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## Matrices

A rectangular array of  $mn$  numbers in the form of  $m$  horizontal lines (called rows) and  $n$  vertical lines (called columns), is called a **matrix** of order  $m \times n$ .

This type of array is enclosed by  $[ ]$  or  $( )$

Each of  $mn$  numbers of a matrix is known as element of a matrix. A matrix is generally denoted by  $A, B, C, \dots$  etc., and its element is denoted by  $a_{ij}$ , where  $a_{ij}$  belongs to the  $i$ th row and  $j$ th column and is called  $(i, j)$ th element of the matrix  $A = [a_{ij}]$ .

An  $m \times n$  matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

e.g.

$$A = \begin{bmatrix} 3 & 2 & 7 \\ 5 & -4 & 6 \\ 4 & 8 & -12 \end{bmatrix} \text{ is a matrix of order } 3 \times 3.$$

### Types of Matrices

#### Row Matrix

A matrix which has only one row and any number of columns, is called a row matrix. e.g.  $A = [27 \ 85 \ 1 \ 4]_{1 \times 4}$  is a row matrix.

#### Column Matrix

A matrix is said to be a column matrix, if it has only one column and any number of rows.

e.g.  $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$  is a column matrix.

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- Elementary Matrix

## Rectangular Matrix

A matrix in which number of rows is not equal to the number of columns or *vice-versa* is called a rectangular matrix.

e.g.  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$  is a rectangular matrix of order  $2 \times 3$ .

## Square Matrix

A matrix in which number of rows is equal to the number of columns, is called a square matrix. The elements  $a_{ij}$  of a square matrix  $A = [a_{ij}]_{m \times m}$  for which  $i = j$  i.e. the elements  $a_{11}, a_{22}, \dots, a_{mm}$  are called the **diagonal elements** and the line along which they lie is called the **principal diagonal** or **leading diagonal** of the matrix.

e.g.  $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}_{3 \times 3}$

is a square matrix of order 3 in which diagonal elements are 1, 2, 1.

## Null Matrix

A matrix of order  $m \times n$  whose all elements are zero, is called a null matrix of order  $m \times n$ .

It is denoted by  $O$ .

e.g.  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are two null matrices of order  $2 \times 2$  and  $2 \times 3$ , respectively.

## Diagonal Matrix

A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero and diagonal elements are not all equal.

If  $d_1, d_2, d_3, \dots, d_n$  are elements of principal diagonal of a diagonal matrix of order  $n \times n$ , then matrix is denoted as  $\text{diag} [d_1, d_2, \dots, d_n]$ .

e.g.  $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$  is a diagonal matrix which is denoted

by  $A = \text{diag} [a, b, c]$ .

**Note** The number of zeroes in a diagonal matrix is given by  $n^2 - n$ , where  $n$  is an order of the matrix.

## Triple Diagonal Matrix

A square matrix  $A$  is said to be a triple diagonal matrix, if all its elements are zero except possibly for those lying on the principle diagonal, the diagonal immediately above as well as below the principle diagonal.

e.g.  $\begin{bmatrix} 5 & -3 & 0 \\ -3 & 4 & -3 \\ 0 & 0 & 4 \end{bmatrix}$  and  $\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \end{bmatrix}$

## Scalar Matrix

A square matrix  $A = [a_{ij}]$  is said to be scalar matrix, if

(a)  $a_{ij} = 0, \forall i \neq j$

(b)  $a_{ij} = k, \forall i = j$ , where  $k \neq 0$

i.e. a diagonal matrix is said to be a scalar matrix, if the elements of principle diagonal are same.

e.g.  $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$  is a scalar matrix.

## Unit Matrix or Identity Matrix

A square matrix  $A = [a_{ij}]$  is said to be a unit matrix or identity matrix, if

(a)  $a_{ij} = 0, \forall i \neq j$

(b)  $a_{ij} = 1, \forall i = j$

i.e. A diagonal matrix, whose elements of principle diagonal are equal to 1 and all remaining elements are zero, is known as unit or identity matrix. It is denoted by  $I$ .

e.g.  $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is a unit matrix of order 3.

## Upper Triangular Matrix

A square matrix  $A = [a_{ij}]$  is known as upper triangular matrix, if

$a_{ij} = 0, \forall i > j$ .

e.g.  $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$  is an upper triangular matrix.

## Lower Triangular Matrix

A square matrix  $A = [a_{ij}]$  is known as lower triangular matrix, if  $a_{ij} = 0, \forall i < j$ .

e.g.  $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$  is a lower triangular matrix.

## Submatrices of a Matrix

A matrix  $B$  obtained by deleting the row (s) or Column (s) or both of a matrix  $A$  is said to be a submatrix of  $A$ .

i.e. The matrix  $B$  constituted by the array of elements, which are left after deleting some rows or columns or both of matrix  $A$  is called submatrix of  $A$ .

(a) **Principle Submatrix** A square submatrix  $B$  of a square matrix  $A$  is called a principle submatrix, if the diagonal elements of  $B$  are also diagonal elements of  $A$ .

(b) **Leading Submatrix** A principle square submatrix  $B$  is said to be a leading submatrix of a square matrix  $A$  if it is obtained by deleting only some of the last rows and the corresponding columns such that the leading elements (i.e.  $a_{11}$ ) is not lost.

### Horizontal Matrix

Any matrix in which the number of columns is more than the number of rows is called a horizontal matrix.

e.g.  $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 8 & 9 & 7 & -2 \\ 2 & -2 & -3 & 4 \end{bmatrix}$  is a horizontal matrix.

### Vertical Matrix

Any matrix in which the number of rows is more than the number of columns is called column matrix.

e.g.  $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$  is a column matrix.

### Equality of Two Matrices

Two matrices  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{m \times n}$  are said to be equal, if

- (i) Number of rows in  $A$  is equal to number of rows in  $B$ .
- (ii) Number of columns in  $A$  is equal to number of columns in  $B$ .
- (iii)  $a_{ij} = b_{ij}, \forall i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$

**Example 1.** If  $3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$ , then the

value of  $z$  is

- (a) 1
- (b) 2
- (c) 3
- (d) 4

**Sol.** (a) Given,  $3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$$3x = x + 4 \Rightarrow 2x = 4 \Rightarrow x = 2$$

and  $3y = 6 + x + y \Rightarrow 2y = 6 + x$

$$\Rightarrow y = \frac{6+x}{2}$$

On putting the value of  $x$ , we get

$$y = \frac{6+2}{2} = \frac{8}{2} = 4$$

Now,

$$3z = -1 + z + w,$$

$$2z = -1 + w$$

$$z = \frac{-1+w}{2} \quad \dots(i)$$

Now,

$$3w = 2w + 3$$

$$\Rightarrow w = 3$$

On putting the value of  $w$  in Eq. (i), we get

$$z = \frac{-1+3}{2} = \frac{2}{2} = 1$$

Hence, the values of  $x, y, z$  and  $w$  are 2, 4, 1 and 3.

## Algebra of Matrices

### Addition and Subtraction of Two Matrices

Let  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{m \times n}$  are two matrices whose orders are same, then

$$A + B = [a_{ij} + b_{ij}], \forall i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n$$

Also,  $A - B = A + (-B)$

e.g. If  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$\text{Then, } A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

### Properties of Matrix Addition

Let  $A, B$  and  $C$  are three matrices of same order, then

- (i) Matrix addition is **commutative**, i.e.  $A + B = B + A$
- (ii) Matrix addition is **associative**,  
i.e.  $(A + B) + C = A + (B + C)$
- (iii) If  $O$  is a null matrix of order  $m \times n$  and  $A + O = A = O + A$ , then  $O$  is known as **additive identity**.
- (iv) If for each matrix  $A = [a_{ij}]_{m \times n}$ , a matrix  $(-A)$  is such that  $A + (-A) = O = (-A) + A$ , then matrix  $(-A)$  is known as **additive inverse** of  $A$ .
- (v) Matrix addition follows cancellation law,  
i.e.  $A + B = A + C \Rightarrow B = C$  (left cancellation law)  
and  $B + A = C + A \Rightarrow B = C$  (right cancellation law)

**Note** Two matrices are said to be conformable for addition or subtraction, if they are of the same order.

**Example 2.** If 1,  $\omega$  and  $\omega^2$  are the cube roots of unity and

$$\begin{bmatrix} 1+\omega & 2\omega \\ -2\omega & -b \end{bmatrix} + \begin{bmatrix} a & -\omega \\ 3\omega & 2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ \omega & 1 \end{bmatrix},$$

then  $a^2 + b^2$  is equal to

- (a)  $1 + \omega^2$
- (b)  $\omega^2 - 1$
- (c)  $1 + \omega$
- (d)  $(1 + \omega)^2$

**Sol.** (c) Given,  $\begin{bmatrix} 1+\omega & 2\omega \\ -2\omega & -b \end{bmatrix} + \begin{bmatrix} a & -\omega \\ 3\omega & 2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ \omega & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1+\omega+a & \omega \\ \omega & 2-b \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ \omega & 1 \end{bmatrix}$$

$$\Rightarrow 1 + \omega + a = 0, 2 - b = 1$$

$$\Rightarrow a = -1 - \omega, b = 1$$

$$\therefore a^2 + b^2 = (-1 - \omega)^2 + 1^2 = 1 + \omega^2 + 2\omega + 1^2$$

$$= 0 + \omega + 1$$

$$[: 1 + \omega + \omega^2 = 0]$$

## Scalar Multiplication

Let  $A = [a_{ij}]$  be any  $m \times n$  matrix and  $k$  be any scalar. Then, the matrix obtained by multiplying each element of  $A$  by  $k$  is called the scalar multiple of  $A$  by  $k$  and is denoted by  $kA$ .

