BINOMIAL THEOREM

1. BINOMIAL EXPRESSION :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For **example** : x – y, $xy + \frac{1}{x}, \frac{1}{z} - 1, \frac{1}{(x-y)^{1/3}} + 3$ etc.

2. TERMINOLOGY USED IN BINOMIAL THEOREM :

Factorial notation : $\underline{|n}$ or n! is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3.2.1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note : $n! = n\,.\,(n-1)!$; $n \in N$

Mathematical meaning of ^nC_r : The term ^nC_r denotes number of ways of combinations (selection) of r things choosen from n distinct things mathematically, $^nC_r = \frac{n!}{(n-r)!\,r!}$, $n \in N$, $r \in W$, $0 \le r \le n$

Note : Other symbols of of ^nC_r are $\binom{n}{r}$ and C(n, r).

Properties related to ^nC_r :

(i) $^nC_r = {^nC_{n-r}}$

Note : If $^nC_x = {^nC_y}$ $\Rightarrow$ Either x = y or x + y = n

(ii) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(iii) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(iv) $^nC_r = \frac{n}{r}\,{^{n-1}C_{r-1}} = \frac{n(n-1)}{r(r-1)}\,{^{n-2}C_{r-2}} = ... = \frac{n(n-1)(n-2).........(n-(r-1))}{r\,(r-1)(r-2).......2\,.1}$

(v) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

3. BINOMIAL THEOREM :

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM.**

If x, y $\in$ R and n$\in$N, then :

$$(x + y)^n = {^nC_0}x^n + {^nC_1}x^{n-1}y + {^nC_2}x^{n-2}y^2 + ... + {^nC_r}x^{n-r}y^r + ... + {^nC_n}y^n = \sum_{r=0}^{n} {^nC_r}x^{n-r}y^r$$

This theorem can be proved by induction.

Observations :

(a) The number of terms in the expansion is (n+1) i.e. one more than the index.

(b) The sum of the indices of x & y in each term is n.

(c) The binomial coefficients (**$^nC_0$, $^nC_1$.....**) of the terms equidistant from the beginning and the end are equal. i.e. **$^nC_r = {^nC_{r-1}}$** (e.g. $^nC_0 = {^nC_n}$, $^nC_1 = {^nC_{n-1}}$, ...)

(d) r^{th} term from the beginning in the expansion of $(x + y)^n$ is same as r^{th} term from end in the expansion of $(y + x)^n$.

(e) r^{th} term from the end in $(x + y)^n$ is **$(n - r + 2)^{th}$ term from the beginning**.

Some important expansions :

(i) $(1 + x)^n = {^nC_0} + {^nC_1}x + {^nC_2}x^2 + \dots\dots + {^nC_n}x^n = \sum_{r=0}^{n} {^nC_r}x^r$

(ii) $(1 - x)^n = {^nC_0} - {^nC_1}x + {^nC_2}x^2 + \dots\dots + (-1)^n . {^nC_n}x^n = \sum_{r=0}^{n} {^nC_r}(-x)^r$

Note : The coefficient of x^r in $(1 + x)^n = {^nC_r}$ & that in $(1-x)^n = (-1)^r . {^nC_r}$

3.1 Pascal's triangle :

$(x+y)^0$	1	1
$(x+y)^1$	$x + y$	1 1
$(x+y)^2$	$x^2 + 2xy + y^2$	1 2 1
$(x+y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$	1 3 3 1
$(x+y)^4$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^3$	1 4 6 4 1

Pascal's triangle

(i) **Pascal's triangle -** A triangular arrangement of numbers as shown. The numbers give the binomial coefficients for the expansion of $(x + y)^n$. The first row is for n = 0, the second for n = 1, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

(ii) Pascal triangle is formed by binomial coefficient.

SOLVED EXAMPLE

Example # 1 : Expand : $(y + 2)^6$.

Solution : $^6C_0y^6 + {^6C_1}y^5.2 + {^6C_2}y^4.2^2 + {^6C_3}y^3.2^3 + {^6C_4}y^2.\ 2^4 + {^6C_5}y^1 .\ 2^5 + {^6C_6} .\ 2^6$.

$= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64$.

Example # 2 : Write first 4 terms of $\left(1 - \frac{2y^2}{5}\right)^7$

Solution : $^7C_0, {^7C_1}\left(-\frac{2y^2}{5}\right), {^7C_2}\left(-\frac{2y^2}{5}\right)^2, {^7C_3}\left(-\frac{2y^2}{5}\right)^3$

Example # 3 : The number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is

(A) 21 (B) 31 (C) 41 (D) 61

Solution : $(1 - 3x + 3x^2 - x^3)^{20} = [(1 - x)^3]^{20} = (1 - x)^{60}$

Therefore number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is **61.**

Problems for Self Practice -1:

(1) Expand $\left(3x^2 - \frac{x}{2}\right)^5$

(2) Write the first three terms in the expansion of $\left(2 - \frac{y}{3}\right)^6$.

Answers :

(1) ${}^5C_0x(3x^2)^5 + {}^5C_1(3x^2)^4\left(-\frac{x}{2}\right) + {}^5C_2(3x^2)^3\left(-\frac{x}{2}\right)^2 + {}^5C_3(3x^2)^2\left(-\frac{x}{2}\right)^3 + {}^5C_4(3x^2)^1\left(-\frac{x}{2}\right)^4 + {}^5C_5\left(-\frac{x}{2}\right)^5$

(2) $64 - 64y + \frac{80}{3}y^2$

3.2 General term :

The general term or the $(r+1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

Note : (i) $(x + y)^n + (x - y)^n = 2[{}^nC_0\, x^n\, y^0 + {}^nC_2\, x^{n-2}\, y^2 + ...]$

(ii) $(x + y)^n - (x - y)^n = 2[{}^nC_1\, x^{n-1}\, y^1 + {}^nC_3\, x^{n-3}\, y^3 + ...]$

SOLVED EXAMPLE

Example # 4 : Find (i) 28^{th} term of $(5x + 8y)^{30}$ (ii) 7^{th} term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

Solution : (i) $T_{27+1} = {}^{30}C_{27}\,(5x)^{30-27}\,(8y)^{27} = \frac{30!}{3!\,27!}\,(5x)^3 \cdot (8y)^{27}$

(ii) 7th term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

$T_{6+1} = {}^9C_6\left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6 = \frac{9!}{3!\,6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3}$

Example # 5 : Find : (a) The coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

Also, find the relation between a and b, so that these coefficients are equal.

Solution : (a) In the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$, the general term is :

$T_{r+1} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r \cdot \frac{a^{11-r}}{b^r} \cdot x^{22-3r}$

putting $22 - 3r = 7$

$\therefore \quad 3r = 15 \Rightarrow \quad r = 5$

$\therefore \quad T_6 = {}^{11}C_5 \frac{a^6}{b^5}.x^7$

Hence the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5 a^6 b^{-5}$. **Ans.**

Note that binomial coefficient of sixth term is ${}^{11}C_5$.

(b) In the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, general term is :

$T_{r+1} = {}^{11}C_r (ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r = (-1)^r\, {}^{11}C_r \frac{a^{11-r}}{b^r}.x^{11-3r}$

putting $11 - 3r = -7$

$\therefore \quad 3r = 18 \Rightarrow r = 6$

$\therefore \quad T_7 = (-1)^6.\ {}^{11}C_6 \frac{a^5}{b^6}.x^{-7}$

Hence the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6 a^5 b^{-6}$. **Ans.**

Also given :

Coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

$\Rightarrow {}^{11}C_5 a^6 b^{-5} = {}^{11}C_6 a^5 b^{-6}$

$\Rightarrow ab = 1 \qquad (\because\ {}^{11}C_5 = {}^{11}C_6)$

which is the required relation between a and b. **Ans.**

Example # 6 : If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively then m is -

(A) 6 (B) 9 (C) 12 (D) 24

Solution : $(1 + x)^m (1 - x)^n = \left[1 + mx + \frac{(m)(m-1).x^2}{2} + ...\right]\left[1 - nx + \frac{n(n-1)}{2}x^2 + ...\right]$

Coefficient of $x = m - n = 3$ ....(i)

Coefficient of $x^2 = -mn + \frac{n(n+1)}{2} + \frac{m(m-1)}{2} = -6$ ....(ii)

Solving (i) and (ii), we get

m = 12 and n = 9.

3.3 Term independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

SOLVED EXAMPLE

Example # 7 : The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is -

(A) 1 (B) $\frac{5}{12}$ (C) ${}^{10}C_1$ (D) none of these

Solution : General term in the expansion is

$${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}.\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$ For constant term, $\frac{3r}{2}=10 \Rightarrow r=\frac{20}{3}$

which is not an integer. Therefore, there will be no constant term. **Ans. (D)**

3.4 Middle term :

The middle term(s) in the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\ x^{n/2}.\ y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

Important Note :

Middle term has greatest binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$\Rightarrow$ ${}^nC_r$ will be maximum
- When $r=\frac{n}{2}$ if n is even
- When $r=\frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd

$\Rightarrow$ The term containing greatest binomial coefficient will be middle term in the expansion of $(1 + x)^n$

SOLVED EXAMPLE

Example # 8 : Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$

Solution : The number of terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+3}{2}\right)^{th}$ are two middle terms. They are given by T_5 and T_6

$\therefore \quad T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4$

$= {}^9C_4 3^5 x^5.\ \frac{x^{12}}{6^4} = \frac{9.8.7.6}{1.2.3.4}.\frac{3^5}{2^4.3^4}x^{17} = \frac{189}{8}x^{17}$

and $\quad T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5$

$= -{}^9C_4 3^4.x^4.\ \frac{x^{15}}{6^5} = \frac{-9.8.7.6}{1.2.3.4}.\frac{3^4}{2^5.3^5}x^{19} = -\frac{21}{16}x^{19}$ **Ans.**

Problems for Self Practice -2:

(1) Find the 7^{th} term of $\left(3x^2 - \frac{1}{3}\right)^{10}$

(2) Find the term independent of x in the expansion :

(i) $\left(2x^2 - \frac{3}{x^3}\right)^{25}$ (ii) $\left(x^2 - \frac{3}{x}\right)^{9}$

(3) Find the middle term in the expansion of : (a) $\left(\frac{2x}{3} - \frac{3}{2x}\right)^{6}$ (b) $\left(2x^2 - \frac{1}{x}\right)^{7}$

(4) Find the middle term(s) in the expansion of $(1 + 3x + 3x^2 + x^3)^{2n}$

(5) Find the coefficient of x^{-1} in $(1 + 3x^2 + x^4)\left(1 + \frac{1}{x}\right)^{8}$

Answers : **(1)** $\frac{70}{3}x^8$ **(2)** (i) $\frac{25!}{10!\ 5!}2^{15}3^{10}$; (ii) 28.3^7

(3) (a) –20; (b) $-560x^5$, $280x^2$ **(4)** ${}^{6n}C_{3n} \cdot x^{3n}$ **(5)** 232

---

4. APPLICATION OF BINOMIAL THEOREM :

4.1 Numerically greatest term :

Let numerically greatest term in the expansion of $(a + b)^n$ be T_{r+1}.

$$\Rightarrow \begin{cases} |T_{r+1}| \geq |T_r| \\ |T_{r+1}| \geq |T_{r+2}| \end{cases} \text{ where } T_{r+1} = {}^nC_r a^{n-r} b^r$$

Solving above inequalities we get $\frac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \leq r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

Case I : When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer equal to m, then T_m and T_{m+1} will be numerically greatest term.

