3 Quadratic Equation and Inequations (Inequalities)

Section-A : JEE Advanced/ IIT-JEE

A

1. -5050 2. $-4, 7$ 3. 2 4. 1 5. 4

6. $-1/4, -1/4$ 7. $\dfrac{\left[k+\left(n-\dfrac{k(k+1)}{2}\right)-1\right]!}{\left[n-\dfrac{k(k+1)}{2}\right]!\,(k-1)!}$ 8. 4

B

1. T	2. F	3. T	4. F	5. T	6. F

C

1. (c)	2. (d)	3. (a)	4. (c)	5. (b)	6. (b)	7. (c)	8. (a)
9. (c)	10. (b)	11. (d)	12. (a)	13. (c)	14. (a)	15. (d)	16. (d)
17. (a)	18. (c)	19. (a)	20. (b)	21. (c)	22. (a)	23. (b)	24. (a)
25. (d)	26. (c)	27. (a)	28. (b)	29. (a)	30. (b)	31. (a)	32. (a)
33. (d)	34. (b)	35. (c)	36. (c)	37. (c)	38. (d)	39. (c)	

D

1. (c, d)	2. (a, d)	3. (b)	4. (b)	5. (a,b,c)	6. (d)	7. (a)	8. (a, b, c)
9. (a, d)							

E

1. $\frac{3}{2}$ 3. $\frac{5}{4}$ 4. $a^{-1/2}, a^{-4/3}$ 7. 3

8. $q(r-p)^2 - p(r-p)(s-q)+(s-q)^2$; $(q-s)^2 = (r-p)(ps-qr)$ 10. $[-1, 2)\cup[3, \infty)$

11. $m \in \left(-\infty, \dfrac{-15}{2}\right)\cup(30, \infty)$ 12. $x=1, y=0, z=0, w=0$ 16. $[-1, 1)\cup(2, 4]$

17. $\pm 2, \pm\sqrt{2}$ 18. $\{a \pm a\sqrt{2}, -a \pm a\sqrt{6}\}$ 19. $(-2, -1)\cup\left(-\dfrac{2}{3}, -\dfrac{1}{2}\right)$

20. $-4, -1-\sqrt{3}$ 24. $\alpha^2\beta, \alpha\beta^2$ 25. $a > 1$ 27. 1210

H 1. (b)

I 1. 7 2. 2 3. 8 4. 2

Section-B : JEE Main/ AIEEE

1. (a)	2. (a)	3. (a)	4. (a)	5. (a)	6. (d)	7. (b)	8. (c)
9. (c)	10. (b)	11. (c)	12. (d)	13. (b)	14. (c)	15. (b)	16. (c)
17. (b)	18. (c)	19. (b)	20. (d)	21. (b)	22. (a)	23. (b)	24. (b)
25. (d)	26. (b)	27. (a)	28. (c)	29. (b)	30. (a)	31. (c)	

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. Given polynomial :
$(x-1)(x-2)(x-3)\ldots(x-100)$
$= x^{100} - (1+2+3+\ldots+100)x^{99} + (\ldots)x^{98} + \ldots$
Here coeff. of $x^{99} = -(1+2+3+\ldots+100)$

$$= \frac{-100\times 101}{2} = -5050.$$

2. As p and q are real;and one root is $2 + i\sqrt{3}$, other should be $2 - i\sqrt{3}$
Then $p = -$ (sum of roots) $= -4$,
$q =$ product of roots $= 4 + 3 = 7$.

3. The given equation is $x^2 - 3kx + 2e^{2\ln k} - 1 = 0$
Or $x^2 - 3kx + (2k^2 - 1) = 0$
Here product of roots $= 2k^2 - 1$

$\therefore\ 2k^2 - 1 = 7 \Rightarrow\ k^2 = 4\ \Rightarrow k = 2, -2$

Now for real roots we must have $D \ge 0$

$\Rightarrow\ 9k^2 - 4(2k^2 - 1) \ge 0\ \Rightarrow k^2 + 4 \ge 0$

Which is true for all k. Thus $k = 2, -2$

But for $k = -2$, $\ln k$ is not defined

$\therefore$ Rejecting $k = -2$, we get $k = 2$.

4. $\because x = 1$ reduces both the equations to $1 + a + b = 0$

$\therefore$ 1 is the common root. for $a + b = -1$

$\therefore$ Numerical value of $a + b = 1$

5. $\log_7 \log_5 \left(\sqrt{x+5} + \sqrt{x}\right) = 0$

$\Rightarrow\ \log_5 \left(\sqrt{x+5} + \sqrt{x}\right) = 1$ **NOTE THIS STEP**

$\Rightarrow\ \sqrt{x+5} + \sqrt{x} = 5\ \Rightarrow\ x + 5 = 25 + x - 10\sqrt{x}$

$\Rightarrow\ 2 = \sqrt{x} \Rightarrow x = 4$ which satisfies the given equation.

6. Given $x < 0,\ y < 0$

$x + y + \frac{x}{y} = \frac{1}{2}$ and $(x + y).\frac{x}{y} = -\frac{1}{2}$

Let $x + y = a$ and $\frac{x}{y} = b$ .... (1)

$\therefore$ We get $a + b = \frac{1}{2}$ and $ab = -\frac{1}{2}$

Solving these two, we get $a + \left(-\frac{1}{2a}\right) = \frac{1}{2}$

$\Rightarrow\ 2a^2 - a - 1 = 0\ \Rightarrow a = 1, -1/2 \Rightarrow\ b = -1/2, 1$

$\therefore$ (1) $\Rightarrow x + y = 1$ and $\frac{x}{y} = -\frac{1}{2}$

or $x + y = \frac{-1}{2}$ and $\frac{x}{y} = 1$ But $x, y < 0$

$\therefore x + y < 0 \Rightarrow x + y = \frac{-1}{2}$ and $\frac{x}{y} = 1$

On solving, we get $x = -1/4$ and $y = -1/4$.

7. We have

$x_1 + x_2 + \ldots\ldots + x_k = n$ .... (1)

where $x_1 \ge 1, x_2 \ge 2, x_3 \ge 3, \ldots\ldots, x_k \ge k$; all integers

Let $y_1 = x_1 - 1, y_2 = x_2 - 2, \ldots\ldots, y_k = x_k - k$

so that $y_1, y_2, \ldots\ldots, y_k \ge 0$

Substituting the values of $x_1, x_2, \ldots\ldots x_k$ in equation .. (1)

We get $y_1 + y_2 + \ldots y_k = n - (1 + 2 + 3 \ldots + k)$

$$= n - \frac{k(k+1)}{2} \quad .... (2)$$

Now keeping in mind that number of solutions of the equation

$\alpha + 2\beta + 3\gamma + \ldots\ldots q\theta = n$

for $\alpha, \beta, \gamma, \ldots\ldots \theta \in I$ and each is ≥ 0, is given by coeff of x^n in

$(1 + x + x^2 + \ldots\ldots)(1 + x^2 + x^4 + \ldots\ldots)$

$(1 + x^3 + x^6 + \ldots\ldots)\ldots\ldots(1 + x^q + x^{2q} + \ldots\ldots)$

We find that no. of solutions of equation (2)

$=$ coeff of $x^{n - \frac{k(k+1)}{2}}$ in $(1 + x + x^2 + \ldots\ldots)^k$

NOTE THIS STEP

$=$ coeff of $x^{n - \frac{k(k+1)}{2}}$ in $(1 - x)^{-k}$

$=$ coeff of $x^{n - \frac{k(k+1)}{2}}$ in $(1 + {}^kC_1 x + {}^{k+1}C_2 x^2$

$+ {}^{k+2}C_3 x^3 + \ldots\ldots) = {}^{k + \left(n - \frac{k(k+1)}{2}\right) - 1}C_{n - \frac{k(k+1)}{2}}$

$$= \frac{\left[k + \left(n - \frac{k(k+1)}{2}\right) - 1\right]!}{\left[n - \frac{k(k+1)}{2}\right]!(k-1)!}.$$

8. $|x - 2|^2 + |x - 2| - 2 = 0$

Case 1. $x \ge 2$

$\Rightarrow\ (x - 2)^2 + (x - 2) - 2 = 0$

$\Rightarrow\ x^2 - 3x = 0\ \Rightarrow x(x - 3) = 0$

$\Rightarrow\ x = 0, 3$ (0 is rejected as $x \ge 2$)

$\Rightarrow\ x = 3$ ....(1)

Case 2. $x < 2$

$\{-(x - 2)\}^2 - (x - 2) - 2 = 0$

$\Rightarrow\ x^2 + 4 - 4x - x = 0\ \Rightarrow\ (x - 1)(x - 4) = 0$

$\Rightarrow\ x = 1, 4$ (4 is rejected as $x < 2$)

$\Rightarrow\ x = 1$ ....(2)

Therefore, the sum of the roots is 3 + 1 = 4.

B. True/False

1. Consider n numbers, namely 1, 2, 3, 4, ...n.

KEY CONCEPT : Now using A.M. > G.M. for distinct numbers, we get

$\frac{1 + 2 + 3 + 4 \ldots + n}{n} > (1.2.3.4..n)^{1/n}$

$\Rightarrow\ \frac{n(n+1)}{2n} > (n!)^{1/n} \Rightarrow (n!)^{1/n} < \frac{n+1}{2}\ \therefore$ *True* .

2. $2x^2 + 3x + 1 = 0 \Rightarrow x = -1, -1/2$ both are rational

$\therefore$ Statement is FALSE.

3. $f(x) = (x-a)(x-c) + 2(x-b)(x-d)$.
$f(a) = +ve; f(b) = -ve; f(c) = -ve; f(d) = +ve$
$\therefore$ There exists two real and distinct roots one in the interval (a, b) and other in (c, d). Hence, (True).

4. Consider $N = n_1 + n_2 + n_3 + ... + n_p$, where N is an even number. Let k numbers among these p numbers be odd, then $p - k$ are even numbers. Now sum of $(p-k)$ even numbers is even and for N to be an even number, sum of k odd numbers must be even which is possible only when k is even.
$\therefore$ The given statement is false.

5. $P(x).Q(x) = (ax^2 + bx + c)(-ax^2 + bx + c)$
$\Rightarrow D_1 = b^2 - 4ac$ and $D_2 = b^2 + 4ac$
clearly, $D_1 + D_2 = 2b^2 \geq 0$
$\therefore$ atleast one of D_1 and D_2 is (+ ve). Hence, atleast two real roots.
Thus, (True)

6. As x and y are positive real numbers and m and n are positive integers

$$\therefore \frac{1+x^{2n}}{2} \geq (1 \times x^{2n})^{1/2} \text{ and } \frac{1+y^{2m}}{2} \geq (1 \times y^{2m})^{1/2}$$

{For two +ve numbers A.M. $\geq$ G.M.}

$$\Rightarrow \left(\frac{1+x^{2n}}{2}\right) \geq x^n \quad ....(1)$$

$$\text{and } \left(\frac{1+y^{2m}}{2}\right) \geq y^m \quad ....(2)$$

Multiplying (1) and (2), we get

$$\frac{(1+x^{2n})(1+y^{2m})}{4} \geq x^n y^m \Rightarrow \frac{1}{4} \geq \frac{x^n y^m}{(1+x^{2n})(1+y^{2m})}$$

Hence the statement is false.

C. MCQs with ONE Correct Answer

1. **(c)** ℓ, m, n are real, $\ell \neq m$
Given equation is

$$(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$$

$$D = 25(\ell + m)^2 + 8(\ell - m)^2 > 0, \ell, m \in R$$

$\therefore$ Roots are real and unequal.

2. **(d)** The given equations are
$x + 2y + 2z = 1$ ....(1)
and $2x + 4y + 4z = 9$ ....(2)
Subtracting (1) × (2) from (2), we get 0 =7 (not possible)
$\therefore$ No solution.

