x+1}{x^5 + x + 1} + c \text{ or } \frac{x^5}{x^5 + x + 1} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Sol 22: } \int \left(\frac{a^2 \tan^2 x + b^2}{a^4 \tan^2 x + b^4} \right) dx &\Rightarrow \frac{1}{(a^2 + b^2)} \int \frac{b^2(a^2 + b^2) + a^2(b^2 + a^2) \tan^2 x}{a^4 \tan^2 x + b^4} dx \\
 &\Rightarrow \frac{1}{a^2 + b^2} \int \left(\frac{a^2 b^2 (1 + \tan^2 x)}{a^4 \tan^2 x + b^4} + 1 \right) dx \\
 &\Rightarrow \frac{1}{a^2 + b^2} \int \left(\frac{a^2 b^2 + b^4 + (a^2 b^2 + a^4) \tan^2 x}{a^4 \tan^2 x + b^4} \right) dx \\
 &\Rightarrow \frac{1}{a^2 + b^2} \int \left(\left(\frac{a^2}{b^2} \right) \sec^2 x \times \frac{1}{\left(\frac{a^4}{b^4} \tan^2 x + 1 \right)} + 1 \right) dx
 \end{aligned}$$

$$\Rightarrow \frac{1}{a^2 + b^2} x + \frac{1}{a^2 + b^2} \int \frac{\left(\frac{a}{b}\right)^2 \sec^2 x}{1 + \left(\frac{a^2}{b^2} \tan x\right)^2} dx$$

$$\text{Put } \frac{a^2}{b^2} \tan x = t \Rightarrow \frac{a^2}{b^2} \sec^2 x dx = dt$$

$$\therefore \frac{1}{a^2 + b^2} x + \frac{1}{a^2 + b^2} \int \frac{dt}{1 + t^2}$$

$$\text{Or } \frac{1}{a^2 + b^2} \left[x + \tan^{-1} \frac{a^2}{b^2} \tan x \right] + c$$

$$\text{Sol 23: } \int \frac{\cos^2 x}{1 + \tan x} dx = \int \frac{\cos^3 x}{\sin x + \cos x} dx$$

$$= \frac{1}{4} \int \frac{3 \cos x}{\sin x + \cos x} dx + \frac{1}{4} \int \frac{\cos 3x}{\sin x + \cos x} dx$$

$$\begin{aligned}
&= \frac{1}{4} \int \frac{\cos x \cos 2x - \sin x \sin 2x}{(\sin x + \cos x)} dx + \frac{3}{4} \int \frac{\cos x}{\sin x + \cos x} dx \\
&= \frac{1}{4} \int \left[(\cos 2x - \sin 2x) + \left(\frac{\sin 2x \cos x - \cos 2x \sin x}{\sin x + \cos x} \right) \right] dx \\
&\quad + \frac{3}{4} \int \frac{\cos x}{\sin x + \cos x} dx \\
&= \frac{1}{4} \int (\cos 2x - \sin 2x) dx + \frac{1}{4} \int dx \\
&\quad - \frac{1}{4} \int \frac{\cos x dx}{\sin x + \cos x} + \frac{3}{4} \int \frac{\cos x dx}{\sin x + \cos x} \\
&= \frac{1}{8} (\sin 2x + \cos 2x) + \frac{x}{4} + \frac{1}{2} \int \frac{\cos x dx}{\sin x + \cos x} \\
&= \frac{1}{8} (\sin 2x + \cos 2x) + \frac{x}{4} \\
&\quad + \frac{1}{4} \int \left(\frac{\cos x + \sin x + \cos x - \sin x}{\cos x + \sin x} \right) dx \\
&= \frac{1}{8} (\sin 2x + \cos 2x) + \frac{x}{4} + \frac{1}{4} \int \left[1 + \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right) \right] dx \\
&= \frac{1}{8} (\sin 2x + \cos 2x) + \frac{x}{2} + \frac{1}{4} \log(\cos x + \sin x) + c
\end{aligned}$$

Sol 24:

$$I = (x \log x - x) \sin^{-1} x - \int \frac{x \log x}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx \quad \dots(i)$$

[Integrating by parts taking $\sin^{-1}x$ as u]

Now in order to evaluate $\int \frac{x \log x}{\sqrt{1-x^2}} dx$

Put $x = \sin \theta$, then $dx = \cos \theta d\theta$

$$\begin{aligned}
\therefore \int \frac{x \log x}{\sqrt{1-x^2}} dx &= \int \sin \theta \log \sin \theta d\theta \\
&= -\cos \theta \log \sin \theta - \int -\cos \theta \cot \theta d\theta \\
&= -\cos \theta \log \sin \theta + \int \frac{\cos^2 \theta}{\sin \theta} d\theta \\
&= -\cos \theta \log \sin \theta + \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta \\
&= -\cos \theta \log \sin \theta + \int (\operatorname{cosec} \theta - \sin \theta) d\theta \\
&= -\cos \theta \log \sin \theta + \log |\operatorname{cosec} \theta - \cot \theta| + \cos \theta \\
&= -\sqrt{1-x^2} \log x + \log \left(\frac{1 - \sqrt{1-x^2}}{x} \right) + \sqrt{1-x^2}
\end{aligned}$$

$$\text{Again, } \int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2}$$

$\therefore$ from (i),

$$I = (x \log x - x) \sin^{-1} x + \sqrt{1-x^2} \log x - \log \left(\frac{1 - \sqrt{1-x^2}}{x} \right) + c$$

$$\begin{aligned}
\text{Sol 25: } \int \frac{(x^2+1)e^x}{(x+1)^2} dx &= \int e^x \frac{(x^2-1)+2}{(x+1)^2} dx \\
&= \int e^x \left[\frac{x^2-1}{(x+1)} + \frac{2}{(x+1)^2} \right] dx = \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx \\
&= \int e^x [f(x) + f'(x)] dx, \text{ where } f(x) = \frac{x-1}{x+1} \\
&= e^x f(x) + c = e^x \left(\frac{x-1}{x+1} \right) + c
\end{aligned}$$

Sol 26: [Here $e^{f(z)}$ occurs, where $f(x) = \sin x$]

$\therefore$ Put $z = f(x) = \sin x$

Put $z = \sin x$, then $dz = \cos x dx$

$$\text{Now, } I = \int \frac{e^{\sin x}}{\cos^3 x} (x \cos^3 x - \sin x) dz$$

$$\begin{aligned}
&= \int e^{\sin x} (x - \tan x \sec^2 x) dz \\
&= \int e^z \left[\sin^{-1} z - \frac{z}{\sqrt{1-z^2}} \cdot \frac{1}{1-z^2} \right] dz \quad [\because \sin x = z] \\
&= \int e^z \left[\sin^{-1} z + \frac{1}{\sqrt{1-z^2}} - \frac{1}{\sqrt{1-z^2}} - \frac{z}{(1-z^2)^{3/2}} \right] dz \\
&= \int e^z \left[\left(\sin^{-1} z - \frac{1}{\sqrt{1-z^2}} \right) + \left\{ \frac{1}{\sqrt{1-z^2}} - \frac{z}{(1-z^2)^{3/2}} \right\} \right] dz \\
&= \int e^z [f(z) + f'(z)] dz, \text{ where } f(z) = \sin^{-1} z - \frac{1}{\sqrt{1-z^2}} \\
&= e^z f(z) + c = e^z \left(\sin^{-1} z - \frac{1}{\sqrt{1-z^2}} \right) + c = e^{\sin x} (x - \sec x) + c
\end{aligned}$$

$$\text{Sol 27: } I = \int \frac{dx}{\sqrt{1-(x^2+2x)}} = \int \frac{dx}{\sqrt{2-(x^2+2x+1)}}$$

$$= \int \frac{dx}{\sqrt{2-(1+x)^2}} = \int \frac{dx}{\sqrt{(2)^2 - (1+x)^2}} \quad \dots(i)$$