Thus, if  $A = [a_{ij}]_{m \times n}$ , then  $kA = [ka_{ij}]_{m \times n}$ .

e.g. If  $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ , then  $2A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \\ 2 & 6 & 2 \end{bmatrix}$

## Properties of Scalar Multiplication

If  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{m \times n}$  are two matrices and  $\lambda$  and  $\mu$  are two scalars, then

- (i)  $\lambda(A + B) = \lambda A + \lambda B$       (ii)  $(\lambda + \mu)A = \lambda A + \mu A$   
 (iii)  $(\lambda)\mu A = \mu(\lambda A) = \lambda(\mu A)$       (iv)  $(-\lambda)A = -(\lambda A) = \lambda(-A)$

## Multiplication of Two Matrices

If  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{n \times p}$  are two matrices such that the number of columns of  $A$  is equal to the number of rows of  $B$ , then a matrix  $C = [c_{ij}]_{m \times p}$  of order  $m \times p$  is known as product of matrices  $A$  and  $B$ , where

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

On multiplication of matrices is denoted by  $C = AB$

e.g. If  $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 5 & 2 \\ 1 & 7 \end{bmatrix}$ , then

$$AB = \begin{bmatrix} 2 \times 5 + 1 \times 1 & 2 \times 2 + 1 \times 7 \\ 3 \times 5 + 5 \times 1 & 3 \times 2 + 5 \times 7 \end{bmatrix} = \begin{bmatrix} 11 & 11 \\ 20 & 41 \end{bmatrix}$$

**Note** In multiplication of two matrices  $A$  and  $B$ , the order roles an important role.

## Properties of Multiplication of Matrices

Let  $A = [a_{ij}]_{m \times n}$ ,  $B = [b_{ij}]_{n \times p}$  and  $C = [c_{ij}]_{p \times m}$  are three matrices, then

- (i) Generally, matrix multiplication is not commutative,  
 i.e.  $AB \neq BA$   
 (ii) Matrix multiplication is associative,  
 i.e.  $A(BC) = (AB)C$   
 (iii) Matrix multiplication is distributive over matrix addition.  
 i.e.  $A(B + C) = AB + AC$   
 (iv) If  $A$  is a  $m \times n$  matrix and  $I_n$  is an identity matrix of order  $n \times n$  and  $I_m$  is an identity matrix of order  $m \times m$ , then

$$I_m A = A \quad \text{and} \quad A I_n = A$$

In particular, if  $A$  is a square matrix of order  $n$ , then

$$A I_n = I_n A = A$$

(v)  $AB = O$  does not necessarily imply that  $A = O$  or  $B = O$  or both  $A$  and  $B$  are  $O$ .

e.g. If  $A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \neq O$  and  $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq O$

But  $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

**Example 3.** If  $A = \begin{bmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{bmatrix}$ , ( $\theta = \frac{\pi}{24}$ )

and  $A^5 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  where  $i = \sqrt{-1}$ , then which one of the following is not true? (JEE Main 2020)

- (a)  $a^2 - d^2 = 0$       (b)  $a^2 - c^2 = 1$   
 (c)  $a^2 - b^2 = 1$       (d)  $0 \leq a^2 + b^2 \leq 1$

**Sol.** (c) Since, matrix  $A = \begin{bmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} \cos 2\theta & i \sin 2\theta \\ i \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} \cos 3\theta & i \sin 3\theta \\ i \sin 3\theta & \cos 3\theta \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} \cos 4\theta & i \sin 4\theta \\ i \sin 4\theta & \cos 4\theta \end{bmatrix}$$

$$\Rightarrow A^5 = \begin{bmatrix} \cos 5\theta & i \sin 5\theta \\ i \sin 5\theta & \cos 5\theta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow a = d = \cos 5\theta, \quad b = c = i \sin 5\theta$$

$$\therefore a^2 - b^2 = \cos^2 5\theta - i^2 \sin^2 5\theta = 1$$

Hence, option (c) is correct.

**Example 4.** Let  $\alpha$  be a root of the equation  $x^2 + x + 1 = 0$

and the matrix  $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix}$ , then the matrix  $A^{31}$  is

equal to (JEE Main 2020)

- (a)  $A^3$       (b)  $I_3$       (c)  $A^2$       (d)  $A$

**Sol.** (a) It is given that ' $\alpha$ ' is the root of equation  $x^2 + x + 1 = 0$ , so  $\alpha^2 + \alpha + 1 = 0$  and  $\alpha^3 = 1$ .

$$\text{Now, matrix } A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

[ $\because \alpha^3 = 1$ ]

$$\therefore A^2 = \left( \frac{1}{\sqrt{3}} \right)^2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1+1+1 & 1+\alpha+\alpha^2 & 1+\alpha^2+\alpha \\ 1+\alpha+\alpha^2 & 1+\alpha^2+\alpha^4 & 1+\alpha^3+\alpha^3 \\ 1+\alpha^2+\alpha & 1+\alpha^3+\alpha^3 & 1+\alpha^4+\alpha^2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

[ $\because \alpha^3=1$  and  $\alpha^2+\alpha+1=0$ ]

$$\begin{aligned} \therefore A^4 &= A^2 \cdot A^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \\ \therefore A^8 &= I = A^{16} = A^{20} = A^{24} = A^{28} \\ \therefore A^{31} &= A^{28} \cdot A^3 = I \cdot A^3 = A^3 \Rightarrow A^{31} = A^3 \end{aligned}$$

## Trace of a Matrix

Let  $A = [a_{ij}]_{n \times n}$  be a square matrix. Then, the sum of all diagonal elements of  $A$  is called the trace of  $A$  and is denoted by  $\text{tr}(A)$ .

Thus,  $\text{tr}(A) = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$

e.g.  $A = \begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$ , then  $\text{tr}(A) = 2 + 3 + 4 = 9$

## Properties of Trace of a Matrix

Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are two square matrix of order  $n$ , then

- (i)  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
- (ii)  $\text{tr}(AB) = \text{tr}(BA)$
- (iii)  $\text{tr}(\lambda A) = \lambda \text{tr}(A)$ , where  $\lambda$  is a scalar.
- (iv)  $\text{tr}(A') = \text{tr}(A)$

**Example 5.** Consider three matrices  $X = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$ ,  $Y = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$

and  $Z = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix}$ . Then, the value of the sum

$$\text{tr}(X) + \text{tr}\left(\frac{XYZ}{2}\right) + \text{tr}\left(\frac{X(YZ)^2}{4}\right) + \text{tr}\left(\frac{X(YZ)^3}{8}\right) + \dots \text{ to } \infty \text{ is}$$

- (a) 6      (b) 9      (c) 12      (d) None of these

**Sol.** (a) Here,  $YZ = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned} \therefore \text{tr}(X) + \text{tr}\left(\frac{XYZ}{2}\right) + \text{tr}\left(\frac{X(YZ)^2}{4}\right) + \text{tr}\left(\frac{X(YZ)^3}{8}\right) + \dots \\ = \text{tr}(X) + \text{tr}\left(\frac{X}{2}\right) + \text{tr}\left(\frac{X}{4}\right) + \dots \\ = \text{tr}(X) + \frac{1}{2} \text{tr}(X) + \frac{1}{4} \text{tr}(X) + \dots \\ = \text{tr}(X) \left[ 1 + \frac{1}{2} + \frac{1}{4} + \dots \right] \\ = \text{tr}(X) \frac{1}{1 - \frac{1}{2}} \\ = 2 \text{tr}(X) = 2(2 + 1) = 6 \end{aligned}$$

## Transpose of a Matrix

If  $A = [a_{ij}]_{m \times n}$  is a matrix of order  $m \times n$ , then the transpose of  $A$  can be obtained by changing all rows to columns and all columns to rows i.e. transpose of  $A = [a_{ji}]_{n \times m}$ . It is denoted by  $A'$ ,  $A^T$  or  $A^t$ .

e.g. If  $A = \begin{bmatrix} 1 & 3 & 5 \\ 6 & 9 & 4 \end{bmatrix}$ , then  $A' = \begin{bmatrix} 1 & 6 \\ 3 & 9 \\ 5 & 4 \end{bmatrix}$

## Properties of Transpose

If  $A$  and  $B$  are two matrices and  $k$  is a scalar, then

- (i)  $(A')' = A$
- (ii)  $(A + B)' = A' + B'$
- (iii)  $(kA)' = kA'$
- (iv)  $(AB)' = B' A'$  (reversal law)

**Note** If  $A, B$  and  $C$  are any three matrices conformable for multiplication, then  $(ABC)' = C' B' A'$

**Example 6.** Let  $a, b, c \in \mathbb{R}$  be all non-zero and satisfy

$a^3 + b^3 + c^3 = 2$ . If the matrix  $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$  satisfies  $A^T A = I$ ,

then a value of  $abc$  can be

- (a)  $-\frac{1}{3}$       (b)  $\frac{1}{3}$       (c) 3      (d)  $\frac{2}{3}$  (JEE Main 2020)

**Sol.** (a) Given matrix,  $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

$$\begin{aligned} \text{So, } A^T A &= \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \\ &= \begin{bmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ac + ba + cb \\ ba + cb + ac & b^2 + c^2 + a^2 & bc + ca + ab \\ ca + ab + bc & cb + ac + ba & c^2 + a^2 + b^2 \end{bmatrix} \end{aligned}$$

$\therefore A^T A = I$  [given]

$\therefore a^2 + b^2 + c^2 = 1$  and  $ab + bc + ca = 0$  ...(i)

Now, as  $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow (a + b + c)^2 = 1 \Rightarrow a + b + c = \pm 1$  ...(ii)

As we know,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$\Rightarrow 2 - 3abc = (a + b + c)(1 - 0)$

[from Eq. (i) and  $a^3 + b^3 + c^3 = 2$  (given)]

$\Rightarrow -3abc = (a + b + c) - 2 \Rightarrow abc = \frac{2 - (a + b + c)}{3}$

If  $a + b + c = 1 \Rightarrow abc = \frac{1}{3}$ , or

If  $a + b + c = -1 \Rightarrow abc = 1$

Now, from the options  $abc = \frac{1}{3}$

Hence, option (b) is correct.

**Example 7.** Let  $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$ . If  $AA^T = I_3$ , then  $|p|$  is

- (a)  $\frac{1}{\sqrt{5}}$  (b)  $\frac{1}{\sqrt{2}}$  (c)  $\frac{1}{\sqrt{3}}$  (d)  $\frac{1}{\sqrt{6}}$  (JEE Main 2019)

**Sol.** (b) Given,  $AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix} \begin{bmatrix} 0 & p & p \\ 2q & q & -q \\ r & -r & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 + 4q^2 + r^2 & 0 + 2q^2 - r^2 & 0 - 2q^2 + r^2 \\ 0 + 2q^2 - r^2 & p^2 + q^2 + r^2 & p^2 - q^2 - r^2 \\ 0 - 2q^2 + r^2 & p^2 - q^2 - r^2 & p^2 + q^2 + r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that, if two matrices are equal, then corresponding elements are also equal, so

$$4q^2 + r^2 = 1 = p^2 + q^2 + r^2, \quad \dots \text{(i)}$$

$$2q^2 - r^2 = 0 \Rightarrow r^2 = 2q^2 \quad \dots \text{(ii)}$$

$$\text{and } p^2 - q^2 - r^2 = 0 \quad \dots \text{(iii)}$$

Using Eqs. (ii) and (iii), we get

$$p^2 = 3q^2 \quad \dots \text{(iv)}$$

Using Eqs. (ii) and (iv) in Eq. (i), we get

$$4q^2 + 2q^2 = 1$$

$$\Rightarrow 6q^2 = 1 \Rightarrow 2p^2 = 1 \quad [\text{using Eq. (iv)}]$$

$$p^2 = \frac{1}{2} \Rightarrow |p| = \frac{1}{\sqrt{2}}$$

## Special Types of Matrices

### Nilpotent Matrix

If  $A^k = O$  and  $A^{k+1} \neq O$ , where  $k$  is a positive integer and  $O$  is a null matrix, then  $A$  is called nilpotent matrix,  $k$  is called the index of the nilpotent matrix  $A$ .