3. **(a)** $u = x^2 + 4y^2 + 9z^2 - 6yx - 3zx - 2xy$

$$= \frac{1}{2}[2x^2 + 8y^2 + 18z^2 - 12yz - 6zx - 4xy]$$

$$= \frac{1}{2}[(x^2 - 4xy + 4y^2) + (4y^2 + 9z^2 - 12yz) + (x^2 + 9z^2 - 6zx)]$$

$$= \frac{1}{2}[(x-2y)^2 + (2y-3z)^2 + (3z-x)^2] \geq 0$$

$\therefore$ u is always non-negative.

4. **(c)** As $a, b, c > 0$, a, b, c should be real (note that order relation is not defined in the set of complex numbers)
$\therefore$ Roots of equation are either real or complex conjugate.
Let α, β be the roots of $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} = -ve, \quad \alpha\beta = \frac{c}{a} = +ve$$

$\Rightarrow$ Either both α, β are – ve (if roots are real) or both α, β have – ve real parts (if roots are complex conjugate)

5. **(b)** The given equation is

$$(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) = 0$$

$$\Rightarrow 3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$$

Discriminant $= 4(a+b+c)^2 - 12(ab+bc+ca)$

$$= 4[a^2 + b^2 + c^2 - ab - bc - ca]$$

$$= 2[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0 \quad \forall a, b, c$$

$\therefore$ Roots of given equation are always real.

6. **(b)** Let $y = 2\log_{10} x - \log_x 0.01$

$$= 2\log_{10} x - \frac{\log_{10} 0.01}{\log_{10} x} = 2\log_{10} x + \frac{2}{\log_{10} x}$$

$$= 2\left[\log_{10} x + \frac{2}{\log_{10} x}\right]$$

[Here $x > 1 \Rightarrow \log_{10} x > 0$]
Now since sum of a real + ve number and its reciprocal is always greater than or equal to 2.
$\therefore$ $y \geq 2 \times 2 \Rightarrow y \geq 4$, $\therefore$ Least value of y is 4.

7. **(c)** As $(x^2 + px + 1)$ is a factor of $ax^3 + bx + c$, we can assume that zeros of $x^2 + px + 1$ are α, β and that of $ax^3 + bx + c$ be α, β, γ so that
$\alpha + \beta = -p$ .... (i)
$\alpha\beta = 1$ .... (ii)
and $\alpha + \beta + \gamma = 0$ .... (iii)

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{b}{a} \quad .... (iv)$$

$$\alpha\beta\gamma = \frac{-c}{a} \quad .... (v)$$

Solving (ii) and (v) we get $\gamma = -c/a$.
Also from (i) and (iii) we get $\gamma = p$
$\therefore$ $p = \gamma = -c/a$
Using equations (i), (ii) and (iv) we get

$$1 + \gamma(-p) = \frac{b}{a}$$

$$\Rightarrow 1 + \left(-\frac{c}{a}\right)\left(\frac{c}{a}\right) = \frac{b}{a} \quad \text{(using } \gamma = p = -c/a)$$

$$\Rightarrow 1 - \frac{c^2}{a^2} = \frac{b}{a} \Rightarrow a^2 - c^2 = ab$$

$\therefore$ (c) is the correct answer.

8. (a) $|x|^2 - 3|x| + 2 = 0$

Case I : $x < 0$ then $|x| = -x$

$\Rightarrow$ $x^2 + 3x + 2 = 0 \Rightarrow (x+1)(x+2) = 0$

$x = -1, -2$ (both acceptable as < 0)

Case II: $x > 0$ then $|x| = x$

$\Rightarrow$ $x^2 - 3x + 2 = 0 \Rightarrow (x-1)(x-2) = 0$

$x = 1, 2$ (both acceptable as > 0)

$\therefore$ There are 4 real solutions.

9. (c) Let the distance of school from $A = x$

$\therefore$ The distance of the school form $B = 60 - x$

Total distance covered by 200 students

$= 2[150x + 50(60 - x)] = 2[100x + 3000]$

This is min., when $x = 0$

$\therefore$ school should be built at town A.

10. (b) if $p = 5, q = 3, r = 2$

max $(p, q) = 5$; max $(p, q, r) = 5$

$\Rightarrow$ max $(p, q) =$ max (p, q, r)

$\therefore$ (a) is not true. Similarly we can show that (c) is not true.

Also min (p,q) $= \frac{1}{2}(p + q - |p - q|)$

Let $p < q$ then LHS $= p$

and R.H.S. $= \frac{1}{2}(p + q - q + p) = p$

Similarly, we can prove that (*b*) is true for $q < p$ too.

11. (d) Given expression $x^{12} - x^9 + x^4 - x + 1 = f(x)$ (say)

For x < 0 put $x = -y$ where $y > 0$

then we get $f(x) = y^{12} + y^9 + y^4 + y + 1 > 0$ for $y > 0$

For $0 < x < 1$, $x^9 < x^4 \Rightarrow -x^9 + x^4 > 0$

Also $1 - x > 0$ and $x^{12} > 0$

$\Rightarrow$ $x^{12} - x^9 + x^4 + 1 - x > 0 \Rightarrow f(x) > 0$

For $x > 1$

$f(x) = x(x^3 - 1)(x^8 + 1) + 1 > 0$

So $f(x) > 0$ for $-\infty < x < \infty$.

12. (a) Given equation is $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$

Clearly $x \neq 1$ for the given eq. to be defined. If $x - 1 \neq 0$, we can cancel the common term $\frac{-2}{x-1}$ on both sides to get $x = 1$, but it is not possible. So given eq. has no roots.

$\therefore$ (*a*) is the correct answer.

13. (c) Given that $a^2 + b^2 + c^2 = 1$ ....(1)

We know $(a + b + c)^2 \geq 0$

$\Rightarrow$ $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \geq 0$

$\Rightarrow$ $2(ab + bc + ca) \geq -1$ [Using (1)]

$\Rightarrow$ $ab + bc + ca \geq -1/2$ ....(2)

Also we know that

$\frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \geq 0$

$\Rightarrow$ $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$

$\Rightarrow$ $ab + bc + ca \leq 1$ [Using (1)] ....(2)

Combining (2) and (3), we get

$-1/2 \leq ab + bc + ca \leq 1 \therefore ab + bc + ca \in [-1/2, 1]$

$\therefore$ (c) is the correct answer.

14. (a) First of all for log $(x - 1)$ to be defined, $x - 1 > 0$

$\Rightarrow$ $x > 1$ ....(1)

Now, $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow$ $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow$ $\log_{0.3}(x-1) < \frac{1}{2}\log_{0.3}(x-1)$

$\Rightarrow$ $2\log_{0.3}(x-1) < \log_{0.3}(x-1)$

$\Rightarrow$ $\log_{0.3}(x-1)^2 < \log_{0.3}(x-1)$

$\Rightarrow$ $(x-1)^2 > (x-1)$ **NOTE THIS STEP**

[The inequality is reversed since base lies between 0 and 1]

$\Rightarrow$ $(x-1)^2 - (x-1) > 0 \Rightarrow (x-1)(x-2) > 0$ ....(2)

Combining (1) and (2) we get $x > 2$

$\therefore$ $x \in (2, \infty)$

15. (d) From the first method,

$q = \alpha\beta, r = \alpha^4 + \beta^4$ ...(3)

Product of the roots of the equation

$x^2 - 4qx + (2q^2 - r) = 0$

$= 2q^2 - r = 2\alpha^2\beta^2 - \alpha^4 - \beta^4 = -(\alpha^2 - \beta^2)^2$ [From (3)]

$= -$ (positive quantity) $= -$ ve quantity

$\Rightarrow$ one root is positive and other is negative.

16. (d) **KEY CONCEPT :** If $f(\alpha)$ and $f(\beta)$ are of opposite signs then there must lie a value γ between α and β such that $f(\gamma) = 0$.

a, b, c are real numbers and $a \neq 0$.

As α is a root of $a^2x^2 + bx + c = 0$

$\therefore a^2\alpha^2 + b\alpha + c = 0$ ....(1)

Also β is a root of $a^2x^2 - bx - c = 0$ $\therefore$

$a^2\beta^2 - b\beta - c = 0$ .... (2)

Now, let $f(x) = a^2x^2 + 2bx + 2c$

Then $f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = a^2\alpha^2 + 2(b\alpha + c)$

$= a^2\alpha^2 + 2(-a^2\alpha^2)$ [Using eq. (1)]

$= -a^2\alpha^2$.

and $f(\beta) = a^2\beta^2 + 2b\beta + 2c$

$= a^2\beta^2 + 2(b\beta + c)$

$= a^2\beta^2 + 2(a^2\beta^2)$ [Using eq. (2)]

$= 3a^2\beta^2 > 0$.

Since $f(\alpha)$ and $f(\beta)$ are of opposite signs and γ is a root of equation $f(x) = 0$

$\therefore$ γ must lie between α and β

Thus $\alpha < \gamma < \beta$. $\therefore$ (d) is the correct option.

17. (a) The given eq. is $\sin(e^x) = 5^x + 5^{-x}$

We know 5^x and 5^{-x} both are +ve real numbers using

AM $\geq$ GM $\therefore 5^x + \frac{1}{5^x} \geq 2 \Rightarrow 5^x + 5^{-x} \geq 2$

$\therefore$ R.H.S. of given eq. ≥ 2

While $\sin e^x \in [-1,1]$ i.e. LHS $\in [-1,1]$

$\therefore$ The equation is not possible for any real value of x. Hence (a) is the correct answer.

18. (c) α, β are roots of the equation $(x-a)(x-b) = c, c \neq 0$

$\therefore$ $(x-a)(x-b) - c = (x-\alpha)(x-\beta)$

$\Rightarrow$ $(x-\alpha)(x-\beta) + c = (x-a)(x-b)$

$\Rightarrow$ roots of $(x-\alpha)(x-\beta) + c = 0$ are a and b.

$\therefore$ (c) is the correct option.

19. (a) We have

$$y = 5x^2 + 2x + 3 = 5\left(x + \frac{1}{5}\right)^2 + \frac{14}{5} > 2, \ \forall x \in R$$

while $y = 2\sin x \leq 2, \forall x \in R$

$\Rightarrow$ The two curves do not meet at all.

20. (b) For real roots $q^2 - 4pr \geq 0$

$\Rightarrow$ $\left(\frac{p+r}{2}\right)^2 - 4pr \geq 0$ $\quad$ ($\because p, q, r$ are in A.P.)

$\Rightarrow$ $p^2 + r^2 - 14pr \geq 0 \Rightarrow \frac{p^2}{r^2} - 14\frac{p}{r} + 1 \geq 0$

$\Rightarrow$ $\left(\frac{p}{r} - 7\right)^2 - 48 \geq 0 \Rightarrow \left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$

21. (c) For the equation $px^2 + qx + 1 = 0$ to have real roots

$D \geq 0 \quad \Rightarrow q^2 \geq 4p$

If $p = 1$ then $q^2 \geq 4 \Rightarrow q = 2, 3, 4$

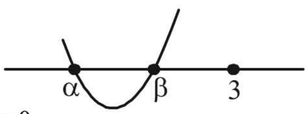
If $p = 2$ then $q^2 \geq 8 \Rightarrow q = 3, 4$

If $p = 3$ then $q^2 \geq 12 \Rightarrow q = 4$

If $p = 4$ then $q^2 \geq 16 \Rightarrow q = 4$

$\therefore$ No. of req. equations = 7.

22. (a) **KEY CONCEPT :** If both roots of a quadratic equation $ax^2 + bx + c = 0$ are less than k then $af(k) > 0, D \geq 0, \alpha + \beta < 2k$.



$f(x) = x^2 - 2ax + a^2 + a - 3 = 0$,

$f(3) > 0, \alpha + \beta < 6, D \geq 0$.

$\Rightarrow$ $a^2 - 5a + 6 > 0, a < 3, -4a + 12 \geq 0$

$\Rightarrow$ $a < 2$ or $a > 3, a < 3, a < 3 \Rightarrow a < 2$.