Let $z = 1 + x$, then $dz = dx$

From (i),

$$I = \int \frac{dz}{\sqrt{(\sqrt{2})^2 - z^2}} = \sin^{-1} \frac{z}{\sqrt{2}} + c = \sin^{-1} \left(\frac{1+x}{\sqrt{2}} \right) + c$$

Sol 28:
$$\int \frac{dx}{\sec x + \operatorname{cosec} x} = \int \left(\frac{\sin x \cos x}{\sin x + \cos x} \right) dx$$

$$= \frac{1}{2} \int \left(\frac{1 + 2 \sin x \cos x - 1}{\sin x + \cos x} \right) dx$$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{(\sin x + \cos x)} dx$$

$$= \frac{1}{2} \int (\sin x + \cos x) dx - \frac{1}{2\sqrt{2}} \int \frac{1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} dx$$

$$= \frac{1}{2} [\sin x - \cos x] - \frac{1}{2\sqrt{2}} \int \frac{dx}{\sin \left(x + \frac{\pi}{4} \right)}$$

$$= \frac{1}{2} [\sin x - \cos x] - \frac{1}{2\sqrt{2}} \int \frac{\sec^2 \left(\frac{x}{2} + \frac{\pi}{8} \right)}{2 \tan \left(\frac{x}{2} + \frac{\pi}{8} \right)} dx$$

$$= \frac{1}{2} [\sin x - \cos x] - \frac{1}{2\sqrt{2}} \log \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) + c$$

$$= \frac{1}{2} \left[\sin x - \cos x - \frac{1}{\sqrt{2}} \log \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right] + c$$

Sol 29
$$I = \int \sqrt{2x^2 + 3x + 4} dx = \int \sqrt{2 \left(x^2 + \frac{3}{2}x + 2 \right)} dx$$

$$= \sqrt{2} \int \sqrt{x^2 + \frac{3}{2}x + 2} dx$$

$$= \sqrt{2} \int \sqrt{x^2 + 2 \cdot x \cdot \frac{3}{4} + \left(\frac{3}{4} \right)^2 - \left(\frac{3}{4} \right)^2 + 2} dx$$

$$= \sqrt{2} \int \sqrt{\left(x + \frac{3}{4} \right)^2 + \frac{23}{16}} dx \quad \dots(i)$$

Let, $z = x + \frac{3}{4}$, then $dz = dx$. Let $\sqrt{\frac{23}{16}} = \frac{\sqrt{3}}{4} = a$

Then from (i), $I = \sqrt{2} \int \sqrt{z^2 + a^2} dz$

Now,

$$\int \sqrt{z^2 + a^2} dz = \frac{z}{2} \sqrt{z^2 + a^2} + \frac{a^2}{2} \log \left(z + \sqrt{z^2 + a^2} \right) + c$$

$$= \frac{\left(-x + \frac{3}{4} \right)}{2} \sqrt{\left(x + \frac{3}{4} \right)^2 + \frac{23}{16}}$$

$$+ \frac{23}{32} \log \left[x + \frac{3}{4} + \sqrt{\left(x + \frac{3}{4} \right)^2 + \frac{23}{16}} \right] + c$$

$$I = \frac{4x + 3}{8} \sqrt{2x^2 + 3x + 4}$$

$$+ \frac{23}{16\sqrt{2}} \log \left[\frac{4x + 3}{4} + \frac{\sqrt{2x^2 + 3x + 4}}{\sqrt{2}} \right] + c$$

Sol 30: Let $z = x^n + 1$, then $dz = nx^{n-1} dx$

Now, $I = \int \frac{dx}{x(x^n + 1)} = \int \frac{dx}{nx^{n-1} \cdot x(x^n + 1)}$

$$= \frac{1}{n} \int \frac{dz}{x^n(x^n + 1)} = \frac{1}{n} \int \frac{dz}{(z-1)z} \quad \dots(i)$$

Let $\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1) + Bz}{z(z-1)} \quad \dots(ii)$

$\therefore A(z-1) + Bz = 1 \quad \dots(iii)$

Putting $z = 0$ we get, $-A = 1$

$\therefore A = -1$

Putting $z = 1$, we get $B = 1$

$\therefore$ From $I = \frac{1}{n} \int \frac{dz}{z(z-1)} = \frac{1}{n} \int \left(-\frac{1}{z} + \frac{1}{z-1} \right) dz$

$$= \frac{1}{n} [-\log |z| + \log |z-1|] + c$$

$$= \frac{1}{n} \log \left| \frac{z-1}{z} \right| + c = \frac{1}{n} \log \left| \frac{x^n}{x^n + 1} \right| + c$$

Sol 31:
$$\int \frac{\cos x - \sin x}{16 - 9(1 + \sin 2x)} dx$$

$$\int \frac{(\cos x - \sin x)}{16 - 9(\sin x + \cos x)^2} dx$$

Let $3(\sin x + \cos x) = t$

$$\Rightarrow (3\cos x - 3\sin x)dx = dt$$

$$\frac{1}{3} \int \frac{dt}{16-t^2} = \frac{1}{2.4} \cdot \frac{1}{3} \log \left| \frac{4+t}{4-t} \right| = \left(\frac{1}{24} \log \frac{4+3\cos x+3\sin x}{4-3\cos x-3\sin x} \right)$$

Sol 32: $\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{1+3\sin 2x} dx$

$$\sqrt{2} \int \frac{\cos x - \sin x}{(\sqrt{\sin 2x})(1+3\sin 2x)} dx$$

$$\sqrt{2} \int \left[\frac{(\cos 2x \cos x + \sin^2 x \sin x) + (\cos 2x \sin x - \sin 2x \cos x)}{(\sqrt{\sin 2x})(1+3\sin 2x)} \right] dx$$

$$2 \int \left[\frac{\cos 2x(\sin x + \cos x) - \sin 2x(\cos x - \sin x)}{\sqrt{2}\sin 2x(1+3\sin 2x)} \right] dx$$

$$\Rightarrow \int \frac{1}{\frac{(\sin x + \cos x)^2 + 2\sin 2x}{(\sin x + \cos x)^2}} \times \left[\frac{(\sin x + \cos x)2 \cdot \cos 2x - 2\sin^2 x(\cos x - \sin x)}{(\sin x + \cos x)^2 \sqrt{2}\sin 2x} \right] dx$$

$$\Rightarrow \int \left(\frac{1}{1 + \frac{2\sin 2x}{(\sin x + \cos x)^2}} \right) \left(\frac{2\sin 2x}{(\sin x + \cos x)} \right) dx$$

$$= \tan^{-1} \left(\frac{\sqrt{2}\sin 2x}{\sin x + \cos x} \right) + c$$

Sol 33: $\int \frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2(x^2 + 1)^2} dx$

$$\therefore \frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2(x^2 + 1)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} + \frac{Ex^2 + Fx + G}{(x^2 + 1)^2}$$

$$\int \frac{(4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7)}{x^2(x^2 + 1)^2} dx$$

$$= \int \left(\frac{4}{x} + \left(-\frac{7}{x^2} \right) + \frac{6}{x^2 + 1} + \frac{6(1-x^2)}{(1+x^2)^2} \right) dx$$

$$= 4\log x + \frac{7}{x} + 6\tan^{-1}x + 6 \int \left(\frac{1-x^2}{(1+x^2)^2} \right) dx$$