Case II : When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note : In order to obtain the term having numerically greatest coefficient in $(ax + by)^n$ put $x = y = 1$ and proceed as discussed above (a and b are given)

SOLVED EXAMPLE

Example # 9 : Find numerically greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \frac{1}{5}$

Solution : Using $\frac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\frac{3}{-5x}\right|} - 1 \le r \le \frac{11+1}{1+\left|\frac{3}{-5x}\right|}$$

solving we get $2 \le r \le 3$

$\therefore$ r = 2, 3

so, the greatest terms are T_{2+1} and T_{3+1}.

$\therefore$ Greatest term (when r = 2)

$T_3 = {}^{11}C_2.3^9\,(-5x)^2 = 55.3^9 = T_4$

From above we say that the value of both greatest terms are equal. **Ans.**

Example # 10 : Given T_3 in the expansion of $(1 - 3x)^6$ has maximum numerical value. Find the range of 'x'.

Solution : Using $\frac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{6+1}{1+\left|\frac{1}{-3x}\right|} - 1 \le 2 \le \frac{7}{1+\left|\frac{1}{-3x}\right|}$$

Let |x| = t

$$\frac{21t}{3t+1} - 1 \le 2 \le \frac{21t}{3t+1}$$

$$\begin{cases} \frac{21t}{3t+1} \le 3 \\ \frac{21t}{3t+1} \ge 2 \end{cases} \Rightarrow \begin{cases} \frac{4t-1}{3t+1} \le 0 \Rightarrow t \in \left[-\frac{1}{3}, \frac{1}{4}\right] \\ \frac{15t-2}{3t+1} \ge 0 \Rightarrow t \in \left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{2}{15}, \infty\right) \end{cases}$$

Common solution $t \in \left[\frac{2}{15}, \frac{1}{4}\right] \Rightarrow x \in \left[-\frac{1}{4}, -\frac{2}{15}\right] \cup \left[\frac{2}{15}, \frac{1}{4}\right]$

<u>Problems for Self Practice</u> -3:

(1) Find the value of numerically greatest term in the expansion of $(3 - 2x)^9$, when x = 1.

(2) In the expansion of $\left(\frac{1}{2} + \frac{2x}{3}\right)^n$ when $x = -\frac{1}{2}$, it is known that 3rd term is the greatest term. Find the possible integral values of n.

Answers : (1) $T_4 = -489888$ and $T_5 = 489888$

(2) n = 4, 5, 6

4.2 Rational terms/Irrational terms :

SOLVED EXAMPLE

Example # 11 : Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution : The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^{r} = {}^{1000}C_r\, 3^{\frac{1000-r}{2}}\, 2^{\frac{r}{2}}$$

The above term will be rational if exponents of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is 501 **Ans.**

4.3 Last digit/Last two digits/Last three digits :

SOLVED EXAMPLE

Example # 12 : Find the last two digits of the number $(17)^{10}$.

Solution : $(17)^{10} = (289)^5 = (290 - 1)^5$

$= {}^5C_0\,(290)^5 - {}^5C_1\,(290)^4 + ... + {}^5C_4\,(290)^1 - {}^5C_5\,(290)^0$

$= {}^5C_0\,(290)^5 - {}^5C_1\,.\,(290)^4 + ...{}^5C_3\,(290)^2 + 5 \times 290 - 1$

= A multiple of 1000 + 1449

Hence, last two digits are 49

Note : We can also conclude that last three digits are 449.

Example # 13 : Find the last three digits in 11^{50}.

Solution : Expansion of $(10 + 1)^{50} = {}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + ... + {}^{50}C_{48} 10^2 + {}^{50}C_{49} 10 + {}^{50}C_{50}$

$$= \underbrace{{}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + ...+ {}^{50}C_{47} 10^3}_{1000K} + 49 \times 25 \times 100 + 500 + 1$$

$\Rightarrow$ 1000 K + 123001

$\Rightarrow$ Last 3 digits are 001.

4.4 Remainder :

SOLVED EXAMPLE

Example # 14 : Show that $9^n + 7$ is divisible by 8, where n is a positive integer.

Solution : $9^n + 7 = (1 + 8)^n + 7$

$= {}^nC_0 + {}^nC_1\,.\,8 + {}^nC_2\,.\,8^2 + ... + {}^nC_n\,8^n + 7.$

$= 8.\,C_1 + 8^2.\,C_2 + ... + C_n\,.\,8^n + 8.$

$= 8\lambda$, where λ is a positive integer

Hence, $9^n + 7$ is divisible by 8.

Example # 15 : What is the remainder when 5^{99} is divided by 13.

Solution : $5^{99} = 5.5^{98} = 5.\,(25)^{49} = 5\,(26 - 1)^{49}$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + ... + {}^{49}C_{48}\,(26)^1 - {}^{49}C_{49}\,(26)^0]$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + ...+ {}^{49}C_{48}\,(26)^1 - 1]$

$= 5\,[{}^{49}C_0(26)^{49} - {}^{49}C_1(26)^{48} + ... + {}^{49}C_{48}\,(26)^1 - 13] + 60$

= 13 (k) + 52 + 8 (where k is a positive integer)

= 13 (k + 4) + 8

Hence, remainder is 8.

Example # 16 : Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Solution : When 2222 is divided by 7 it leaves a remainder 3.

So adding & subtracting 3^{5555}, we get :

$$E = \underbrace{2222^{5555} - 3^{5555}}_{E_1} + \underbrace{3^{5555} + 5555^{2222}}_{E_2}$$

For E_1 : Now since 2222–3 = 2219 is divisible by 7, therefore E_1 is divisible by 7

($\because$ $x^n - a^n$ is divisible by x –a)

For E_2 : 5555 when devided by 7 leaves remainder 4.

So adding and subtracting 4^{2222}, we get :

$E_2 = 3^{5555} + 4^{2222} + 5555^{2222} - 4^{2222}$

$= (243)^{1111} + (16)^{1111} + (5555)^{2222} - 4^{2222}$

Again $(243)^{1111} + 16^{1111}$ and $(5555)^{2222} - 4^{2222}$ are divisible by 7

($\because$ $x^n + a^n$ is divisible by x + a when n is odd)

Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

Example # 17 : Which number is larger $(1.01)^{1000000}$ or 10,000 ?

Solution : By Binomial Theorem

$(1.01)^{1000000} = (1 + 0.01)^{1000000}$

$= 1 + {}^{1000000}C_1\,(0.01)$ + other positive terms

$= 1 + 1000000 \times 0.01$ + other positive terms

$= 1 + 10000$ + other positive terms

Hence $(1.01)^{1000000} > 10,000$

Problems for Self Practice -4:

(1) The sum of all rational terms in the expansion of $(3^{1/5} + 2^{1/3})^{15}$ is

(2) Find the last digit, last two digits and last three digits of the number $(81)^{25}$.

(3) Prove that

(i) $5^{25} - 3^{25}$ is divisible by 2.

(ii) $3^{2n+1} + 2^{n+2}$ is divisible by 7.

(4) Find the remainder when the number

(i) 9^{100} is divided by 8. (ii) 7^{103} is divided by 25 .

(5) Which number is larger $(1.2)^{4000}$ or 800

Answers : (1) 59 (2) 1, 01, 001

(4) (i) 1; (ii) 18 (5) $(1.2)^{4000}$.

5. PROPERTIES OF BINOMIAL COEFFICIENTS :

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots\ldots + C_nx^n = \sum_{r=0}^{n} {}^nC_r r^r$; $n \in N$ ....(i)

where $C_0, C_1, C_2, \ldots\ldots\ldots\ldots C_n$ are called combinatorial (binomial) coefficients.

(a) The sum of all the binomial coefficients is 2^n.

Put x = 1, in (i) we get

$C_0 + C_1 + C_2 + \ldots\ldots\ldots\ldots + C_n = 2^n \Rightarrow \sum_{r=0}^{n} {}^nC_r = 0$ ....(ii)

(b) Put x=–1 in (i) we get

$C_0 - C_1 + C_2 - C_3 \ldots\ldots\ldots + C_n = 0 \Rightarrow \sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0$ ...(iii)

(c) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

From (ii) & (iii), $C_0 + C_2 + C_4 + \ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(f) ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}.\frac{n-1}{r-1}\,{}^{n-2}C_{r-2} = \ldots\ldots = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots\ldots 1}$

(g) ${}^nC_r = \frac{r+1}{n+1}.\,{}^{n+1}C_{r+1}$

SOLVED EXAMPLE

Example # 18 : Prove that :

(i) $C_0 + 3C_1 + 3^2C_2 + \ldots\ldots\ldots + 3^n\, C_n = 4^n$.

(ii) $C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n = n\,.\,2^{n-1}$

(iii) $C_0 + 2C_1 + 3.\,C_2 + \ldots\ldots + (n+1)\,C_n = 2^{n-1}\,(n+2)$.

(iv) $C_0 - 3C_1 + 5C_2 - \ldots\ldots(-1)^n(2n+1)C_n = 0$

Solution : (i) $(1+x)^n = C_0 + C_1\,x + C_2x^2 + \ldots\ldots\ldots + C_nx^n$

put x = 3

$C_0 + 3\,.\,C_1 + 3^2\,.\,C_2 + \ldots\ldots\ldots + 3^n\,.\,C_n = \mathbf{4^n}$

(ii) L.H.S. $= \sum_{r=1}^{n} r.\,{}^nC_r = \sum_{r=1}^{n} r.\frac{n}{r}.\,{}^{n-1}C_{r-1} = n\sum_{r=1}^{n} {}^{n-1}C_{r-1} = n.\left[{}^{n-1}C_0 + {}^{n-1}C_1 + \ldots\ldots + {}^{n-1}C_{n-1}\right]$

$= n\,.\,2^{n-1}$

Aliter : (Using method of differentiation)

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n$ ....(A)

Differentiating (A), we get

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + n.C_nx^{n-1}$.

Put x = 1, $C_1 + 2C_2 + 3C_3 + \ldots\ldots + n.C_n = n.2^{n-1}$

(iii) <u>**I Method : By Summation**</u>

L.H.S. $= {}^nC_0 + 2.\,{}^nC_1 + 3\,.\,{}^nC_2 + \ldots\ldots + (n+1).\,{}^nC_n$.

