CONTINUITY, DIFFERENTIABILITY & DIFFERENTIATION

CONTINUITY

1. DEFINITION

A function $f(x)$ is said to be continuous at x = a; where a $\in$ domain of $f(x)$, if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

i.e., LHL = RHL = value of a function at x = a

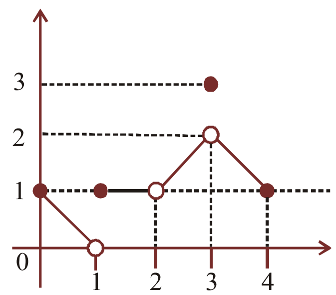
or $\lim_{x \to a} f(x) = f(a)$

1.1 Reasons of discontinuity

If $f(x)$ is not continuous at x = a, we say that $f(x)$ is discontinuous at x = a.

There are following possibilities of discontinuity :

1. $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but they are not equal.
2. $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exists and are equal but not equal to $f(a)$.
3. $f(a)$ is not defined.
4. At least one of the limits does not exist. Geometrically, the graph of the function will exhibit a break at the point of discontinuity.



The graph as shown is discontinuous at x = 1, 2 and 3.

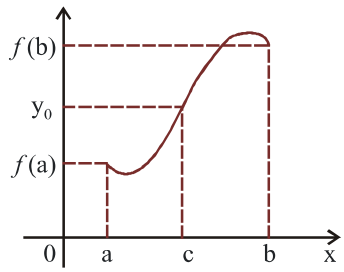
2. PROPERTIES OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be continuous functions at x = a. Then,

1. $c f(x)$ is continuous at x = a, where c is any constant.
2. $f(x) \pm g(x)$ is continuous at x = a.
3. $f(x) . g(x)$ is continuous at x = a.
4. $f(x)/g(x)$ is continuous at x = a, provided $g(a) \neq 0$.-
5. If $f(x)$ is continuous on [a, b] such that $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one solution of equation $f(x) = 0$ in the open interval (a, b).

3. THE INTERMEDIATE VALUE THEOREM

Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, their exits a number c between a and b such that $f(c) = y_0$.



The Function f, being continuous on (a,b) takes on every value between $f(a)$ and $f(b)$

NOTES :

That a function f which is continuous in [a, b] possesses the following properties :

(i) If $f(a)$ and $f(b)$ possess opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).

(ii) If K is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation $f(x) = K$ in the open interval (a, b).

4. CONTINUITY IN AN INTERVAL

(a) A function f is said to be continuous in (a, b) if f is continuous at each and every point $\in$ (a, b).

(b) A function f is said to be continuous in a closed interval [a, b] if :

(1) f is continuous in the open interval (a, b) and

(2) f is right continuous at 'a' i.e. $\lim_{x\to a^+} f(x) = f(a) =$ a finite quantity.

(3) f is left continuous at 'b'; i.e. $\lim_{x\to b^-} f(x) = f(b) =$ a finite quantity.

5. CONTINUOUS OF COMPOSITE FUNCTIONS

Theorem 5.1

Suppose $a < c < b, f:[a,b] \to R$ is a function and $\lim_{x\to c} f(x) = l$. Further suppose that g: R → R is continuous at l. Then $\lim_{x\to c}(gof)(x)$ exists and is equal to g(l).

Proof : Write p = g(l) and ε > 0. Then there exists δ > 0 such that y∈(l - δ, l + δ) ⇒ |g(y) - p| < ε (since g is continuous at l). Since f(x) → l as x → c, corresponding to this δ > 0 there exists η > 0 such that

$$x \in (c-\eta, c+\eta) \Rightarrow |f(x) - l| < \delta$$
$$\Rightarrow l - \delta < f(x) < l + \delta$$
$$\Rightarrow |g(f(x)) - p| < \varepsilon$$
$$\Rightarrow |(gof)(x) - p| < \varepsilon$$

Hence

$$\lim_{x\to c}(gof)(x) = p = g(l)$$

Theorem 5.2

Suppose a < c < b and f : [a, b] → R and g: R → R are, respectively, continuous at c and f(c). Then gof: [a,b] → R is continuous c.

Proof : By Theorem 5.1

$$\lim_{x\to c}(gof)(x) = g(f(c)) = (gof)(c)$$

Hence gof is continuous at c.

Note : If c = a or b, in a similar way as Theorem we can show that gof is continuous at a or b.

Theorem 5.3

Suppose f : [a, b] → R is continuous, f([a, b]) ⊂ [p, q] and g:[p, q] → R is continuous. Then gof : [a, b] → R is continuous.

Proof : Suppose c ∈ [a, b]. Since f is continuous on [a, b] and g is continuous on [p, q] it follows that gof is continuous at c (by Theorem and the note). Hence gof is continuous on [a, b].