Put $x = \tan \theta$

$$\Rightarrow dx = \sec^2 \theta d\theta$$

$$= 4\log x + \frac{7}{x} + 6\tan^{-1}x + 6 \int \frac{1-\tan^2 \theta}{(1+\tan^2 \theta)} d\theta$$

$$= 4\log x + \frac{7}{x} + 6\tan^{-1}x + 6 \int \cos 2\theta d\theta$$

$$= 4\log x + \frac{7}{x} + 6\tan^{-1}x + 6 \frac{1}{2} \sin 2\theta$$

$$\therefore \sin 2\theta = \frac{2\tan \theta}{1+\tan^2 \theta} = \frac{2x}{1+x^2}$$

$$\therefore I = 4\log x + \frac{7}{x} + 6\tan^{-1}x + \frac{6x}{1+x^2} + c$$

Sol 34: $\int \frac{dx}{\cos^3 x - \sin^3 x}$

$$\int \frac{dx}{(\cos x - \sin x)(1 + \cos x \sin x)}$$

$$= \int \frac{(\cos x - \sin x)^2 + 2\sin x \cos x}{(\cos x - \sin x)(1 + \cos x \sin x)} dx$$

$$\int \frac{(\cos x - \sin x)}{(1 + \cos x \sin x)} dx + 2 \int \frac{\sin x \cos x}{(\cos x - \sin x)(1 + \cos x \sin x)} dx$$

$$= \frac{2}{3} \int \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} dx + \frac{2}{3} \int \frac{(\cos x - \sin x)}{2 - (\sin x + \cos x)^2} dx$$

$$= \frac{2}{3} \tan^{-1}(\sin x + \cos x) + \frac{1}{3\sqrt{2}} \log \left| \frac{\sqrt{2} + \sin x + \cos x}{\sqrt{2} - \sin x - \cos x} \right| + c$$

Sol 35: $\int \frac{x^2}{(x \cos x - \sin x)(x \sin x + \cos x)} dx$

$$\int \frac{(x \cos x - \sin x)^2 + (x \sin x + \cos x)^2}{(x \cos x - \sin x)(x \sin x + \cos x)} dx$$

$$\int \frac{x \cos x - \sin x}{x \sin x + \cos x} dx + \int \frac{x \sin x + \cos x}{x \cos x - \sin x} dx$$

$$+ \int \frac{-1}{(x \cos x - \sin x)(x \sin x + \cos x)} dx$$

$$= \int \left(\frac{x \cos x + \sin x - \sin x}{x \sin x + \cos x} \right) dx + \int \frac{x \sin x + \sin x - \cos x}{x \cos x - \sin x} dx$$

$$+ \int \frac{\cos^2 x + \sin^2 x - 1}{(x \sin x + \cos x)(x \cos x - \sin x)} dx$$

$$= \log \left| \frac{x \sin x + \cos x}{x \cos x - \sin x} \right| + c$$

Sol 36: $\int \cos 2\theta \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) d\theta$

$$\Rightarrow \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) \int \cos 2\theta d\theta$$

$$- \int \left(\frac{d \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right)}{d\theta} \int \cos 2\theta d\theta \right) d\theta$$

$$\Rightarrow \frac{1}{2} \sin 2\theta \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) - \int \left(\frac{2}{\cos 2\theta} \times \frac{\sin 2\theta}{2} \right) d\theta$$

$$\Rightarrow \frac{1}{2} \sin 2\theta \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) - \int \tan 2\theta d\theta$$

$$\Rightarrow \frac{1}{2} \sin 2\theta \log \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right) - \frac{1}{2} \log(\sec 2\theta) d\theta$$

$$= \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) \left(\frac{(1 - \tan \theta \times \sec^2 \theta)}{(1 - \tan \theta)^2} - (1 + \tan \theta) \sec^2 \theta \right)$$

$$= \frac{(1 - \tan \theta)}{(1 - \tan \theta)} \times \frac{2 \sec^2 \theta}{(1 - \tan \theta)^2} = \frac{2(1 + \tan^2 \theta)}{(1 - \tan^2 \theta)} = \frac{2}{\cos 2\theta}$$

Sol 37: $A \rightarrow s; B \rightarrow p; C \rightarrow q; D \rightarrow r$

(A) $\int \frac{x^4 - 1}{x^3 \sqrt{x^2 + \frac{1}{x^2} + 1}} dx$

$$\int \frac{x - x^{-3}}{\sqrt{x^2 + \frac{1}{x^2} + 1}} dx$$

Put $x^2 + \frac{1}{x^2} + 1 = t$

$$\left(2x - \frac{2}{x^3} \right) dx = \frac{dt}{2} \text{ or } \left(x - \frac{1}{x^3} \right) dx = \frac{dt}{2}$$

$$\frac{1}{2} \int \frac{dt}{\sqrt{t}} = \sqrt{t} + c = \sqrt{x^2 + \frac{1}{x^2} + 1} + c = \sqrt{\frac{x^4 + x^2 + 1}{x}} + c$$

(B) $\int \frac{x^2 - 1}{x \sqrt{1 + x^4}} dx$

$$\int \frac{x^2 - 1}{x^2 \sqrt{x^2 + \frac{1}{x^2}}} dx \Rightarrow \int \frac{(1 - x^{-2})}{\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}} dx$$

Put $x + \frac{1}{x} = t$

$$\Rightarrow (1 - x^{-2}) dx = dt$$

$$\int \frac{dt}{\sqrt{t^2 - 2}}$$

Put $t = \sqrt{2} \sec \theta$

$$\int \frac{\sqrt{2} \sec \theta \tan \theta d\theta}{\sqrt{2} \tan \theta} = \log |\sec \theta + \tan \theta|$$

$$= \log \left| \frac{x^2 + 1}{\sqrt{2}x} + \frac{\sqrt{x^4 + 1}}{\sqrt{2}x} \right| + c$$

Or $\log \left| \frac{x^2 + 1}{x} + \frac{\sqrt{x^4 + 1}}{x} \right| + c$

(C) $\int \frac{(1 + x^2)}{x^2 \left(\frac{1}{x} - x \right) \sqrt{x^2 + \frac{1}{x^2}}} dx$

$$\int \frac{(1 + x^{-2})}{\left(\frac{1}{x} - x \right) \sqrt{\left(x - \frac{1}{x} \right)^2 + 2}} dx$$