23. (b) Given $c < 0 < b$ and $\alpha + \beta = -b$ ....(1)

$\alpha\beta = c$ ....(2)

From (2), $c < 0 \Rightarrow \alpha\beta < 0 \Rightarrow$ either α is -ve or β is -ve and second ;quantity is positive.

from (1), $b > 0 \Rightarrow -b < 0 \Rightarrow \alpha + \beta < 0 \Rightarrow$ the sum is negative

$\Rightarrow$ modules of nengative quantity is > modulus of positive quantity but $\alpha < \beta$ is given. Therefore, it is clear that α is negative and β is positive and modulus of α is greater than modulus of $\beta \Rightarrow \alpha < 0 < \beta < |\alpha|$

24. (a) As A.M. $\geq$ G.M. for positive real numbers, we get

$$\frac{(a+b)+(c+d)}{2} \geq \sqrt{(a+b)(c+d)} \Rightarrow M \leq 1$$

(Putting values)

Also $(a+b)(c+d) > 0$ $\quad$ [$\therefore$ a,b,c,d > 0]

$\therefore$ $0 \leq M \leq 1$

25. (d) The given equation is $(x-a)(x-b) - 1 = 0, b > a$.

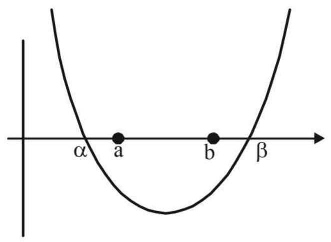
or $x^2 - (a+b)x + ab - 1 = 0$

Let $f(x) = x^2 - (a+b)x + ab - 1$

Since coeff. of x^2 i.e. $1 > 0$, $\therefore$ it represents upward parabola, intersecting x - axis at two points. (corresponding to two real roots, D being +ve). Also $f(a) = f(b) = -1 \Rightarrow$ curve is below x-axis at a and b

$\Rightarrow$ a and b both lie between the roots.

Thus the graph of given eq^n is as shown.



from graph it is clear that one root of the equation lies in $(-\infty, a)$ and other in (b, ∞).

26. (c) Let α, α^2 be the roots of $3x^2 + px + 3$.

$\therefore$ $\alpha + \alpha^2 = -p/3$ and $\alpha^3 = 1$

$\Rightarrow$ $(\alpha - 1)(\alpha^2 + \alpha + 1) = 0 \Rightarrow \alpha = 1$ or $\alpha^2 + \alpha = -1$

If $\alpha = 1, p = -6$ which is not possible as $p > 0$

If $\alpha^2 + \alpha = -1 \Rightarrow -p/3 = -1 \Rightarrow p = 3$.

27. (a) We have

$$\frac{(a_1 + a_2 + ... + a_{n-1} + 2a_n)}{n} \geq (a_1 a_2 ... a_{n-1} 2a_n)^{1/n}$$

[Using A.M. $\geq$ G.M.]

$\Rightarrow$ $a_1 + a_2 + a_3 + ... + a_{n-1} + 2a_n \geq n(2c)^{1/n}$

28. (b) For $x < -2$, $|x+2| = -(x+2)$ and the inequality becomes

$x^2 + x + 2 + x > 0 \quad \Rightarrow (x+1)^2 + 1 > 0$

which is valid $\forall \ x \in R$ but $x < -2$

$\therefore$ $x \in (-\infty, -2)$ ....(1)

For $x \geq 2, |x+2| = x+2$ and the inequality becomes

$x^2 - x - 2 + x > 0 \Rightarrow x^2 > 2 \Rightarrow x > \sqrt{2}$ or $x < -\sqrt{2}$

i.e., $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

but $x \geq -2 \Rightarrow x \in [-2, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ ....(2)

From (1) and (2)

$x \in (-\infty, -2) \cup [-2, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

$\Rightarrow$ $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

29. (a) Let $a = \sqrt{x^2 + x}$ and $b = \frac{\tan^2 \alpha}{\sqrt{x^2 + x}}$

then using AM $\geq$ GM, we get $\frac{a+b}{2} \geq \sqrt{ab}$

$\Rightarrow a+b \geq 2\sqrt{ab}$

$\Rightarrow \sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}} \geq 2\sqrt{\tan^2\alpha} = 2\tan\alpha$

$[\because \alpha \in (0, \pi/2)]$

30. **(b)** **KEY CONCEPT :** $f(x) = ax^2 + bx + c$ has same sign as that of a if $D < 0$.

$x^2 + 2ax + 10 - 3a > 0 \,\forall x$

$\Rightarrow D < 0 \Rightarrow 4a^2 - 4(10-3a) < 0 \Rightarrow a^2 + 3a - 10 < 0$

$\Rightarrow (a+5)(a-2) < 0 \Rightarrow a \in (-5, 2)$

31. **(a)** $x^2 + px + q = 0$

Let roots be α and α^2

$\Rightarrow \alpha + \alpha^2 = -p, \alpha\alpha^2 = q \Rightarrow \alpha = q^{1/3}$

$\therefore (q)^{1/3} + (q^{1/3})^2 = -p$

Taking cube of both sides, we get

$q + q^2 + 3q(q^{1/3} + q^{2/3}) = -p^3$

$\Rightarrow q + q^2 - 3pq = -p^3 \Rightarrow p^3 + q^2 - q(3p-1) = 0$

32. **(a)** $\because$ a, b, c are sides of a triangle and $a \neq b \neq c$

$\therefore |a-b| < |c| \Rightarrow a^2 + b^2 - 2ab < c^2$

Similarly, we have

$b^2 + c^2 - 2bc < a^2$; $c^2 + a^2 - 2ca < b^2$

On adding, we get

$a^2 + b^2 + c^2 < 2(ab + bc + ca)$

$\Rightarrow \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2$ ....(1)

$\because$ Roots of the given equation are real

$\therefore (a+b+c)^2 - 3\lambda(ab+bc+ca) \geq 0$

$\Rightarrow \frac{a^2+b^2+c^2}{ab+bc+ca} \geq 3\lambda - 2$ ....(2)

From (1) and (2), we get $3\lambda - 2 < 2 \Rightarrow \lambda < \frac{4}{3}$.

33. **(d)** As α, β are the roots of $x^2 - px + r = 0$

$\therefore \alpha + \beta = p$ ....(1)

and $\alpha\beta = r$ ....(2)

Also $\frac{\alpha}{2}, 2\beta$ are the roots of $x^2 - qx + r = 0$

$\therefore \frac{\alpha}{2} + 2\beta = q$ or $\alpha + 4\beta = 2q$ ....(3)

Solving (1) and (3) for α and β, we get

$\beta = \frac{1}{3}(2q - p)$ and $\alpha = \frac{2}{3}(2q - q)$

Substituting values of α and β, in equation (2),

we get $\frac{2}{9}(2p-q)(2q-p) = r$.

34. **(b)** Given that $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$

$\Rightarrow (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = q$

$\Rightarrow -p^3 - 3\alpha\beta(-p) = q \Rightarrow \alpha\beta = \frac{p^3+q}{3p}$

Now for required quadratic equation,

sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{p^2 - 2\left(\frac{p^3+q}{3p}\right)}{\frac{p^3+q}{3p}}$

$= \frac{3p^3 - 2p^3 - 2q}{p^3+q} = \frac{p^3 - 2q}{p^3+q}$

and product of roots $= \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$

$\therefore$ Required equation is $x^2 - \left(\frac{p^3-2q}{p^3+q}\right)x + 1 = 0$

or $(p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$

35. **(c)** We have $(2x)^{\ell n 2} = (3y)^{\ell n 3}$

$\Rightarrow \ell n2 . \ell n2x = \ell n3 . \ell n3y$

$\Rightarrow \ell n2 . \ell n2x = \ell n3 . (\ell n3 + \ell ny)$ ...(1)

Also given $3^{\ell nx} = 2^{\ell ny}$

$\Rightarrow \ell nx . \ell n3 = \ell ny . \ell n2 \Rightarrow \ell ny = \frac{\ell nx . \ell n3}{\ell n2}$

Substituting this value of ℓny in equation (1), we get

$\ell n2 . \ell n2x = \ell n3 \left[\ell n3 + \frac{\ell nx . \ell n3}{\ell n2}\right]$

$\Rightarrow (\ell n2)^2 \ell n2x = (\ell n3)^2 \ell n2 + (\ell n3)^2 \ell nx$

$\Rightarrow (\ell n2)^2 \ell n2x = (\ell n3)^2 (\ell n2 + \ell nx)$

$\Rightarrow (\ell n2)^2 \ell n2x - (\ell n3)^2 \ell n2x = 0$

$\Rightarrow [(\ell n2)^2 - (\ell n3)^2] \ell n2x = 0 \Rightarrow \ell n2x = 0$

$\Rightarrow 2x = 1$ or $x = \frac{1}{2}$

36. **(c)** $\because \alpha, \beta$ are the roots of $x^2 - 6x - 2 = 0$

$\therefore \alpha^2 - 6\alpha - 2 = 0$

$\Rightarrow \alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0$

$\Rightarrow \alpha^{10} - 2\alpha^8 = 6\alpha^9$ ...(1)

Similarly $\beta^{10} - 2\beta^8 = 6\beta^9$ ...(2)

From equation (1) and (2)

$\alpha^{10} - \beta^{10} - 2(\alpha^8 - \beta^8) = 6(\alpha^9 - \beta^9)$

$\Rightarrow a_{10} - 2a_8 = 6a_9 \Rightarrow \frac{a_{10} - 2a_8}{2a_9} = 3$

37. **(b)** Let α be the common root of given equations, then

$\alpha^2 + b\alpha - 1 = 0$ ...(1)

and $\alpha^2 + \alpha + b = 0$ ...(2)

Subtracting (2) from (1), we get

$(b-1)\alpha - (b+1) = 0$

or $\alpha = \dfrac{b+1}{b-1}$

Substituting this value of α in equation (1), we get

$$\left(\frac{b+1}{b-1}\right)^2 + b\left(\frac{b+1}{b-1}\right) - 1 = 0 \text{ or } b^3 + 3b = 0$$

$\Rightarrow b = 0, i\sqrt{3}, -i\sqrt{3}$

38. **(d)** Quadratic equation with real coefficients and purely imaginary roots can be considered as

$p(x) = x^2 + a = 0$ where $a > 0$ and $a \in R$

The $p[p(x)] = 0 \Rightarrow (x^2 + a)^2 + a = 0$

$\Rightarrow x^4 + 2ax^2 + (a^2 + a) = 0$

$$\Rightarrow x^2 = \frac{-2a \pm \sqrt{4a^2 - 4a^2 - 4a}}{2}$$

$\Rightarrow x^2 = -a \pm \sqrt{a}\, i$

$\Rightarrow x = \sqrt{-a \pm \sqrt{a}\, i} = \alpha \pm i\beta$ where $\alpha, \beta \neq 0$

$\therefore p[p(x)] = 0$ has complex roots which are neither purely real nor purely imaginary.

39. **(c)** $x^2 - 2x \sec\theta + 1 = 0 \Rightarrow x = \sec\theta \pm \tan\theta$

and $x^2 + 2x\tan\theta - 1 = 0 \Rightarrow x = -\tan\theta \pm \sec\theta$

$\because -\dfrac{\pi}{6} < \theta < -\dfrac{\pi}{12}$

$\Rightarrow \sec\dfrac{\pi}{6} > \sec\theta > \sec\dfrac{\pi}{12}$

and $-\tan\dfrac{\pi}{6} < \tan\theta < -\dfrac{\tan\pi}{12}$

also $\tan\dfrac{\pi}{12} < -\tan\theta < \tan\dfrac{\pi}{6}$

α_1, β_1 are roots of $x^2 - 2x\sec\theta + 1 = 0$

and $\alpha_1 > \beta_1$

$\therefore \alpha_1 = \sec\theta - \tan\theta$ and $\beta_1 = \sec\theta + \tan\theta$

α_2, β_2 are roots of $x^2 + 2x\tan\theta - 1 = 0$ and $\alpha_2 > \beta_2$

$\therefore \alpha_2 = -\tan\theta + \sec\theta$, $\beta_2 = -\tan\theta - \sec\theta$

$\therefore \alpha_1 + \beta_2 = \sec\theta - \tan\theta - \tan\theta - \sec\theta = -2\tan\theta$

D. MCQs with ONE or MORE THAN ONE Correct

1. **(c,d)** Let $y = \dfrac{(x-a)(x-b)}{(x-c)}$

$\Rightarrow (x-c)y = x^2 - (a+b)x + ab$

$\Rightarrow x^2 - (a+b+y)x + ab + cy = 0$

Here, $\Delta = (a+b+y)^2 - 4(ab+cy)$

$= y^2 + 2y(a+b-2c) + (a-b)^2$

Since x is real and y assumes all real values.