$= \sum_{r=0}^{n} (r+1)\,.\,{}^nC_r = \sum_{r=0}^{n} r.\,{}^nC_r + \sum_{r=0}^{n} {}^nC_r$

$= n\sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r = n\,.\,2^{n-1} + 2^n = 2^{n-1}\,(n+2)$. RHS

II Method : By Differentiation

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$
Multiplying both sides by x,
$x(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_n x^{n+1}$.
Differentiating both sides
$(1 + x)^n + x\, n (1 + x)^{n-1} = C_0 + 2.\, C_1x + 3 .\, C_2x^2 + ... + (n + 1)C_nx^n$.
putting x = 1, we get
$C_0 + 2.C_1 + 3 .\, C_2 + ... + (n + 1)\, C_n = 2^n + n .\, 2^{n-1}$
$C_0 + 2.C_1 + 3 .\, C_2 + ... + (n + 1)\, C_n = 2^{n-1}(n + 2)$ **Proved**

(iv) $T_r = (-1)^r(2r + 1)^nC_r = 2(-1)^r r .\, {}^nC_r + (-1)^r\, {}^nC_r$

$$\Sigma T_r = 2\sum_{r=1}^{n}(-1)^r.r.\frac{n}{r}.{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r\ {}^nC_r = 2\sum_{r=1}^{n}(-1)^r.{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r.\,{}^nC_r$$

$$= 2\left[{}^{n-1}C_0 - {}^{n-1}C_1 + ...\right] + \left[{}^nC_0 - {}^nC_1 + ...\right] = 0$$

Example # 19 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + c_nx^n$, then show that

(i) $$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

(ii) $$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ... + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1}.$$

Solution : (i) $$\text{L.H.S.} = \sum_{r=0}^{n}\frac{C_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n}\frac{n+1}{r+1}\,{}^nC_r$$

$$= \frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1} = \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}\right] = \frac{1}{n+1}\left[2^{n+1}-1\right]$$

Aliter : (Using method of integration)

Integrating (A), we get

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}$$ (where C is a constant)

Put x = 0, we get, $C = -\frac{1}{n+1}$

$$\therefore \quad \frac{(1+x)^{n+1}-1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}$$

Put x = 1, we get $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ...\frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Put x = –1, we get $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... = \frac{1}{n+1}$

(ii) **I Method : By Summation**

L.H.S. = $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ... + (-1)^n . \frac{C_n}{n+1}$

$= \sum_{r=0}^{n} (-1)^r . \frac{^nC_r}{r+1} = \frac{1}{n+1} \sum_{r=0}^{n} (-1)^r . {}^{n+1}C_{r+1}$ $\left\{\frac{n+1}{r+1} . {}^nC_r = {}^{n+1}C_{r+1}\right\}$

$= \frac{1}{n+1} [{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - ....+ (-1)^n . {}^{n+1}C_{n+1}]$

$= \frac{1}{n+1} [- {}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + ... + (-1)^n . {}^{n+1}C_{n+1} + {}^{n+1}C_0]$

$= \frac{1}{n+1}$ = R.H.S. , since $\left\{- {}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + ... + (-1)^n \; {}^{n+1}C_{n+1} = 0\right\}$

II Method : By Integration

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$.

Integrating both sides, within the limits – 1 to 0.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0} = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ... + C_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\frac{1}{n+1} - 0 = 0 - \left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + ... + (-1)^{n+1}\frac{C_n}{n+1}\right]$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1} \quad \text{Proved}$$

Example # 20 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$, then prove that

(i) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

(ii) $C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2} C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1. C_0^2 + 3 . C_1^2 + 5. C_2^2 + ... + (2n + 1) . C_n^2 . = 2n. {}^{2n-1}C_n + {}^{2n}C_n$.

Solution : (i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$. ....(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n x^0$ ....(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + ... + C_nx^n) (C_0x^n + C_1x^{n-1} + ... + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of x^n,

$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

(ii) From the product of (i) and (ii) comparing coefficients of x^{n-2} or x^{n+2} both sides,

$C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$.

(iii) **I Method : By Summation**

L.H.S. = $1. C_0^2 + 3. C_1^2 + 5. C_2^2 + ... + (2n + 1) C_n^2$.

$= \sum_{r=0}^{n} (2r+1) {}^nC_r^2 = \sum_{r=0}^{n} 2.r . ({}^nC_r)^2 + \sum_{r=0}^{n} ({}^nC_r)^2 = 2 \sum_{r=1}^{n} . n . {}^{n-1}C_{r-1} {}^nC_r + {}^{2n}C_n$

$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ...{}^nC_n x^n$ ....(i)

$(x + 1)^{n-1} = {}^{n-1}C_0 x^{n-1} + {}^{n-1}C_1 x^{n-2} + ...+{}^{n-1}C_{n-1}x^0$ ....(ii)

Multiplying (i) and (ii) and comparing coeffcients of x^n.

$^{n-1}C_0 \cdot {}^nC_1 + {}^{n-1}C_1 \cdot {}^nC_2 + ... + {}^{n-1}C_{n-1} \cdot {}^nC_n = {}^{2n-1}C_n$

$$\sum_{r=0}^{n} {}^{n-1}C_{r-1} \cdot {}^nC_r = {}^{2n-1}C_n$$

Hence, required summation is $2n.\ {}^{2n-1}C_n + {}^{2n}C_n$ = R.H.S.

II Method : By Differentiation

$(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + C_3x^6 + ...+ C_n x^{2n}$

Multiplying both sides by x

$x(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + ... + C_nx^{2n+1}$.

Differentiating both sides

$x \cdot n (1 + x^2)^{n-1} \cdot 2x + (1 + x^2)^n = C_0 + 3.\ C_1x^2 + 5.\ C_2 x^4 + ... + (2n + 1) C_n x^{2n}$ ....(i)

$(x^2 + 1)^n = C_0 x^{2n} + C_1 x^{2n-2} + C_2 x^{2n-4} + ... + C_n$ ....(ii)

Multiplying (i) & (ii)

$(C_0 + 3C_1x^2 + 5C_2x^4 + ... + (2n + 1) C_n x^{2n}) (C_0 x^{2n} + C_1x^{2n-2} + ... + C_n)$

$= 2n x^2 (1 + x^2)^{2n-1} + (1 + x^2)^{2n}$

comparing coefficient of x^{2n},

$C_0^2 + 3C_1^2 + 5C_2^2 + ...+ (2n + 1) C_n^2 = 2n \cdot {}^{2n-1}C_{n-1} + {}^{2n}C_n$.

$C_0^2 + 3C_1^2 + 5C_2^2 + ...+ (2n + 1) C_n^2 = 2n \cdot {}^{2n-1}C_n + {}^{2n}C_n$. Proved

Example # 21: Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n.\ {}^{2n}C_n$

Solution : $(1 - x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - ....+(-1)^n\, {}^{2n}C_{2n}x^{2n}$ ....(i)

and $(x + 1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} +...+{}^{2n}C_{2n}$ ....(ii)

Multiplying (i) and (ii), we get

$(x^2 - 1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1x + ... + (-1)^n\, {}^{2n}C_{2n}x^{2n}) \times ({}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + ... + {}^{2n}C_{2n})$ ....(iii)

Now, coefficient of x^{2n} in R.H.S.

$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ...... + (-1)^n ({}^{2n}C_{2n})^2$

$\because$ General term in L.H.S., $T_{r+1} = {}^{2n}C_r(x^2)^{2n-r}(-1)^r$

Putting $2(2n - r) = 2n$

$\therefore$ $r = n$

$\therefore$ $T_{n+1} = {}^{2n}C_nx^{2n}(-1)^n$

Hence coeffiecient of x^{2n} in L.H.S. $= (-1)^n.{}^{2n}C_n$

But (iii) is an identity, therefore coefficient of x^{2n} in R.H.S. = coefficient of x^{2n} in L.H.S.

$\Rightarrow$ $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n.\ {}^{2n}C_n$

Example # 22: Prove that : ${}^nC_0.{}^{2n}C_n - {}^nC_1.\ {}^{2n-2}C_n + {}^nC_2.{}^{2n-4}C_n + ... = 2^n$

Solution : L.H.S. = Coefficient of x^n in $[{}^nC_0(1 + x)^{2n} - {}^nC_1(1 + x)^{2n-2} ......]$

= Coefficient of x^n in $[(1 + x)^2 - 1]^n$

= Coefficient of x^n in $x^n(x + 2)^n = 2^n$

Example# 23: Find the summation of the following series –

(i) ${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

(ii) ${}^nC_3 + 2 \cdot {}^{n+1}C_3 + 3.\ {}^{n+2}C_3 + ... + n \cdot {}^{2n-1}C_3$

Solution : (i) **I Method :** Using property, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + ... + {}^nC_m$ $\{\because {}^mC_m = {}^{m+1}C_{m+1}\}$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + ... + {}^nC_m$

$= {}^{m+3}C_{m+1} + ... + {}^nC_m \;\; = {}^nC_{m+1} + {}^nC_m = \mathbf{{}^{n+1}C_{m+1}}$

II Method

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

The above series can be obtained by writing the coefficient of x^m in

$(1 + x)^m + (1 + x)^{m+1} + ... + (1 + x)^n$

Let $S = (1 + x)^m + (1 + x)^{m+1} + ... + (1 + x)^n$

$$= \frac{(1+x)^m\left[(1+x)^{n-m+1} - 1\right]}{x} = \frac{(1+x)^{n+1} - (1+x)^m}{x}$$

$$= \text{coefficient of } x^m \text{ in } \frac{(1+x)^{n+1}}{x} - \frac{(1+x)^m}{x} = {}^{n+1}C_{m+1} + 0 = {}^{n+1}C_{m+1}$$

(ii) ${}^nC_3 + 2 . {}^{n+1}C_3 + 3 . {}^{n+2}C_3 + ... + n . {}^{2n-1}C_3$

The above series can be obatined by writing the coefficient of x^3 in

$(1 + x)^n + 2 . (1 + x)^{n+1} + 3 . (1 + x)^{n+2} + ... + n . (1 + x)^{2n-1}$

Let $S = (1 + x)^n + 2 . (1 + x)^{n+1} + 3. (1 + x)^{n+2} + ... + n (1 + x)^{2n-1}$ .....(i)

$(1 + x)S = (1 + x)^{n+1} + 2 (1 + x)^{n+2} + ... + (n - 1) (1 + x)^{2n-1} + n(1 + x)^{2n}$ ....(ii)

Subtracting (ii) from (i)

$- xS = (1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + ... + (1 + x)^{2n-1} - n(1 + x)^{2n}$

$$= \frac{(1+x)^n\left[(1+x)^n - 1\right]}{x} - n(1+x)^{2n}$$

$$S = \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

$x^3 : S$ (coefficient of x^3 in S)

$$x^3 : \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

Hence, required summation of the series is $-{}^{2n}C_5 + {}^nC_5 + n . {}^{2n}C_4$

<u>Problems for Self Practice</u> -5:

(1) If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, $n \in N$. Prove that

(i) $C_0 + 3C_1 + 5C_2 + ... + (2n + 1) C_n = 2^n (n + 1)$

(ii) $3C_0 - 8C_1 + 13C_2 - 18C_3 + ...$ upto $(n + 1)$ terms = 0, if $n \ge 2$.

(iii) $2C_0 + 2^2\frac{C_1}{2} + 2^3\frac{C_2}{3} + 2^4\frac{C_3}{4} + ... + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{n+1} - 1}{n+1}$

(iv) ${}^nC_0 . {}^{n+1}C_n + {}^nC_1 . {}^nC_{n-1} + {}^nC_2 . {}^{n-1}C_{n-2} + ... + {}^nC_n . {}^1C_0 = 2^{n-1} (n + 2)$

(v) $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + ... + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{((n+1)!)^2}$

(vi) ${}^2C_2 + {}^3C_2 + ... + {}^nC_2 = {}^{n+1}C_3$

6. MULTINOMIAL THEOREM :

Using binomial theorem, we have $(x + a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r,\ n \in N$

$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} x^{n-r} a^r = \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r$, where s + r = n

This result can be generalized in the following form.

$(x_1 + x_2 + ... + x_k)^n = \sum_{r_1+r_2+....+r_k=n} \frac{n!}{r_1!r_2!....r_k!} x_1^{r_1} x_2^{r_2} ....x_k^{r_k}$

The general term in the above expansion $\frac{n!}{r_1!r_2!r_3!.....r_k!} . x_1^{r_1} x_2^{r_2} x_3^{r_3} ......x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + ... + r_k = n$ because each solution of this equation gives a term in the above expansion.