### Periodic Matrix

If  $A^{k+1} = A$ , where  $k$  is a positive integer, then  $A$  is known as periodic matrix,  $k$  is known as period of matrix  $A$ .

For  $k = 1$ ,  $A^2 = A$ .

### Idempotent Matrix

If  $A^2 = A$ , then square matrix  $A$  is known as idempotent matrix.

### Involutory Matrix

If  $A^2 = I$ , where  $I$  is an identity matrix, then  $A$  is called an involutory matrix.

### Symmetric Matrix

If for a square matrix  $A$ ,  $A' = A$ , then  $A$  is known as symmetric matrix.

i.g. If  $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ ,

then  $A' = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \Rightarrow A' = A$

Thus, matrix  $A$  is a symmetric matrix.

### Skew-symmetric Matrix

A square matrix  $A = [a_{ij}]_{m \times m}$  is said to be a skew-symmetric matrix, if

(a)  $a_{ij} = -a_{ji}, \forall i, j$

(b) Each element of diagonal is zero.

OR

A square matrix  $A$  is said to be a skew-symmetric matrix, if  $A' = -A$ .

e.g. If  $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$ ,

then  $A' = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & -3 \\ 3 & -3 & 0 \end{bmatrix} = -A$

Hence,  $A$  is a skew-symmetric matrix.

**Note** Every square matrix can be uniquely expressed as the sum of symmetric and skew-symmetric matrix.

i.e.  $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

### Orthogonal Matrix

If the product of a square matrix and its transpose  $A'$  is an identity matrix, then matrix  $A$  is said to be an orthogonal matrix.

i.e.  $AA' = I = A' A$

e.g. Let  $A = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$

$\therefore A' = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

Also,  $AA' = \frac{1}{2\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

$$= \frac{1}{8} \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow AA' = I$

Similarly,  $A' A = I$

Thus,  $A$  is an orthogonal matrix.

If  $A$  and  $B$  are two orthogonal matrices, then  $AB$  will also be an orthogonal matrix.

**Example 8.** If  $A, B$  and  $A + B$  are idempotent matrices, then  $AB$  is equal to

- (a)  $BA$  (b)  $-BA$  (c)  $I$  (d)  $O$

**Sol.** (b) Since,  $A, B$  and  $A + B$  are idempotent matrix

$$\therefore A^2 = A; B^2 = B \text{ and } (A + B)^2 = A + B$$

$$\text{Now, consider } (A + B)^2 = A + B$$

$$\Rightarrow A^2 + B^2 + AB + BA = A + B$$

$$\Rightarrow A + B + AB + BA = A + B \Rightarrow AB = -BA$$

**Example 9.** If  $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ , then  $A$  is a/an (JEE Main 2021)

- (a) idempotent matrix (b) nilpotent matrix  
(c) involutory matrix (d) None of these

**Sol.** (b) Since,  $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$   
 $\therefore A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$   
 $\Rightarrow A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Thus,  $A$  is a nilpotent matrix of order 2.

**Example 10.** If  $A$  and  $B$  are symmetric matrices of the same order and  $X = AB + BA$  and  $Y = AB - BA$ , then  $(XY)^T$  is equal to

- (a)  $XY$  (b)  $YX$   
(c)  $-YX$  (d) None of these

**Sol.** (c) Since,  $A$  and  $B$  are symmetric matrices

$$\therefore X = AB + BA$$

will be a symmetric matrix and  $Y = AB - BA$  will be a skew-symmetric matrix.

$$\text{Thus, we get } X^T = X \text{ and } Y^T = -Y$$

$$\text{Now, consider } (XY)^T = Y^T X^T = (-Y)(X) = -YX$$

**Example 11.** If  $A$  is a symmetric matrix and  $B$  is a

skew-symmetric matrix such that  $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$  then  $AB$  is

equal to (JEE Main 2019)

- (a)  $\begin{bmatrix} -4 & -2 \\ -1 & 4 \end{bmatrix}$  (b)  $\begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$  (c)  $\begin{bmatrix} 4 & -2 \\ 1 & -4 \end{bmatrix}$  (d)  $\begin{bmatrix} -4 & 2 \\ 1 & 4 \end{bmatrix}$

**Sol.** (b) Given matrix  $A$  is a symmetric and matrix  $B$  is a skew-symmetric.

$$\therefore A^T = A \text{ and } B^T = -B$$

$$\text{Since, } A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} \quad (\text{given}) \quad \dots (i)$$

On taking transpose both sides, we get

$$(A + B)^T = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}^T$$

$$\Rightarrow A^T + B^T = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} \quad \dots (ii)$$

$$\text{Given, } A^T = A \text{ and } B^T = -B$$

$$\Rightarrow A - B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$$

On solving Eqs. (i) and (ii), we get

$$A = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\text{So, } AB = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$$

**Example 12.** If  $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ , then the matrix  $A$  is

- (a) symmetric matrix  
(b) skew-symmetric matrix  
(c) identity matrix  
(d) orthogonal matrix

**Sol.** (d) Here,  $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$   
 $\Rightarrow A' = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}' = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$   
 $\therefore A'A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}' \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$   
 $= \begin{bmatrix} (\cos \alpha)(\cos \alpha) + (\sin \alpha)(\sin \alpha) & (\cos \alpha)(\sin \alpha) + (-\sin \alpha)(\cos \alpha) \\ (\sin \alpha)(\cos \alpha) + (\cos \alpha)(-\sin \alpha) & (\sin \alpha)(\sin \alpha) + (\cos \alpha)(\cos \alpha) \end{bmatrix}$   
 $= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & \sin \alpha \cdot \cos \alpha - \sin \alpha \cdot \cos \alpha \\ \sin \alpha \cdot \cos \alpha - \sin \alpha \cdot \cos \alpha & \cos^2 \alpha + \sin^2 \alpha \end{bmatrix}$   
 $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$

So,  $A$  is an orthogonal matrix.

## Elementary Row Transformations

Any one of the following operations on a matrix is called an elementary row (or column) transformation.

(i) Interchanging any two rows (or columns). This transformation is indicated by

$$R_i \leftrightarrow R_j \text{ (or } C_i \leftrightarrow C_j)$$

(ii) Multiplication of the elements of any row (or column) by a non-zero scalar quantity. This transformation is indicated as

$$R_i \leftrightarrow kR_i \text{ (or } C_i \leftrightarrow kC_i)$$

(iii) Addition of a constant multiple of the elements of any row to the corresponding element of any other row. This transformation is indicated as

$$R_i \rightarrow R_i + kR_j.$$

## Elementary Matrix

A matrix obtained from an identity matrix by a single elementary operation is called an elementary matrix.

## Inverse of a Matrix using Elementary Row Transformation

Let  $A = IA$

If matrix  $A$  (LHS) is reduced to  $I$  by elementary row transformation, then suppose  $I$  (RHS) is reduced to  $P$  and not change  $A$  in RHS i.e., after transformation, we get  $I = PA$ , then  $P$  is the inverse of  $A$ ,

$$\therefore P = A^{-1}$$

**Example 13.** The inverse of  $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$  by elementary

row transformation is

$$(a) \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$$

$$(b) \frac{1}{5} \begin{bmatrix} 2 & 0 & 3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$$

$$(c) -\frac{1}{5} \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$$

(d) None of the above

**Sol.** (c) Let  $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ .

We know that,  $A = IA$

$$\therefore \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad [\text{using } R_1 \rightarrow R_1 + R_2 - R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -5 \\ 0 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -3 & -3 & 4 \end{bmatrix} A \quad [\text{using } R_2 \leftrightarrow R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 3/5 & 3/5 & -4/5 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A$$

$$[\text{using } R_2 \rightarrow -\frac{1}{5}R_2 \text{ and } R_3 \rightarrow \frac{-1}{5}R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 1/5 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A$$

$$[\text{using } R_2 \rightarrow R_2 - 2R_3 \text{ and } R_1 \rightarrow R_1 - 4R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A$$

$$[\text{using } R_1 \rightarrow R_1 - R_2]$$

$$\Rightarrow I_3 = -\frac{1}{5} \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix} A$$

$$\therefore A^{-1} = -\frac{1}{5} \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix} \quad [:\ AA^{-1} = I_3]$$

# Practice Exercise

## ROUND I Topically Divided Problems

### Types of Algebra of Matrix

- The non-zero values of  $x$  satisfying the matrix equation  $x \begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2 \begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2 \begin{bmatrix} (x^2+8) & 24 \\ (10) & 6x \end{bmatrix}$  is  
 (a) 1 (b) 2 (c) 3 (d) 4
- If  $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ , then value of  $\alpha$  for which  $A^2 = B$  is  
 (a) 1 (b) -1  
 (c) 4 (d) No real values
- Let  $A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$ , ( $\alpha \in R$ ) such that  $A^{32} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Then, a value of  $\alpha$  is  
 (JEE Main 2019, 8 April)  
 (a)  $\frac{\pi}{32}$  (b) 0  
 (c)  $\frac{\pi}{64}$  (d)  $\frac{\pi}{16}$
- The minimum number of zero's in an upper triangular matrix of order  $n \times n$ , is  
 (a)  $\frac{n(n-1)}{2}$  (b)  $\frac{n(n+1)}{2}$   
 (c)  $\frac{n(n^2-1)}{2}$  (d) None of these
- If  $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$  and  $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$ , then the values of  $k, a, b$  are respectively  
 (a) -6, -12, -18  
 (b) -6, 4, 9  
 (c) -6, -4, -9  
 (d) -6, 12, 18
- If  $A = [a_{ij}]_{n \times n}$  be a diagonal matrix with diagonal element all different and  $B = [b_{ij}]_{n \times n}$  be some another matrix. Let  $AB = [c_{ij}]_{n \times n}$ , then  $c_{ij}$  is equal to  
 (a)  $a_{jj} b_{ij}$  (b)  $a_{ii} b_{ij}$   
 (c)  $a_{ij} b_{ij}$  (d)  $a_{ij} b_{ji}$

$$7. \text{ If } A = \frac{1}{\pi} \begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix},$$

$$B = \frac{1}{\pi} \begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix},$$

then  $A - B$  is equal to

- (a)  $I$  (b)  $O$  (c)  $2I$  (d)  $\frac{1}{2}I$
- If matrix  $A = [a_{ij}]_{2 \times 2}$ , where  $a_{ij} = 1$ , if  $i \neq j$  and  $a_{ij} = 0$ , if  $i = j$ , then  $A^2$  is equal to  
 (a)  $I$  (b)  $A$   
 (c)  $O$  (d) None of these
  - If  $A$  is a square matrix such that  $A^2 = I$ , then  $(A - I)^3 + (A + I)^3 - 7A$  is equal to  
 (a)  $A$  (b)  $I - A$   
 (c)  $I + A$  (d)  $3A$
  - If  $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , then which of the following is correct?  
 (a)  $(A + B) \cdot (A - B) \neq A^2 - B^2$   
 (b)  $(A + B) \cdot (A - B) = A^2 - B^2$   
 (c)  $(A + B) \cdot (A - B) = I$   
 (d) None of the above
  - If  $A$  is square matrix such that  $A^2 = A$ , then  $(A + I)^3$  is equal to  
 (a)  $A + 1$  (b)  $7A + I$   
 (c)  $3A + I$  (d)  $A - I$
  - If  $A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$  and  $I$  is the identity matrix of order 2, then  $(I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$  is equal to  
 (a)  $A$  (b)  $I$   
 (c)  $I + A$  (d) None of these