$\therefore \quad \Delta \geq 0$ for all real values of y

$\Rightarrow \quad y^2 + 2y(a+b-2c) + (a-b)^2 \geq 0$

Now we know that the sign of a quad is same as of coeff of y^2 provided its descriminant $B^2 - 4AC < 0$

This will be so if, $4(a+b-2c)^2 - 4(a-b)^2 < 0$

or $4(a+b-2c+a-b)(a+b-2c-a+b) < 0$

$\Rightarrow 16(a-c)(b-c) < 0$

$\Rightarrow 16(c-a)(c-b) < 0$ ....(1)

Now,

If $a < b$ then from inequation (1), we get $c \in (a, b)$

$\Rightarrow a < c < b$

or If $a > b$ then from inequation (1) we get, $c \in (b, a)$

$\Rightarrow b < c < a$ or $a > c > b$

Thus, we observe that both (c) and (d) are the correct answer.

2. **(a, d)** **KEY CONCEPT :** Wavy curve method :

Let $f(x) = (x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$

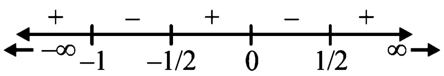
To find sign of $f(x)$, plot $\alpha_1, \alpha_2, ... \alpha_n$ on number line in ascending order of magnitude. Starting from right extreme put + ve, –ve signs alternately. $f(x)$ is positive in the intervals having + ve sign and negative in the intervals having –ve sign.

We have,

$$f(x) = \frac{2x-1}{2x^3+3x^2+x} = \frac{2x-1}{x(2x+1)(x+1)}$$

NOTE THIS STEP : Critical points are $x = 1/2, 0, -1/2, -1$

On number line by wavy method, we have



For $f(x) > 0$, when

$x \in (-\infty, -1) \cup (-1/2, 0) \cup (1/2, \infty)$

Clearly S contains $(-\infty, -3/2)$ and $(1/2, 3)$

3. **(b)** Given that a, b, c are distinct +ve numbers. The expression whose sign is to be checked is $(b+c-a)(c+a-b)(a+b-c) - abc$.

As this expression is symmetric in a, b, c, without loss of generality, we can assume that $a < b < c$.

Then $c - a = +$ ve and $c - b = +$ ve

$\therefore \quad b + c - a = +$ ve and $c + a - b = +$ ve

But $a + b - c$ may be + ve or – ve.

Case I : If $a + b - c = +$ ve then we can say that a, b, c, are such that sum of any two is greater than the 3rd. Consider

$x = a + b - c, \quad y = b + c - a, \quad z = c + a - b$

then x, y, z all are + ve.

and then $a = \dfrac{x+z}{2}, b = \dfrac{y+x}{2}, c = \dfrac{z+y}{2}$

Now we know that A.M. > G.M. for distinct real numbers

$\therefore \quad \dfrac{x+y}{2} > \sqrt{xy}, \dfrac{y+z}{2} > \sqrt{yz}, \dfrac{z+x}{2} > \sqrt{zx}$

$\Rightarrow \left(\frac{x+y}{2}\right)\left(\frac{y+z}{2}\right)\left(\frac{z+x}{2}\right) > xyz$

$\Rightarrow abc > (a+b-c)(b+c-a)(c+a-b)$

$\Rightarrow (b+c-a)(c+a-b)(a+b-c) - abc < 0$

Case II : If $a+b-c =$ -ve then

$(b+c-a)(c+a-b)(a+b-c) - abc$

$= (+\text{ve})(+\text{ve})(-\text{ve}) - (+\text{ve})$

$= (-\text{ve}) - (+\text{ve}) = (-\text{ve})$

$\Rightarrow (b+c-a)(c+a-b)(a+b-c) < abc$

Hence in either case given expression is –ve.

4. **(b)** Given that a, b, c, d, p are real and distinct numbers such that

$(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \le 0$

$\Rightarrow (a^2p^2+b^2p^2+c^2p^2) - (2abp+2bcp+2cdp) + (b^2+c^2+d^2) \le 0$

$\Rightarrow (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) \le 0$

$\Rightarrow (ap-b)^2 + (bp-c)^2 + (cp-d)^2 \le 0$

Being sum of perfect squares, LHS can never be –ve, therefore the only possibility is

$(ap-b)^2 + (bp-c)^2 + (cp-d)^2 = 0$

Which is possible only when each term is zero individually i.e.

$ap-b=0;\ bp-c=0;\ cp-d=0$

$\Rightarrow \frac{b}{a} = p; \frac{c}{b} = p; \frac{d}{c} = p \Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = p$

$\Rightarrow a, b, c, d$ are in G.P.

5. **(a, b, c)** The given equation is, $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$

For $x > 0$, taking log on both sides to the base x, we get

$\frac{3}{4}(\log_2 x)^2 + (\log_2 x) - \frac{5}{4} = \log_x \sqrt{2} = \frac{1}{2}\log_x 2$

Let $\log_2 x = y$, then we get, $\frac{3}{4}y^2 + y - \frac{5}{4} = \frac{1}{2y}$

$\Rightarrow 3y^3 + 4y^2 - 5y - 2 = 0$

$\Rightarrow (y-1)(y+2)(3y+1) = 0 \Rightarrow y = 1, -2, -1/3$

$\Rightarrow \log_2 x = 1, -2, -1/3 \Rightarrow x = 2, 2^{-2}, 2^{-1/3}$

$\Rightarrow x = 2, \frac{1}{4}, \frac{1}{2^{1/3}}$ (All accepted as > 0)

$\therefore$ There are three real solution in which one is irrational.

6. **(d)** Let $x_1, x_2, ..., x_n$ be the n +ve numbers According to the question,

$x_1 x_2 x_3 .... x_n = 1$ ....(1)

We know for +ve no.'s A.M. ≥ G.M.

$\Rightarrow \frac{x_1 + x_2 + ... + x_n}{n} \ge \sqrt[n]{x_1 x_2 ... x_n}$

$\Rightarrow \frac{x_1 + x_2 + ... + x_n}{n} \ge 1$ [Using eq. (1)]

$\Rightarrow x_1 + x_2 + ... + x_n \ge n$.

7. **(a)** We have $240 = 2^4.3.5.$

Divisors of 240 are

1, 2,	4, 8,	16
3, 6,	12, 24,	48
5, 10,	20, 40,	80,
15, 30,	60, 120,	240

Out of these divisors just 4 divisors viz., 2, 6, 10, 30 are of the form 4n +2.

8. **(a, b, c)**

$3^x = 4^{x-1} \Rightarrow x\log 3 = 2(x-1)\log 2$

$\Rightarrow x = \frac{2\log 2}{2\log 2 - \log 3} \Rightarrow x = \frac{2\log_3 2}{2\log_3 2 - 1} = \frac{2}{2 - \log_2 3}$

Also $x = \frac{1}{1 - \frac{1}{2}\log_2 3} = \frac{1}{1 - \log_4 3}$

9. **(a, d)** $\alpha x^2 - x + \alpha = 0$ has distinct real roots.

$\therefore D > 0 \Rightarrow 1 - 4\alpha^2 > 0$

$\Rightarrow \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ ...(i)

Also $|x_1 - x_2| < 1$

$\Rightarrow (x_1 - x_2)^2 < 1 \Rightarrow (x_1 + x_2)^2 - 4x_1x_2 < 1$

$\Rightarrow \frac{1}{\alpha^2} - 4 < 1 \Rightarrow \frac{1}{\alpha^2} < 5$ or $\alpha^2 > \frac{1}{5}$

$\Rightarrow \alpha \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$ ...(ii)

Combining (i) and (ii)

$S = \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

$\therefore$ Subsets of S can be $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ and $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$.

E. Subjective Problems

1. $4^x - 3^{x-1/2} = 3^{x+1/2} - \frac{(2^2)^x}{2}$

$\Rightarrow 4^x - \frac{3^x}{\sqrt{3}} = 3^x\sqrt{3} - \frac{4^x}{2}$

$\Rightarrow \frac{3}{2}\cdot 4^x = 3^x\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) \Rightarrow \frac{3}{2}\cdot 4^x = 3^x \frac{4}{\sqrt{3}}$

$\Rightarrow \frac{4^{x-1}}{4^{1/2}} = \frac{3^{x-1}}{\sqrt{3}} \Rightarrow 4^{x-3/2} = 3^{x-3/2}$

$$\Rightarrow \left(\frac{4}{3}\right)^{x-3/2} = 1 \Rightarrow x - \frac{3}{2} = 0 \Rightarrow \quad x = 3/2$$

2. $\text{RHS} = (m-1, n+1) + x^{m-n-1}(m-1, n)$

$$= \frac{(1-x^{m-1})(1-x^{m-2})\ldots(1-x^{m-n-1})}{(1-x)(1-x^2)\ldots(1-x^{n+1})}$$

$$+ x^{m-n-1}\left[\frac{(1-x^{m-1})(1-x^{m-2})\ldots(1-x^{m-n})}{(1-x)(1-x^2)\ldots(1-x^n)}\right]$$

$$= \frac{(1-x^{m-1})(1-x^{m-2})\ldots(1-x^{m-n})}{(1-x)(1-x^2)\ldots(1-x^n)}$$

$$\left[\frac{1-x^{m-n-1}}{1-x^{n+1}} + x^{m-n-1}\right]$$

$$\left[\frac{1-x^{m-n-1}+x^{m-n-1}-x^m}{1-x^{n+1}}\right]$$

$$= \frac{(1-x^m)(1-x^{m-1})\ldots(1-x^{m-n})}{(1-x)(1-x^2)\ldots(1-x^n)(1-x^{n+1})}$$

$= (m, n+1) = \text{L.H.S.}$ Hence Proved

3. $\sqrt{x+1} = 1 + \sqrt{x-1}$

Squaring both sides, we get

$x + 1 = 1 + x - 1 + 2\sqrt{x-1} \Rightarrow 1 = 2\sqrt{x-1}$.

$\Rightarrow 1 = 4(x-1)$

$\Rightarrow x = 5/4$

4. Given $a > 0$, so we have to consider two cases : $a \neq 1$ and $a = 1$. Also it is clear that $x > 0$ and $x \neq 1$, $ax \neq 1$, $a^2x \neq 1$.

Case I : If $a > 0, \neq 1$

then given equation can be simplified as

$$\frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{3}{2+\log_a x} = 0$$

Putting $\log_a x = y$, we get

$2(1+y)(2+y) + y(2+y) + 3y(1+y) = 0$

$\Rightarrow 6y^2 + 11y + 4 = 0 \Rightarrow y = -4/3$ and $-1/2$

$\Rightarrow \log_a x = -4/3$ and $\log_a x = -1/2$

$\Rightarrow x = a^{-4/3}$ and $x = a^{-1/2}$

Case II : If $a = 1$ then equation becomes

$2\log_x 1 + \log_x 1 + 3\log_x 1 = 6\log_x 1 = 0$

which is true $\forall\, x > 0, \neq 1$

Hence solution is if $a = 1, x > 0, \neq 1$

if $a > 0, \neq 1$; $x = a^{-1/2}, a^{-4/3}$

5. Let $x = \dfrac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{50+38+5\sqrt{3}-10\sqrt{76+10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{75+1+10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{(5\sqrt{3})^2+(1)^2+2\times 5\sqrt{3}\times 1}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{(5\sqrt{3}+1)^2}}$$

$= \dfrac{26-15\sqrt{3}}{3(26-15\sqrt{3})} = \dfrac{1}{3}$, which is a rational number.