The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular cases :

(i) $(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x + y + z + u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

SOLVED EXAMPLE

Example # 24 : Find the coefficient of $x^2 y^3 z^4 w$ in the expansion of $(x - y - z + w)^{10}$

Solution : $(x - y - z + w)^{10} = \sum_{p+q+r+s=10} \frac{n!}{p!q!r!s!} (x)^p (-y)^q (-z)^r (w)^s$

We want to get $x^2y^3z^4w$ this implies that p = 2, q = 3, r = 4, s = 1

$\therefore$ Coefficient of $x^2y^3z^4w$ is $\frac{10!}{2!\,3!\,4!\,1!} (-1)^3(-1)^4 = -12600$ **Ans.**

Example # 25 : Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3.

Solution : Total number of terms = ${}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $x^2y^3 = \frac{10!}{2! \times 3! \times 5!} = 2520$ **Ans.**

Example # 26 : Find the coefficient of x^5 in the expansion of $(2 - x + 3x^2)^6$.

Solution : The general term in the expansion of $(2 - x + 3x^2)^6 = \frac{6!}{r!s!t!} 2^r (-x)^s (3x^2)^t$,

where r + s + t = 6.

$= \frac{6!}{r!s!t!} 2^r \times (-1)^s \times (3)^t \times x^{s+2t}$

For the coefficient of x^5, we must have s + 2t = 5.

But, r + s + t = 6,

$\therefore$ s = 5 – 2t and r = 1 + t, where $0 \le r, s, t \le 6$.

Now t = 0 $\Rightarrow$ r = 1, s = 5.

t = 1 $\Rightarrow$ r = 2, s = 3.

t = 2 $\Rightarrow$ r = 3, s = 1.

Thus, there are three terms containing x^5 and coefficient of x^5

$$= \frac{6!}{1!\,5!\,0!} \times 2^1 \times (-1)^5 \times 3^0 + \frac{6!}{2!\,3!\,1!} \times 2^2 \times (-1)^3 \times 3^1 + \frac{6!}{3!\,1!\,2!} \times 2^3 \times (-1)^1 \times 3^2$$

= –12 – 720 – 4320 = –5052. **Ans.**

Example 27 : If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then prove that (a) $a_r = a_{2n-r}$ (b) $\sum_{r=0}^{n-1} a_r = \frac{1}{2}(3^n - a_n)$

Solution : (a) We have

$$\left(1+x+x^2\right)^n = \sum_{r=0}^{2n} a_r x^r \quad ....(A)$$

Replace x by $\frac{1}{x}$

$$\therefore \quad \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} a_r \left(\frac{1}{x}\right)^r \Rightarrow \left(x^2+x+1\right)^n = \sum_{r=0}^{2n} a_r x^{2n-r}$$

$$\Rightarrow \quad \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r} \quad \{\text{Using (A)}\}$$

Equating the coefficient of x^{2n-r} on both sides, we get

$a_{2n-r} = a_r$ for $0 \le r \le 2n$.

Hence $a_r = a_{2n-r}$.

(b) Putting x = 1 in given series, then

$a_0 + a_1 + a_2 + ...+ a_{2n} = (1+1+1)^n$

$a_0 + a_1 + a_2 + ...+ a_{2n} = 3^n$ ....(1)

But $a_r = a_{2n-r}$ for $0 \le r \le 2n$

$\therefore$ series (1) reduces to

$2(a_0 + a_1 + a_2 + ...+ a_{n-1}) + a_n = 3^n$.

$\therefore$ $a_0 + a_1 + a_2 + ...+ a_{n-1} = \frac{1}{2}(3^n - a_n)$

Problems for Self Practice -6:

(1) The number of terms in the expansion of $(a + b + c + d + e + f)^n$ is

(2) Find the coefficient of

(i) $x^3 y^4 z^2$ in the expansion of $(2x - 3y + 4z)^9$

(ii) $x^2 y^5$ in the expansion of $(3 + 2x - y)^{10}$.

(iii) x^4 in the expansion of $(1 + x - 2x^2)^7$

Answers : (1) ${}^{n+5}C_n$ (2) (i) $\frac{9!}{3!\,4!\,2!} 2^3\,3^4\,4^2$ (ii) –272160 (iii) – 91

7. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... \infty$ provided $|x| < 1$.

Note :

(i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. (n+1) & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, ... {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - .... \infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + ... \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... \infty$

(e) $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + \frac{(-1)^r(r+1)(r+2)}{2!}x^r + ...$

(f) $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + ... + \frac{(r+1)(r+2)}{2!}x^r + ...$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of 1/x, which then will be small.

7.1 Approximations :

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 ....$$

If **x < 1**, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately.

This is an approximate value of $(1 + x)^n$

SOLVED EXAMPLE

Example # 28 : Prove that the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

Solution: $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)......(-n-r+1)}{r!}(-x)^r$$

$$= (-1)^r \frac{n(n+1)(n+2)......(n+r-1)}{r!}(-x)^r = \frac{n(n+1)(n+2)......(n+r-1)}{r!}x^r$$

$$= \frac{(n-1)!\,n(n+1)......(n+r-1)}{(n-1)!\,r!}x^r$$

Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\,r!} = {}^{n+r-1}C_r$ Proved

Example # 29 : If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}}$

Solution : $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \dfrac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \dfrac{1}{2}\left(2-\dfrac{19}{6}x\right)\left(1+\dfrac{x}{4}\right)^{-1/2} = \dfrac{1}{2}\left(2-\dfrac{19}{6}x\right)\left(1-\dfrac{x}{8}\right)$

$= \dfrac{1}{2}\left(2-\dfrac{x}{4}-\dfrac{19}{6}x\right) = 1-\dfrac{x}{8}-\dfrac{19}{12}x = 1-\dfrac{41}{24}x$ **Ans.**

Example # 30 : The value of cube root of 1001 upto five decimal places is –

(A) 10.03333 (B) 10.00333 (C) 10.00033 (D) none of these

Solution : $(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1+\dfrac{1}{1000}\right)^{1/3} = 10\left\{1+\dfrac{1}{3}.\dfrac{1}{1000}+\dfrac{1/3(1/3-1)}{2!}\dfrac{1}{1000^2}+.....\right\}$

= 10{1 + 0.0003333 – 0.00000011 + ...} = 10.00333 **Ans. (B)**

Example # 31 : The sum of $1+\dfrac{1}{4}+\dfrac{1.3}{4.8}+\dfrac{1.3.5}{4.8.12}+....\infty$ is -

(A) $\sqrt{2}$ (B) $\dfrac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Solution : Comparing with $1 + nx + \dfrac{n(n-1)}{2!}x^2 + ...$

$nx = 1/4$ ....(i)

and $\dfrac{n(n-1)x^2}{2!} = \dfrac{1.3}{4.8}$

or $\dfrac{nx(nx-x)}{2!} = \dfrac{3}{32} \Rightarrow \dfrac{1}{4}\left(\dfrac{1}{4}-x\right) = \dfrac{3}{16}$ (by (i))

$\Rightarrow \left(\dfrac{1}{4}-x\right) = \dfrac{3}{4} \Rightarrow x = \dfrac{1}{4}-\dfrac{3}{4} = -\dfrac{1}{2}$ ....(ii)

putting the value of x in (i)

$n(-1/2) = 1/4 \Rightarrow n = -1/2$

$\therefore$ sum of series = $(1 + x)^n = (1 - 1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$ **Ans. (A)**

<u>Problems for Self Practice</u> -7:

(1) Find the possible set of values of x for which expansion of $(3 - 2x)^{1/2}$ is valid in ascending powers of x.

(2) If $y = \dfrac{2}{5} + \dfrac{1.3}{2!}\left(\dfrac{2}{5}\right)^2 + \dfrac{1.3.5}{3!}\left(\dfrac{2}{5}\right)^3 + ...$, then find the value of $y^2 + 2y$

(3) The coefficient of x^{100} in $\dfrac{3-5x}{(1-x)^2}$ is

Answers : (1) $x \in \left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$ (2) 4 (3) –197

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : General Term, Coefficient of x^k in $(ax+b)^n$ and Middle term

A-1. Expand the following :

(i) $\left(\frac{2}{x}-\frac{x}{2}\right)^5$, $(x \neq 0)$ (ii) $\left(\sqrt{\frac{x}{a}}-\sqrt{\frac{a}{x}}\right)^4$ $(x > 0)$

A-2. In the binomial expansion of $\left(\sqrt[5]{2}+\frac{1}{\sqrt[5]{3}}\right)^n$, the ratio of the 6th term from the begining to the 6th term from the end is 36 : 1 ; find n.

A-3. Find the coefficient of

(i) x^6y^3 in $(x + 3y)^9$ (ii) $a^5 b^7$ in $(a + 2b)^{12}$

A-4. Find the co-efficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ and find the relation between 'a' & 'b' so that these co-efficients are equal. (where $a, b \neq 0$).

A-5. Find the terms independent of 'x' in the expansion of the expression,

$(1 + x + 2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

A-6. If $(x + (x^4 - 1)^{1/2})^7 + (x - (x^4 - 1)^{1/2})^7$ is a polynomial of degree 'n', then find n.

A-7. Given positive integers r > 1, n > 2 and the coefficients of $(3r)^{th}$ and $(r + 2)^{th}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal then prove that n = 2r.

A-8. (i) Find the coefficient of x^5 in $(1 + 2x)^6 . (1 - x)^7$

(ii) Find the coefficient x^4 in $(1 + 2x)^4 (2 - x)^5$

A-9. Find the middle term(s) in the expansion of

(i) $\left(\frac{x}{y}-\frac{y}{x}\right)^7$ (ii) $(1 - 3x + 3x^2 - x^3)^{2n}$

A-10. Prove that the co-efficient of the middle term in the expansion of $(1 + x)^{2n}$ is equal to the sum of the co-efficients of middle terms in the expansion of $(1 + x)^{2n-1}$.

Section (B) : Numerically/Algebrically Greatest terms & Remainder

B-1. Find numerically greatest term(s) in the expansion of $(3 + 5x)^{10}$ when $x = \frac{1}{25}$

B-2. Find the term in the expansion of $(2x - 5)^6$ which have

(i) Greatest binomial coefficient (ii) Greatest numerical coefficient

(iii) Algebrically greatest coefficient (iv) Algebrically least coefficient

B-3. Given T_3 in the expansion of $(1 - 3x)^6$ has maximum numerical value. Find the range

B-4. Find the digit at the unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$

B-5. (i) Find the remainder when 7^{98} is divided by 5

(ii) Find the remainder when $(6^n - 5n)$ is divided by 25 $(n \in N)$

(iii) Find the last three digits in 11^{50}.