$x - \frac{1}{x} = t \Rightarrow (1 + x^{-2}) dx = dt$

$$= - \int \frac{dt}{t \sqrt{t^2 + 2}}$$

$t = \sqrt{2} \tan \theta$

$$dt = \sqrt{2} \sec^2 \theta d\theta$$

$$= - \int \frac{\sqrt{2} \sec^2 \theta d\theta}{\sqrt{2} \tan \theta \sqrt{2} \sec \theta} - \frac{1}{\sqrt{2}} \int \operatorname{cosec} \theta$$

$$= \frac{1}{\sqrt{2}} \log |\operatorname{cosec} \theta - \cot \theta| + c$$

$$\Rightarrow - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{x^4 + 1} - \sqrt{x}}{(x^2 - 1)} \right| + c$$

(D) $\int \frac{\sqrt{1 + x^4 + x^2}}{1 + x^4} dx$

$$= \int \left(\frac{1}{\sqrt{1 + \frac{1}{x^4}}} \right) \frac{1}{2 \sqrt{\left(\sqrt{1 + \frac{1}{x^4}} - 1 \right)}} \times \frac{1}{2 \sqrt{1 + \frac{1}{x^4}}} \times \left(\frac{-4}{x^5} \right)$$

$$- \tan^{-1} \left(\sqrt{\sqrt{1 + \frac{1}{x^4}} - 1} \right) + c$$

$$\begin{aligned} \therefore \frac{d \tan^{-1} \left(\sqrt{\sqrt{1 + \frac{1}{x^4}} - 1} \right)}{dx} \\ = \frac{1}{1 + \left(\sqrt{1 + \frac{1}{x^4}} - 1 \right)} \times \left(\sqrt{\sqrt{1 + \frac{1}{x^4}} - 1} \right) \\ = \left(\frac{1}{\sqrt{1 + \frac{x}{4}}} \right) \left(\frac{1}{2\sqrt{1 + \frac{1}{x^4}} - 1} \right) \frac{1}{2\sqrt{1 + \frac{1}{x^4}}} \times \frac{-4}{x^5} \end{aligned}$$

Exercise 2

Single Correct Choice Type

Sol 1: (D) $\int \frac{\tan^{-1} x - \cot^{-1} x}{\tan^{-1} x + \cot^{-1} x} dx$

$$\therefore \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\begin{aligned} I &= \int \left(\frac{2 \tan^{-1} x - \frac{\pi}{2}}{\frac{\pi}{2}} \right) dx = \frac{4}{\pi} \int \tan^{-1} x dx - \int dx \\ &= \frac{4}{\pi} \int \tan^{-1} x dx - \int \left[\left(\frac{d \tan^{-1} x}{dx} \right) \int dx \right] dx - x + c \\ &= \frac{4}{\pi} x \tan^{-1} x - \frac{4}{x} \int \left(\frac{x}{1+x^2} \right) dx - x + c \\ &= \frac{4}{\pi} x \tan^{-1} x - \frac{2}{\pi} \log(1+x^2) - x + c \end{aligned}$$

Sol 2: (A) $\int \frac{x^2 - 4}{x^4 + 24x^2 + 16} dx = \int \frac{x^2 - 4}{x^2 \left(x^2 + 24 + \frac{16}{x^2} \right)} dx$

$$= \int \frac{\left(1 - \frac{4}{x^2} \right)}{\left(x^2 + \frac{16}{x^2} + 8 \right) + 16} dx = \int \frac{\left(1 - \frac{4}{x^2} \right)}{\left(x + \frac{4}{x} \right)^2 + (4)^2} dx$$

Put $x + \frac{4}{x} = t$

$$\Rightarrow \left(1 - \frac{4}{x^2} \right) dx = dt$$

$$\therefore \int \frac{dt}{t^2 (4)^2} = \frac{1}{4} \tan^{-1} \frac{t}{4} + c$$

$$\therefore I = \frac{1}{4} \tan^{-1} \left(\frac{x + \frac{4}{x}}{4} \right) + c$$

Or $I = \frac{1}{4} \tan^{-1} \left(\frac{x^2 + 4}{4x} \right) + c$

Sol 3: (D) $\int \frac{(x-1)^2}{x^4 + 2x^2 + 1} dx = \int \frac{x^2 - 1 - 2x + 2}{(x^2 + 1)^2} dx$

$$= \int \frac{(x^2 - 1)}{x^2 \left(x + \frac{1}{x} \right)^2} dx + \int \frac{-2x}{(x^2 + 1)^2} dx + \int \frac{2dx}{(1 + x^2)^2}$$

Put $x^2 + 1 = t$ and Put $x = \tan \theta$

$$dx = \sec^2 \theta d\theta$$

$$= \int \frac{\left(1 - \frac{1}{x^2} \right)}{\left(x + \frac{1}{x} \right)^2} dx + \int -\frac{dt}{t^2} + 2 \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta}$$

Put $x + \frac{1}{x} = u$

$$= \int \frac{du}{u^2} + \frac{1}{x^2 + 1} + 2 \int \cos^2 \theta d\theta$$

$$= -\frac{1}{\left(x + \frac{1}{x} \right)} + \frac{1}{(x^2 + 1)} + \frac{2}{2} \int (\cos 2\theta + 1) d\theta$$

$$= \frac{(1-x)}{x^2 + 1} + \frac{1}{2} \sin 2\theta + \theta + c$$

$$= \frac{(1-x)}{1+x^2} + \frac{1}{2} \frac{2 \times x}{1+x^2} + \tan^{-1} x + c$$

$$= \frac{(1-x)}{1+x^2} + \frac{x}{1+x^2} + \tan^{-1} x + c = \frac{1}{1+x^2} + \tan^{-1} x + c$$

Sol 4: (A) $\int \frac{x^4 - 4}{x^2 \sqrt{x^4 + x^2 + 4}} dx$

$$\int \frac{x^4 - 4}{x^2 \times x \sqrt{x^2 + 1 + \frac{4}{x^2}}} dx$$

$$\int \frac{\left[x - \frac{4}{x^3}\right]}{\sqrt{x^2 + 1 + \frac{4}{x^2}}} dx$$

$$\text{Put } x^2 + \frac{4}{x^2} + 1 = t \Rightarrow \left(2x - \frac{8}{x^3}\right) dx = dt$$

$$= \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \cdot 2\sqrt{t} + c = \sqrt{x^2 + 1 + \frac{4}{x^2}} + c$$

$$= \frac{\left(\sqrt{x^4 + x^2 + 4}\right)}{x} + c$$

$$\text{Sol 5: (A)} \int \left(\frac{\sec x + \tan x - 1}{\tan x - \sec x + 1}\right) dx$$

$$= \int \frac{\sec x + \tan x - (\sec^2 x - \tan^2 x)}{(\tan x - \sec x + 1)} dx$$

$$= \int \frac{(\sec x + \tan x)[1 - \sec x + \tan x]}{[\tan x - \sec x + 1]} dx$$

$$= \int (\sec x + \tan x) dx$$

$$= \log|\sec x + \tan x| + \log \sec x + c$$

$$s = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$s_x = x + 2x^2 + 3x^3 + \dots$$

$$s(1-x) = 1 + x + x^2 + x^3 + \dots$$

$$s(1-x) = \frac{1}{1-x} \therefore s = \frac{1}{(1-x)^2}$$

$$\therefore \int \frac{1}{(1-x)^2} dx = \frac{(1-x)^{-1}}{-1 \times -1} + c = (1-x)^{-1} + c$$

$$\text{Sol 6: (C)} \int e^x \frac{(x^2 - 3)}{(x+3)^2} dx = \int e^x \frac{(x^2 - 9 + 6)}{(x+3)^2} dx$$