Theorem 5.4

Suppose α∈R, a < c < b, f: [a, b] → [0, ∞) and $\lim_{x\to c} f(x) = l$, then $\lim_{x\to c} f^{\alpha}(x) = l^{\alpha}$.

Proof : Since f(x) ≥ 0 on [a, b], by Theorem it follows that $\lim_{x\to c} f(x) = l \ge 0$. Write $g(x) = x^{\alpha}\,\forall x \ge 0$. By Example we get g is continuous at l. Therefore by Theorem it follows that

$$\lim_{x\to c} f^{\alpha}(x) = \lim_{x\to c}(f(x))^{\alpha} = \lim_{x\to c} g(f(x)) = g(l) = l^{\alpha}$$

6. A LIST OF CONTINUOUS FUNCTIONS

	Function f (x)	Interval in which f (x) is continuous
1.	constant c	$(-\infty, \infty)$
2.	x^n, n is an integer ≥ 0	$(-\infty, \infty)$
3.	x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
4.	$\|x-a\|$	$(-\infty, \infty)$
5.	$P(x) = a_0x^n + a_1x^{n-1} + ... + a_n$	$(-\infty, \infty)$
6.	$\frac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial in x	$(-\infty, \infty) - \{x ; q(x)=0\}$
7.	sin x	$(-\infty, \infty)$
8.	cos x	$(-\infty, \infty)$
9.	tan x	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$
10.	cot x	$(-\infty, \infty) - \{n\pi : n \in I\}$
11.	sec x	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$

12.	cosec x	$(-\infty, \infty) - \{n\pi : n \in I\}$
13.	e^x	$(-\infty, \infty)$
14.	$\log_e x$	$(0, \infty)$

7. TYPES OF DISCONTINUITIES

Type-1 : (Removable type of discontinuities)

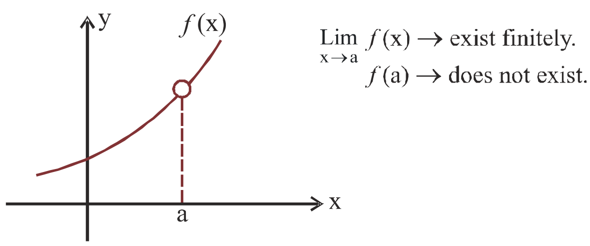
In case, $\underset{x \to c}{\text{Limit}}\ f(x)$ exists but is not equal to $f(c)$ then the function is said to have a **removable discontnuity or discontinuity of the first kind.** In this case, we can redefine the function such that $\underset{x \to c}{\text{Limit}}\ f(x) = f(c)$ and make it continuous at $x = c$. Removable type of discontinuity can be further classified as :

(a) Missing Point Discontinuity :

Where $\underset{x \to a}{\text{Limit}}\ f(x)$ exists finitely but $f(a)$ is not defined.

E.g. $f(x) = \dfrac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at $x = 1$, and

$f(x) = \dfrac{\sin x}{x}$ has a missing point discontinuity at $x = 0$.



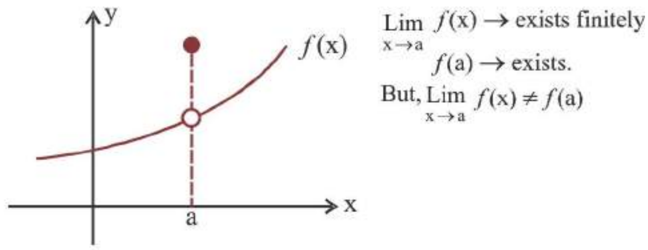
missing point discontinuity at x = a

(b) Isolated Point Discontinuity :

Where $\underset{x \to a}{\text{Limit}}\ f(x)$ exists & $f(a)$ also exists but; $\underset{x \to a}{\text{Limit}} \neq f(a)$.

e.g. $f(x) = \dfrac{x^2 - 16}{x - 4}$, $x \neq 4$ and $f(4) = 9$ has an isolated point discontinuity at $x = 4$.

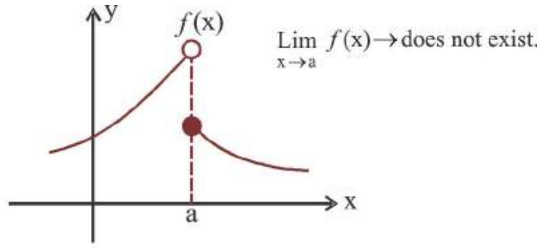
Similarly $f(x) = [x] + [-x] = \begin{bmatrix} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{bmatrix}$ has an isolated point discontinuity at all $x \in I$.