13. If  $\omega \neq 1$  is the complex cube root of unity and matrix

$$H = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}, \text{ then } H^{70} \text{ is equal to}$$

- (a)  $H$  (b)  $0$  (c)  $-H$  (d)  $H^2$

14. For each real number  $x$  such that  $-1 < x < 1$ , let

$$A(x) = \begin{bmatrix} 1 & -x \\ 1-x & 1-x \\ -x & 1 \\ 1-x & 1-x \end{bmatrix} \text{ and } z = \frac{x+y}{1+xy}. \text{ Then,}$$

- (a)  $A(z) = A(x) + A(y)$  (b)  $A(z) = A(x) [A(y)]^{-1}$   
 (c)  $A(z) = A(x) \cdot A(y)$  (d)  $A(z) = A(x) - A(y)$

15. If  $A = \text{diag} \{2, -1, 3\}$  and  $B = \text{diag} \{-1, 3, 2\}$ , then  $A^2 B$  is

- (a)  $\text{diag} \{5, 4, 11\}$  (b)  $\text{diag} \{-4, 3, 18\}$   
 (c)  $\text{diag} \{3, 1, 8\}$  (d)  $B$

16. If  $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$  and  $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ , then the sum

of the elements of the matrix  $3X - 4Y$  is equal to

- (a) 14 (b) 16 (c) 12 (d) 25

17. If  $P = \begin{bmatrix} i & 0 & -i \\ 0 & -i & i \\ -i & i & 0 \end{bmatrix}$  and  $Q = \begin{bmatrix} -i & i \\ 0 & 0 \\ i & -i \end{bmatrix}$ , then  $PQ$  is

equal to

- (a)  $\begin{bmatrix} -2 & 2 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$  (b)  $\begin{bmatrix} 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$  (c)  $\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$  (d)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. A square matrix  $P$  satisfies  $P^2 = I - P$ , where  $I$  is the identity matrix. If  $P^n = 5I - 8P$ , then  $n$  is equal to

- (a) 4 (b) 5 (c) 6 (d) 7

19. Let  $A = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$

and  $B = \begin{bmatrix} \cos^2 \phi & \sin \phi \cos \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$ , then  $AB = O$ , if

- (a)  $\theta = n\phi, n = 0, 1, 2, \dots$   
 (b)  $\theta + \phi = n\pi, n = 0, 1, 2, \dots$   
 (c)  $\theta = \phi + (2n + 1)\frac{\pi}{2}, n = 0, 1, 2, \dots$   
 (d)  $\theta = \phi + n\frac{\pi}{2}, n = 0, 1, 2, \dots$

20. If  $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  and  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , then which one of the

following holds for all  $n \geq 1$ , by the principle of Mathematical Induction

- (a)  $A^n = nA + (n-1)I$  (b)  $A^n = 2^{n-1} A + (n-1)I$   
 (c)  $A^n = nA - (n-1)I$  (d)  $A^n = 2^{n-1} A - (n-1)I$

21. Let  $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$  and

$$2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}. \text{ If } t_r(A) \text{ denotes the sum of}$$

all diagonal elements of the matrix  $A$ , then

$t_r(A) - t_r(B)$  has value equal to (JEE Main 2021)

- (a) 1 (b) 2  
 (c) 0 (d) 3

22. If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  satisfies the equation

$$x^2 - (a+d)x + k = 0, \text{ then}$$

- (a)  $k = bc$  (b)  $k = ad$   
 (c)  $k = a^2 + b^2 + c^2 + d^2$  (d)  $k = ad - bc$

23. If  $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ , then  $A^n$  is equal to

- (a)  $\begin{bmatrix} -\cos n\theta & \sin n\theta \\ -\sin n\theta & \sin n\theta \end{bmatrix}$  (b)  $\begin{bmatrix} \sin n\theta & \sin n\theta \\ -\sin n\theta & \sin n\theta \end{bmatrix}$   
 (c)  $\begin{bmatrix} \sin n\theta & -\sin n\theta \\ \sin n\theta & \sin n\theta \end{bmatrix}$  (d) None of these

24. If  $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ , then  $A^2 + A$  is equal to

- (a)  $A + I$  (b)  $A^2 + I$   
 (c)  $A(A + I)$  (d) None of these  
 (where  $I$  is  $3 \times 3$  unit matrix)

25. If  $A$  and  $B$  are square matrices of the same order such that  $AB = BA$ , then  $AB^n$  is equal to

- (a)  $A^n B$  (b)  $AB^n$   
 (c)  $AB$  (d)  $B^n A$

26. Elements of a matrix  $A$  of order  $10 \times 10$  are defined as  $a_{ij} = \omega^{i+j}$  (where,  $\omega$  is cube root of unity), then  $\text{tr}(A)$  of the matrix is

- (a) 0 (b)  $\omega$   
 (c) 3 (d) None of these

## Transpose and Special Types of Matrix

27. The total number of matrices  $A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}$ ,

( $x, y \in \mathbb{R}, x \neq y$ ) for which  $A^T A = 3I_3$  is (JEE Main 2019)

- (a) 2 (b) 4  
 (c) 3 (d) 6

28. If  $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$ ,  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  and  $Q = PAP^T$  and

$X = P^T Q^{2005} P$ , then  $X$  is

- (a)  $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$   
 (b)  $\begin{bmatrix} 4 + 2005\sqrt{3} & 6015 \\ 2005 & 4 - 2005\sqrt{3} \end{bmatrix}$   
 (c)  $\begin{bmatrix} 1 & 0 \\ 2005 & 1 \end{bmatrix}$   
 (d)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

29. Let  $M = [a_{uv}]_{n \times n}$  be a matrix, where

$$a_{uv} = \sin(\theta_u - \theta_v) + i \cos(\theta_u - \theta_v),$$

then  $\bar{M}$  is equal to

- (a)  $\bar{M}$  (b)  $-\bar{M}$  (c)  $\bar{M}^T$  (d)  $-\bar{M}^T$

30. If  $A$  and  $B$  are matrices of same order, then  $(AB' - BA')$  is a

- (a) skew-symmetric matrix (b) null matrix  
 (c) symmetric matrix (d) unit matrix

31. The matrix  $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$  is

- (a) unitary (b) orthogonal  
 (c) nilpotent (d) involutory

32. If  $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$  is the sum of a symmetric matrix

$B$  and skew-symmetric matrix,  $C$ , then  $B$  is

- (a)  $\begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$  (b)  $\begin{bmatrix} 0 & 2 & -2 \\ -2 & 5 & -2 \\ 2 & 2 & 0 \end{bmatrix}$   
 (c)  $\begin{bmatrix} 6 & 6 & 7 \\ -6 & 2 & -5 \\ -7 & 5 & 1 \end{bmatrix}$  (d)  $\begin{bmatrix} 0 & 6 & -2 \\ 2 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}$

33. If  $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and  $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , then

$$E^2 F + F^2 E$$

- (a)  $F$  (b)  $E$   
 (c) 0 (d) None of these

34. If  $A$  and  $B$  are two invertible matrices and both are symmetric and commute each other, then

- (a) both  $A^{-1}B$  and  $A^{-1}B^{-1}$  are symmetric  
 (b) neither  $A^{-1}B$  nor  $A^{-1}B^{-1}$  are symmetric  
 (c)  $A^{-1}B$  is symmetric but  $A^{-1}B^{-1}$  is not symmetric  
 (d)  $A^{-1}B^{-1}$  is symmetric but  $A^{-1}B$  is not symmetric

35. The matrix  $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$  is

- (a) idempotent (b) nilpotent  
 (c) involutory (d) orthogonal

36. If  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$  and  $A^T A = AA^T = I$ , then  $xy$  is

equal to

- (a) -1 (b) 1 (c) 2 (d) -2

37. If  $A$  and  $B$  are symmetric matrices of the same order and  $X = AB + BA$  and  $Y = AB - BA$ , then  $(XY)^T$  is equal to

- (a)  $XY$  (b)  $YX$   
 (c)  $-YX$  (d) None of these

38. An orthogonal matrix is

- (a)  $\begin{bmatrix} \cos \alpha & 2 \sin \alpha \\ -2 \sin \alpha & \cos \alpha \end{bmatrix}$  (b)  $\begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$   
 (c)  $\begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$  (d)  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

39. If  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , then  $A^T + A = I_2$ , if

- (a)  $\theta = n\pi, n \in Z$  (b)  $\theta = (2n + 1)\pi/2, n \in Z$   
 (c)  $\theta = 2n\pi + \pi/3, n \in Z$  (d) None of these

40. If for the matrix,  $A = \begin{bmatrix} 1 & -\alpha \\ \alpha & \beta \end{bmatrix}$ ,  $AA^T = I_2$ , then the

value of  $\alpha^4 + \beta^4$  is

(JEE Main 2021)

- (a) 1 (b) 3 (c) 2 (d) 4

41.  $A$  and  $B$  are two square matrices such that  $A^2 B = BA$  and if  $(AB)^{10} = A^k B^{10}$ . Then,  $k$  is

- (a) 1001 (b) 1023  
 (c) 1042 (d) None of these

42. If  $A$  and  $B$  are two square matrices such that  $AB = B$  and  $BA = A$ , then for  $n \in N$   $(A + B)^n$  is

- (a)  $2^n (A + B)$  (b)  $2^{n-1} (A + B)$   
 (c)  $2^{n+1} (A + B)$  (d)  $2^{n/2} (A + B)$

43. Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  be a matrix such that

$$a + d = 1 = ad - bc, \text{ then } A^{2010} \text{ is}$$

- (a)  $I_2$  (b)  $-I_2$   
 (c)  $A$  (d) null matrix

44. Let  $A, B$ , are square matrices of same order satisfying  $AB = A$  and  $BA = B$ , then

$(A^{2010} + B^{2010})^{2011}$  equals

- (a)  $A + B$  (b)  $2010 (A + B)$   
 (c)  $2011 (A + B)$  (d)  $2^{2011} (A + B)$

## ROUND II Mixed Bag

1. Let  $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}$  and  $Q = [q_{ij}]$  be two  $3 \times 3$  matrices such that  $Q - P^5 = I_3$ . Then,  $\frac{q_{21} + q_{31}}{q_{32}}$  is equal to

(JEE Main 2019)

(a) 10      (b) 135      (c) 9      (d) 15

2. If  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$  is a matrix satisfying the equation  $AA^T = 9I$ , where  $I$  is  $3 \times 3$  identity matrix, then the ordered pair  $(a, b)$  is equal to (JEE Main 2015)
- (a) (2, -1)      (b) (-2, 1)  
(c) (2, 1)      (d) (-2, -1)

