20.BINOMIAL EXPONENTIAL & LOGARITHMIC SERIES

1. BINOMIAL THEOREM : The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **Binomial Theorem**. If $x, y \in R$ and $n \in N$, then ;

$$(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2}y^2 + ... + {}^nC_r\, x^{n-r}y^r + ... + {}^nC_n y^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r.$$

This theorem can be proved by Induction .www.MathsBySuhag.com , www.TekoClasses.com

OBSERVATIONS :(i) The number of terms in the expansion is $(n + 1)$ i.e. one or more than the index .**(ii)** The sum of the indices of x & y in each term is n **(iii)** The binomial coefficients of the terms ${}^nC_0, {}^nC_1$ .... **equidistant** from the beginning and the end are equal.

2. IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE :

(i) General term **(ii)** Middle term **(iii)** Term independent of x & **(iv)** Numerically greatest term

(i) The general term or the $(r + 1)^{th}$ term in the expansion of $(x + y)^n$ is given by ; $T_{r+1} = {}^nC_r\, x^{n-r} . y^r$

(ii) The middle term(s) is the expansion of $(x + y)^n$ is (are) :

(a) If n is even , there is only one middle term which is given by ; $T_{(n+2)/2} = {}^nC_{n/2} . x^{n/2} . y^{n/2}$

(b) If n is odd , there are two middle terms which are : $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(iii) Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

(iv) To find the Numerically greatest term is the expansion of $(1 + x)^n$, $n \in N$ find $\frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^r}{{}^nC_{r-1} x^{r-1}} = \frac{n-r+1}{r} x$. Put the absolute value of x & find the value of r Consistent with the inequality $\frac{T_{r+1}}{T_r} > 1$.

Note that the Numerically greatest term in the expansion of $(1 - x)^n$, $x > 0$, $n \in N$ is the same as the greatest term in $(1 + x)^n$.

3. If $\left(\sqrt{A} + B\right)^n = I + f$, where I & n are positive integers, n being odd and $0 < f < 1$, then $(I + f) . f = K^n$ where $A - B^2 = K > 0$ & $\sqrt{A} - B < 1$.
If n is an even integer, then $(I + f)(1 - f) = K^n$.

4. BINOMIAL COEFFICIENTS : (i) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(ii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

(iii) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!\, n!}$

(iv) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + ... + C_{n-r}.C_n = \frac{(2n)!}{(n+r)!\,(n-r)!}$

REMEMBER : (i) $(2n)! = 2^n . n! \,[1. 3. 5 ...... (2n - 1)]$

5. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES

If $n \in Q$, then $(1 + x)^n = 1 + n\,x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + ...\infty$ Provided $|x| < 1$.

Note : (i) When the index n is a positive integer the number of terms in the expansion of $(1 + x)^n$ is finite i.e. $(n + 1)$ & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3 ..... {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1 + x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the Coefficient of the general term .

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - .... \infty$ **(b)** $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + ... \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \infty$ **(d)** $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... \infty$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convinient to expand in powers of $\frac{1}{x}$, which then will be small.

6. APPROXIMATIONS : $(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 .....$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its squares and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately. This is an approximate value of $(1 + x)^n$.

7. EXPONENTIAL SERIES : www.MathsBySuhag.com , www.TekoClasses.com

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$; where x may be any real or complex & $e = \underset{n \to \infty}{\text{Limit}} \left(1 + \frac{1}{n}\right)^n$

(ii) $a^x = 1 + \frac{x}{1!} \ln a + \frac{x^2}{2!} \ln^2 a + \frac{x^3}{3!} \ln^3 a + ...\infty$ where $a > 0$

Note : (a) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

(b) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(c) $e + e^{-1} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...\infty\right)$ **(d)** $e - e^{-1} = 2\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + ...\infty\right)$

(e) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm**.

8. LOGARITHMIC SERIES :

(i) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$ where $-1 < x \le 1$

(ii) $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$ where $-1 \le x < 1$

(iii) $\ln \frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\infty\right)$ $|x| < 1$

Remember : (a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... \infty = \ln 2$ **(b)** $e^{\ln x} = x$

(c) $\ln 2 = 0.693$ **(d)** $\ln 10 = 2.303$