Isolated point discontinuity at x = a

Type-2 : (Non-Removable type of discontinuities)

In case, $\underset{x \to a}{\text{Limit}}\ f(x)$ does not exist, then it is not possible to make the function continuous by redefining it. Such discontinuities are known as **non-removable discontinuity or discontinuity of the 2nd kind.** Non-removable type of discontinuity can be further classified as :



non-removable discontinuity at x = a

(a) Finite Discontinuity :

e.g., $f(x) = x - [x]$ at all integral x; $f(x) = \tan^{-1}\dfrac{1}{x}$ at $x = 0$ and

$f(x) = \dfrac{1}{1 + 2^{\frac{1}{x}}}$ at $x = 0$ (note that $f(0^+) = 0$; $f(0^-) = 1$)

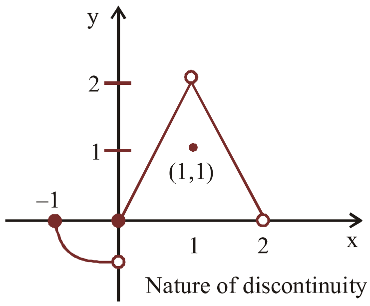
(b) Infinite Discontinuity :

e.g., $f(x) = \dfrac{1}{x-4}$ or $g(x) = \dfrac{1}{(x-4)^2}$ at $x = 4$; $f(x) = 2^{\tan x}$ at $x = \dfrac{\pi}{2}$ and $f(x) = \dfrac{\cos x}{x}$ at $x = 0$.

(c) Oscillatory Discontinuity :

e.g., $f(x) = \sin\dfrac{1}{x}$ at $x = 0$.

In all these cases the value of $f(a)$ of the function at x = a (point of discontinuity) may or may not exist but $\underset{x\to a}{\text{Limit}}$ does not exist.



Nature of discontinuity

From the adjacent graph note that

– f is continuous at x = –1

– f has isolated discontinuity at x = 1

– f has missing point discontinuity at x = 2

– f has non-removable (finite type) discontinuity at the origin.

NOTES :

(a) In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at x = a and LHL at x = a is called the **jump of discontinuity.** A function having a finite number of jumps in a given interval I is called a piece wise continuous or sectionally continuous function in this interval.

(b) All Polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their domains.

(c) If $f(x)$ is continuous and $g(x)$ is discontinuous at x = a then the product function $\phi(x) = f(x) . g(x)$ is not necessarily be discontinuous at x = a. e.g.

$$f(x) = x \text{ and } g(x) = \begin{bmatrix} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{bmatrix}$$

(d) If $f(x)$ and $g(x)$ both are discontinuous at x = a then the product function $\phi(x) = f(x) . g(x)$ is not necessarily be discontinuous at x = a . e.g.

$$f(x) = -g(x) = \begin{bmatrix} 1 & x \geq 0 \\ -1 & x < 0 \end{bmatrix}$$

(e) Point functions are to be treated as discontinuous eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

(f) A continuous function whose domain is closed must have a range also in closed interval.

(g) If f is continuous at x = a and g is continuous at x = $f(a)$ then the composite $g[f(x)]$ is continous at x = a E.g $f(x) = \frac{x\sin x}{x^2+2}$ and $g(x) = |x|$ are continuous at x = 0, hence the composite $(gof)(x) = \left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0.

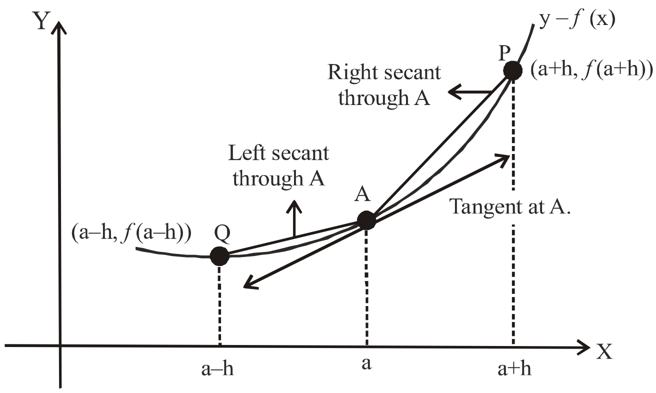
DIFFERENTIABILITY

8. DEFINITION

Let $f(x)$ be a real valued function defined on an open interval (a, b) where c ∈ (a, b). Then $f(x)$ is said to be differentiable or derivable at x = c,

iff, $\lim\limits_{x\to c} \frac{f(x)-f(c)}{(x-c)}$ exists finitely.

This limit is called the derivative or differentiable coefficient of the function $f(x)$ at x = c, and is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))_{x=c}$.



- Slope of Right hand secant = $\frac{f(a+h)-f(a)}{h}$ as $h \to 0$, P → A and secant (AP) → tangent at A

⇒ Right hand derivative $= \lim\limits_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$

= Slope of tangent at A (when approached from right) $f'(a^+)$.

- Slope of Left hand secant = $\dfrac{f(a-h)-f(a)}{-h}$ as h $\to 0$, Q $\to$ A and secant AQ $\to$ tangent at A

$\Rightarrow$ Left hand derivative $= \lim\limits_{h\to 0}\left(\dfrac{f(a-h)-f(a)}{-h}\right)$

= Slope of tangent at A (when approached from left) $f'(a^-)$.