3. If  $A = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ , then  $B^T A^T$  is
- (a) null matrix  
(b) an identity matrix  
(c) scalar but not an identity matrix  
(d) such that  $T_r(B^T A^T) = 4$

4. If  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  and  $A^{16} = \lambda I$ , then the value of  $\lambda$  is
- (a) 2      (b) 1      (c) 0      (d) 3

5. Let  $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ ,  $i = \sqrt{-1}$ , then the system of linear equations  $A^8 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$  has
- (JEE Main 2021)

(a) a unique solution  
(b) Infinitely many solution  
(c) No solution  
(d) Exactly two solutions

6. Let  $A$  is a  $3 \times 3$  matrix and  $A = [a_{ij}]_{3 \times 3}$ . If for every column matrix  $X$ , if  $X'AX = 0$  and  $a_{23} = -2009$ , then  $a_{32} = \dots$
- (a) 2009      (b) -2009  
(c) 0      (d) 2008

7. If  $A = \begin{bmatrix} -1+i\sqrt{3} & -1-i\sqrt{3} \\ 2i & 2i \\ 1+i\sqrt{3} & 1-i\sqrt{3} \\ 2i & 2i \end{bmatrix}$ ,  $i = \sqrt{-1}$  and

$f(x) = x^2 + 2$ , then  $f(A)$  is equal to

(a)  $\begin{pmatrix} 5-i\sqrt{3} \\ 2 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$       (b)  $\begin{pmatrix} 3-i\sqrt{3} \\ 2 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$   
(c)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$       (d)  $(2+i\sqrt{3}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. Matrix  $A$  such that  $A^2 = 2A - I$ , where  $I$  is the identity matrix. Then, for  $n \geq 2$ ,  $A^n$  is equal to
- (a)  $nA - (n-1)I$   
(b)  $nA - I$   
(c)  $2^{n-1}A - (n-1)I$   
(d)  $2^{n-1}A - I$

9. If  $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ , then  $I + 2A + 3A^2 + \dots \infty$

(a)  $\begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix}$       (b)  $\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$   
(c)  $\begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$       (d)  $\begin{bmatrix} 5 & 2 \\ -3 & -8 \end{bmatrix}$

10. Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ , if  $u_1$  and  $u_2$  are column matrices

such that  $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , then  $u_1 + u_2$  is

equal to

(a)  $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$       (b)  $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$   
(c)  $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$       (d)  $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

11. If  $M$  is a  $3 \times 3$  matrix satisfying

$$M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

Then, the sum of the diagonal entries of  $M$  is

(a) 9      (b) 8      (c) 10      (d) 11

12. If  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$  and  $I$  is the unit matrix of order

3, then  $A^2 + 2A^4 + 4A^6$  is equal to

(a)  $7A^8$       (b)  $7A^7$       (c)  $8I$       (d)  $6I$

13. If  $\omega$  is a complex cube root of unity and  $A = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ ,

then  $A^{50}$  is equal to

(a)  $\omega^2 A$       (b)  $\omega A$       (c)  $A$       (d) 0

14. If  $A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ , then  $A^4$  is equal to

(a)  $27A$       (b)  $81A$       (c)  $243A$       (d)  $729A$



# Solutions

## Round I

1. Given that,

$$\begin{aligned} x \begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2 \begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} &= 2 \begin{bmatrix} (x^2 + 8) & 24 \\ 10 & 6x \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 2x^2 & 2x \\ 3x & x^2 \end{bmatrix} + \begin{bmatrix} 16 & 10x \\ 8 & 8x \end{bmatrix} &= \begin{bmatrix} 2x^2 + 16 & 48 \\ 20 & 12x \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 2x^2 + 16 & 2x + 10x \\ 3x + 8 & x^2 + 8x \end{bmatrix} &= \begin{bmatrix} 2x^2 + 16 & 48 \\ 20 & 12x \end{bmatrix} \end{aligned}$$

By equality of two matrices,

$$3x + 8 = 20 \Rightarrow 3x = 12 \Rightarrow x = 4$$

2. Given that,  $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 + 0 & 0 + 0 \\ \alpha + 1 & 0 + 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix}$$

Also,  $B = A^2$  (given)

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix}$$

Clearly, this is not satisfied by any real value of  $\alpha$ .

3. Given, matrix  $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \alpha - \sin^2 \alpha & -\cos \alpha \sin \alpha - \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha + \cos \alpha \sin \alpha & -\sin^2 \alpha + \cos^2 \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Similarly,  $A^n = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}$ ,  $n \in \mathbb{N}$

$$\Rightarrow A^{32} = \begin{bmatrix} \cos(32\alpha) & -\sin(32\alpha) \\ \sin(32\alpha) & \cos(32\alpha) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (\text{given})$$

So,  $\cos(32\alpha) = 0$  and  $\sin(32\alpha) = 1$

$$\Rightarrow 32\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{64}$$

4.

The minimum number of zeros in upper triangular matrix	Order
1	$2 \times 2$
$1 + 2 = 3$	$3 \times 3$
$1 + 2 + 3 = 6$	$4 \times 4$
$1 + 2 + 3 + 4 = 10$	$5 \times 5$
$1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2}$	$n \times n$

$$5. \therefore kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$\Rightarrow k \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$\Rightarrow 2k = 3a, \quad 3k = 2b, \quad -4k = 24$$

$$\Rightarrow a = \frac{2k}{3}, \quad b = \frac{3k}{2}, \quad k = -6$$

$$\therefore a = -4, \quad b = -9, \quad k = -6$$

6.  $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$  (in general)

And in a diagonal matrix non-diagonal elements are

$$\text{zero. i.e. } a_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ a_{ii}, & \text{if } i = j \end{cases}$$

So,  $c_{ij} = a_{ii} b_{ij}$

7. Given that,  $A = \frac{1}{\pi} \begin{bmatrix} \sin^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$

$$\text{and } B = \frac{1}{\pi} \begin{bmatrix} -\cos^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$$

$$\text{then } A - B = \frac{1}{\pi} \begin{bmatrix} \sin^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$$

$$- \frac{1}{\pi} \begin{bmatrix} -\cos^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$$

$$= \frac{1}{\pi} \begin{bmatrix} \sin^{-1}(\pi x) + \cos^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) - \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) - \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) + \tan^{-1}(\pi x) \end{bmatrix}$$

$$= \frac{1}{\pi} \begin{bmatrix} \frac{\pi}{2} & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix}$$

$$\left[ \because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \text{ and } \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \right]$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} I$$

8. Given that,  $a_{ij} = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}$

then  $A = [a_{ij}]_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix}$   
 $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

9. Given that,  $A^2 = I$  ... (i)

Now, we have

$$\begin{aligned} (A - I)^3 + (A + I)^3 - 7A &= (A)^3 - I^3 - 3AI(A - I) + (A)^3 + I^3 \\ &\quad + 3A \cdot I(A + I) - 7A \\ &= 2(A)^3 - 3A(A - I) + 3A(A + I) - 7A \\ &= 2(A^2 \cdot A) - 3A(A - I - A - I) - 7A \\ &= 2(I \cdot A) - 3A(-2I) - 7A \quad [\text{from Eq. (i)}] \\ &= 2A + 6A \cdot I - 7A \\ &= 2A + 6A - 7A = 8A - 7A = A \end{aligned}$$

10. Given that,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^2 &= A \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0+1 & 0+1 \\ 0+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and } B^2 &= B \cdot B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0-1 & 0+0 \\ 0+0 & -1+0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

$$\therefore A^2 - B^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\begin{aligned} \text{and } (A + B)(A - B) &= \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 4+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \end{aligned}$$

Hence,  $(A + B)(A - B) \neq A^2 - B^2$

11. Given that,  $A^2 = A$  ... (i)

$$\begin{aligned} \text{Now, we have } (I + A)^3 &= (I)^3 + (A)^3 + 3I \cdot A(I + A) \\ &= I + A^2 \cdot A + 3A(I + A) \\ & \quad [\because I^3 = I, I \cdot A = A] \end{aligned}$$

$$= I + A \cdot A + 3A(I + A) \quad [\because A^2 = A]$$

$$= I + A^2 + 3(A \cdot I + A^2)$$

$$= I + A + 3(A + A)$$

$$[\because A^2 = A]$$

$$= I + A + 3(2A)$$

$$= I + A + 6A = 7A + I$$

12. Here,  $A = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$ , where  $t = \tan\left(\frac{\alpha}{2}\right)$

$$\text{Now, } \cos \alpha = \frac{1 - \tan^2\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)} = \frac{1 - t^2}{1 + t^2}$$

$$\text{and } \sin \alpha = \frac{2 \tan\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)} = \frac{2t}{1 + t^2}$$

$$\text{Now, we have } (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$= \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} \right] \begin{bmatrix} \frac{1 - t^2}{1 + t^2} & \frac{-2t}{1 + t^2} \\ \frac{2t}{1 + t^2} & \frac{1 - t^2}{1 + t^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \begin{bmatrix} \frac{1 - t^2}{1 + t^2} & \frac{-2t}{1 + t^2} \\ \frac{2t}{1 + t^2} & \frac{1 - t^2}{1 + t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1 - t^2 + 2t^2}{1 + t^2} & \frac{-2t + t(1 - t^2)}{1 + t^2} \\ \frac{-t(1 - t^2) + 2t}{1 + t^2} & \frac{2t^2 + 1 - t^2}{1 + t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1 + t^2}{1 + t^2} & \frac{-2t + t - t^3}{1 + t^2} \\ \frac{-t + t^3 + 2t}{1 + t^2} & \frac{2t^2 + 1 - t^2}{1 + t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1 + t^2}{1 + t^2} & \frac{-t(1 + t^2)}{1 + t^2} \\ \frac{t(1 + t^2)}{1 + t^2} & \frac{1 + t^2}{1 + t^2} \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \quad \dots (i)$$

$$\text{Now, } I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & -t+0 \\ t+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \quad \dots (ii)$$

On putting the value of  $t$  in both equations, we get

$$\begin{bmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{bmatrix}$$

$$\therefore (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} = I + A$$

13. Clearly,  $H^2 = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$

$$H^3 = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^3 & 0 \\ 0 & \omega^3 \end{bmatrix}$$

$\therefore H^{70} = \begin{bmatrix} \omega^{70} & 0 \\ 0 & \omega^{70} \end{bmatrix} = \begin{bmatrix} \omega^{69} \cdot \omega & 0 \\ 0 & \omega^{69} \cdot \omega \end{bmatrix}$

$$= \begin{bmatrix} (\omega^3)^{23} \cdot \omega & 0 \\ 0 & (\omega^3)^{23} \cdot \omega \end{bmatrix}$$

$$= \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = H \quad [\because \omega^3 = 1]$$