$$= \int e^x \left(\frac{x-3}{x+3} + \frac{6}{(x+3)^2} \right) dx$$

$$= \int e^x (f(x) + f'(x)) dx = e^x f(x) + c$$

$$= e^x \frac{(x-3)}{(x+3)} + c$$

$$\text{Sol 7: (B)} I = \int \sqrt{\frac{1 - \cos x}{\cos \alpha - \cos x}} dx$$

$$I = \int \frac{\sqrt{2} \sin \frac{x}{2}}{\sqrt{2 \cos^2 \frac{\alpha}{2} - 2 \cos^2 \frac{x}{2}}} dx$$

$$= \int \frac{\sqrt{2} \sin \frac{x}{2}}{\sqrt{2} \cos \frac{\alpha}{2} \sqrt{1 - \left(\frac{\cos \frac{x}{2}}{\cos \frac{\alpha}{2}}\right)^2}} dx$$

$$= \int \frac{\frac{1}{2} \times 2x \left(\frac{-\sin \frac{x}{2}}{\cos \frac{\alpha}{2}}\right)}{\sqrt{1 - \left(\frac{\cos \frac{x}{2}}{\cos \frac{\alpha}{2}}\right)^2}} dx$$

$$\text{Let } \frac{\cos \frac{x}{2}}{\cos \frac{\alpha}{2}} = t \therefore I = \int -\frac{2}{\sqrt{1-t^2}} dt$$

$$= 2 \cos^{-1}(t) + c = 2 \cos^{-1} \left(\frac{\cos \frac{x}{2}}{\cos \frac{\alpha}{2}} \right) + c$$

$$\text{Sol 8: (B)} I = \int \frac{3x^4 - 1}{(x^4 + x + 1)^2} dx$$

$$= \int \frac{4x^4 + x - (x^4 + x + 1)}{(x^4 + x + 1)^2} dx$$

$$= \int \frac{x(4x^2 - 1)}{(x^4 + x + 1)^2} - \frac{1}{(x^4 + x + 1)} dx$$

$$\text{Let } \int \frac{1}{x^4 + x + 1} = I_1$$

$$\therefore I = x \int \frac{(4x^3 + 1)}{(x^4 + x + 1)^2} - \int \frac{4x^3 + 1}{(x^4 + x + 1)^2} dx - I_1$$

$$= x \times \frac{(x^4 + x + 1)^{-1}}{-1} - \int \frac{(x^4 + x + 1)^{-1}}{-1} dx - I_1$$

$$= \frac{-x}{(x^4 + x + 1)} + c + I_1 - I_1$$

$$\text{Sol 9: (C)} \int e^{3x} \cos 4x dx$$

$$\text{Let } 3x = t$$

$$\begin{aligned}
 \therefore I &= \frac{1}{3} \int e^t \cos \frac{4t}{3} dt \\
 &= \frac{1}{3} \int e^t \cos \frac{4t}{3} dt = \frac{1}{3} \int e^t \left(\cos \frac{4t}{3} - \frac{4}{3} \sin \frac{4t}{3} \right) dt \\
 &\quad + \frac{4}{9} \int e^t \left(\sin \frac{4t}{3} \right) dt \\
 &= \frac{1}{3} \int \left[e^t \cos \frac{4t}{3} \right] + \frac{4}{9} \int e^t \left(\sin \frac{4t}{3} + \frac{4}{3} \cos \frac{4t}{3} \right) dt \\
 &\quad - \int \frac{16}{27} e^t \cos \frac{4t}{3} dt
 \end{aligned}$$

$$\frac{25}{9} I = \frac{1}{3} e^t \cos \frac{4t}{3} + \frac{4}{9} e^t \sin \frac{4t}{3} + c$$

$$I = \frac{e^t}{25} \left(3 \frac{4t}{3} + 4 \sin \frac{4t}{3} \right) + c$$

$$\therefore 3A = 4B$$

Sol 10: (C)

$$\begin{aligned}
 I &= \int \frac{p + x^{p+2q-1} + qx^{p+2q-1} - q(x^{q-1} + x^{p+2q-1})}{(x^{p+q} + 1)^2} dx \\
 &= \int \frac{(p+q)x^{p+q-1}x^q - qx^{q-1}(x^{p+q}) + 1}{(x^{p+q} + 1)^2} dx
 \end{aligned}$$

$$\text{It is of the form } \frac{uv' - vu'}{u^2}$$

$$\therefore \text{Where } u = x^{p+q} + 1 \text{ and } v = -x^q$$

$$\therefore I = \frac{v}{u} + c = \frac{-x^q}{x^{p+q} + 1} + c$$

Sol 11: (B) Let $f^{-1}(x) = t$

$$\therefore f(f^{-1}(x)) = x$$

$$\therefore f(t) = x$$

$$\therefore \int f^{-1}(x) dx$$

$$\int t dx = \int t f'(t) dt$$

$$= t f(t) - \int f(t) dt = t f(t) - g(t)$$

$$= f^{-1}(x) (x - g)(f^{-1}(x))$$

$$\text{Sol 12: (B)} \left(\frac{x}{(x^4 - 1)^4} \right)^{1/3} dx$$

$$\begin{aligned}
 I &= \int \left(\frac{x}{(x^4 - 1)^4} \right)^{1/3} dx = \int \left(\frac{x}{x^{16} \left(1 - \frac{1}{x^4} \right)^4} \right)^{1/3} dx \\
 &= \int \frac{1}{\left(1 - \frac{1}{x^4} \right)^{4/3} x^5} dx
 \end{aligned}$$

$$\text{Let } 1 - \frac{1}{x^4} = t \quad \therefore \frac{4}{x^5} dx = dt$$

$$\therefore I = \int \frac{1}{t^{4/3}} \times \frac{1}{4} dt = \frac{1}{4} \times \frac{t^{-1/3}}{-\frac{1}{3}} + c = -\frac{3}{4} t^{-1/3} + c$$

$$= -\frac{3}{4} \left(1 - \frac{1}{x^4} \right)^{-1/3} + c = -\frac{3}{4} \left(\frac{x^4}{x^4 - 1} \right)^{1/3} + c$$

Sol 13: (A) $I = \int e^u \sin 2x dx$

When $u = x$

$$I = \int e^x \sin 2x dx = \int e^x (\sin 2x + 2 \cos 2x - 2 \cos 2x) dx$$

$$= \int e^x (\sin 2x + 2 \cos 2x) - 2 \int e^x \cos 2x$$

$$= \int e^x (\sin 2x + 2 \cos 2x)$$

$$- 2 \int e^x (\cos 2x - 2 \sin 2x + 2 \sin 2x) dx$$

$$\therefore 5I = \int e^x (\sin 2x + 2 \cos 2x) dx$$

$$- 2 \int e^x (\cos 2x - 2 \sin 2x) dx$$

$$= e^x \sin 2x - 2e^x \cos 2x + c$$

When $u = \sin x$

$$I = \int 2e^{\sin x} \cos x \sin x dx$$

$$\text{Put } \sin x = t \quad \therefore \cos x dx = dt$$

$$\therefore I = 2 \int t e^t dt \text{ which is solvable}$$

Previous Years' Questions

Sol 1: (C) Let $I = \int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$

$$= \int \frac{(\cos^2 x + \cos^4) \cdot \cos x dx}{(\sin^2 x + \sin^4 x)}$$

$$\text{Put } \sin x = t \Rightarrow \cos x dx = dt$$

$$\therefore I = \int \frac{[(1-t^2) + (1-t^2)^2]}{t^2 + t^4} dt$$

$$\Rightarrow I = \int \frac{1-t^2+1-2t^2+t^4}{t^2+t^4} dt$$

$$\Rightarrow I = \int \frac{2-3t^2+t^4}{t^2(t^2+1)} dt.$$

Using partial fraction for,

$$\frac{y^2-3y+2}{y(y+1)} = 1 + \frac{A}{y} + \frac{B}{y+1} \quad (\text{where } y = t^2)$$