Thus, $f(x)$ is differentiable at x = c.

$$\Leftrightarrow \lim_{x\to c}\frac{f(x)-f(c)}{(x-c)} \text{ exists finitely}$$

$$\Leftrightarrow \lim_{x\to c^-}\frac{f(x)-f(c)}{(x-c)} = \lim_{x\to c^+}\frac{f(x)-f(c)}{(x-c)}$$

$$\Leftrightarrow \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

Hence, $\lim\limits_{x\to c^-}\dfrac{f(x)-f(c)}{(x-c)} = \lim\limits_{h\to 0}\dfrac{f(c-h)-f(c)}{-h}$ is called the **left hand derivative** of $f(x)$ at x = c and is denoted by $f'(c^-)$ or L$f'(c)$.

While, $\lim\limits_{x\to c^+}\dfrac{f(x)-f(c)}{x-c} = \lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$ is called the **right hand derivative** of $f(x)$ at x = c and is denoted by $f'(c^+)$ or R$f'(c)$.

If $f'(c^-) \neq f'(c^+)$, we say that $f(x)$ is not differentiable at x = c.

9. DIFFERENTIABILITY IN A SET

1. A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b), if it is differentiable at each point of (a, b).
2. A function $f(x)$ defined on closed interval [a, b] is said to be differentiable or derivable. "If f is derivable in the open interval (a, b) and also the end points a and b, then f is said to be derivable in the closed interval [a, b]".

i.e., $\lim\limits_{x\to a^+}\dfrac{f(x)-f(a)}{x-a}$ and $\lim\limits_{x\to b^-}\dfrac{f(x)-f(b)}{x-b}$, both exist.

A function f is said to be a differentiable function if it is differentiable at every point of its domain.

NOTES :

1. If $f(x)$ and $g(x)$ are derivable at x = a then the functions $f(x)+g(x)$, $f(x)-g(x)$, $f(x).g(x)$ will also be derivable at x = a and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at x = a.
2. If $f(x)$ is differentiable at x = a and $g(x)$ is not differentiable at x = a, then the product function F (x) = $f(x).g(x)$ can still be differentiable at x = a. E.g. $f(x) = x$ and $g(x) = |x|$.
3. If $f(x)$ and $g(x)$ both are not differentiable at x = a then the product function; $F(x) = f(x).g(x)$ can still be differentiable at x = a. E.g., $f(x) = |x|$ and $g(x) = |x|$.
4. If $f(x)$ and $g(x)$ both are not differentiable at x = a then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function. E.g., $f(x) = |x|$ and $g(x) = -|x|$.

10. RELATION B/W CONTINUITY & DIFFERENTIABILITY

In the previous section we have discussed that if a function is differentiable at a point, then it should be continuous at that point and a discontinuous function cannot be differentiable. This fact is proved in the following theorem.

Theorem : If a function is differentiable at a point, it is necessarily continuous at that point. But the converse is not necessarily true,

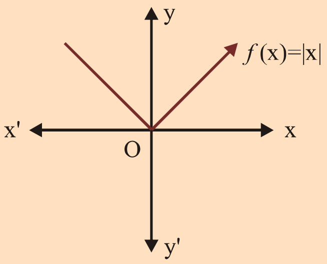
or $f(x)$ is differentiable at x = c

$\Rightarrow$ $f(x)$ is continuous at x = c.

NOTES :

Converse : The converse of the above theorem is not necessarily true i.e., a function may be continuous at a point but may not be differentiable at that point.

e.g., The function $f(x) = |x|$ is continuous at x = 0 but it is not differentiable at x = 0, as shown in the figure.



The figure shows that sharp edge at x = 0 hence, function is not differentiable but continuous at x = 0.

NOTES :

(a) Let $f'^{+}(a) = p$ & $f'^{-}(a) = q$ where p & q are finite then:

(i) p = q $\Rightarrow f$ is derivable at x = a

$\Rightarrow f$ is continuous at x = a.

(ii) p ≠ q $\Rightarrow f$ is not derivable at x = a.

It is very important to note that f may be still continuous at x = a.

In short, for a function f:

Differentiable $\Rightarrow$ **Continuous;**

Not Differentiable $\not\Rightarrow$ **Not Continuous**

(i.e., function may be continuous)

But,

Not Continuous $\Rightarrow$ **Not Differentiable.**

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

Theorem 2 : Let f and g be real functions such that fog is defined if g is continuous at x = a and f is continuous at g (a), show that fog is continuous at x = a.

DIFFERENTIATION

11. DEFINITION

The rate of change of one quantity with respect to some another quantity has a great importance. For example, the rate of change of displacement of a particle with respect to time is called its velocity and the rate of change of velocity is called its acceleration.