14. We have,  $A(x) = \frac{1}{1-x} \begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix} \quad \dots(i)$

$\therefore A(y) = \frac{1}{1-y} \begin{bmatrix} 1 & -y \\ -y & 1 \end{bmatrix} \quad \dots(ii)$

and  $A(z) = \frac{1}{1 - \frac{(x+y)}{1+xy}} \begin{bmatrix} 1 & -\frac{(x+y)}{1+xy} \\ -\frac{(x+y)}{1+xy} & 1 \end{bmatrix}$

$$= \frac{1+xy}{1+xy-x-y} \begin{bmatrix} 1 & -\frac{(x+y)}{1+xy} \\ -\frac{(x+y)}{1+xy} & 1 \end{bmatrix}$$

$$= \frac{1+xy}{(1-x)(1-y)} \begin{bmatrix} 1 & -\frac{(x+y)}{1+xy} \\ -\frac{(x+y)}{1+xy} & 1 \end{bmatrix}$$

$$= \frac{1}{(1-x)(1-y)} \begin{bmatrix} 1+xy & -(x+y) \\ -(x+y) & 1+xy \end{bmatrix} \quad \dots(iii)$$

Now, consider

$$A(x) \cdot A(y) = \frac{1}{(1-x)(1-y)} \cdot \begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix} \begin{bmatrix} 1 & -y \\ -y & 1 \end{bmatrix}$$

$$= \frac{1}{(1-x)(1-y)} \cdot \begin{bmatrix} 1+xy & -(x+y) \\ -(x+y) & 1+xy \end{bmatrix} \quad \dots(iv)$$

From Eqs. (iii) and (iv), we get

$$A(z) = A(x) \cdot A(y).$$

15.  $A = \text{diag} \{2, -1, 3\}$  and  $B = \text{diag} \{-1, 3, 2\}$

$$\Rightarrow A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then,  $A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$\therefore A^2 B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 18 \end{bmatrix}$

$$= \text{diag} \{-4, 3, 18\}$$

16.  $X + Y + X - Y = 2X$

$$\therefore 2X = (X + Y) + (X - Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \therefore X = \frac{1}{2} \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Again,  $2Y = (X + Y) - (X - Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$\therefore Y = \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

$$3X - 4Y = 3 \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} - 4 \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ -1 & 8 \end{bmatrix}$$

Sum of element of  $3X - 4Y = 7 + 0 - 1 + 8 = 14$

17. Since,  $P = \begin{bmatrix} i & 0 & -i \\ 0 & -i & i \\ -i & i & 0 \end{bmatrix}$  and  $Q = \begin{bmatrix} -i & i \\ 0 & 0 \\ i & -i \end{bmatrix}$

$$\therefore PQ = \begin{bmatrix} i & 0 & -i \\ 0 & -i & i \\ -i & i & 0 \end{bmatrix} \begin{bmatrix} -i & i \\ 0 & 0 \\ i & -i \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 - i^2 & i^2 + i^2 \\ i^2 & -i^2 \\ i^2 & -i^2 \end{bmatrix} = \begin{bmatrix} 1+1 & -1-1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$$

18.  $\therefore P^3 = P(I - P) \quad (\because P^2 = I - P)$

$$= PI - P^2 = PI - (I - P) = P - I + P = 2P - I$$

Now,  $P^4 = P \cdot P^3$

$$\Rightarrow P^4 = P(2P - I)$$

$$\Rightarrow P^4 = 2P^2 - P$$

$$\Rightarrow P^4 = 2I - 2P - P$$

$$\Rightarrow P^4 = 2I - 3P \text{ and } P^5 = P(2I - 3P)$$

$$\Rightarrow P^5 = 2P - 3(I - P)$$

$$\Rightarrow P^5 = 5P - 3I$$

Also,  $P^6 = P(5P - 3I)$

$$\Rightarrow P^6 = 5P^2 - 3P$$

$$\Rightarrow P^6 = 5(I - P) - 3P$$

$$\Rightarrow P^6 = 5I - 8P$$

So,  $n = 6$

**Alternate Solution**

$$\therefore P^n = 5I - 8P = 5(I - P) - 3P$$

$$= P(5P - 3I) \quad [\because P^2 = I - P]$$

$$= P(2P - 3P^2) = P^2(2I - 3P)$$

$$= P^2[2(I - P) - P]$$

$$= P^2[2P^2 - P] = P^3[2P - I]$$

$$= P^4[I - P] = P^4 \cdot P^2 = P^6$$

$$\Rightarrow n = 6$$

$$\begin{aligned}
19. AB &= \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos^2 \phi & \sin \phi \cos \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix} \\
&= \begin{bmatrix} \cos^2 \theta \cos^2 \phi + \sin \theta \cos \phi \cos \theta \sin \phi & \cos^2 \theta \sin \phi \cos \phi + \sin^2 \theta \sin \theta \cos \theta \\ \cos^2 \phi \cos \theta \sin \theta + \sin^2 \theta \sin \phi \cos \phi & \cos \theta \sin \theta \sin \phi \cos \phi + \sin^2 \theta \sin^2 \phi \end{bmatrix} \\
&= \begin{bmatrix} \cos \theta \cos \phi \cos(\theta - \phi) & \sin \phi \cos \theta \cos(\theta - \phi) \\ \sin \theta \cos \phi \cos(\theta - \phi) & \sin \theta \sin \phi \cos(\theta - \phi) \end{bmatrix} \\
\therefore AB &= O \\
\Rightarrow \cos(\theta - \phi) &= 0 \\
\Rightarrow \cos(\theta - \phi) &= \cos(2n + 1) \frac{\pi}{2} \\
\Rightarrow \theta &= (2n + 1) \frac{\pi}{2} + \phi, \text{ where } n = 0, 1, 2, \dots
\end{aligned}$$

$$\begin{aligned}
20. A^2 &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \\
A^3 &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \\
\therefore A^n &= \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \\
nA &= \begin{bmatrix} n & 0 \\ n & n \end{bmatrix}, (n-1)I = \begin{bmatrix} n-1 & 0 \\ n & n-1 \end{bmatrix} \\
nA - (n-1)I &= \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n
\end{aligned}$$

$$\begin{aligned}
21. A + 2B &= \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix} \quad \dots(i) \\
2A - B &= \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix} \\
4A - 2B &= \begin{bmatrix} 4 & -2 & 10 \\ 4 & -2 & 12 \\ 0 & 2 & 4 \end{bmatrix} \quad \dots(ii)
\end{aligned}$$

Adding Eqs. (i) and (ii), we get

$$\begin{aligned}
5A &= \begin{bmatrix} 5 & 0 & 10 \\ 10 & -5 & 15 \\ -5 & 5 & 5 \end{bmatrix} \\
A &= \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow t_r(A) &= 1 - 1 + 1 = 1 \\
\Rightarrow t_r(B) &= 0 - 1 + 0 = -1 \\
\therefore t_r(A) - t_r(B) &= 2
\end{aligned}$$

$$\begin{aligned}
22. \text{ We have, } A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
\Rightarrow A^2 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} \\
(a+d) \cdot A &= \begin{bmatrix} (a+d) \cdot a & (a+d) \cdot b \\ (a+d) \cdot c & (a+d) \cdot d \end{bmatrix} \\
&= \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} \\
\therefore x^2 - (a+d)x + k &= 0 \\
\Rightarrow \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} bc - ad + k & 0 \\ 0 & bc - ad + k \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow k &= ad - bc
\end{aligned}$$

$$\begin{aligned}
23. \text{ Since, } A &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \\
\therefore A^2 &= A \cdot A \\
&= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \\
\text{Similarly, } A^n &= \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
24. A &= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}, \\
A^2 &= A \times A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1+0+0 & 0+0+(-1) & -1+0+(-1) \\ 2+2+0 & 0+1+3 & -2+3+3 \\ 0+2+0 & 0+1+1 & 0+3+1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & -2 \\ 4 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix} \\
\therefore A + I &= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \\
A^2 + A &= \begin{bmatrix} 1 & -1 & -2 \\ 4 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1+1 & -1+0 & -2+(-1) \\ 4+2 & 4+1 & 4+3 \\ 2+0 & 2+1 & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ 6 & 5 & 7 \\ 2 & 3 & 5 \end{bmatrix} \\
A(A+1) &= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}
\end{aligned}$$

25. Given that,  $AB = BA$  ... (i)

We want to prove that  $AB^n = B^n A$  ... (ii)

For  $n = 1$ , Eq. (ii) is obviously true.

Let Eq. (ii) be true for a positive integer  $n = m$ .

i.e.  $AB^m = B^m A$  ... (iii)

Then, for  $n = m + 1$ ,  $AB^{m+1} = A(B^m B) = (AB^m)B$

(associative law of matrix multiplication)

$$= (B^m A)B \quad [\text{using Eq. (iii)}]$$

$$= B^m (AB) = B^m (BA) \quad [\text{using Eq. (i)}]$$

$$= (B^m B)A = B^{m+1} A$$

Hence, by induction Eq. (ii) is true for all  $n \in N$ .

26.  $\text{tr}(A) = (a_{11} + a_{22} + \dots + a_{10}a_{10})$   
 $= (\omega^2 + \omega^4 + \omega^6 + \dots + \omega^{20})$   
 $= \omega^2(1 + \omega^2 + \omega^4 + \dots + \omega^{18})$   
 $= \omega^2[(1 + \omega^2 + \omega) + \dots + (1 + \omega + \omega^2) + 1]$   
 $= \omega^2 [0 + \dots + 0 + 1] = \omega^2$

27. Given matrix  $A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}$ , ( $x, y \in R, x \neq y$ )

for which  $A^T A = 3I_3$

$$\Rightarrow \begin{bmatrix} 0 & 2x & 2x \\ 2y & y & -y \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Here, two matrices are equal, therefore equating the corresponding elements, we get

$$8x^2 = 3 \text{ and } 6y^2 = 3$$

$$\Rightarrow x = \pm \sqrt{\frac{3}{8}} \text{ and } y = \pm \frac{1}{\sqrt{2}}$$

$\therefore$  There are 2 different values of  $x$  and  $y$  each.

So, 4 matrices are possible such that  $A^T A = 3I_3$ .

28.  $Q^2 = Q \cdot Q = PAP^T \cdot PAP^T$   
 $= PA(P^T \cdot P)AP^T = PA \cdot I \cdot AP^T = PA^2 P^T$   
 Similarly,  $Q^3 = PA^3 P^T$

$$Q^{2005} = PA^{2005} P^T$$

Now,  $X = P^T Q^{2005} P$   
 $= P^T (PA^{2005} P^T) P$   
 $= (P^T P) A^{2005} (P^T P)$   
 $= I \cdot A^{2005} \cdot I = A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

29. Given,  $M = [a_{uv}]_{n \times n} = [\sin(\theta_u - \theta_v) + i \cos(\theta_u - \theta_v)]$   
 $\Rightarrow \bar{M} = [\sin(\theta_u - \theta_v) - i \cos(\theta_u - \theta_v)]$   
 $\Rightarrow (\bar{M})^T = [\sin(\theta_v - \theta_u) - i \cos(\theta_v - \theta_u)]$

$$= [-\sin(\theta_u - \theta_v) - i \cos(\theta_u - \theta_v)]$$

$$= -[\sin(\theta_u - \theta_v) + i \cos(\theta_u - \theta_v)] = -M$$

30. Let  $P = AB' - BA'$  be any matrix.

Now,  $P' = (AB' - BA')' = (AB)' - (BA)'$   
 $= (B')'(A') - (A')'(B')$  [ $\because (AB)' = B'A'$   
 $\therefore (A')' = A'$ ]

$$= -[AB' - BA'] = -P$$

$$\therefore P' = -P$$

$P$  is a skew-symmetric matrix.