B-6. Prove that $(99^{50} + 100^{50}) < (101)^{50.}$

Section (C):Summation of series, Variable upper index & Product of binomial coefficients

C-1. If $C_0, C_1, C_2, ..., C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$, then prove the following :

(a) $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + ... + \frac{n.C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

(b) $(C_0+C_1)(C_1+C_2)(C_2+C_3) ..... (C_{n-1}+C_n) = \frac{C_0.C_1.C_2....C_{n-1}(n+1)^n}{n!}$

C-2. If $C_0, C_1, C_2, ..., C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$, then prove the following :

(a) $C_1 + 2C_2 + 3C_3 +.....+ n . C_n = n . 2^{n-1}$

(b) $C_0 + 2C_1 + 3C_2 +.....+ (n+1)C_n = (n+2)2^{n-1}$

(c) $C_0 - 3C_1 + 5C_2 - ....(-1)^n(2n + 1)C_n = 0$

(d) $\frac{(3.2-1)}{2}C_1 + \frac{3^2.2^2-1}{2^2}C_2 + \frac{3^3.2^3-1}{2^3}C_3 + ... + \frac{3^n.2^n-1}{2^n}C_n = \frac{2^{3n}-3^n}{2^n}$

C-3. If $C_0, C_1, C_2, ..., C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$, then prove the following :

(a) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(b) $2 . C_o + \frac{2^2.C_1}{2} + \frac{2^3.C_2}{3} + \frac{2^4.C_3}{4} + ... \frac{2^{n+1}.C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(c) $C_o - \frac{C_1}{2} + \frac{C_2}{3} - ...... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

C-4. Prove the following identities using the theory of permutation where $C_0, C_1, C_2, ..., C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$:

(a) $C_0^2 + C_1^2 + C_2^2 +.....+ C_n^2 = \frac{(2n)!}{n!\,n!}$

(b) $C_0 C_3 + C_1 C_4 + ... + C_{n-3} C_n = \frac{(2n)!}{(n+3)!(n-3)!}$

(c) $C_oC_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r} C_n = \frac{2n!}{(n-r)!\,(n+r)!}$

(d) $C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2 = 0$ or $(-1)^{n/2} C_{n/2}$ according as n is odd or even.

(e) $\sum_{r=0}^{n-2}({}^nC_r.{}^nC_{r+2}) = \frac{(2n)!}{(n-2)!(n+2)!}$

(f) $1 . C_o^2 + 3 . C_1^2 + 5 . C_2^2 + ... + (2n+1)\, C_n^2 = \frac{(n+1)\,(2n)!}{n!\,n!}$

(g) ${}^{100}C_{10} + 5.\, {}^{100}C_{11} + 10 .\, {}^{100}C_{12} + 10 .\, {}^{100}C_{13} + 5.\, {}^{100}C_{14} + {}^{100}C_{15} = {}^{105}C_{90}$

C-5. Prove that

(i) ${}^{25}C_{10} + {}^{24}C_{10} + ... + {}^{10}C_{10} = {}^{26}C_{11}$

(ii) ${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + ... + {}^rC_r = {}^{n+1}C_{r+1}$

C-6. If $\binom{n}{r}$ denotes ${}^nC_r$, then prove that

$$2^{15}\binom{30}{0}\binom{30}{15} - 2^{14}\binom{30}{1}\binom{29}{14} + 2^{13}\binom{30}{2}\binom{28}{13} - ....-2^0\binom{30}{15}\binom{15}{0} = \binom{30}{15}$$

Section (D) : Negative & fractional index, Multinomial theorem

D-1. (i) Find the coefficient of $a^5 b^4 c^7$ in the expansion of $(bc + ca + ab)^8$.

(ii) Find the coefficient of $a^2b^3c^4d$ in the expansion of $(a - 2b - c + d)^{10}$

D-2. In the expantion of $(1 - 2x + x^3)^5$, find

(i) Sum of coefficients of x

(ii) Sum of coefficients of odd powers of x

(iii) Coefficient of x^7

D-3. Find the co-efficient of x^6 in the expansion of $(1 - 2x)^{-5/2}$.

D-4. (i) Find the coefficient of x^{10} in $\frac{4+2x-x^2}{(1+x)^3}$

(ii) Find the coefficient of x^{98} in $\frac{6+5x}{(1-x)^2}$

D-5. If 'x' is so small such that x^2 and higher powers of ' x ' can be neglected, find the value of

$$\frac{(1-2x)^{1/3} + (1+5x)^{-3/2}}{(9+x)^{1/2}}$$

PART-II : OBJECTIVE QUESTIONS

Section (A) : General Term, Coefficient of x^k in $(ax + b)^n$ and Middle term

A-1. If 4th term in the expansion of $\left(ax+\frac{1}{x}\right)^n$ is $\frac{5}{2}$, then

(A) $a = \frac{1}{2}$, n = 6 (B) $a = -\frac{1}{2}$, n = 6 (C) a = 1, n = 5 (D) $a = \frac{1}{2}$, n = 5

A-2. If the constant term of the binomial expansion $\left(2x-\frac{1}{x}\right)^n$ is – 160, then n is equal to -

(A) 4 (B) 6 (C) 8 (D) 10

A-3. The value of, $\frac{18^3+7^3+3.18.7.25}{3^6+6.243.2+15.81.4+20.27.8+15.9.16+6.3.32+64}$ is :

(A) 1 (B) 2 (C) 3 (D) none

A-4. In the expansion of $\left(3-\sqrt{\frac{17}{4}+3\sqrt{2}}\right)^{15}$, the 11th term is a:

(A) positive integer (B) positive irrational number
(C) negative integer (D) negative irrational number.

A-5. If the second term of the expansion $\left[a^{1/13}+a^{3/2}\right]^n$ is $14a^{5/2}$, then the value of $\frac{{}^nC_5}{{}^nC_4}$ is:

(A) 2 (B) 3 (C) 4 (D) 6

A-6. The value of k, for which the coefficients of the $(3k+2)^{th}$ and $(4k-1)^{th}$ terms in the expansion of $(1+x)^{20}$ are equal, is

(A) 3 (B) 1 (C) 5 (D) 8

A-7. The expression, $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$ is a polynomial of degree

(A) 6 (B) 8 (C) 3 (D) 4

A-8. The co-efficient of x in the expansion of $(1-2x^3+3x^5)\left(1+\frac{1}{x}\right)^8$ is :-

(A) 56 (B) 65 (C) 154 (D) 62

A-9. The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is -

(A) 1 (B) $\frac{5}{12}$ (C) ${}^{10}C_1$ (D) none of these

A-10. The term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^3$ is:

(A) – 3 (B) 0 (C) 1 (D) 3

A-11. The coefficient of x^{52} in the expansion $\sum_{m=0}^{100}{}^{100}C_m\,(x-3)^{100-m}.\,2^m$ is :

(A) ${}^{100}C_{47}$ (B) ${}^{100}C_{48}$ (C) $-{}^{100}C_{52}$ (D) $-{}^{100}C_{100}$

A-12. If $k\in R$ and the middle term of $\left(\frac{k}{2}+2\right)^8$ is 1120, then value of k is:

(A) 3 (B) 2 (C) – 3 (D) – 4

Section (B) : Numerically/Algebrically Greatest terms & Remainder

B-1. Numerically greatest term in the binomial expansion of $(a + 2x)^9$ when a = 1 & $x = \frac{1}{3}$ is :

(A) 3^{rd} & 4^{th} (B) 4^{th} & 5^{th} (C) only 4^{th} (D) only 5^{th}

B-2. In the expansion of $\left(\sqrt{2} + \sqrt[4]{3}\right)^{100}$, the number of terms free from radicals is:

(A) 25 (B) 26 (C) 27 (D) 78

B-3. The last two digits of the number 19^{100} are:

(A) 81 (B) 43 (C) 29 (D) 01

B-4. The remainder when 27^{40} is divided by 12 is :

(A) 3 (B) 2 (C) 8 (D) 9

Section (C):Summation of series, Variable upper index & Product of binomial coefficients

C-1. The value of $\sum_{r=1}^{10} r^2 \cdot \frac{{}^{26}C_r}{{}^{26}C_{r-1}}$ is equal to

(A) 900 (B) 1000 (C) 1100 (D) 1200

C-2. $\sum_{r=0}^{2019} \frac{{}^{2020}C_r}{{}^{2020}C_r + {}^{2020}C_{r+1}} =$

(A) 1010 (B) $\frac{2021}{2}$ (C) 2020 (D) $\frac{2019}{2}$

C-3. The value of the expression $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{K=0}^{10} (-1)^K \frac{{}^{10}C_K}{2^K}\right)$ is :

(A) –1 (B) 1 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

C-4. $\sum_{r=0}^{7} (3r+2).{}^7C_r =$

(A) 1600 (B) 1599 (C) 1400 (D) 1399

C-5. $\frac{{}^{2021}C_0}{1} + \frac{{}^{2021}C_1}{2} + \frac{{}^{2021}C_2}{3} + \ldots\ldots + \frac{{}^{2021}C_{2020}}{2021} =$

(A) $\frac{2^{2021}-1}{2021}$ (B) $\frac{2^{2021}-1}{1011}$

(C) $\frac{3^{2021}-1}{2021}$ (D) $\frac{3^{2021}-1}{1011}$

C-6. The value of $\sum_{r=0}^{50}(-1)^r.\left(\frac{^{50}C_r}{r+1}\right)$ is :

(A) $\frac{1}{50}$ (B) $\frac{50}{51}$ (C) $\frac{1}{51}$ (D) $\frac{51}{50}$

C-7. The value of $\binom{50}{0}\binom{50}{1}+\binom{50}{1}\binom{50}{2}+..........+\binom{50}{49}\binom{50}{50}$ is, where $^nC_r = \binom{n}{r}$

(A) $\binom{100}{50}$ (B) $\binom{100}{51}$ (C) $\binom{50}{25}$ (D) $\binom{50}{25}^2$

C-8. The value of $^{50}C_4 + \sum_{r=1}^{6} {^{56-r}C_3}$ is :

(A) $^{56}C_4$ (B) $^{56}C_3$ (C) $^{55}C_3$ (D) $^{55}C_4$

Section (D) : Negative & fractional index, Multinomial theorem

D-1. The coefficient of x^5 in the expansion of $(2 - x + 3x^2)^6$ is

(A) 5000 (B) –5000 (C) –5052 (D) 5052

D-2. If $(r + 1)^{th}$ term is the first negative term in the expansion of $(1 + x)^{7/2}$, then the value of r (where | x | < 1) is

(A) 3 (B) 4 (C) 5 (D) 6

D-3. The coefficient of x^5 in $(1 + 2x + 3x^2 +.....)^{-3/2}$ is :

(A) 21 (B) 25 (C) 26 (D) none of these

D-4. Coefficient x^n in the expansion of $(1 - 9x + 20x^2)^{-1}$ is

(A) $5^n - 4^n$ (B) $4^n - 5^n$ (C) $5^{n+1} - 4^{n+1}$ (D) $5^{n+1} - 4^{n+1} - 2$

PART-III : MATCH THE COLUMN

1.

	Column – I		Column – II
(A)	If $11^n + 21^n$ is divisible by 16, then n can be	(p)	4
(B)	The remainder when 3^{37} is divided by 80 is less than	(q)	5
(C)	If the number of dissimilar terms in the expansion of $(x + y + z)^{2n+1} - (x + y - z)^{2n+1}$ ($n \in N$) is $an^2 + bn + c$, then a + b + c is	(r)	6
(D)	The coefficient of x^4 in the expression $(1 + 2x + 3x^2 + 4x^3 + ...\text{up to } \infty)^{1/2}$ is c, ($c \in N$), then c + 1 (where | x | < 1) is	(s)	7

Exercise # 2

PART-I : OBJECTIVE QUESTIONS

1. Consider the following statements :

S_1 : The term independent of x in $(1 + x)^m\left(1+\frac{1}{x}\right)^n$ is ${}^{m+n}C_n$

S_2 : $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)\ldots\ldots(1 + x + x^2 + \ldots\ldots + x^{100})$ when written in the ascending power of x then the highest exponent of x is 5000.