The rate of change of a quantity 'y' with respect to another quantity 'x' is called the derivative or differential coefficient of y with respect to x.

Derivative at a Point

The derivative of a function at a point x = a is defined by

$f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ (provided the limit exists and is finite)

The above definition of derivative is also called derivative by first principle.

(1) Geometrical meaning of derivatives at a point : Consider the curve y = f(x). Let f(x) be differentiable at x = c. Let P(c,f(c)) be a point on the curve and Q(x, f(x)) be a neighbouring point on the curve. Then,

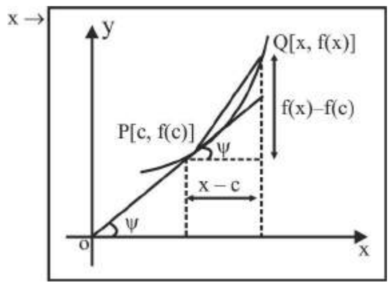
Slope of the chord $PQ = \dfrac{f(x)-f(c)}{x-c}$. Taking limit as $Q \to P$, i.e., we get

$$\lim_{Q\to P}(\textit{Slope of the chord } PQ) = \lim_{x\to e} \frac{f(x)-f(c)}{x-c} \quad ..(i)$$

As $Q \to P$, chord PQ becomes tangent at P.

Therefore from (i), we have

slope of the tangent at $P = \lim_{x\to c}\dfrac{f(x)-f(c)}{x-c} = \left(\dfrac{df(x)}{dx}\right)_{x=c}$



NOTES :

Thus, the derivatives of a function at point x = c is the slope of the tangetn to curve , y = f(x) at point (c, f(c)).

(2) Physical interpretation at a point : Let a particle moves in a straight line OX starting from O towards X. Clearly, the position of the particle at any instant would depend upon the time elapsed. In other words, the distance of the particle from O will be some function f of time t.

Let at any time $t = t_0$, the particle be at P and after a further time h, it is at Q so that $OP = f(t_0)$ and $OQ = f(t_0 + h)$. hence, the average speed of the particle during the journey from P to Q is $\frac{PQ}{h}$, i.e., $\frac{f(t_0+h)-f(t_0)}{h} = f(t_0, h)$. Taking the limit of $f(t_0, h)$ as $h \to 0$, we get its instantaneous speed to be $\lim_{h\to 0}\frac{f(t_0+h)-f(t_0)}{h}$, which is simply f($t_0$).

Thus, if f(t) gives the distance of a moving particle at time t, then the disivative of f at t = t_0 represents the instantaneous speed of the particle at the point P, i.e., at time t = t_0.

Important Tips

* $\frac{dy}{dx}$ is $\frac{d}{dx}(y)$ in which $\frac{d}{dx}$ is simply a symbol of operation and 'd' divided by dx.

* If $f(x_0) = \infty$, the function is said to have an infinite dervative at the point x_0. In this case the line tangent to the curve of y = f(x) at the point x_0 is perpendicular to the x-axis.

(a) Let us consider a function y = f(x) defined in a certain interval. It has a definite value for each value of the independent variable x in this interval.

Now, the ratio of the increment of the function to the increment in the independent variable,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Now, as $\Delta x \to 0$, $\Delta y \to 0$ and $\frac{\Delta y}{\Delta x} \to$ finite quantity, then derivative f(x) exists and is denoted by y′ or f'(x) or $\frac{dy}{dx}$

Thus, $f'(x) = \lim_{x\to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

(if it exits)

for the limit to exist,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

(Right Hand derivative) (Left Hand derivative)

(b) The derivative of a given function f at a point x = a of its domain is defined as :

$\underset{h\to 0}{\text{Limit}}\frac{f(a+h)-f(a)}{h}$, provided the limit exists & is denoted by f'(a).

Note that alternatively, we can define

$f'(a) = \underset{x\to a}{\text{Limit}}\frac{f(x)-f(a)}{x-a}$, provided the limit exists.

This method is called first principle of finding the derivative of f(x).

12. DERIVATIVE OF STANDARD FUNCTION

(i) $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$; $x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \cdot \ln a \ (a > 0)$

(iv) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a |x|) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

(x) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cdot \cot x$

(xi) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(xii) $\frac{d}{dx}(\text{constant}) = 0$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

(xiv) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

(xv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \in R$

(xvi) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, \quad x \in R$

(xvii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$

(xviii) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$

(xix) **Results :**

If the inverse functions f & g are defined by $y = f(x)$ & $x = g(y)$. Then $g(f(x)) = x$.

$\Rightarrow$ $g'(f(x)) \,.\, f'(x) = 1$.