31. We have,  $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$   
 $\therefore A^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$   
 $= \begin{bmatrix} \frac{1}{2} - \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2} & -\frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

$\therefore$  Matrix  $A$  is nilpotent.

32. Given,  $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$

and symmetric matrix,  $B = \frac{A + A'}{2}$

$$\therefore B = \frac{1}{2} \left\{ \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 9 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix} \right\} = \begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$$

33.  $F$  is unit matrix  $\Rightarrow F^2 = F$

and  $E^2 F + F^2 E = E^2 + E$

Also,  $E^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\therefore E^2 + E = E.$$

34. Consider,  $(A^{-1}B)^T = B^T (A^{-1})^T$

$$= B^T (A^T)^{-1} = B A^{-1}$$

$$[\because A^T = A \text{ and } B^T = B]$$

$$= A^{-1}B \quad [\because AB = BA \Rightarrow A^{-1}(AB)A^{-1}$$

$$= A^{-1}(BA)A^{-1} \Rightarrow BA^{-1} = A^{-1}B]$$

$\Rightarrow A^{-1}B$  is symmetric.

Now, consider

$$(A^{-1}B^{-1})^T = ((BA)^{-1})^T = ((AB)^{-1})^T \quad [\because AB = BA]$$

$$= (B^{-1}A^{-1})^T = (A^{-1})^T (B^{-1})^T$$

$$= (A^T)^{-1} (B^T)^{-1} = A^{-1} B^{-1}$$

$\Rightarrow A^{-1}B^{-1}$  is also symmetric.

35. Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

Then,  $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence,  $A$  is nilpotent matrix of index 2.

36. Since,  $A$  is orthogonal, each row is orthogonal to the other rows.

$\Rightarrow R_1 \cdot R_3 = 0$

$\Rightarrow x + 4 + 2y = 0$

Also,  $R_2 \cdot R_3 = 0$

$\Rightarrow 2x + 2 - 2y = 0$

On solving, we get

$x = -2, y = -1$

$\therefore xy = 2$

37. Since,  $A$  and  $B$  are symmetric matrices

$\therefore X = AB + BA$  will be a symmetric matrix

and  $Y = AB - BA$  will be a skew-symmetric matrix.

Thus, we get  $X^T = X$  and  $Y^T = -Y$

Now, consider  $(XY)^T = Y^T X^T$   
 $= (-Y)(X) = -YX$

38.  $\begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}^T$   
 $= \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$   
 $= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & -\cos \alpha \sin \alpha + \sin \alpha \cos \alpha \\ -\sin \alpha \cos \alpha + \cos \alpha \sin \alpha & \sin^2 \alpha + \cos^2 \alpha \end{bmatrix}$   
 $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$  If  $A$  is an orthogonal matrix, then

$AA^T = I.$

$\therefore \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$  is an orthogonal matrix.

39. Given,  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

$\therefore A^T + A = I_2$

$\Rightarrow \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 \cos \theta & 0 \\ 0 & 2 \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \cos \theta = \frac{1}{2}$

$\therefore \theta = 2n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$

40.  $\begin{bmatrix} 1 & -\alpha \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ -\alpha & \beta \end{bmatrix} = \begin{bmatrix} 1 + \alpha^2 & \alpha - \alpha\beta \\ \alpha - \alpha\beta & \alpha^2 + \beta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$1 + \alpha^2 = 1$

$\alpha^2 = 0$

$\alpha^2 + \beta^2 = 1$

$\beta^2 = 1$

$\alpha^4 = 0$

$\beta^4 = 1$

$\alpha^4 + \beta^4 = 1$

41. Here,  $(AB)(AB) = A(BA)B = A(A^2B)B = A^3B^2$

Now,  $(AB)(AB)(AB) = (A^3B^2)AB$

$= A^3B^2AB = A^3B(BA)B$

$= A^3B(A^2B)B = A^3(BA) \cdot AB^2$

$= A^3(A^2B) \cdot AB$

$= A^5BAB^2 = A^5 \cdot A^2B \cdot B^2$

$= A^7 \cdot B^3$

So,  $(AB)^n = A^{2^n - 1} \cdot B^n$

$\therefore (AB)^{10} = A^{2^{10} - 1} \cdot B^{10} \Rightarrow k = 2^{10} - 1 = 1023$

42. We have,  $(A + B)^2 = A^2 + AB + BA + B^2$

$\left[ \begin{array}{l} \because A^2 = BA \cdot BA = B \cdot B \cdot A = B \cdot A = A \\ \text{and } B^2 = ABA \cdot B = A \cdot AB = AB = B \end{array} \right]$

$= A + B + A + B$

$= 2(A + B)$

Also,  $(A + B)^3 = (A + B)^2 \cdot (A + B)$

$= 2(A + B) \cdot (A + B)$

$= 2(A + B)^2 = 2 \times 2(A + B)$

$= 4(A + B)$

Hence,  $(A + B)^n = 2^{n-1}(A + B)$

43. Here,  $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

$= \begin{bmatrix} a^2 + ad - 1 & b(a + d) \\ c(a + d) & d^2 + ad - 1 \end{bmatrix}$

$= \begin{bmatrix} a(a + d) - 1 & b \\ c & d(a + d) - 1 \end{bmatrix}$

$= \begin{bmatrix} a - 1 & b \\ c & d - 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A^2 = A - I \Rightarrow A^3 = A^2 - A = (A - I) - A = -I$

$\therefore A^3 = -I$

$\Rightarrow A^{2010} = (A^3)^{670} = (-I)^{670} = I_2$

44.  $\left. \begin{array}{l} AB = A \\ BA = B \end{array} \right\} \Rightarrow \left. \begin{array}{l} A^2 = A \\ B^2 = B \end{array} \right\} \Rightarrow \left. \begin{array}{l} A^n = A \\ B^n = B \end{array} \right\}$

and  $(A + B)^2 = A^2 + B^2 + AB + BA$

$= A + B + A + B = 2(A + B)$

$\therefore (A^{2010} + B^{2010})^{2011} = (A + B)^{2011} = 2^{2011}(A + B)$

## Round II

1. Given matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P = X + I \text{ (let)}$$

$$\text{Now, } P^5 = (I + X)^5$$

$$= I + {}^5C_1(X) + {}^5C_2(X^2) + {}^5C_3(X^3) + \dots$$

$$[\because I^n = I, I \cdot A = A \text{ and } (a+x)^n = {}^nC_0a^n + {}^nC_1a^{n-1}x + \dots + T^n C_n x^n]$$

$$\text{Here, } X^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } X^3 = X^2 \cdot X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow X^4 = X^5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{So, } P^5 = I + 5 \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} + 10 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 15 & 1 & 0 \\ 135 & 15 & 1 \end{bmatrix}$$

$$\text{and } Q = I + P^5 = \begin{bmatrix} 2 & 0 & 0 \\ 15 & 2 & 0 \\ 135 & 15 & 2 \end{bmatrix} = [q_{ij}]$$

$$\Rightarrow q_{21} = 15, q_{31} = 135 \text{ and } q_{32} = 15$$

$$\text{Hence, } \frac{q_{21} + q_{31}}{q_{32}} = \frac{15 + 135}{15} = \frac{150}{15} = 10$$

2. Given,  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$   $A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}$$

It is given that

$$AA^T = 9I$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

On comparing, we get

$$a + 4 + 2b = 0 \Rightarrow a + 2b = -4 \quad \dots(i)$$

$$2a + 2 - 2b = 0 \Rightarrow a - b = -1 \quad \dots(ii)$$

$$\text{and } a^2 + 4 + b^2 = 9 \quad \dots(iii)$$

On solving Eqs. (i) and (ii), we get

$$a = -2, b = -1$$

This satisfies Eq. (iii)

Hence,  $(a, b) \equiv (-2, -1)$

$$3. \because B^T A^T = (AB)^T = \left( \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} 8-7 & 2-2 \\ -28+28 & -7+8 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence,  $B^T A^T$  is an identity matrix.

4.  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^8 = I \cdot I = I$$

$$\Rightarrow A^{16} = I \cdot I = I$$

$$\therefore A^{16} = \lambda I$$

$$\Rightarrow I = \lambda I$$

$$\therefore \lambda = 1$$

5.  $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$

$$A^2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = 2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^4 = 2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} 2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= 8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^8 = 8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} 8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 128 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^8 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$128 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$128 \begin{bmatrix} x & -y \\ -x & y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$x - y = \frac{1}{16} \text{ and } -x + y = \frac{1}{2}$$

Hence, the linear equation has no solution.

6. Let  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$  and  $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

$\therefore X' \cdot A \cdot X = 0$   
 $\Rightarrow a_{11} X_1^2 + a_{22} X_2^2 + a_{33} X_3^2 + (a_{12} + a_{21}) X_1 X_2$   
 $+ (a_{13} + a_{31}) X_1 X_3 + (a_{23} + a_{32}) X_2 X_3 = 0$

This is true  $X_i$   
 $a_{11} = a_{22} = a_{33} = 0$   
 $a_{12} + a_{21} = 0$   
 $a_{13} + a_{31} = 0$   
 $a_{23} + a_{32} = 0$   
 $\Rightarrow -2009 + a_{32} = 0$   
 $\Rightarrow a_{32} = 2009$

7. Thus,  $A = \begin{bmatrix} \frac{\omega}{i} & \frac{\omega^2}{i} \\ -\frac{\omega^2}{i} & -\frac{\omega}{i} \end{bmatrix} = \frac{\omega}{i} \begin{bmatrix} 1 & \omega \\ -\omega & -1 \end{bmatrix}$

$\therefore A^2 = -\omega^2 \begin{bmatrix} 1 - \omega^2 & 0 \\ 0 & 1 - \omega^2 \end{bmatrix}$   
 $= -\begin{bmatrix} -\omega^2 + \omega^4 & 0 \\ 0 & -\omega^2 + \omega^4 \end{bmatrix}$   
 $= \begin{bmatrix} -\omega^2 + \omega & 0 \\ 0 & -\omega^2 + \omega \end{bmatrix}$