6. $x^2 + y^2 - 2x \geq 0 \Rightarrow x^2 - 2x + 1 + y^2 \geq 1$

$\Rightarrow (x-1)^2 + y^2 \geq 1$ which represents the boundary and exterior region of the circle with centre at (1,0) and radius as 1.

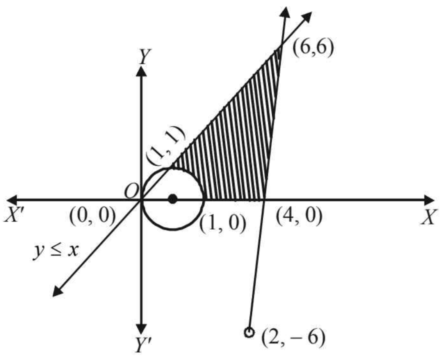
For $3x - y \leq 12$, the corresponding equation is $3x - y = 12$; any two points on it can be taken as (4, 0), (2, – 6). Also putting (0, 0) in given inequation, we get $0 \leq 12$ which is true.

$\therefore$ given inequation represents that half plane region of line $3x - y = 12$ which contains origin.

For $y \leq x$, the corresponding equation $y = x$ has any two points on it as (0, 0) and (1, 1). Also putting (2, 1) in the given inequation, we get $1 \leq 2$ which is true, so $y \leq x$ represents that half plane which contains the points (2, 1).

$y \geq 0$ represents upper half cartesian plane.

Combining all we find the solution set as the shaded region in the graph.



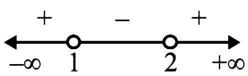
7. There are two parts of this question

$(5x-1) < (x+1)^2$ and $(x+1)^2 < (7x-3)$

Taking first part

$(5x-1) < (x+1)^2 \Rightarrow 5x - 1 < x^2 + 2x + 1$

$\Rightarrow x^2 - 3x + 2 > 0 \Rightarrow (x-1)(x-2) > 0$



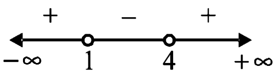
(using wavy method)

$\Rightarrow x < 1$ or $x > 2$ ....(1)

Taking second part

$(x+1)^2 < (7x-3) \Rightarrow x^2 - 5x + 4 < 0$

$\Rightarrow \quad (x-1)(x-4) < 0$



(using wavy method)

$\Rightarrow \quad 1 < x < 4$ ....(2)

Combining (1) and (2) [taking common solution], we get $2 < x < 4$ but x is an integer therefore $x = 3$.

8. $\because \quad \alpha, \beta$ are the roots of $x^2 + px + q = 0$

$\therefore \quad \alpha + \beta = -p, \quad \alpha\beta = q$

$\because \quad \gamma, \delta$ are the roots of $x^2 + rx + s = 0$

$\therefore \quad \gamma + \delta = -r, \gamma\delta = s$

Now, $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$

$= \quad [\alpha^2 - (\gamma+\delta)\alpha + \gamma\delta][\beta^2 - (\gamma+\delta)\beta + \gamma\delta]$

$= \quad [\alpha^2 + r\alpha + s][\beta^2 + r\beta + s]$

[$\because \alpha, \beta$ are roots of $x^2 + px + q = 0$

$\therefore \alpha^2 + p\alpha + q = 0$ and $\beta^2 + p\beta + q = 0$]

$= \quad [(r-p)\alpha + (s-q)][(r-p)\beta + (s-q)]$

$= \quad (r-p)^2\alpha\beta + (r-p)(s-q)(\alpha+\beta) + (s-q)^2$

$= \quad q(r-p)^2 - p(r-p)(s-q) + (s-q)^2$

Now if the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ have a common root say α, then $\alpha^2 + p\alpha + q = 0$ and $\alpha^2 + r\alpha + s = 0$

$$\Rightarrow \quad \frac{\alpha^2}{ps - qr} = \frac{\alpha}{q-s} = \frac{1}{r-p}$$

$$\Rightarrow \quad \alpha^2 = \frac{ps - qr}{r-p} \quad \text{and} \quad \alpha = \frac{q-s}{r-p}$$

$\Rightarrow \quad (q-s)^2 = (r-p)(ps - qr)$ which is the required condition.

9. Given that $n^4 < 10^n$ for a fixed +ve integer $n \geq 2$.

To prove that $(n+1)^4 < 10^{n+1}$

Proof : Since $n^4 < 10^n \Rightarrow 10n^4 < 10^{n+1}$ ....(1)

So it is sufficient to prove that $(n+1)^4 < 10n^4$

$$\text{Now} \quad \left(\frac{n+1}{n}\right)^4 = \left(1+\frac{1}{n}\right)^4 \leq \left(1+\frac{1}{2}\right)^4 \quad [\because n \geq 2]$$

$$= \frac{81}{16} < 10$$

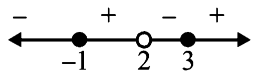
$\Rightarrow \quad (n+1)^4 < 10n^4$ .... (2)

From (1) and (2), $(n+1)^4 < 10^{n+1}$

10. $y = \sqrt{\dfrac{(x+1)(x-3)}{(x-2)}}$

y will take all real values if $\dfrac{(x+1)(x-3)}{(x-2)} \geq 0$

By wavy method



$x \in [-1, 2) \cup [3, \infty)$

[2 is not included as it makes denominator zero, and hence y an undefined number.]

11. The given equations are $3x + my - m = 0$ and $2x - 5y - 20 = 0$

Solving these equations by cross product method, we get

$$\frac{x}{-20m - 5m} = \frac{y}{-2m + 60} = \frac{1}{-15 - 2m} \qquad \textbf{NOTE THIS STEP}$$

$$\Rightarrow \quad x = \frac{25m}{2m+15}, y = \frac{2m-60}{2m+15}$$

$$\text{For} \quad x > 0 \Rightarrow \frac{25m}{2m+15} > 0$$

$$\Rightarrow m < -\frac{15}{2} \text{ or } m > 0 \qquad \text{....(1)}$$

$$\text{For} \quad y > 0 \Rightarrow \frac{2(m-30)}{2m+15} > 0$$

$$\Rightarrow m < -\frac{15}{2} \text{ or } m > 30 \qquad \text{....(2)}$$

Combining (1) and (2), we get the common values of m as follows :

$$m < -\frac{15}{2} \text{ or } \ m > 30 \quad \therefore \ m \in \left(-\infty, \frac{-15}{2}\right) \cup (30, \infty)$$

12. The given system is

$x + 2y + z = 1$ ....(1)

$2x - 3y - \omega = 2$ ....(2)

where $x, y, z, \omega \geq 0$

Multiplying eqn. (1) by 2 and subtracting from (2), we get

$7y + 2z + \omega = 0 \Rightarrow \omega = -(7y + 2z)$

Now if $y, z > 0$, $\omega < 0$ (not possible)

If $y = 0, z = 0$ then $x = 1$ and $\omega = 0$.

$\therefore$ The only solution is $x = 1, y = 0, z = 0, \omega = 0$.

13. $e^{\sin x} - e^{-\sin x} - 4 = 0$

Let $e^{\sin x} = y$ then $e^{-\sin x} = 1/y$

$\therefore$ Equation becomes, $y - \dfrac{1}{y} - 4 = 0$

$\Rightarrow y^2 - 4y - 1 = 0 \Rightarrow y = 2 + \sqrt{5}, 2 - \sqrt{5}$

But y is real +ve number,

$\therefore \quad y \neq 2 - \sqrt{5} \Rightarrow y = 2 + \sqrt{5}$

$\Rightarrow \quad e^{\sin x} = 2 + \sqrt{5} \Rightarrow \sin x = \log_e(2 + \sqrt{5})$

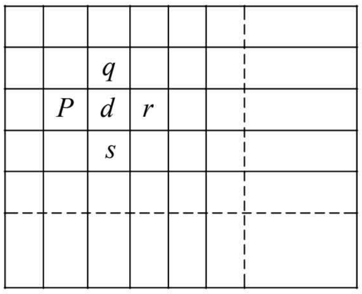
But $\quad 2 + \sqrt{5} > e \Rightarrow \log_e(2 + \sqrt{5}) > \log_e e$

$\Rightarrow \quad \log_e(2+\sqrt{5})>1 \qquad$ Hence, $\sin x > 1$

Which is not possible.

$\therefore$ Given equation has no real solution.

14. For any square there can be at most 4, neighbouring squares.



Let for a square having largest number d, p, q, r, s be written then

According to the question,

$$p+q+r+s=4d$$

$$\Rightarrow \quad (d-p)+(d-q)+(d-r)+(d-s)=0$$

Sum of four +ve numbers can be zero only if these are zero individually

$\therefore \quad d-p=0=d-q=d-r=d-s$

$\Rightarrow \quad p=q=r=s=d$

$\Rightarrow$ all the numbers written are same.

Hence Proved.

15. Let α, β be the roots of eq. $ax^2+bx+c=0$

According to the question, $\beta=\alpha^n$

Also $\quad \alpha+\beta=-b/a$; $\alpha\beta=c/a$

$$\alpha\beta=\frac{c}{a} \Rightarrow \alpha.\alpha^n=\frac{c}{a} \Rightarrow \alpha=\left(\frac{c}{a}\right)^{\frac{1}{n+1}}$$

then $\alpha+\beta=-b/a \Rightarrow \alpha+\alpha^n=\dfrac{-b}{a}$

or $\quad \left(\dfrac{c}{a}\right)^{\frac{1}{n+1}}+\left(\dfrac{c}{a}\right)^{\frac{n}{n+1}}=\dfrac{-b}{a}$

$$\Rightarrow \quad a.\left(\frac{c}{a}\right)^{\frac{1}{n+1}}+a.\left(\frac{c}{a}\right)^{\frac{n}{n+1}}+b=0$$

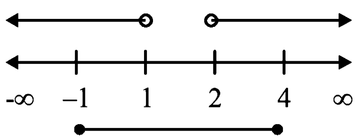
$$\Rightarrow \quad a^{\frac{n}{n+1}}c^{\frac{1}{n+1}}+a^{\frac{1}{n+1}}c^{\frac{n}{n+1}}+b=0$$

$$\Rightarrow \quad (a^nc)^{\frac{1}{n+1}}+(ac^n)^{\frac{1}{n+1}}+b=0$$

Hence Proved.

16. $x^2-3x+2>0, \quad x^2-3x-4\le 0$

$\Rightarrow (x-1)(x-2)>0$ and $(x-4)(x+1)<0$



$\Rightarrow \quad x\in(-\infty,1)\cup(2,\infty)$ and $x\in[-1,4]$

$\therefore$ Common solution is $[-1,1)\cup(2,4]$

17. The given equation is

$$(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10 \qquad ....(1)$$

Let $(5+2\sqrt{6})^{x^2-3}=y \qquad ....(2)$

then $(5-2\sqrt{6})^{x^2-3}=\left(\dfrac{(5-2\sqrt{6})(5+2\sqrt{6})}{5+2\sqrt{6}}\right)^{x^2-3}$

$$=\left(\frac{25-24}{5+2\sqrt{6}}\right)^{x^2-3}=\left(\frac{1}{5+2\sqrt{6}}\right)^{x^2-3}=\frac{1}{y}\text{ (Using (2))}$$

$\therefore$ The given equation (1) becomes $y+\dfrac{1}{y}=10$

$$\Rightarrow \quad y^2-10y+1=0 \Rightarrow y=\frac{10\pm\sqrt{100-4}}{2}=\frac{10\pm4\sqrt{6}}{2}$$

$\Rightarrow \quad y=5+2\sqrt{6}$ or $5-2\sqrt{6}$

Consider, $y=5+2\sqrt{6}$

$$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3}=(5+2\sqrt{6})$$

$\Rightarrow x^2-3=1 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

Again consider

$$y=5-2\sqrt{6}=\frac{1}{5+2\sqrt{6}}=(5+2\sqrt{6})^{-1}$$

$$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3}=(5+2\sqrt{6})^{-1} \Rightarrow x^2-3=-1$$

$\Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$

Hence the solutions are $2, -2, \sqrt{2}, -\sqrt{2}$.