This result can also be written as, if $\frac{dy}{dx}$ exists & $\frac{dy}{dx} \neq 0$, then

$\frac{dx}{dy} = 1/\left(\frac{dy}{dx}\right)$ or $\frac{dy}{dx}.\frac{dx}{dy} = 1$ or $\frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right)\left[\frac{dx}{dy} \neq 0\right]$

13. THEOREMS ON DERIVATIVES

If u and v are derivable functions of x, then,

(i) Term by term differentiation : $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) Multiplication by a constant $\frac{d}{dx}(K\,u) = K\frac{du}{dx}$, where K is any constant

(iii) **"Product Rule"** $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$ known as

In general,

(a) If $u_1, u_2, u_3, u_4, \ldots, u_n$ are the functions of x, then

$$\frac{d}{dx}(u_1 . u_2 . u_3 . u_4 \ldots . u_n)$$

$$= \left(\frac{du_1}{dx}\right)(u_2\, u_3\, u_4 \ldots u_n) + \left(\frac{du_2}{dx}\right)(u_1\, u_3\, u_4 \ldots u_n)$$

$$+ \left(\frac{du_3}{dx}\right)(u_1\, u_2\, u_4 \ldots u_n) + \left(\frac{du_4}{dx}\right)(u_1\, u_2\, u_3\, u_5 \ldots u_n)$$

$$+ \ldots + \left(\frac{du_n}{dx}\right)(u_1\, u_2\, u_3 \ldots u_{n-1})$$

(iv) **"Quotient Rule"** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$ where $v \neq 0$

known as

(b) **Chain Rule :** If $y = f(u)$, $u = g(w)$, $w = h(x)$

then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dw}.\frac{dw}{dx}$

or $\frac{dy}{dx} = f'(u).g'(w).h'(x)$

NOTES :

In general if $y = f(u)$ then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

14. METHODS OF DIFFERENTIATION

14.1 Derivative by using Trigonometrical Substitution

Using trigonometrical transformations before differentiation shorten the work considerably. Some important results are given below :

(i) $\sin 2x = 2 \sin x \cos x = \frac{2\tan x}{1+\tan^2 x}$

(ii) $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$

(iii) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$, $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

(iv) $\sin 3x = 3\sin x - 4\sin^3 x$

(v) $\cos 3x = 4\cos^3 x - 3\cos x$

(vi) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

(vii) $\tan\left(\dfrac{\pi}{4} + x\right) = \dfrac{1 + \tan x}{1 - \tan x}$

(viii) $\tan\left(\dfrac{\pi}{4} - x\right) = \dfrac{1 - \tan x}{1 + \tan x}$

(ix) $\sqrt{(1 \pm \sin x)} = \left|\cos\dfrac{x}{2} \pm \sin\dfrac{x}{2}\right|$

(x) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left(\dfrac{x \pm y}{1 \mp xy}\right)$

(xi) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

(xii) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\,\sqrt{1-y^2}\right\}$

(xiii) $\sin^{-1}x + \cos^{-1}x = \tan^{-1}x + \cot^{-1}x = \sec^{-1}x + \text{cosec}^{-1}x = \pi/2$

(xiv) $\sin^{-1}x = \text{cosec}^{-1}(1/x)$; $\cos^{-1}x = \sec^{-1}(1/x)$; $\tan^{-1}x = \cot^{-1}(1/x)$

NOTES :

Some standard substitutions :

Expressions	**Substitutions**
$\sqrt{(a^2 - x^2)}$	$x = a\sin\theta$ or $a\cos\theta$
$\sqrt{(a^2 + x^2)}$	$x = a\tan\theta$ or $a\cot\theta$
$\sqrt{(x^2 - a^2)}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\left(\dfrac{a+x}{a-x}\right)}$ or $\sqrt{\left(\dfrac{a-x}{a+x}\right)}$	$x = a\cos\theta$ or $a\cos 2\theta$
$\sqrt{(a-x)(x-b)}$ or $\sqrt{\left(\dfrac{a-x}{x-b}\right)}$ or $\sqrt{\left(\dfrac{x-b}{a-x}\right)}$	$x = a\cos^2\theta + b\sin^2\theta$
$\sqrt{(x-a)(x-b)}$ or $\sqrt{\left(\dfrac{x-a}{x-b}\right)}$ or $\sqrt{\left(\dfrac{x-b}{x-a}\right)}$	$x = a\sec^2\theta - b\tan^2\theta$
$\sqrt{(2ax - x^2)}$	$x = a(1 - \cos\theta)$

14.2 Logarithmic Differentiation

To find the derivative of :

If $y = \{f_1(x)\}^{f_2(x)}$ or $y = f_1(x).f_2(x).f_3(x)...$

or $y = \dfrac{f_1(x).f_2(x).f_3(x)...}{g_1(x).g_2(x).g_3(x)...}$

then it is convenient to take the logarithm of the function first and then differentiate. This is called derivative of the logarithmic function.