$f(x) = x^2 + 2$   
 $\therefore f(A) = A^2 + 2 = \begin{bmatrix} -\omega^2 + \omega & 0 \\ 0 & -\omega^2 + \omega \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$   
 $= [-\omega^2 + \omega + 2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$   
 $= (3 + 2\omega) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (2 + i\sqrt{3}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. Given,  $A^2 = 2A - I$   
 Now,  $A^3 = A^2 \cdot A = 2A^2 - IA$   
 $= 2A^2 - A = 2(2A - I) - A$   
 $= 3A - 2I = 3A - (3 - 1)I$   
 $\dots \dots \dots$   
 $\dots \dots \dots$

$A^n = nA - (n - 1)I$   
 9.  $A^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

$\therefore I + 2A + 3A^2 + \dots = I + 2A$   
 $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$

10. Let  $u_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$  and  $u_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$   
 $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$   $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  [given]

We have,  $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and  $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a_1 \\ 2a_1 + b_1 \\ 3a_1 + 2b_1 + c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and  $\begin{bmatrix} a_2 \\ 2a_2 + b_2 \\ 3a_2 + 2b_2 + c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow a_1 = 1, 2a_1 + b_1 = 0, 3a_1 + 2b_1 + c_1 = 0$   
 and  $a_2 = 0, 2a_2 + b_2 = 1, 3a_2 + 2b_2 + c_2 = 0$   
 $\Rightarrow a_1 = 1, b_1 = -2, c_1 = 1$   
 and  $a_2 = 0, b_2 = 1, c_2 = -2$

$\therefore u_1 + u_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

11. Let  $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$   
 $\therefore M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$  and  $M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 - a_2 \\ b_1 - b_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

and  $\begin{bmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$   
 $\Rightarrow a_2 = -1, b_2 = 2, c_2 = 3, a_1 - a_2 = 1, b_1 - b_2 = 1,$   
 $c_1 - c_2 = -1, a_1 + a_2 + a_3 = 0, b_1 + b_2 + b_3 = 0,$   
 $c_1 + c_2 + c_3 = 12$

$\Rightarrow a_1 = 0, b_2 = 2, c_3 = 7$   
 $\therefore a_1 + b_2 + c_3 = 0 + 2 + 7 = 9$   
 Hence, the sum of diagonal elements is 9.

12.  $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$   
 $\Rightarrow A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$A^4 = A^2 \cdot A^2 = I_3 \cdot I_3 = I_3$   
 $A^6 = A^2 \cdot A^4 = I_3 \cdot I_3 = I_3$   
 $\therefore A^2 + 2A^4 + 4A^6 = I_3 + 2I_3 + 4I_3 = 7I_3 = 7(I_3)^4$   
 $= 7(A^2)^4 = 7A^8$

13. Given,  $A = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$

$$A^2 = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^3 & 0 \\ 0 & \omega^3 \end{bmatrix}$$

Similarly,  $A^{50} = \begin{bmatrix} \omega^{50} & 0 \\ 0 & \omega^{50} \end{bmatrix}$

$$= \begin{bmatrix} (\omega^3)^{16} \omega^2 & 0 \\ 0 & (\omega^3)^{16} \omega^2 \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} = \omega A$$

14. Given,  $A = 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Now,  $A^2 = 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 9A$

$\therefore A^4 = A^2 \cdot A^2 = 9A \cdot 9A = 81 \cdot 9A = 729A$

15.  $\therefore A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & a \\ 4 & b \end{bmatrix} = \begin{bmatrix} 2 & -1+a \\ 6 & -1+b \end{bmatrix}$

$$\Rightarrow (A+B)^2 = \begin{bmatrix} 2 & -1+a \\ 6 & -1+b \end{bmatrix} \begin{bmatrix} 2 & -1+a \\ 6 & -1+b \end{bmatrix}$$

$$= \begin{bmatrix} -2+6a & -1+a-b+ab \\ 6+6b & -5+6a-2b+b^2 \end{bmatrix}$$

and  $A^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Also,  $B^2 = \begin{bmatrix} 1 & a \\ 4 & b \end{bmatrix} \begin{bmatrix} 1 & a \\ 4 & b \end{bmatrix} = \begin{bmatrix} 1+4a & a+ab \\ 4+4b & 4a+b^2 \end{bmatrix}$

Given,  $(A+B)^2 = A^2 + B^2$

$$\therefore \begin{bmatrix} -2+6a & -1+a-b+ab \\ 6+6b & -5+6a-2b+b^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1+4a & a+ab \\ 4+4b & 4a+b^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2+6a & -1+a-b+ab \\ 6+6b & -5+6a-2b+b^2 \end{bmatrix} = \begin{bmatrix} 4a & a+ab \\ 4+4b & -1+4a+b^2 \end{bmatrix}$$

On comparing both sides, we get

$$-2+6a=4a \text{ and } 6+6b=4+4b$$

$\therefore a=1 \text{ and } b=-1$

16. Here,  $C$ ,  $A$  and  $C^T$  are matrices of order  $n \times 1$ ,  $n \times n$  and  $1 \times n$ , respectively.

Let  $C^T AC = k$

Then,  $(C^T AC)^T = C^T A^T (C^T)^T = C^T A^T C = C^T (-A) C$

$$= -C^T AC = -k$$

$\Rightarrow k = -k \Rightarrow k = 0$

Hence,  $C^T AC$  is a null matrix, which shows that  $C^T AC$  is a zero matrix of order 1.

17.  $\therefore S^3 = S \cdot S \cdot S$

$$= (ABCD)(ABCD)(ABCD)$$

$$= (ABC)(DAB)(CDA)(BCD)$$

$$= D^T C^T B^T A^T = (BCD)^T A^T = (A^T)^T A^T$$

$$= AA^T = ABCD = S$$

$\therefore S^3 = S$  and  $S^4 = S^2$

18. We know that a matrix  $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$  will be orthogonal if  $AA^T = I$ , which implies

$$\Sigma a_i^2 = \Sigma b_i^2 = \Sigma c_i^2 = 1$$

and  $\Sigma a_i b_i = \Sigma b_i c_i = \Sigma c_i a_i = 0$

Now, from the given options, only  $\frac{1}{7} \begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{bmatrix}$

satisfies these conditions.

Hence,  $\frac{1}{7} \begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{bmatrix}$  is an orthogonal matrix.

19. Given,  $A_1, A_3, \dots, A_{2n-1}$  are skew-symmetric matrices of same order.

$$\Rightarrow (A_1)^T = -A_1, (A_3)^T = -A_3, \dots, (A_{2n-1})^T = -A_{2n-1}$$

Now,  $B = \sum_{r=1}^n (2r-1)(A_{2r-1})^{2r-1}$

$$B = A_1 + 3(A_3)^3 + 5(A_5)^5 + \dots + (2n-1)(A_{2n-1})^{2n-1}$$

$$B^T = (A_1)^T + 3[(A_3)^3]^T + 5[(A_5)^5]^T + \dots$$

$$+ \dots (2n-1)[(A_{2n-1})^{2n-1}]^T$$

$$= -A_1 - 3(A_3)^3 - 5(A_5)^5 - \dots$$

$$- (2n-1)(A_{2n-1})^{2n-1}$$

$$= -B$$

Hence,  $B$  is a skew-symmetric matrix.

20. Given matrix,  $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix} \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (x^2+1)^2+x^2 & x(x^2+2) \\ x(x^2+2) & x^2+1 \end{bmatrix}$$

$\therefore A^4 = [a_{ij}]$  and  $a_{11} = 109$

$$\Rightarrow (x^2+1)^2 + x^2 = 109$$

$$\Rightarrow x^2 = 9$$

$\therefore a_{22} = x^2 + 1$

$$= 9 + 1 = 10$$

21. Let a  $3 \times 3$  matrix  $A = [a_{ij}]_{3 \times 3}, i, j \in \{1, 2, 3\}$

So, sum of diagonal elements of  $AA^T$

$$\begin{aligned} &= \text{trace of } AA^T = \text{tr}([a_{ij}][a_{ji}]) \\ &= a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2 + a_{31}^2 \\ &\quad + a_{32}^2 + a_{33}^2 \\ &= 3 \end{aligned} \quad (\text{given})$$

$\therefore a_{ij} \in \{-1, 0, 1\}$

So, possible combinations are

0, 0, 0, 0, 0, 0, 1, 1, 1

0, 0, 0, 0, 0, 0, -1, 1, 1

0, 0, 0, 0, 0, 0, -1, -1, 1

0, 0, 0, 0, 0, 0, -1, -1, -1

$\therefore$  Number of such matrices such that  $\text{tr}(AA^T) = 3$  is

$$\begin{aligned} &= {}^9C_6 \times 2 \times 2 \times 2 \\ &= \frac{9 \times 8 \times 7}{3 \times 2} \times 2 \times 2 \times 2 \\ &= 84 \times 8 \\ &= 672 \end{aligned}$$

Hence, answer is 672.

22.  $\therefore A^T A = I$

$$\begin{aligned} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ac + ab + bc \\ ab + bc + ca & b^2 + c^2 + a^2 & bc + ca + ab \\ ca + ab + bc & bc + ca + ab & c^2 + a^2 + b^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$\therefore a^3 + b^3 + c^3 - 3abc$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \quad \dots(i)$$

and  $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$$= 1 + 2 \times 0 = 1$$

$\Rightarrow a + b + c = 1 \quad \dots(ii)$

[ $\therefore a, b, c$  are positive real numbers]

From Eqs. (i) and (ii), we get

$$a^3 + b^3 + c^3 - 3 \times 1 = 1 \times (1 - 0) \quad [\therefore abc = 1]$$

$\Rightarrow a^3 + b^3 + c^3 = 4$

23. Let  $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$  where  $a \in \{0, 1, 2\}$

$$\text{Then, } M^T M = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

Sum of the diagonal entries in  $M^T M = 7$

$$\Rightarrow (a_1^2 + a_4^2 + a_7^2) + (a_2^2 + a_5^2 + a_8^2) + (a_3^2 + a_6^2 + a_9^2) = 7$$

It is possible when

**Case I** Seven  $a_{is}$  are 1 and two  $a_{is}$  are zero, which can be done in

$${}^9C_2 = 36$$

**Case II** One  $a_{is}$  is 2 and three  $a_{is}$  are 1 and five  $a_{is}$  are zero.

$$\text{It can be done in } {}^9C_1 \cdot {}^8C_3 = \frac{9!}{5!3!} = 504$$

$\therefore$  Total = 540

24.  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} a - b \\ c - d \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and  $A^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a - b \\ c - d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -a + 2b \\ -c + 2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now,  $a - b = -1 \quad \dots(i)$

$$c - d = 2 \quad \dots(ii)$$

$$-a + 2b = 1 \quad \dots(iii)$$

$$-c + 2d = 0 \quad \dots(iv)$$

On adding Eqs. (i), (ii), (iii) and (iv), we get

$$b + d = 2$$

On adding Eqs. (i) and (ii), we get

$$(a + c) - (b + d) = 1$$

$$\Rightarrow a + c - 2 = 1 \Rightarrow a + c = 3$$

Hence,  $a + b + c + d = 2 + 3 = 5$