18. The given equation is,

$x^2-2a|x-a|-3a^2=0$

Here two cases are possible.

Case I : $x-a>0$ then $|x-a|=x-a$

$\therefore$ Eq. becomes

$x^2-2a(x-a)-3a^2=0$

or $\quad x^2-2ax-a^2=0 \Rightarrow x=\dfrac{2a\pm\sqrt{4a^2+4a^2}}{2}$

$\Rightarrow x=a\pm a\sqrt{2}$

Case II : $x-a<0$ then $|x-a|=-(x-a)$

$\therefore$ Eq. becomes

$x^2+2a(x-a)-3a^2=0$

or $\quad x^2+2ax-5a^2=0 \Rightarrow x=\dfrac{-2a\pm\sqrt{4a^2+20a^2}}{2}$

$\Rightarrow \quad x = \dfrac{-2a \pm 2a\sqrt{6}}{2} \quad \Rightarrow \quad x = -a \pm a\sqrt{6}$

Thus the solution set is $\{a \pm a\sqrt{2}, -a \pm a\sqrt{6}\}$

19. We are given $\dfrac{2x}{2x^2+5x+2} > \dfrac{1}{x+1}$

$\Rightarrow \quad \dfrac{2x}{2x^2+5x+2} - \dfrac{1}{x+1} > 0$

$\Rightarrow \quad \dfrac{2x^2+2x-2x^2-5x-2}{(2x^2+5x+2)(x+1)} > 0$

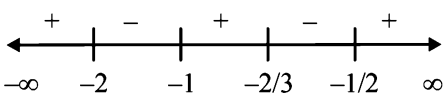
$\Rightarrow \quad \dfrac{-3x-2}{(2x+1)(x+1)(x+2)} > 0 \Rightarrow \dfrac{(3x+2)}{(x+1)(x+2)(2x+1)} < 0$

$\Rightarrow \quad \dfrac{(3x+2)(x+1)(x+2)(2x+1)}{(x+1)^2(x+2)^2(2x+1)^2} < 0$

$\Rightarrow \quad (3x+2)(x+1)(x+2)(2x+1) < 0$ ....(1)

NOTE THIS STEP : Critical pts are $x = -2/3, -1, -2, -1/2$

On number line



Clearly Inequality (1) holds for,

$x \in (-2, -1) \cup (-2/3, -1/2)$

$[as\ x \neq -2, -1, -2/3, -1/2]$

20. The Given equation is,

$|x^2+4x+3| + 2x + 5 = 0$

Now there can be two cases.

Case I : $x^2+4x+3 \geq 0 \Rightarrow (x+1)(x+3) \geq 0$

$\Rightarrow \quad x \in (-\infty, -3] \cup [-1, \infty)$ ....(i)

Then given equation becomes,

$\Rightarrow \quad x^2+6x+8 = 0$

$\Rightarrow \quad (x+4)(x+2) = 0 \Rightarrow x = -4, -2$

But $x = -2$ does not satisfy (i), hence rejected

$\therefore \quad x = -4$ is the sol.

Case II : $x^2+4x+3 < 0$

$\Rightarrow \quad (x+1)(x+3) < 0$

$\Rightarrow \quad x \in (-3, -1)$ ....(ii)

Then given equation becomes,

$-(x^2+4x+3) + 2x + 5 = 0$

$\Rightarrow -x^2 - 2x + 2 = 0 \Rightarrow x^2 + 2x - 2 = 0$

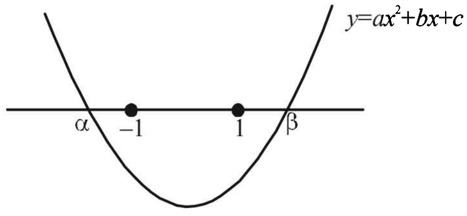
$\Rightarrow \quad x = \dfrac{-2 \pm \sqrt{4+8}}{2} \quad \Rightarrow \quad x = -1+\sqrt{3}, -1-\sqrt{3}$

Out of which $x = -1-\sqrt{3}$ is sol.

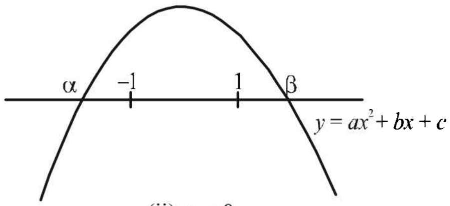
Combining the two cases we get the solutions of given equation as $x = -4, -1-\sqrt{3}$.

21. Given that for $a, b, c \in R$, $ax^2+bx+c=0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$. There may be two cases depending upon value of a, as shown below.

In each of cases (i) and (ii) $af(-1) < 0$ and $af(1) < 0$



(i) $a > 0$



(ii) $a < 0$

$\Rightarrow \quad a(a-b+c) < 0$ and $a(a+b+c) < 0$

Dividing by a^2 (> 0), we get

$$1 - \frac{b}{a} + \frac{c}{a} < 0 \qquad ....(1)$$

and $$1 + \frac{b}{a} + \frac{c}{a} < 0 \qquad ....(2)$$

Combining (1) and (2) we get

$1 + \left|\dfrac{b}{a}\right| + \dfrac{c}{a} < 0$ **or** $1 + \dfrac{c}{a} + \left|\dfrac{b}{a}\right| < 0$ **Hence Proved.**

22. $a^2 = p^2 + s^2, b^2 = (1-p)^2 + q^2$

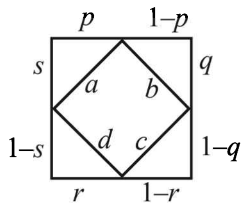
$c^2 = (1-q)^2 + (1-r)^2, d^2 = r^2 + (1-s)^2$

$\therefore \quad a^2+b^2+c^2+d^2 = \{p^2+(1-p)^2\} + \{q^2-(1-q)^2\}$
$+ \{r^2+(1-r)^2\} + \{s^2+(1-s)^2\}$

where p, q, r, s all vary in the interval [0, 1].

Now consider the function

$y^2 = x^2 + (1-x)^2, \; 0 \leq x \leq 1,$



$2y\dfrac{dy}{dx} = 2x - 2(1-x) = 0$

$\Rightarrow \quad x = \dfrac{1}{2}$ which $\dfrac{d^2y}{dx^2} = 4$ i.e.+ive

Hence y is minimum at $x = \dfrac{1}{2}$ and its minimum value is $\dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{2}$.

Clearly value is maximum at the end pts which is 1.

$\therefore$ Minimum value of $a^2+b^2+c^2+d^2 = \dfrac{1}{2}+\dfrac{1}{2}+\dfrac{1}{2}+\dfrac{1}{2} = 2$

and maximum value is $1+1+1+1 = 4$. Hence proved.

23. We know that,

$(\alpha-\beta)^2 = [(\alpha+\delta) - (\beta+\delta)]^2$

$\Rightarrow \quad (\alpha+\beta)^2 - 4\alpha\beta = (\alpha+\delta+\beta+\delta)^2 - 4(\alpha+\delta)(\beta+\delta)$

$\Rightarrow \quad \frac{b^2}{a^2} - \frac{4c}{a} = \frac{B^2}{A^2} - \frac{4C}{A} \Rightarrow \frac{4ac-b^2}{a^2} = \frac{4AC-B^2}{A^2}$

[Here $\alpha+\beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$,

$(\alpha+\delta)(\beta+\delta)$

$= -\frac{B}{A}$ and $(\alpha+\delta)(\beta+\delta) = \frac{C}{A}$]

Hence proved.

24. Divide the equation by a^3, we get

$x^2 + \frac{b}{a}.\frac{c}{a}.x + \left(\frac{c}{a}\right)^3 = 0$

$\Rightarrow x^2 - (\alpha+\beta).(\alpha\beta)x + (\alpha\beta)^3 = 0$

$\Rightarrow x^2 - \alpha^2\beta x - \alpha\beta^2 x + (\alpha\beta)^3 = 0$

$\Rightarrow x(x-\alpha^2\beta) - \alpha\beta^2(x-\alpha^2\beta) = 0$

$\Rightarrow (x-\alpha^2\beta)(x-\alpha\beta^2) = 0$

$\Rightarrow x = \alpha^2\beta, \alpha\beta^2$ which is the required answer.

25. The given equation is,

$x^2 + (a-b)x + (1-a-b) = 0, a, b \in R$

For this eq^n to have unequal real roots $\forall\, b$

$D > 0$

$\Rightarrow \quad (a-b)^2 - 4(1-a-b) > 0$

$\Rightarrow \quad a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$

$\Rightarrow \quad b^2 + b(4-2a) + a^2 + 4a - 4 > 0$

Which is a quadratic expression in b, and it will be true $\forall\, b \in R$ if discriminant of above eq^n less than zero.

i.e., $(4-2a)^2 - 4(a^2+4a-4) < 0$

$\Rightarrow \quad (2-a)^2 - (a^2+4a-4) < 0$

$\Rightarrow \quad 4 - 4a + a^2 - a^2 - 4a + a < 0$

$\Rightarrow \quad -8a + 8 < 0$

$\Rightarrow \quad a > 1$

26. Given that a, b, c are positive real numbers. To prove that

$(a+1)^7(b+1)^7(c+1)^7 > 7^7 a^4b^4c^4$

Consider L.H.S. $= (1+a)^7.(1+b)^7.(1+c)^7$

$= [(1+a)(1+b)(1+c)]^7$

$[1 + a + b + c + ab + bc + ca + abc]^7$

$> [a + b + c + ab + bc + ca + abc]^7$ ....(1)

Now we know that AM $\geq$ GM using it for +ve no's a, b, c, ab, bc, ca and abc, we get

$\frac{a+b+c+ab+bc+ca+abc}{7} \geq (a^4b^4c^4)^{1/7}$

$\Rightarrow \quad (a+b+c+ab+bc+ca+abc)^7 \geq 7^7(a^4b^4c^4)a$

From (1) and (2), we get

$[(1+a)(1+b)(1+c)]^7 > 7^7a^4b^4c^4$

Hence Proved.

27. Roots of $x^2 - 10cx - 11d = 0$ are a and b

$\Rightarrow a + b = 10c$ and $ab = -11d$

Similarly c and d are the roots of $x^2 - 10ax - 11b = 0$

$\Rightarrow c + d = 10a$ and $cd = -11b$

$\Rightarrow \quad a + b + c + d = 10(a+c)$ and $abcd = 121\, bd$

$\Rightarrow \quad b + d = 9(a+c)$ and $ac = 121$

Also we have $a^2 - 10ac - 11d = 0$ and $c^2 - 10ac - 11b = 0$

$\Rightarrow \quad a^2 + c^2 - 20ac - 11(b+d) = 0$

$\Rightarrow \quad (a+c)^2 - 22 \times 121 - 99(a+c) = 0$

$\Rightarrow \quad a + c = 121$ or -22

For $a + c = -22$, we get $a = c$

$\therefore$ rejecting this value we have $a + c = 121$

$\therefore \quad a + b + c + d = 10(a+c) = 1210$

H. Assertion & Reason Type Questions

1. **(b)** As $a, b, c, p, q, \in R$ and the two given equations have exactly one common root

$\Rightarrow$ Either both equations have real roots or both eqations have imaginary roots

$\Rightarrow$ Either $\Delta_1 \geq 0$ and $\Delta_2 \geq 0$ or $\Delta_1 \leq 0$ and $\Delta_2 \leq 0$

$\Rightarrow p^2 - q \geq 0$ and $b^2 - ac \geq 0$

or $p^2 - q \leq 0$ and $b^2 - ac \leq 0$

$\Rightarrow (p^2 - q)(b^2 - ac) \geq 0$

$\therefore$ Statement 1 is true.