Important Notes (Alternate methods)

1. If $y = \{f(x)\}^{g(x)} = e^{g(x)\ln f(x)}$ ((variable)$^{\text{variable}}$) $\{\because x = e^{\ln x}\}$

$$\therefore \frac{dy}{dx} = e^{g(x)\ln f(x)}.\left\{g(x).\frac{d}{dx}\ln f(x) + \ln f(x).\frac{d}{dx}g(x)\right\}$$

$$= \{f(x)\}^{g(x)}.\left\{g(x).\frac{f'(x)}{f(x)} + \ln f(x).\,g'(x)\right\}$$

2. If $y = \{f(x)\}^{g(x)}$

$\therefore \dfrac{dy}{dx} =$ Derivative of y treating $f(x)$ as constant + Derivative of y treating $g(x)$ as constant

$$= \{f(x)\}^{g(x)}.\ln f(x).\frac{d}{dx}g(x) + g(x)\{f(x)\}^{g(x)-1}.\frac{d}{dx}f(x)$$

$$= \{f(x)\}^{g(x)}.\ln f(x).g'(x) + g(x).\{f(x)\}^{g(x)-1}.f'(x)$$

14.3 Implict Differentiation : $\phi(x, y) = 0$

(i) In order to find dy/dx in the case of implicit function, we differentiate each term w.r.t. x, regarding y as a function of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit function, both x & y are present.

Alternate Method : If $f(x, y) = 0$

then $$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{\text{diff. of f w.r.t. x treating y as constant}}{\text{diff. of f w.r.t. y treating x as constant}}$$

14.4 Parametric Differentiation

If $y = f(t)$ & $x = g(t)$ where t is a Parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{...(1)}$$

NOTES :

1. $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

2. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} \quad \left(\because \frac{dy}{dx} \text{ in terms of t}\right)$

$$= \frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right).\frac{1}{f'(t)} \quad \{\text{From (1)}\}$$

$$= \frac{f''(t)g'(t) - g''(t)f'(t)}{\{f'(t)\}^3}$$

14.5 Derivative of a Function w.r.t. another Function

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

14.6 Derivative of Infinite Series

If taking out one or more than one terms from an infinite series, it remains unchanged. Such that

(A) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$

then $y = \sqrt{f(x) + y} \Rightarrow (y^2 - y) = f(x)$

Differentiating both sides w.r.t. x, we get $(2y - 1)\frac{dy}{dx} = f'(x)$

(B) If $y = \{f(x)\}^{\{f(x)\}^{\{f(x)\}^{.......\infty}}}$ then $y = \{f(x)\}^y \Rightarrow y = e^{y \ln f(x)}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{y\{f(x)\}^{y-1}.f'(x)}{1 - \{f(x)\}^y.\ell n\, f(x)} = \frac{y^2 f'(x)}{f(x)\{1 - y\,\ell n\, f(x)\}}$$

15. DERIVATIVE OF ORDER TWO & THREE

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b), is a certain function $f'(x)$ [or (dy/dx) or y'] & is called the first derivative of y w.r.t. x. If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w.r.t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Similarly, the 3rd order derivative of y w.r.t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ it is also denoted by $f'''(x)$ or y'''.

Some Standard Results :

(i) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}.a^n.(ax+b)^{m-n}$, $m \geq n$.

(ii) $\frac{d^n}{dx^n}x^n = n!$

(iii) $\frac{d^n}{dx^n}(e^{mx}) = m^n.e^{mx}$, $m \in R$

(iv) $\frac{d^n}{dx^n}(\sin(ax+b)) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$, $n \in N$

(v) $\frac{d^n}{dx^n}(\cos(ax+b)) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$, $n \in N$

(vi) $\frac{d^n}{dx^n}\{e^{ax}\sin(bx+c)\} = r^n.e^{ax}.\sin(bx + c + n\phi)$, $n \in N$

where $r = \sqrt{(a^2 + b^2)}$, $\phi = \tan^{-1}(b/a)$.

(vii) $\frac{d^n}{dx^n}\{e^{ax}.\cos(bx+c)\} = r^n.e^{ax}.\cos(bx + c + n\phi)$, $n \in N$

where $r = \sqrt{(a^2 + b^2)}$, $\phi = \tan^{-1}(b/a)$.