$$\Rightarrow A = 2, B = -6$$

$$\therefore \frac{y^2-3y+2}{y(y+1)} = 1 + \frac{2}{y} - \frac{6}{y+1}$$

$\therefore$ Eq. (i) reduces to,

$$I = \int \left(1 + \frac{2}{t^2} - \frac{6}{1+t^2} \right) dt = t - \frac{2}{t} - 6 \tan^{-1}(t) + c$$

$$= \sin x - \frac{2}{\sin x} - 6 \tan^{-1}(\sin x) + c$$

Sol 2: (A) Given, $f(x) = \frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$

$$\therefore ff(x) = \frac{f(x)}{[1+f(x)^n]^{1/n}} = \frac{x}{(1+2x^n)^{1/n}}$$

$$\text{and } fff(x) = \frac{x}{(1+3x^n)^{1/n}}$$

$$\therefore g(x) = \underbrace{\text{fofo...of}}_{n \text{ times}}(x) = \frac{x}{(1+nx^n)^{1/n}}$$

$$\text{Let } I = \int x^{n-2} g(x) dx = \int \frac{x^{n-1} dx}{(1+nx^n)^{1/n}} = \frac{1}{n^2} \int \frac{n^2 x^{n-1} dx}{(1+nx^n)^{1/n}}$$

$$= \frac{1}{n^2} \int \frac{d(1+nx^n)}{(1+nx^n)^{1/n}} dx$$

$$\therefore I = \frac{1}{n(n-1)} (1+nx^n)^{1-\frac{1}{n}} + c$$

Sol 3: (C) Since, $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx$

$$J = \int \frac{e^{3x}}{1 + e^{2x} + e^{4x}} dx$$

$$\therefore J - I = \int \frac{(e^{3x} - e^x)}{1 + e^{2x} + e^{4x}} dx$$

$$\text{Put } e^x = u \Rightarrow e^x dx = du$$

$$\therefore J - I = \int \frac{(u^2 - 1)}{1 + u^2 + u^4} du$$

$$\dots(i) = \int \frac{\left(1 - \frac{1}{u^2}\right)}{1 + \frac{1}{u^2} + u^2} du = \int \frac{\left(1 - \frac{1}{u^2}\right)}{\left(u + \frac{1}{u}\right)^2 - 1} du$$

$$\text{Put } u + \frac{1}{u} = t$$

$$\Rightarrow \left(1 - \frac{1}{u^2}\right) du = dt = \int \frac{dt}{t^2 - 1}$$

$$= \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + c = \frac{1}{2} \log \left| \frac{u^2 - u + 1}{u^2 + u + 1} \right| + c$$

$$= \frac{1}{2} \log \left| \frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right| + c$$

Sol 4: Given, $f(x) = \int \left(\frac{2 \sin x - \sin 2x}{x^3} \right) dx$

On differentiating w.r.t. x , we get

$$f'(x) = \frac{2 \sin x - \sin 2x}{x^3} = \frac{2 \sin x}{x} \left(\frac{1 - \cos x}{x^2} \right)$$

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} 2 \left(\frac{\sin x}{x} \right) \left(\frac{2 \sin^2 \frac{x}{2}}{x^2} \right)$$

$$= 4 \cdot 1 \cdot \lim_{x \rightarrow 0} \left(\frac{\sin^2 \frac{x}{2}}{4 \times \left(\frac{x^2}{2} \right)} \right) = 1$$

Sol 5: (i) Let $I = \int \sqrt{1 + \sin \frac{x}{2}} dx$

$$\Rightarrow I = \int \sqrt{\cos^2 \frac{x}{4} + \sin^2 \frac{x}{4} + 2 \sin \frac{x}{4} \cos \frac{x}{4}} dx$$

$$\Rightarrow I = \int \left(\cos \frac{x}{4} + \sin \frac{x}{4} \right) dx$$

$$= 4 \sin \frac{x}{4} - 4 \cos \frac{x}{4} + c$$

(ii) Let $I = \int \frac{x^2}{\sqrt{1-x}} dx$

Put $1-x = t^2 \Rightarrow -dx = 2t dt$

$$\begin{aligned} \therefore I &= \int \frac{(1-t^2)^2 \cdot (-2t) dt}{t} \\ &= -2 \int (1-2t^2+t^4) dt = -2 \left(t - \frac{2t^3}{3} + \frac{t^5}{5} \right) + c \\ &= -2 \left\{ \sqrt{1-x} - \frac{2}{3}(1-x)^{3/2} + \frac{1}{5}(1-x)^{5/2} \right\} \end{aligned}$$

Sol 6: Let $I = \int (e^{\log x} + \sin x) \cos x dx$

$$\begin{aligned} &= \int (x + \sin x) \cos x dx \\ \therefore I &= \int x \cos x dx + \frac{1}{2} \int (\sin 2x) dx \\ &= (x \cdot \sin x - \int 1 \cdot \sin x dx) - \frac{\cos 2x}{4} + c \\ &= x \sin x + \cos x - \frac{\cos 2x}{4} + c \end{aligned}$$

Sol 7: Let $I = \int \frac{(x-1)e^x}{(x+1)^3} dx$

$$\begin{aligned} \Rightarrow I &= \int \left\{ \frac{x+1-2}{(x+1)^3} \right\} e^x dx \\ &= \int \left\{ \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3} \right\} e^x dx \\ &= \int e^x \cdot \frac{1}{(x+1)^2} dx - 2 \int e^x \cdot \frac{1}{(x+1)^3} dx \end{aligned}$$

Applying integration by parts,

$$\begin{aligned} &= \left\{ \frac{1}{(x+1)^2} \cdot e^x - \int e^x \cdot \frac{-2}{(x+1)^3} dx \right\} \\ &- 2 \int e^x \cdot \frac{1}{(x+1)^3} dx = \frac{e^x}{(x+1)^2} + c \end{aligned}$$

Sol 8: Let $I = \int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^2 \cdot x^3 \left(1 + \frac{1}{x^4}\right)^{3/4}}$