Also we have $\alpha\beta = q$ and $\frac{\alpha}{\beta} = \frac{c}{a}$

$\therefore \frac{\alpha\beta}{\alpha/\beta} = \frac{q}{c} \times a \Rightarrow \beta^2 = \frac{qa}{c}$

As $\beta \neq 1$ or $-1 \Rightarrow \beta^2 \neq 1 \Rightarrow \frac{qa}{c} \neq 1$ or $c \neq qa$

Again, as exactly one root α is common, and $\beta \neq 1$

$\therefore \alpha + \beta \neq \alpha + \frac{1}{\beta} \Rightarrow \frac{-2b}{a} \neq -2p \Rightarrow b \neq ap$

$\therefore$ Statement 2 is correct.

But Statement 2 is not a correct explanation of Statement 1.

I. Integer Value Correct Type

1. **(7)** The given system of equations is

$3x - y - z = 0$

$-3x + z = 0$

$-3x + 2y + z = 0$

Let $x = p$ where p is an integer, then $y = 0$ and $z = 3p$

But $x^2 + y^2 + z^2 \leq 100 \Rightarrow p^2 + 9p^2 \leq 100$

$\Rightarrow \quad p^2 \leq 10 \Rightarrow p = 0, \pm 1, \pm 2 \pm 3$

i.e. p can take 7 different values.

$\therefore$ Number of points (x, y, z) are 7.

2. **(2)** The given equation is

$x^2 - 8kx + 16(k^2 - k + 1) = 0$

$\because$ Both the roots are real and distinct

$\therefore \quad D > 0 \Rightarrow (8k)^2 - 4 \times 16(k^2 - k + 1) > 0$

$\Rightarrow \quad k > 1$ ...(i)

$\because$ Both the roots are greater than or equal to 4

$\therefore$ $\alpha+\beta > 8$ and $f(4) \ge 0$

$\Rightarrow$ $k > 1$ ...(ii)

and $16 - 32k + 16(k^2 - k + 1) \ge 0$

$\Rightarrow$ $k^2 - 3k + 2 \ge 0 \Rightarrow (k-1)(k-2) \ge 0$

$\Rightarrow$ $k \in (-\infty, 1] \cup [2, \infty)$ ...(iii)

Combining (i), (ii) and (iii), we get $k \ge 2$ or the smallest value of $k = 2$.

3. **(8)** $\because a > 0, \therefore a^{-5}, a^{-4}, 3a^{-3}, 1, a^8, a^{10} > 0$

Using AM $\ge$ GM for positive real numbers we get

$$\frac{\frac{1}{a^5}+\frac{1}{a^4}+\frac{1}{a^3}+\frac{1}{a^3}+\frac{1}{a^3}+1+a^8+a^{10}}{8} \ge$$

$$\left(\frac{1}{a^5}.\frac{1}{a^4}.\frac{1}{a^3}.\frac{1}{a^3}.\frac{1}{a^3}.1.a^8.a^{10}\right)^{\frac{1}{8}}$$

$$\Rightarrow \frac{1}{a^5}+\frac{1}{a^4}+\frac{3}{a^3}+1+a^8+a^{10} \ge 8(1)^{\frac{1}{8}}$$

4. **(2)** We have $x^4 - 4x^3 + 12x^2 + x - 1 = 0$

$\Rightarrow x^4 - 4x^3 + 6x^2 - 4x + 1 + 6x^2 + 5x - 2 = 0$

$\Rightarrow (x-1)^4 + 6x^2 + 5x - 2 = 0$

$\Rightarrow (x-1)^4 = -6x^2 - 5x + 2$

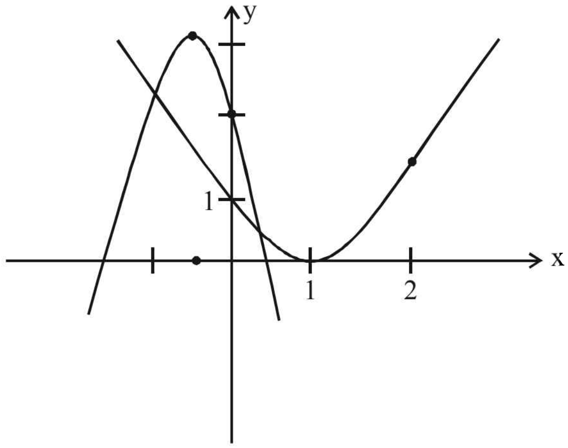
To solve the above polynomial, it is equivalent to find the intersection points of the curves $y = (x-1)^4$ and

$y = -6x^2 - 5x + 2$ or $y = (x-1)^4$ and $\left(x+\frac{5}{12}\right)^2 = -\frac{1}{6}\left(y - \frac{73}{24}\right)$

The graph of above two curves as follows.

Clearly they have two points of intersection.

Hence the given polynomial has two real roots.



Section-B JEE Main/ AIEEE

1. **(a)** We have $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$;

$\Rightarrow \alpha$ & β are roots of equation, $x^2 = 5x - 3$

or $x^2 - 5x + 3 = 0$ $\therefore \alpha + \beta = 5$ and $\alpha\beta = 3$

Thus, the equation having $\frac{\alpha}{\beta}$ & $\frac{\beta}{\alpha}$ as its roots is

$$x^2 - x\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)+\frac{\alpha\beta}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - x\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)+1 = 0 \text{ or } 3x^2 - 19x + 3 = 0$$

2. **(a)** Let α, β and γ, δ be the roots of the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ respectively.

$\therefore \alpha + \beta = -a, \alpha\beta = b$ and $\gamma + \delta = -b, \gamma\,\delta = a$.

Given $|\alpha - \beta| = |\gamma - \delta| \Rightarrow (\alpha - \beta)^2 = (\gamma - \delta)^2$

$\Rightarrow (\alpha+\beta)^2 - 4\alpha\beta = (\gamma+\delta)^2 - 4\gamma\delta$

$\Rightarrow a^2 - 4b = b^2 - 4a \Rightarrow (a^2 - b^2) + 4(a - b) = 0$

$\Rightarrow a + b + 4 = 0$ $(\because a \ne b)$

3. **(a)** Product of real roots $= \frac{9}{t^2} > 0, \ \forall\ t \in R$

$\therefore$ Product of real roots is always positive.

4. **(a)** $p + q = -p$ and $pq = q \Rightarrow q\,(p-1) = 0$

$\Rightarrow q = 0$ or $p = 1$.

If $q = 0$, then $p = 0$. i.e. $p = q$

$\therefore p = 1$ and $q = -2$.

5. **(a)** $\because (a-b)^2 + (b-c)^2 + (c-a)^2 > 0$

$\Rightarrow 2(a^2 + b^2 + c^2 - ab - bc - ca) > 0$

$\Rightarrow 2 > 2(ab + bc + ca) \Rightarrow ab + bc + ca < 1$

6. **(d)** $ax^2 + bx + c = 0, \ \alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

As for given conditon, $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$$\alpha + \beta = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} - \frac{b}{a} = \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}}$$

On simplification $2a^2c = ab^2 + bc^2$

$\Rightarrow \frac{2a}{b} = \frac{c}{a} + \frac{b}{c}$ $\Rightarrow \frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in A.P.

$\therefore \frac{a}{c}, \frac{b}{a}, \& \frac{c}{b}$ are in H.P.

7. **(b)** Let the roots of given equation be α and 2α then

$$\alpha + 2\alpha = 3\alpha = \frac{1-3a}{a^2 - 5a + 3}$$

$$\& \ \alpha.2\alpha = 2\alpha^2 = \frac{2}{a^2 - 5a + 3} \Rightarrow \alpha = \frac{1-3a}{3\left(a^2 - 5a + 3\right)}$$

$\therefore\ 2\left[\frac{1}{9}\frac{(1-3a)^2}{(a^2-5a+3)^2}\right]=\frac{2}{a^2-5a+3}$

$\frac{(1-3a)^2}{(a^2-5a+3)}=9$ or $9a^2-6a+1=9a^2-45a+27$

or $39a=26$ *or* $a=\frac{2}{3}$

8. (c) $x^2-3|x|+2=0 \Rightarrow |x|^2-3|x|+2=0$

$(|x|-2)(|x|-1)=0$

$|x|=1,2$ or $x=\pm1,\pm2$ $\therefore$ No.of solution $=4$

9. (c) $y=x+\frac{1}{x}$ or $\frac{dy}{dx}=1-\frac{1}{x^2}$

For max. or min., $1-\frac{1}{x^2}=0\Rightarrow x=\pm1$

$\frac{d^2y}{dx^2}=\frac{2}{x^3}\Rightarrow\left(\frac{d^2y}{dx^2}\right)_{x=2}=2$ (+ve minima) $\therefore x=1$

10. (b) Let two numbers be a and b then $\frac{a+b}{2}=9$ and

$\sqrt{ab}=4$

$\therefore$ Equation with roots a and b is

$x^2-(a+b)x+ab=0\ \Rightarrow x^2-18x+16=0$

11. (c) Let the second root be α.

Then $\alpha+(1-p)=-p\Rightarrow\alpha=-1$

Also $\alpha.(1-p)=1-p$

$\Rightarrow(\alpha-1)(1-p)=0\Rightarrow p=1[\because\alpha=-1]$

$\therefore$ Roots are $\alpha=-1$ and $p-1=0$

12. (d) 4 is a root of $x^2+px+12=0$

$\Rightarrow 16+4p+12=0\Rightarrow p=-7$

Now, the equation $x^2+px+q=0$

has equal roots.

$\therefore p^2-4q=0\Rightarrow q=\frac{p^2}{4}=\frac{49}{4}$

13. (b) $\tan\left(\frac{P}{2}\right),\tan\left(\frac{Q}{2}\right)$ are the roots of $ax^2+bx+c=0$

$\tan\left(\frac{P}{2}\right)+\tan\left(\frac{Q}{2}\right)=-\frac{b}{a},\ \tan\left(\frac{P}{2}\right)\cdot\tan\left(\frac{Q}{2}\right)=\frac{c}{a}$

$$\frac{\tan\left(\frac{P}{2}\right)+\tan\left(\frac{Q}{2}\right)}{1-\tan\left(\frac{P}{2}\right)\tan\left(\frac{Q}{2}\right)}=\tan\left(\frac{P}{2}+\frac{Q}{2}\right)=1$$

$\Rightarrow\frac{-\frac{b}{a}}{1-\frac{c}{a}}=1\Rightarrow -\frac{b}{a}=\frac{a}{a}-\frac{c}{a}$

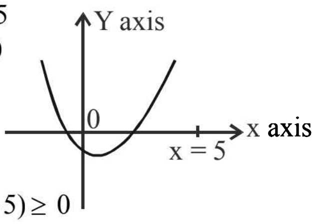
$\Rightarrow -b=a-c$ or $c=a+b$.

14. (c) both roots are less than 5

then (i) Discriminant ≥ 0

(ii) $p(5)>0$

(iii) $\frac{\text{Sum of roots}}{2}<5$



Hence (i) $4k^2-4(k^2+k-5)\geq 0$

$4k^2-4k^2-4k+20\geq 0$

$4k\leq 20\Rightarrow k\leq 5$

(ii) $f(5)>0\,;25-10k+k^2+k-5>0$

or $k^2-9k+20>0$

or $k(k-4)-5(k-4)>0$

or $(k-5)(k-4)>0$

$\Rightarrow k\in(-\infty,4)\cup(-\infty,5)$

(ii) $\frac{\text{Sum of roots}}{2}=-\frac{b}{2a}=\frac{2k}{2}<5$

The interection of (i), (ii) & (iii) gives $k\in(-\infty,4)$.