S_3 : If $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^xC_y$ then x = n and y = r + 1

S_4 : If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n}$, then $a_0 + a_2 + a_4 + \ldots\ldots + a_{2n} = \frac{3^n-1}{2}$

State, in order, whether S_1, S_2, S_3, S_4 are true or false

(A) TFTF (B) TTTT (C) FFFF (D) TFTF

2. The sum of the coefficients of all the integral powers of x in the expansion of $\left(1+2\sqrt{x}\right)^{40}$ is :

(A) $3^{40} + 1$ (B) $3^{40} – 1$ (C) $\frac{1}{2}(3^{40} – 1)$ (D) $\frac{1}{2}(3^{40} + 1)$

3. The coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$ is :

(A) 70 (B) 112 (C) 105 (D) 210

4. Find the complete set of positive values of x such that fourth term in the expansion of $(5 + 3x)^{10}$, is greatest.

(A) $\frac{5}{8}\le x\le\frac{20}{21}$ (B) $\frac{7}{8}\le x\le\frac{20}{21}$ (C) $\frac{5}{8}\le x\le\frac{25}{21}$ (D) none of these

5. If the sum of the co-efficients in the expansion of $(1 + 2x)^n$ is 6561, then find which term is the greatest term in the expansion for x = 1/2.

(A) 4^{th} (B) 5^{th} (C) 6^{th} (D) None of these

6. Let $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$. The greatest value of the integer which divides f(n) for all n is :

(A) 27 (B) 9 (C) 3 (D) None of these

7. If $(1 + x)^n = \sum_{r=0}^{n} a_r x^r$ and $b_r = 1 + \frac{a_r}{a_{r-1}}$ and $\prod_{r=1}^{n} b_r = \frac{(101)^{100}}{100!}$, then n equals to :

(A) 99 (B) 100 (C) 101 (D) 102

8. The sum $\frac{1}{1!(n-1)!}+\frac{1}{2!(n-2)!}+\ldots\ldots\frac{1}{1!(n-1)!}$ is equal to :

(A) $\frac{1}{n!}(2^{n-1} – 1)$ (B) $\frac{2}{n!}(2^n – 1)$ (C) $\frac{2}{n!}(2^{n-1} – 1)$ (D) none

9. If $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots + a_{15}x^{15}$, then a_{10} equals to :

(A) 99 (B) 101 (C) 100 (D) 110

10. If $\sum_{r=0}^{n} (-1)^r . {}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\text{........ to m terms}\right] = k\left(1-\frac{1}{2^{mn}}\right)$, then find the value of k.

(A) $\frac{1}{2^n+1}$ (B) $\frac{1}{2^n-1}$ (C) $\frac{1}{2^{n+1}-1}$ (D) $\frac{1}{2^{n-1}-1}$

11. If $a_n = \sum_{r=0}^{n}\frac{1}{{}^nC_r}$, the value of $\sum_{r=0}^{n}\frac{n-2r}{{}^nC_r}$ is :

(A) $\frac{n}{2}a_n$ (B) $\frac{1}{4}a_n$ (C) na_n (D) 0

12. Co-efficient of α^t in the expansion of ,

$(\alpha + p)^{m-1} + (\alpha + p)^{m-2}(\alpha + q) + (\alpha + p)^{m-3}(\alpha + q)^2 + ...(\alpha + q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is :

(A) $\frac{{}^mC_t\left(p^t-q^t\right)}{p-q}$ (B) $\frac{{}^mC_t\left(p^{m-t}-q^{m-t}\right)}{p-q}$ (C) $\frac{{}^mC_t\left(p^t+q^t\right)}{p-q}$ (D) $\frac{{}^mC_t\left(p^{m-t}+q^{m-t}\right)}{p-q}$

13. The sum of: $3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + ...$ upto (n+1) terms is :

(A) zero (B) 1 (C) 2 (D) none of these

14. The sum $\sum_{r=0}^{n} (r + 1) C_r^2$ is equal to :

(A) $\frac{(n+2)(2n-1)!}{n!(n-1)!}$ (B) $\frac{(n+2)(2n+1)!}{n!(n-1)!}$ (C) $\frac{(n+2)(2n+1)!}{n!(n+1)!}$ (D) $\frac{(n+2)(2n-1)!}{n!(n+1)!}$

15. The co-efficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + ... + (1 + x)^{30}$ is :

(A) ${}^{51}C_5$ (B) ${}^9C_5$ (C) ${}^{31}C_6 - {}^{21}C_6$ (D) ${}^{30}C_5 + {}^{20}C_5$

16. The number of terms in the expansion of $\left(x^2+1+\frac{1}{x^2}\right)^n$, $n \in N$, is :

(A) 2n (B) 3n (C) 2n + 1 (D) 3n + 1

17. If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + ... + a_{40}x^{40}$, then $a_0 + a_2 + a_4 .... + a_{38}$ is equal to :

(A) $2^{19}(2^{30} + 1)$ (B) $2^{20}(2^{19} - 1)$ (C) $2^{39} - 2^{19}$ (D) $2^{39} + 2^{19}$

18. The coefficient of x^{49} in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)....\left(x-\frac{1}{2^{49}}\right)$ is equal to -

(A) $-2\left(1-\frac{1}{2^{50}}\right)$ (B) +ve coefficient of x

(C) –ve coefficient of x (D) $-2\left(1-\frac{1}{2^{49}}\right)$

19. The largest real value for x such that $\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{8}{3}$ is -

(A) $2\sqrt{2}-5$ (B) $2\sqrt{2}+5$ (C) $-2\sqrt{2}-5$ (D) $-2\sqrt{2}+5$

PART-II : NUMERICAL QUESTIONS

1. If the 6^{th} term in the expansion of $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ is 5600, then x =

2. The number of values of 'x' for which the fourth term in the expansion,

$\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}} + \frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336, is :

3. If second, third and fourth terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, then n is equal to

4. In the expansion of $\left(x^3 - \frac{1}{x^2}\right)^n$, $n \in N$, if the sum of the coefficients of x^5 and x^{10} is 0, then n is :

5. Let the co-efficients of x^n in $(1 + x)^{2n}$ & $(1 + x)^{2n-1}$ be P & Q respectively, then $\left(\frac{P+Q}{Q}\right)^5 =$

6. The index 'n' of the binomial $\left(\frac{x}{5} + \frac{2}{5}\right)^n$ if the 9^{th} term of the expansion has numerically the greatest coefficient ($n \in N$), is :

7. Which term is the numerically greatest term in the expansion of $(2x + 5y)^{34}$, when x = 3 & y = 2 ?

8. If {x} denotes the fractional part of 'x', then $82\left\{\frac{3^{1001}}{82}\right\} =$

9. The number of values of 'r' satisfying the equation, ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$ is :

10. If n is a positive integer & $C_k = {}^nC_k$, find the value of $\left(\sum_{k=1}^{n} \frac{k^3}{n(n+1)^2.(n+2)}\left(\frac{C_k}{C_{k-1}}\right)^2\right)^{-1}$ is :

11. The value of λ if $\sum_{m=97}^{100} {}^{100}C_m \,.\, {}^mC_{97} = 2^{\lambda} \,.\, {}^{100}C_{97}$, is :

12. If $(1 + x + x^2 + ... + x^p)^6 = a_0 + a_1x + a_2x^2 + ... + a_{6p}x^{6p}$, then the value of :

$\frac{1}{p(p+1)^6}[a_1 + 2a_2 + 3a_3 + ... + 6p\, a_{6p}]$ is :

13. If $\sum_{r=0}^{n} \frac{2r+3}{r+1}.{}^nC_r = \frac{(n+k).2^{n+1}-1}{n+1}$ then 'k' is

14. If $\sum_{r=0}^{n} \frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{1}{a(n+b)}$, then a + b is

15. If $({}^{2n}C_1)^2 + 2.({}^{2n}C_2)^2 + 3.({}^{2n}C_3)^2 + ... + 2n.({}^{2n}C_{2n})^2 = 18 \,.\, {}^{4n-1}C_{2n-1}$, then n is :

16. The value of p, for which coefficient of x^{50} in the expression $(1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + ... + 1001\, x^{1000}$ is equal to ${}^{1002}C_p$, is :

17. If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1 + \frac{y^2}{kx^2}$

18. Let a and b be the coefficient of x^3 in $(1 + x + 2x^2 + 3x^3)^4$ and $(1 + x + 2x^2 + 3x^3 + 4x^4)^4$ respectively. Find the value of (a – b).

19. Find the index n of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ if the 9th term of the expansion has numerically the greatest coefficient $(n \in N)$.

20. Find the sum of the roots (real or complex) of the equation $x^{2001} + \left(\frac{1}{2}-x\right)^{2001} = 0$.

21. Let $a = \left(4^{1/401}-1\right)$ and let $b_n = {}^nC_1 + {}^nC_2 \cdot a + {}^nC_3 \cdot a^2 + ... + {}^nC_n \cdot a^{n-1}$.
Find the value of $(b_{2006} - b_{2005})$

22. Let the coefficient of x^{49} in the polynomial

$\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\cdot\frac{C_2}{C_1}\right)\left(x-3^2\cdot\frac{C_3}{C_2}\right)................\left(x-50^2\cdot\frac{C_{50}}{C_{49}}\right)$ {where $C_r = {}^{50}C_r$} is 'k', then find the value of $\frac{-k}{1000}$

23. Find the value of $\dfrac{\sum_{K=0}^{n} {}^nC_K \sin Kx.\cos(n-K)x}{2^n \sin nx}$

24. Find the value of $({}^nC_0 + {}^nC_1 + {}^nC_2 + ...+ {}^nC_n)^2 - (1 + {}^{2n}C_1 + {}^{2n}C_2 + ... + {}^{2n}C_{2n})$

PART - III : ONE OR MORE THAN ONE CORRECT QUESTION

1. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

(A) the number of irrational terms is 19
(B) middle term is irrational
(C) the number of rational terms is 2
(D) 9th term is rational

2. $7^9 + 9^7$ is divisible by :

(A) 16 (B) 24 (C) 64 (D) 72

3. Let $a_n = \frac{1000^n}{n!}$ for $n \in N$, then a_n is greatest, when

(A) n = 997 (B) n = 998 (C) n = 999 (D) n = 1000

4. The sum of the series $\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r(a-r)$ is equal to

(A) 5 if a = 5 (B) –5 if a = 5 (C) –5 if a = –5 (D) 5 if a = –5

5. ${}^nC_0 - 2.3\ {}^nC_1 + 3.3^2\ {}^nC_2 - 4.3^3\ {}^nC_3 + ... + (-1)^n (n+1)\ {}^nC_n\ 3^n$ is equal to

(A) $2^n\left(\frac{3n}{2}+1\right)$ if n is even
(B) $2^n\left(n+\frac{3}{2}\right)$ if n is even
(C) $-2^n\left(\frac{3n}{2}+1\right)$ if n is odd
(D) $2^n\left(n+\frac{3}{2}\right)$ if n is odd

6. The sum of the co–efficients in the expansion of $(1 - 2x + 5x^2)^n$ is a and the sum of the co–efficients in the expansion of $(1 + x)^{2n}$ is b. Then :
(A) a = b
(B) $(x - a)^2 + (x - b)^2 = 0$, has real roots
(C) $\sin^2 a + \cos^2 b = 1$
(D) ab = 1

7. Let $P(n) = \sum_{r=0}^{n} \frac{(-1)^r r}{r+1} {}^nC_r$. Now which of the following holds good ?
(A) $|P_{10}|$ is harmonic mean of $|P_9|$ & $|P_{11}|$
(B) $\sum_{r=5}^{10} P(r)P(r-1) = -\frac{6}{55}$
(C) $|P_{10}|$ is arithmetic mean of $|P_9|$ & $|P_{11}|$
(D) $\sum_{r=5}^{10} P(r)P(r-1) = \frac{6}{55}$

