CHAPTER 19

Determinants

PROPERTIES OF DETERMINANTS

(*i*) The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows.

e.g. $$|A'| = |A|$$

(*ii*) If $A = [a_{ij}]_{n \times n}$, $n > 1$ and B be the matrix obtained from A by interchanging two of its rows or columns, then

$$\det(B) = -\det(A)$$

(*iii*) If two rows (or columns) of a square matrix A are proportional, then $|A| = 0$.

(*iv*) $|B| = k|A|$, where B is the matrix obtained from A, by multiplying one row (or column) of A by k.

(*v*) $|kA| = k^n|A|$, where A is a matrix of order $n \times n$.

(*vi*) If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

e.g. $$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

(*vii*) If the same multiple of the elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged, e.g. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{31} & a_{12}+ka_{32} & a_{13} \quad ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(*viii*) If each element of a row (or column) of a determinant is zero, then its value is zero.

(*ix*) If any two rows (or columns) of a determinant are identical, then its value is zero.

(*x*) If r rows (or r columns) become identical, when a is substituted for x, then $(x - a)^{r-1}$ is a factor of given determinant.

IMPORTANT RESULTS ON DETERMINANTS

(*i*) $|AB| = |A||B|$, where A and B are square matrices of the same order.

(*ii*) $|A^n| = |A|^n$.

(*iii*) If A, B and C are square matrices of the same order such that i^{th} columns (or rows) of A is the sum of i^{th} columns (or rows) of B and C and all other columns (or rows) of A, B and C are identical, then $|A| = |B| + |C|$.

(*iv*) $|I_n| = 1$, where I_n is identity matrix of order n.

(*v*) $|O_n| = 0$, where O_n is a zero matrix of order n.

(*vi*) If $\Delta(x)$ has a 3rd order determinant having polynomials as its elements.

(*a*) If $\Delta(a)$ has 2 rows (or columns) proportional, then $(x - a)$ is a factor of $\Delta(x)$.

(*b*) If $\Delta(a)$ has 3 rows (or columns) proportional, then $(x - a)^2$ is a factor of $\Delta(x)$.

(*vii*) A square matrix A is **non-singular**, if $|A| \neq 0$ and **singular**, if $|A| = 0$.

(*viii*) Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.

(*ix*) In general, $|B + C| \neq |B| + |C|$.

(*x*) Determinant of a diagonal matrix = Product of its diagonal elements

(*xi*) If A is a non-singular matrix, then $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$.

(*xii*) Determinant of a orthogonal matrix = 1 or – 1.

(*xiii*) Determinant of a hermitian matrix is purely real.

(*xiv*) If A and B are non-zero matrices and $AB = O$, then it implies $|A| = O$ and $|B| = O$.

Minors and Cofactors

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then the **minor** M_{ij} of the element a_{ij} is the determinant

obtained by deleting the i^{th} row and j^{th} column,

i.e. M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The **cofactor** of the element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Properties of Minors and Cofactors

(*i*) The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero,

i.e. if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$ and so on.

(*ii*) The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) in Δ,

i.e. if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $|A| = \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$.

(*iii*) In general, if $|A| = \Delta$, then $|A| = \sum_{i=1}^{n} a_{ij}\, C_{ij}$ and (adj A) $= \Delta^{n-1}$, where A is a matrix of order $n \times n$.

Applications of Determinants in Geometry

Let the three points in a plane be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

(*i*) Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2}[x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)]$$

(*ii*) If the given points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

(*iii*) Let two points are $A(x_1, y_1)$, $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B, then the equation of line is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

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