15. (b) $x^2+px+q=0$

Sum of roots $=\tan30°+\tan15°=-p$

Product of roots $=\tan30°.\tan15°=q$

$\tan45°=\frac{\tan30°+\tan15°}{1-\tan30°.\tan15°}=\frac{-p}{1-q}=1$

$\Rightarrow -p=1-q\ \Rightarrow q-p=1\quad\therefore\ 2+q-p=3$

16. (c) Equation $x^2-2mx+m^2-1=0$

$(x-m)^2-1=0$ or $(x-m+1)(x-m-1)=0$

$x=m-1,m+1$

$m-1>-2$ and $m+1<4$

$\Rightarrow m>-1$ and $m<3$ or, $-1<m<3$

17. (b) $y=\frac{3x^2+9x+17}{3x^2+9x+7}$

$3x^2(y-1)+9x(y-1)+7y-17=0$

$D\geq 0\ \because$ x is real

$81(y-1)^2-4\times3(y-1)(7y-17)\geq 0$

$\Rightarrow(y-1)(y-41)\leq 0\ \Rightarrow 1\leq y\leq 41$

$\therefore$ Max value of y is 41

18. (c) Let α and β are roots of the equation $x^2+ax+1=0$ So, $\alpha+\beta=-a$ and $\alpha\beta=1$

given $|\alpha-\beta|<\sqrt5\ \Rightarrow\ \sqrt{(\alpha+\beta)^2-4\alpha\beta}<\sqrt5$

$\left(\because\ (\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta\right)$

$\Rightarrow \sqrt{a^2-4} < \sqrt{5} \Rightarrow a^2-4<5$
$\Rightarrow a^2-9<0 \Rightarrow a^2<9 \Rightarrow -3<a<3 \Rightarrow a \in (-3, 3)$

19. **(b)** Statement 2 is $\sqrt{n(n+1)} < n+1, n \geq 2$

$\Rightarrow \sqrt{n} < \sqrt{n+1}, n \geq 2$ which is true

$\Rightarrow \sqrt{2} < \sqrt{3} < \sqrt{4} < \sqrt{5} < ------- \sqrt{n}$

Now $\sqrt{2} < \sqrt{n} \Rightarrow \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{n}}$

$\sqrt{3} < \sqrt{n} \Rightarrow \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{n}}$; $\sqrt{n} \leq \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$

Also $\frac{1}{\sqrt{1}} > \frac{1}{\sqrt{n}}$ $\therefore$ Adding all, we get

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{n} > \frac{n}{\sqrt{n}} = \sqrt{n}$

Hence both the statements are correct and statement 2 is a correct explanation of statement-1.

20. **(d)** Let the roots of equation $x^2 - 6x + a = 0$ be α and 4β and that of the equation $x^2 - cx + 6 = 0$ be α and 3β. Then

$\alpha + 4\beta = 6$; $4\alpha\beta = a$

and $\alpha + 3\beta = c$; $3\alpha\beta = 6$

$\Rightarrow a = 8$

$\therefore$ The equation becomes $x^2 - 6x + 8 = 0$

$\Rightarrow (x-2)(x-4) = 0 \Rightarrow$ roots are 2 and 4

$\Rightarrow \alpha = 2, \beta = 1$ $\therefore$ Common root is 2.

21. **(b)** Given that roots of the equation $bx^2 + cx + a = 0$ are imaginary

$\therefore c^2 - 4ab < 0$ ....(i)

Let $y = 3b^2x^2 + 6bcx + 2c^2$

$\Rightarrow 3b^2x^2 + 6bcx + 2c^2 - y = 0$

As x is real, $D \geq 0$

$\Rightarrow 36b^2c^2 - 12b^2(2c^2 - y) \geq 0$

$\Rightarrow 12b^2(3c^2 - 2c^2 + y) \geq 0$

$\Rightarrow c^2 + y \geq 0 \quad \Rightarrow y \geq -c^2$

But from eqn. (i), $c^2 < 4ab$ or $-c^2 > -4ab$

$\therefore$ we get $y \geq -c^2 > -4ab$

$\Rightarrow y > -4ab$

22. **(a)** Given that $\left|z - \frac{4}{z}\right| = 2$

Now $|z| = \left|z - \frac{4}{z} + \frac{4}{-z}\right| \leq \left|z - \frac{4}{z}\right| + \frac{4}{|z|}$

$\Rightarrow |z| \leq 2 + \frac{4}{|z|} \quad \Rightarrow |z|^2 - 2|z| - 4 \leq 0$

$\Rightarrow \left(|z| - \frac{2+\sqrt{20}}{2}\right)\left(|z| - \frac{2-\sqrt{20}}{2}\right) \leq 0$

$\Rightarrow \left(|z| - (1+\sqrt{5})\right)\left(|z| - (1-\sqrt{5})\right) \leq 0$

$\Rightarrow \left(-\sqrt{5}+1\right) \leq |z| \leq \left(\sqrt{5}+1\right)$

$\Rightarrow |z|_{max} = \sqrt{5}+1$

23. **(b)** $x^2 - x + 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{1-4}}{2}$

$x = \frac{1 \pm \sqrt{3}i}{2}$

$\alpha = \frac{1}{2} + i\frac{\sqrt{3}}{2} = -\omega^2 \quad \beta = \frac{1}{2} - \frac{i\sqrt{3}}{2} = -\omega$

$\alpha^{2009} + \beta^{2009} = (-\omega^2)^{2009} + (-\omega)^{2009} = -\omega^2 - \omega = 1$

24. **(b)** Given equation is $e^{\sin x} - e^{-\sin x} - 4 = 0$

Put $e^{\sin x} = t$ in the given equation, we get

$t^2 - 4t - 1 = 0$

$\Rightarrow t = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$

$\Rightarrow e^{\sin x} = 2 \pm \sqrt{5}$ $\left(\because t = e^{\sin x}\right)$

$\Rightarrow e^{\sin x} = 2 - \sqrt{5}$ and $e^{\sin x} = 2 + \sqrt{5}$

$\Rightarrow e^{\sin x} = 2 - \sqrt{5} < 0$ and $\sin x = \ln\left(2+\sqrt{5}\right) > 1$

So rejected So, rejected

Hence given equation has no solution.

$\therefore$ The equation has no real roots.

25. **(d)** $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0 \quad \forall x \in R \quad (\because x^2 > 0)$

$\Rightarrow f(x)$ is strictly increasing function

$\Rightarrow f(x) = 0$ has only one real root, so two roots are not possible.

26. **(b)** From the given system, we have

$\frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1}$ ($\because$ System has no solution)

$\Rightarrow k^2 + 4k + 3 = 8k$

$\Rightarrow k = 1, 3$

If $k = 1$ then $\frac{8}{1+3} \neq \frac{4.1}{2}$ which is false

And if $k = 3$

then $\frac{8}{6} \neq \frac{4.3}{9-1}$ which is true, therefore $k = 3$

Hence for only one value of k. System has no solution.

27. **(a)** Given equations are

$x^2 + 2x + 3 = 0$ …(i)

$ax^2 + bx + c = 0$ …(ii)

Roots of equation (i) are imaginary roots.

According to the question (ii) will also have both roots same as (i). Thus

$\frac{a}{1} = \frac{b}{2} = \frac{c}{3} = \lambda$ (say) $\Rightarrow a = \lambda, b = 2\lambda, c = 3\lambda$

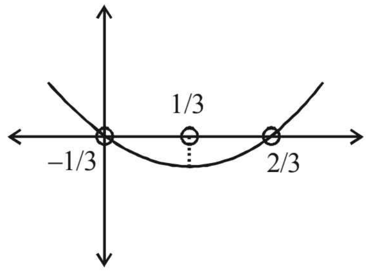
Hence, required ratio is 1 : 2 : 3

28. **(c)** Consider $-3(x-[x])^2 + 2[x-[x]] + a^2 = 0$

$\Rightarrow 3\{x\}^2 - 2\{x\} - a^2 = 0 \quad (\because x - [x] = \{x\})$

$\Rightarrow \quad 3\left(\{x\}^2 - \frac{2}{3}\{x\}\right) = a^2, a \neq 0$

$\Rightarrow \quad a^2 = 3\{x\}\left(\{x\} - \frac{2}{3}\right)$



Now, $\{x\} \in (0,1)$ and $\frac{-2}{3} \leq a^2 < 1$ (by graph)

Since , x is not an integer

$\therefore \quad a \in (-1,1) - \{0\} \Rightarrow a \in (-1,0) \cup (0,1)$

29. (b) Let p, q, r are in AP

$\Rightarrow 2q = p + r$ ...(i)

Given $\frac{1}{\alpha} + \frac{1}{\beta} = 4 \Rightarrow \frac{\alpha+\beta}{\alpha\beta} = 4$

We have $\alpha + \beta = -q/p$ and $\alpha\beta = \frac{r}{p}$

$\Rightarrow \quad \frac{-\frac{q}{p}}{\frac{r}{p}} = 4 \Rightarrow q = -4r$ ....(ii)

From (i), we have

$2(-4r) = p + r \Rightarrow p = -9r$

$q = -4r$

Now $|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= \sqrt{\left(\frac{-q}{p}\right)^2 - \frac{4r}{p}} = \frac{\sqrt{q^2 - 4pr}}{|p|}$

$= \frac{\sqrt{16r^2 + 36r^2}}{|-9r|} = \frac{2\sqrt{13}}{9}$

30. (a) $\alpha, \beta = \frac{6 \pm \sqrt{36+8}}{2} = 3 \pm \sqrt{11}$

$\alpha = 3 + \sqrt{11}, \beta = 3 - \sqrt{11}$

$\therefore a_n = \left(3+\sqrt{11}\right)^n - \left(3-\sqrt{11}\right)^n$

$\frac{a_{10} - 2a_8}{2a_9}$

$= \frac{\left(3+\sqrt{11}\right)^{10} - \left(3-\sqrt{11}\right)^{10} - 2\left(3+\sqrt{11}\right)^8 + 2\left(3-\sqrt{11}\right)^8}{2\left[\left(3+\sqrt{11}\right)^9 - \left(3-\sqrt{11}\right)^9\right]}$

$= \frac{\left(3+\sqrt{11}\right)^8\left[\left(3+\sqrt{11}\right)^2 - 2\right] + \left(3-\sqrt{11}\right)^8\left[2 - \left(3-\sqrt{11}\right)^2\right]}{2\left[\left(3+\sqrt{11}\right)^9 - \left(3-\sqrt{11}\right)^9\right]}$

$= \frac{\left(3+\sqrt{11}\right)^8\left(9+11+6\sqrt{11}-2\right) + \left(3-\sqrt{11}\right)^8\left(2-9-11+6\sqrt{11}\right)}{2\left[\left(3+\sqrt{11}\right)^9 - \left(3-\sqrt{11}\right)^9\right]}$

$= \frac{6\left(3+\sqrt{11}\right)^9 - 6\left(3-\sqrt{11}\right)^9}{2\left[\left(3+\sqrt{11}\right)^9 - \left(3-\sqrt{11}\right)^9\right]} = \frac{6}{2} = 3$

31. (c) $(x^2 - 5x + 5)^{x^2+4x-60} = 1$

Case I

$x^2 - 5x + 5 = 1$ and $x^2 + 4x - 60$ can be any real number

$\Rightarrow x = 1, 4$

Case II

$x^2 - 5x + 5 = -1$ and $x^2 + 4x - 60$ has to be an even number

$\Rightarrow x = 2, 3$

where 3 is rejected because for $x = 3$, $x^2 + 4x - 60$ is odd.

Case III

$x^2 - 5x + 5$ can be any real number and $x^2 + 4x - 60 = 0$

$\Rightarrow x = -10, 6$

$\Rightarrow$ Sum of all values of $x = -10 + 6 + 2 + 1 + 4 = 3$