Put $1 + x^{-4} = t \Rightarrow -\frac{4}{x^5} dx = dt$

$$\therefore I = -\frac{1}{4} \int \frac{dt}{t^{3/4}} = -\frac{1}{4} \cdot \frac{t^{1/4}}{1/4} + c$$

$$= -\left(1 + \frac{1}{x^4}\right)^{1/4} + c = -\frac{(x^4+1)^{1/4}}{x} + c$$

Sol 9: Let $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$

Put $x = \cos^2 \theta \Rightarrow dx = -2 \sin \theta \cos \theta d\theta$

$$\begin{aligned} \therefore I &= \int \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} \cdot (-2 \sin \theta \cos \theta) d\theta \\ &= -\int 2 \tan \frac{\theta}{2} \cdot \sin \theta \cos \theta d\theta = -2 \int 2 \sin^2 \frac{\theta}{2} \cdot \cos \theta d\theta \\ &= -2 \int (1 - \cos \theta) \cos \theta d\theta = -2 \int (\cos \theta - \cos^2 \theta) d\theta \\ &= -2 \int \cos \theta d\theta + \int (1 + \cos 2\theta) d\theta \\ &= -2 \sin \theta + \theta + \frac{\sin 2\theta}{2} + c \\ &= -2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x(1-x)} + c \end{aligned}$$

Sol 10: Let $I = \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$

$$\begin{aligned} &= \int \frac{\sin^{-1} \sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1} \sqrt{x}\right)}{\frac{\pi}{2}} dx \\ &= \frac{2}{\pi} \int \left(2 \sin^{-1} \sqrt{x} - \frac{\pi}{2}\right) dx \\ &= \frac{4}{\pi} \int \sin^{-1} \sqrt{x} dx - x + c \quad \dots(i) \end{aligned}$$

Now, $\int \sin^{-1} \sqrt{x} dx$

Put $x = \sin^2 \theta \Rightarrow dx = \sin 2\theta$

$$\begin{aligned} &= \int \theta \cdot \sin 2\theta d\theta \\ &= -\frac{\theta \cos 2\theta}{2} + \int \frac{1}{2} \cos 2\theta d\theta \\ &= -\frac{\theta}{2} \cos 2\theta + \frac{1}{4} \sin 2\theta \\ &= -\frac{1}{2} \theta (1 - 2 \sin^2 \theta) + \frac{1}{2} \sin \theta \sqrt{1 - \sin^2 \theta} \\ &= -\frac{1}{2} \sin^{-1} \sqrt{x} (1 - 2x) + \frac{1}{2} \sqrt{x} \sqrt{1-x} \quad \dots(ii) \end{aligned}$$

From Eqs. (i) and (ii), we get

$$I = \frac{4}{\pi} \left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2} \right] - x + c$$

$$= \frac{2}{\pi} \left[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x} \right] - x + c$$

Sol 11: Let $I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx$

$$= \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin^2 x} dx = \int \frac{\sqrt{\cot^2 x - 1}}{\sin^2 x} dx$$

Put $\cot x = \sec \theta \Rightarrow -\operatorname{cosec}^2 x dx = \sec \theta \tan \theta d\theta$

$$\therefore I = \int \sqrt{\sec^2 \theta - 1} \cdot \frac{\sec \theta \cdot \tan \theta}{-(1 + \sec^2 \theta)} d\theta$$

$$= -\int \frac{\sec \theta \cdot \tan^2 \theta}{1 + \sec^2 \theta} d\theta$$

$$= -\int \frac{\sin^2 \theta}{\cos \theta + \cos^3 \theta} d\theta$$

$$= -\int \frac{1 - \cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} d\theta$$

$$= -\int \frac{(1 + \cos^2 \theta) - 2\cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} d\theta$$

$$= -\int \sec \theta d\theta + 2 \int \frac{\cos \theta}{1 + \cos^2 \theta} d\theta$$

$$= -\log |\sec \theta + \tan \theta| + 2 \int \frac{\cos \theta}{2 - \sin^2 \theta} d\theta$$

$$= -\log |\sec \theta + \tan \theta| + \int \frac{dt}{2 - t^2}, \text{ where } \sin \theta = t$$

$$= -\log |\sec \theta + \tan \theta| + 2 \cdot \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + \sin \theta}{\sqrt{2} - \sin \theta} \right| + c$$

$$= -\log \left| \cot x + \sqrt{\cot^2 x - 1} \right| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + c$$

Sol 12: Let $I = \int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\ln(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}} \right) dx$

$$\therefore I = I_1 + I_2,$$

where $I_1 = \int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} \right) dx$, $I_2 = \int \frac{\ln(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}} dx$

Now, $I_1 = \int \left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} \right) dx$

Put $x = t^{12} \Rightarrow dx = 12t^{11} dt$

$$\therefore I_1 = 12 \int \frac{t^{11}}{t^4 + t^3} dt = 12 \int \frac{t^8 dt}{t+1}$$

$$= 12 \int (t^7 - t^6 + t^5 - t^4 + t^3 - t^2 + t - 1) dt + 12 \int \frac{dt}{t+1}$$

$$= 12 \left(\frac{t^8}{8} - \frac{t^7}{7} + \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t \right) + 12 \log(t+1)$$

And $I_2 = \int \left\{ \frac{\ln(1 + \sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}} \right\} dx$

Put $x = u^6 \Rightarrow dx = 6u^5 du$

$$\therefore I_2 = \int \frac{\log(1+u)}{u^2 + u^3} 6u^5 du = \int \frac{\log(1+u)}{u^2(1+u)} \cdot 6u^5 du$$

$$6 \int \frac{u^3}{(u+1)} \log(1+u) du = 6 \int \left(\frac{u^3 - 1 + 1}{u+1} \right) \log(1+u) du$$

$$= 6 \int \left(u^2 - u + 1 - \frac{1}{u+1} \right) \log(1+u) du$$

$$= 6 \int (u^2 - u + 1) \log(1+u) du - 6 \int \frac{\log(1+u)}{(u+1)} du$$

$$= 6 \left(\frac{u^3}{3} - \frac{u^2}{2} + u \right) \log(1+u) - \int \frac{2u^3 - 3u^2 + 6u}{u+1} du - 6 \frac{1}{2} [\log(1+u)]^2$$

$$= (2u^3 - 3u^2 + 6u) \log(1+u)$$

$$- \int \left(2u^2 - 5u + \frac{11u}{u+1} \right) du - 3 [\log(1+u)]^2$$

$$= (2u^3 - 3u^2 + 6u) \log(1+u)$$

$$= - \left(\frac{2u^3}{3} - \frac{5}{2}u^2 + 11u - 11 \log(u+1) \right) - 3 [\log(1+u)]^2$$

$$\therefore I = \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3} - 4x^{1/4}$$

$$- 6x^{1/6} - 12x^{1/12} + 12 \log(x^{1/12} + 1)$$

$$+ (2x^{1/2} - 3x^{1/3} + 6x^{1/6} \ln(1 + x^{1/6})) \log(1 + x^{1/6})$$

$$- \left[\frac{2}{3}x^{1/2} - \frac{5}{2}x^{1/3} \ln x^{1/6} - 11 \log(1 + x^{1/6}) \right]$$

$$\begin{aligned}
& -3[\log(1+x^{1/6})]^2 + c \\
& = \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + \frac{4}{3}x^{1/2} - \frac{12}{5}x^{5/12} \\
& + \frac{1}{2}x^{1/3} - 4x^{1/4} - 7x^{1/6} - 12x^{1/12} \\
& + (2x^{1/2} - 3x^{1/3} - 6x^{1/6} + 11)\log(1+x^{1/6}) \\
& + 12\log(1+x^{1/2}) - 3[\log(1+x^{1/6})]^2 + c
\end{aligned}$$