8. Let $(1 + x)^m = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_mx^m$, where $C_r = {}^mC_r$ and $A = C_1C_3 + C_2C_4 + C_3C_5 + C_4C_6 + ...+ C_{m-2}C_m$, then -
(A) $A \geq {}^{2m}C_{m-2}$
(B) $A < {}^{2m}C_{m-2}$
(C) $A > C_0^2 + C_1^2 + C_2^2 + ...C_m^2$
(D) $A < C_0^2 + C_1^2 + C_2^2 + ...C_m^2$

9. Element in set of values of r for which, ${}^{18}C_{r-2} + 2.\ {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ is :
(A) 9 (B) 5 (C) 7 (D) 10

10. If for $n \in I$, n > 10; $1 + (1 + x) + (1 + x)^2 + ... + (1 + x)^n = \sum_{k=0}^{n} a_k.x^k$, $x \neq 0$ then
(A) $\sum_{k=0}^{n} a_k = 2^{n+1}$
(B) $a_{n-2} = \frac{n(n+1)}{2}$
(C) $a_p > a_{p-1}$ for $p < \frac{n}{2}$, $p \in N$
(D) $(a_9)^2 - (a_8)^2 = {}^{n+2}C_{10} ({}^{n+1}C_{10} - {}^{n+1}C_9)$

11. In the expansion of $(x + y + z)^{25}$
(A) every term is of the form ${}^{25}C_r.\ {}^rC_k.\ x^{25-r}.\ y^{r-k}.\ z^k$
(B) the coefficient of $x^8 y^9 z^9$ is 0
(C) the number of terms is 325
(D) none of these

12. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 +.... + a_{20}x^{20}$, then :
(A) $a_1 = 20$ (B) $a_2 = 210$ (C) $a_4 = 8085$ (D) $a_{20} = 2^2.\ 3^7.\ 7$

13. If $n \in N$ and if $(1 + 4x + 4x^2)^n = \sum_{r=0}^{2n} a_r x^r$, where $a_0,a_1,a_2,.......,a_{2n}$ are real numbers, then
(A) $2\sum_{r=0}^{n} a_{2r} = 9^n + 1$ (B) $2\sum_{r=1}^{n} a_{2r-1} = 9^n - 1$ (C) $a_{2n-1} = n.\ 2^{2n}$ (D) $a_{2n-1} = n.\ 2^{2n-1}$

14. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x - 2)^2$, $P_2(x) = ((x - 2)^2 - 2)^2$
$P_3(x) = ((x - 2)^2 - 2)^2 - 2)^2$ .... (In general $P_k(x) = (P_{k-1}(x) - 2)^2$, then the constant term in $P_k(x)$ is
(A) 4 (B) 2 (C) 16 (D) a perfect square

15. If the expansion of $(3x + 2)^{-1/2}$ is valid in ascending powers of x, then x lies in the interval.
(A) (0, 2/3) (B) (–3/2, 3/2) (C) (–2/3, 2/3) (D) $(-\infty, -3/2) \cup (3/2, \infty)$

16. The coefficient of x^4 in $\left(\frac{1+x}{1-x}\right)^2$, $|x| < 1$, is
(A) 4 (B) –4 (C) $10 + {}^4C_2$ (D) 16

PART - IV : COMPREHENSION

Comprehension # 1 (Q. No. 1 to 3)

Let P be a product given by $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$ and Let $S_1 = a_1 + a_2 + \ldots\ldots. + a_n = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum\sum_{i<j<k} a_i.a_j.a_k$ and so on, then it can be shown that

$P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots\ldots + S_n$.

1. The coefficient of x^8 in the expression $(2 + x)^2 (3 + x)^3 (4 + x)^4$ must be

(A) 26 (B) 27 (C) 28 (D) 29

2. The coefficient of x^{203} in the expression $(x - 1)(x^2 - 2)(x^3 - 3) \ldots\ldots\ldots. (x^{20} - 20)$ must be

(A) 11 (B) 12 (C) 13 (D) 15

3. The coefficient of x^{98} in the expression of $(x - 1)(x - 2) \ldots\ldots\ldots (x - 100)$ must be

(A) $1^2 + 2^2 + 3^2 + \ldots\ldots. + 100^2$

(B) $(1 + 2 + 3 + \ldots\ldots. + 100)^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots. + 100^2)$

(C) $\frac{1}{2}[(1 + 2 + 3 + \ldots\ldots. + 100)^2 - (1^2 + 2^2 + 3^2 + \ldots\ldots. + 100^2)]$

(D) None of these

Comprehension # 2 (Q. No. 4 to 6)

Consider, sum of the series $\sum\sum_{0 \le i < j \le n} f(i)\, f(j)$

In the given summation, i and j are not independent.

In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)\, f(j) = \sum_{i=1}^{n}\left(f(i)\left(\sum_{j=1}^{n} f(j)\right)\right)$ i and j are independent. In this summation, three types of terms occur, those when i < j, i > j and i = j.

Also, sum of terms when i < j is equal to the sum of the terms when i > j if f(i) and f(j) are symmetrical.

So, in that case

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0 \le i < j \le n} f(i)f(j) + \sum\sum_{0 \le i < j \le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0 \le i < j \le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow \sum\sum_{0 \le i < j \le n} f(i)f(j) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

When f(i) and f(j) are not symmetrical, we find the sum by listing all the terms.

4. $\sum\sum_{0\le i<j\le n} {}^nC_i \; {}^nC_j$ is equal to-

(A) $\frac{2^{2n} - {}^{2n}C_n}{2}$ (B) $\frac{2^{2n} + {}^{2n}C_n}{2}$ (C) $\frac{2^{2n} - {}^{n}C_n}{2}$ (D) $\frac{2^{2n} + {}^{n}C_n}{2}$

5. $\sum_{m=0}^{n}\sum_{p=0}^{m} {}^nC_m \, . \, {}^mC_p$ is equal to-

(A) 2^n-1 (B) 3^n (C) 3^n-1 (D) 2^n

6. $\sum\sum_{0\le i\le j\le n} \left({}^nC_i + {}^nC_j\right)$ is equal to-

(A) $n2^n$ (B) $(n + 1)2^n$ (C) $(n - 1)2^n$ (D) $(n + 1)2^n-1$

Comprehension # 3 (Q. No. 7 to 9)

The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x.

e.g. [7.2] = 7 ; [6.99] = 6; [–8.9] = –9 ; [–11.11] = –12

g(x) = {x} is called fractional part function, where {x} = x – [x]

e.g. the fractional part of the number 2.1 is 2.1–2 = 0.1 and the fractional part of –3.7 is 0.3 .

Any real number can be expressed as Integral Part + fractional Part i.e. I + f, where $0 \le f < 1$

(x = [x] + {x})

7. If $\left(7+4\sqrt{3}\right)^n = p+\beta$ where n & p are positive integers and β is a proper fraction, find the value of $(1 - \beta)(p + \beta)$.

(A) –1 (B) 0 (C) 1 (D) 2

8. If $\left(9+\sqrt{80}\right)^n = I + f$, where I, n are integers and 0 < f < 1, then which of the following option is **INCORRECT**

(A) I is an odd integer (B) I is an even integer

(C) (I + f) (1 – f) = 1 (D) $1 - f = \left(9-\sqrt{80}\right)^n$

9. Let $P = \left(2+\sqrt{3}\right)^5$ and f = P – [P], where [P] denotes the greatest integer function.

Find the value of $\left(\frac{f^2}{1-f}\right)$.

(A) 721 (B) 722 (C) 723 (D) none of these

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. For r = 0, 1, ..., 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

[IIT-JEE 2010, Paper-2, (5, –2)/79]

(A) $B_{10} - C_{10}$ (B) $A_{10}(B^2_{10} - C_{10}A_{10})$ (C) 0 (D) $C_{10} - B_{10}$

2. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

3. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) 1051 (B) 1106 (C) 1113 (D) 1120

4. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)......(1 + x^{100})$ is

[JEE (Advanced) 2015, P-2 (4, 0) / 80]

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is **[JEE(Advanced)-2016, 3(0)]**

6. Let $X = \left({}^{10}C_1\right)^2 + 2\left({}^{10}C_2\right)^2 + 3\left({}^{10}C_3\right)^2 + ... + 10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, r ∈ {1, 2, ..., 10} denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

7. Suppose $\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals

[JEE(Advanced)-2019, 3(0)]

PART - II : JEE(MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is : **[AIEEE 2011, (4, –1), 120]**

(1) 144 (2) – 132 (3) – 144 (4) 132

2. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012, (4, –1), 120]**

(1) an irrational number
(2) an odd positive integer
(3) an even positive integer
(4) a rational number other than positive integers

3. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is : **[AIEEE - 2013, (4, –¼),120]**

(1) 4 (2) 120 (3) 210 (4) 310

4. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to **[JEE(Main) 2014, (4, – ¼), 120]**

(1) $\left(14, \frac{272}{3}\right)$ (2) $\left(16, \frac{272}{3}\right)$ (3) $\left(16, \frac{251}{3}\right)$ (4) $\left(14, \frac{251}{3}\right)$

5. The sum of coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$ is :
[JEE(Main) 2015, (4, – ¼), 120]

(1) $\frac{1}{2}(3^{50}+1)$ (2) $\frac{1}{2}(3^{50})$ (3) $\frac{1}{2}(3^{50}-1)$ (4) $\frac{1}{2}(2^{50}+1)$

6. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n, x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is :- **[JEE(Main)-2016]**

(1) 729 (2) 64 (3) 2187 (4) 243

7. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is :- **[JEE(Main)-2017]**

(1) $2^{20} - 2^{10}$ (2) $2^{21} - 2^{11}$ (3) $2^{21} - 2^{10}$ (4) $2^{20} - 2^{9}$

8. The sum of the co-efficients of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5 + \left(x-\sqrt{x^3-1}\right)^5$, (x > 1) is - **[JEE(Main)-2018]**

(1) 0 (2) 1 (3) 2 (4) –1

9. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to : **[JEE(Main)- 2019]**

(1) 14 (2) 6 (3) 4 (4) 8

10. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is **[JEE(Main)- 2019]**

(1) 12 (2) 15 (3) 10 (4) 14

11. The coefficient of x^7 in the expression $(1 + x)^{10} + x(1 + x)^9 + x^2(1 + x)^8 + \ldots + x^{10}$ is : **[JEE(Main)-2020]**

(1) 120 (2) 330 (3) 210 (4) 420

Answers

Exercise # 1

PART-I :

SECTION-(A)

A-1. (i) $\left(\frac{2}{x}\right)^5 - 5\left(\frac{2}{x}\right)^3 + 10\left(\frac{2}{x}\right) - 10\left(\frac{x}{2}\right) + 5\left(\frac{x}{2}\right)^3 - \left(\frac{x}{2}\right)^5$

(ii) $\frac{x^2}{a^2} - \frac{4x}{a} + 6 - 4\frac{a}{x} + \frac{a^2}{x^2}$

A-2. n = 20

A-3. (i) ${}^9C_3 .27$ (ii) $2^7 . {}^{12}C_7$

A-4. ${}^{11}C_5 \frac{a^6}{b^5}$, ${}^{11}C_6 \frac{a^5}{b^6}$, a b = 1

A-5. $\frac{17}{54}$

A-6. n = 13

A-8. (i) 171 (ii) –438

A-9. (i) $-\frac{35x}{y}, \frac{35y}{x}$

(ii) $(-1)^{3n} \frac{(6n)!}{(3n)!\,(3n)!} x^{3n}$

SECTION-(B)

B-1. $T_2 = 5 \times (3)^9$

B-2. (i) T_4
(ii) T_5, T_6
(iii) T_5
(iv) T_6

B-3. $x \in \left[-\frac{1}{4}, -\frac{2}{15}\right] \cup \left[\frac{2}{15}, \frac{1}{4}\right]$ of 'x'.