Sol 13: $\frac{x^3+3x+2}{(x^2+1)^2(x+1)} = \frac{x^3+2x+x+2}{(x^2+1)^2(x+1)}$

$$= \frac{x(x^2+1)+2(x+1)}{(x^2+1)^2(x+1)} = \frac{x}{(x^2+1)(x+1)} + \frac{2}{(x^2+1)^2}$$

Again, $\frac{x}{(x^2+1)(x+1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x+1)}$

$$\Rightarrow x = (Ax+B)(x+1) + C(x^2+1)$$

Putting $x = -1$, we get $-1 = 2C \Rightarrow C = -1/2$

Equation coefficient of x^2 , we get

$$0 = A + C \Rightarrow A = -C = 1/2$$

Putting $x = 0$, we obtain

$$0 = B + C \Rightarrow B = -C = 1/2$$

$$\frac{x^3+3x+2}{(x^2+1)^2(x+1)} = \frac{x+1}{2(x^2+1)} - \frac{1}{2(x+1)} + \frac{2}{(x^2+1)^2}$$

$$\begin{aligned}
\therefore I &= \int \frac{x^3+3x+2}{(x^2+1)^2(x+1)} dx \\
&= -\frac{1}{2} \int \frac{dx}{x+1} + \frac{1}{2} \int \frac{x+1}{x^2+1} dx + 2 \int \frac{dx}{(x^2+1)^2} \\
&\Rightarrow I = -\frac{1}{2} \log|x+1| + \frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1}x + 2I_1 \dots (i)
\end{aligned}$$

where $I_1 = \int \frac{dx}{(x^2+1)^2}$

Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

$$\therefore I_1 = \int \frac{\sec^2 \theta d\theta}{(\tan^2 \theta + 1)^2} = \int \cos^2 \theta d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right] = \frac{1}{2} \theta + \frac{1}{2} \cdot \frac{\tan \theta}{(1 + \tan^2 \theta)}$$

$$= \frac{1}{2} \tan^{-1}x + \frac{1}{2} \cdot \frac{x}{(1+x^2)}$$

$\therefore$ From Eq. (i)

$$I = -\frac{1}{2} \log|x+1| + \frac{1}{4} \log|x^2+1| + \frac{3}{2} \tan^{-1}x + \frac{x}{x^2+1} + c$$

Sol 14: Let $I = \int \sin^{-1} \left(\frac{2x+2}{\sqrt{4x^2+8x+13}} \right) dx$

$$= \int \sin^{-1} \left(\frac{2x+2}{\sqrt{(2x+2)^2+9}} \right) dx$$

Put $2x+2 = 3 \tan \theta \Rightarrow 2dx = 3 \sec^2 \theta d\theta$

$$\therefore I = \int \sin^{-1} \left(\frac{3 \tan \theta}{\sqrt{9 \tan^2 \theta + 9}} \right) \cdot \frac{3}{2} \sec^2 \theta d\theta$$

$$= \int \sin^{-1} \left(\frac{3 \tan \theta}{3 \sec \theta} \right) \cdot \frac{3}{2} \sec^2 \theta d\theta$$

$$= \int \sin^{-1} \left(\frac{\sin \theta}{\cos \theta \cdot \sec \theta} \right) \cdot \frac{3}{2} \sec^2 \theta d\theta$$

$$= \frac{3}{2} \int \sin^{-1}(\sin \theta) \cdot \sec^2 \theta d\theta$$

$$= \frac{3}{2} \int \theta \cdot \sec^2 \theta d\theta = \frac{3}{2} [\theta \cdot \tan \theta - \int 1 \cdot \tan \theta d\theta]$$

$$= \frac{3}{2} [\theta \tan \theta - \log \sec \theta] + c$$

$$= \frac{3}{2} \left[\tan^{-1} \left(\frac{2x+2}{3} \right) \cdot \left(\frac{2x+2}{3} \right) - \log \sqrt{1 + \left(\frac{2x+2}{3} \right)^2} \right] + c_1$$

$$= (x+1) \tan^{-1} \left(\frac{2x+2}{3} \right) - \frac{3}{4} \log \left(1 + \left(\frac{2x+2}{3} \right)^2 \right) + c_1$$

$$= (x+1) \tan^{-1} \left(\frac{2x+2}{3} \right) - \frac{3}{4} \log(4x^2 + 8x + 13) + c$$

$$\left(\text{let } \frac{3}{2} \log 3 + c_1 = c \right)$$

Sol 15: For any natural number m , the given integral can be written as,

$$I = \int (x^{3m} + x^{2m} + x^m)(2x^{3m} + 3x^{2m} + 6x^m)^{1/m} dx$$

$$\Rightarrow I = \int (2x^{3m} + 3x^{2m} + 6x^m)^{1/m} (x^{3m-1} + x^{2m-1} + x^{m-1}) dx$$

Put $2x^{3m} + 3x^{2m} + 6x^m = t$

$$\Rightarrow (6mx^{3m-1} + 6mx^{2m-1} + 6mx^{m-1})dx = dt$$

$$\therefore I = \int t^{1/m} \frac{dt}{6m} = \frac{1}{6m} \cdot \frac{t^{\frac{1}{m}+1}}{\left(\frac{1}{m}+1\right)}$$

$$= \frac{1}{6(m+1)} \cdot (2x^{3m} + 3x^{2m} + 6x^m)^{\frac{(m+1)}{m}} + c$$

$$= -\frac{1}{r^{11/2}} \left(\frac{1}{11} + \frac{t^2}{7} \right)$$

$$= -\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + k.$$

Sol 16: (C)

$$J - I = \int \frac{e^x (e^{2x} - 1)}{e^{4x} + e^{2x} + 1} dx = \int \frac{(z^2 - 1)}{z^4 + z + 1} dz \text{ where } z = e^x$$

$$J - I = \int \frac{\left(1 - \frac{1}{z^2}\right) dx}{\left(z + \frac{1}{z}\right) - 1} = \frac{1}{2} \log \left(\frac{e^x + e^{-x} - 1}{e^x + e^{-x} + 1} \right)$$

$$\therefore J - I = \frac{1}{2} \log \left(\frac{e^x + e^{-x} - 1}{e^x + e^{-x} + 1} \right) + c.$$

$$\textbf{Sol 17: (C)} \quad I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$$

$$\text{Let } \sec x + \tan x = t$$

$$\Rightarrow \sec x - \tan x = 1/t$$

$$\sec x (\sec x + \tan x) dx = dt$$

$$\sec x dx = \frac{dt}{t}, \frac{1}{2} \left(t + \frac{1}{t} \right) = \sec x$$

$$I = \frac{1}{2} \int \frac{\left(t + \frac{1}{t}\right) dt}{t^{9/2}} = \frac{1}{2} \int \left(t^{-9/2} + t^{-11/2} \right) dt$$

$$= \frac{1}{2} \left[\frac{t^{-9/2+1}}{-\frac{9}{2}+1} + \frac{t^{-11/2+1}}{-\frac{11}{2}+1} \right]$$

$$= \frac{1}{2} \left[\frac{t^{-7/2}}{-\frac{7}{2}} + \frac{t^{-9/2}}{-\frac{9}{2}} \right]$$

$$= -\frac{1}{7} t^{-7/2} - \frac{1}{11} t^{-9/2}$$

$$= -\frac{1}{7} \frac{1}{t^{7/2}} - \frac{1}{11} \frac{1}{t^{9/2}}$$