B-4. 1

B-5. (i) 4
(ii) 1
(iii) 001

SECTION-(D)

D-1. (i) 280 (ii) –149200

D-2. (i) 0 (ii) 512
(iii) 20

D-3. $\frac{15015}{16}$

D-4. (i) 109 (ii) 1084

D-5. $\frac{2}{3} - \frac{149}{54}x$

PART-II

SECTION-(A)

A-1.	(A)	**A-2.**	(B)
A-3.	(A)	**A-4.**	(B)
A-5.	(A)	**A-6.**	(A)
A-7.	(A)	**A-8.**	(C)
A-9.	(D)	**A-10.**	(B)
A-11.	(B)	**A-12.**	(B)

SECTION-(B)

B-1.	(B)	**B-2.**	(B)
B-3.	(D)	**B-4.**	(D)

SECTION-(C)

C-1.	(C)	**C-2.**	(A)
C-3.	(B)	**C-4.**	(A)
C-5.	(B)	**C-6.**	(C)
C-7.	(B)	**C-8.**	(A)

SECTION-(D)

D-1.	(C)	**D-2.**	(C)
D-3.	(D)	**D-4.**	(C)

PART-III

1. (A) → (q, s), (B) → (p, q, r, s), (C) → (p), (D) → (q)

Exercise # 2

PART-I

1.	(A)	**2.**	(D)
3.	(D)	**4.**	(A)
5.	(B)	**6.**	(B)
7.	(B)	**8.**	(C)
9.	(B)	**10.**	(B)
11.	(D)	**12.**	(B)
13.	(A)	**14.**	(A)
15.	(C)	**16.**	(C)
17.	(C)	**18.**	(A)
19.	(A)		

PART-II

1.	10	**2.**	2
3.	5	**4.**	15
5.	32	**6.**	12
7.	22	**8.**	3
9.	2	**10.**	12
11.	3	**12.**	3
13.	2	**14.**	5
15.	9	**16.**	50
17.	2	**18.**	0
19.	12	**20.**	500
21.	1024	**22.**	22.10
23.	0.50	**24.**	0

PART - III

1.	(A, B, C, D)	**2.**	(A, C)
3.	(C, D)	**4.**	(A, C)
5.	(A, C)	**6.**	(A, B, C)
7.	(A, D)	**8.**	(B, D)
9.	(A, C, D)	**10.**	(B, C, D)
11.	(A, B)	**12.**	(A, B, C)
13.	(A, B, C)	**14.**	(A, D)
15.	(A, C)	**16.**	(C, D)

PART - IV

1.	(D)	**2.**	(C)
3.	(C)	**4.**	(A)
5.	(B)	**6.**	(A)
7.	(C)	**8.**	(B)
9.	(B)		

Exercise # 3

PART - I

1.	(D)	**2.**	6
3.	(C)	**4.**	8
5.	5	**6.**	646
7.	6.20		

PART - II

1.	(3)	**2.**	(1)
3.	(3)	**4.**	(2)
5.	(1)	**6.**	Bonus
7.	(1)	**8.**	(3)
9.	(4)	**10.**	(2)
11.	(2)		

SUBJECTIVE QUESTIONS

1. Find the co-efficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$.

2. Prove that the co-efficient of x^{15} in $(1 + x + x^3 + x^4)^n$ is $\sum_{r=0}^{5} {}^nC_{15-3r}\,{}^nC_r$.

3. If n is even natural and coefficient of x^r in the expansion of $\frac{(1+x)^n}{1-x}$ is 2^n, $(|x| < 1)$, then prove that $r \geq n$

4. Find the coefficient of x^n in polynomial $(x + {}^{2n+1}C_0)(x + {}^{2n+1}C_1)........(x + {}^{2n+1}C_n)$.

5. Find the value of $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\,{}^rC_p\,2^p\right)$.

Comprehension (Q-6 to Q.7)

For k, n $\in$ N, we define

$B(k, n) = 1.2.3......... k + 2.3.4........(k+1) + ...+ n(n + 1).......(n + k - 1)$, $S_0(n) = n$ and $S_k(n) = 1^k + 2^k + ... + n^k$.

To obtain value B(k, n), we rewrite B(k, n) as follows

$$B(k,n) = k!\left[{}^kC_k + {}^{k+1}C_k + {}^{k+2}C_k + ...+ {}^{n+k-1}C_k\right] = k!\left({}^{n+k}C_{k+1}\right)$$

$$= \frac{n(n+1)........(n+k)}{k+1}$$

where ${}^nC_k = \frac{n!}{k!\,(n-k)!}$

6. Prove that $S_2(n) + S_1(n) = B(2, n)$

7. Prove that $S_3(n) + 3S_2(n) = B(3, n) - 2B(1, n)$

8. If $(1 + x)^p = 1 + {}^pC_1 x + {}^pC_2 x^2 + ...+ {}^pC_p x^p$, $p \in N$, then show that ${}^{k+1}C_1 S_k(n) + {}^{k+1}C_2 S_{k-1}(n) + ...+ {}^{k+1}C_k S_1(n) + {}^{k+1}C_{k+1} S_0(n) = (n + 1)^{k+1} - 1$

9. Show that $25^n - 20^n - 8^n + 3^n$, $n \in I^+$ is divisible by 85.

10. Prove that ${}^nC_1 ({}^nC_2)^2 ({}^nC_3)^3........({}^nC_n)^n \leq \left(\frac{2^n}{n+1}\right)^{{}^{n+1}C_2}$.

11. If p is nearly equal to q and n > 1, show that $\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \left(\frac{p}{q}\right)^{1/n}$. Hence find the approximate value of $\left(\frac{99}{101}\right)^{1/6}$.

12. If $(18x^2 + 12x + 4)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then prove that

$a_r = 2^n 3^r \left({}^{2n}C_r + {}^nC_1\,{}^{2n-2}C_r + {}^nC_2\,{}^{2n-4}C_r + ...\right)$

13. Prove that : $1^2.C_0 + 2^2.C_1 + 3^2.C_2 + 4^2.C_3 + ... + (n+1)^2C_n = 2^{n-2}(n+1)(n+4)$.

14. If $(1 - x)^{-n} = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$, find the value of, $a_0 + a_1 + a_2 + ... + a_n$.

15. Find the remainder when $32^{32^{32}}$ is divided by 7.

16. If n is an integer greater than 1, show that : $a - {}^nC_1(a-1) + {}^nC_2(a-2) - ..... + (-1)^n(a - n) = 0$.

17. If $(1 + x)^n = p_0 + p_1x + p_2x^2 + p_3x^3 + ...$, then prove that :

(a) $p_0 - p_2 + p_4 - .... = 2^{n/2}\cos\dfrac{n\pi}{4}$ (b) $p_1 - p_3 + p_5 - .... = 2^{n/2}\sin\dfrac{n\pi}{4}$

18. Prove the following $4C_0 + \dfrac{4^2}{2}.C_1 + \dfrac{4^3}{3}C_2 + ... + \dfrac{4^{n+1}}{n+1}C_n = \dfrac{5^{n+1}-1}{n+1}$

19. Show that if the greatest term in the expansion of $(1 + x)^{2n}$ has also the greatest co-efficient, then 'x' lies between, $\dfrac{n}{n+1}$ & $\dfrac{n+1}{n}$.

20. Prove that if 'p' is a prime number greater than 2, then $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$ is divisible by p, where [.] denotes greatest integer function.

21. Given $s_n = 1 + q + q^2 + ... + q^n$ & $S_n = 1 + \dfrac{q+1}{2} + \left(\dfrac{q+1}{2}\right)^2 + ... + \left(\dfrac{q+1}{2}\right)^n$, $q \neq 1$,

prove that ${}^{n+1}C_1 + {}^{n+1}C_2.s_1 + {}^{n+1}C_3.s_2 + ... + {}^{n+1}C_{n+1}.s_n = 2^n.S_n$.

22. If $(1+x)^{15} = C_0 + C_1.x + C_2.x^2 + ... + C_{15}.x^{15}$, then find the value of : $C_2 + 2C_3 + 3C_4 + ... + 14C_{15}$

23. Prove that, $\dfrac{1}{2}{}^nC_1 - \dfrac{2}{3}{}^nC_2 + \dfrac{3}{4}{}^nC_3 - \dfrac{4}{5}{}^nC_4 + ... + \dfrac{(-1)^{n+1}n}{n+1}.{}^nC_n = \dfrac{1}{n+1}$

24. If $(1+x)^n = \sum_{r=0}^{n} C_r.x^r$, then prove that;

$$\frac{2^2.C_0}{1.2} + \frac{2^3.C_1}{2.3} + \frac{2^4.C_2}{3.4} + ... + \frac{2^{n+2}.C_n}{(n+1)(n+2)} = \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}$$

25. Prove that $\sum_{r=0}^{n} r^2\,{}^nC_r\,p^r q^{n-r} = npq + n^2p^2$, if $p + q = 1$.

26. Prove that : $(n-1)^2.C_1 + (n-3)^2.C_3 + (n-5)^2.C_5 + ... = n(n+1)2^{n-3}$

27. Prove that ${}^nC_r + 2\,{}^{n+1}C_r + 3\,{}^{n+2}C_r + ... + (n+1)\,{}^{2n}C_r = {}^nC_{r+2} + (n+1)\,{}^{2n+1}C_{r+1} - {}^{2n+1}C_{r+2}$

28. Show that, $\sqrt{3} = 1 + \dfrac{1}{3} + \dfrac{1}{3}.\dfrac{3}{6} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9}.\dfrac{7}{12} + ...$

29. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, show that for $m \geq 2$

$C_0 - C_1 + C_2 - .... + (-1)^{m-1}C_{m-1} = (-1)^{m-1}\,{}^{n-1}C_{m-1}$.

30. If $a_0, a_1, a_2, ...$ be the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x, then prove that:

(i) $a_0 a_1 - a_1 a_2 + a_2 a_3 - .... = 0$

(ii) $a_0a_2 - a_1a_3 + a_2a_4 - ..... + a_{2n-2}\, a_{2n} = a_{n+1}$

(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; where $E_1 = a_0 + a_3 + a_6 + ...$; $E_2 = a_1 + a_4 + a_7 + ...$ & $E_3 = a_2 + a_5 + a_8 + ...$

31. $\sum_{k=1}^{3n} {}^{6n}C_{2k-1}\,(-3)^k$ is equal to :

32. Prove that

(i) $n^n\left(\frac{n+1}{2}\right)^{2n}$ is greater than or equal to $\left(\frac{n+1}{2}\right)^3$

(ii) $n^n\left(\frac{n+1}{2}\right)^{2n}$ is greater than or equal to $(n!)^3$

Answers

1.	60	**4.**	2^{2n}	**5.**	$4^n - 3^n$	**11.**	$\frac{1198}{1202}$
14.	$\frac{(2n)!}{(n!)^2}$	**15.**	4	**22.**	212993	**31.**	0