16. DIFFERENTIATION OF DETERMINANTS

If $F(X)=\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$,

where *f, g, h, ℓ, m, n, u, v, w* are differentiable function of x then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

17. L' HOSPITAL'S RULE

If $f(x)$ & $g(x)$ are functions of x such that :

(i) $\lim\limits_{x\to a} f(x)=0=\lim\limits_{x\to a} g(x)$ or $\lim\limits_{x\to a} f(x)=\infty=\lim\limits_{x\to a} g(x)$ and

(ii) Both $f(x)$ & $g(x)$ are continuous at x = a and

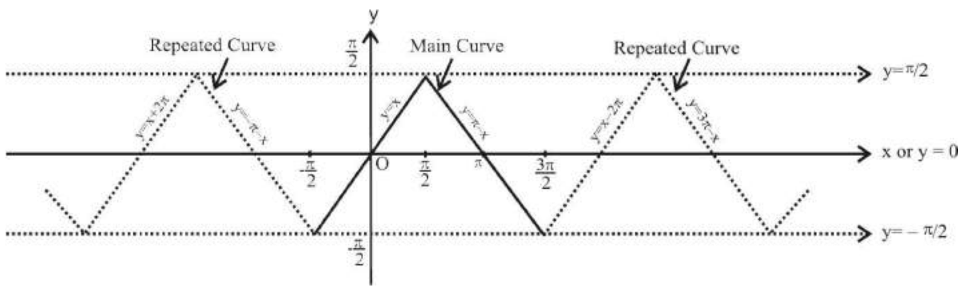
(iii) Both $f(x)$ & $g(x)$ are differentiable at x = a and

(iv) Both $f'(x)$ & $g'(x)$ are continuous at x = a, Then

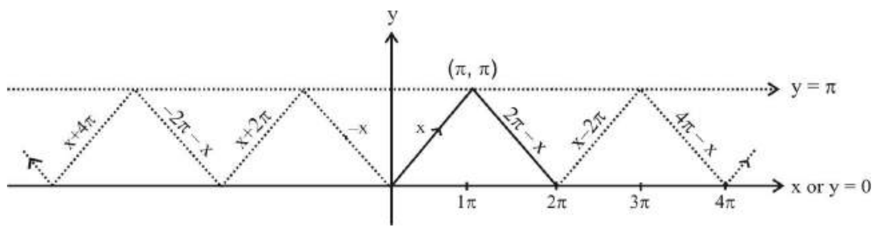
$$\underset{x\to a}{\text{Limit}}\frac{f(x)}{g(x)}=\underset{x\to a}{\text{Limit}}\frac{f'(x)}{g'(x)}=\underset{x\to a}{\text{Limit}}\frac{f''(x)}{g''(x)}$$ & so on till indeterminant form vanishes..

18. ANALYSIS & GRAPHS OF SOME USEFUL FUNCTION

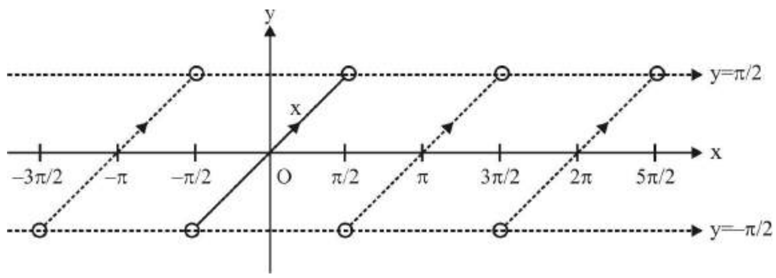
(i) $y=\sin^{-1}(\sin x)$ $\quad x\in R\,;\, y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$



(ii) $y=\cos^{-1}(\cos x)$ $\quad x\in R\,;\, y\in[0,\pi]$



(iii) $y=\tan^{-1}(\tan x)$ $\quad x\in R-\left\{x : x=(2n+1)\frac{\pi}{2}, n\in Z\right\}; y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



19. LEIBNITZ THEOREM

19.1 If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + u v_n$$

where u_r and v_r represent r^{th} derivatives of u and u respectively.

19.2 $$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)dt\right] = \frac{d}{dx}[h(x)].f[h(x)] - \frac{d}{dx}[g(x)].f[g(x)]$$

SOLVED EXAMPLES

Example – 1

If f is a real-valued function satisfying the relation

$f(x)+2f\left(\frac{1}{x}\right)=3x$ for all real $x \neq 0$, then $\lim_{x\to 0}(\sin x)\, f(x)$

is equal to

(a) 1 (b) 2

(c) 0 (d) ∞

Ans. (b)

Sol. We have $f(x)+2f\left(\frac{1}{x}\right)=3x$ ... (1)

Replacing x with 1/x, we have

$$2f(x)+f\left(\frac{1}{x}\right)=\frac{3}{x} \quad \text{... (2)}$$

From Eqs. (1) and (2) we get

$$f(x)=\frac{2}{x}-x$$

Therefore

$$\lim_{x\to 0}(\sin x)\, f(x)=\lim_{x\to 0}\left(\frac{2\sin x}{x}-x\sin x\right)$$

$=2(1)-0=2.$

Example – 2

Show that $f(x)=\begin{cases} 5x-4, & \text{when } 0<x\le 1 \\ 4x^3-3x, & \text{when } 1<x<2 \end{cases}$

is continuous at $x = 1$.

Sol. We have,

$(\text{LHL at } x=1)=\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} 5x-4$

$[\because f(x)=5x-4, \text{ when } x\le 1]$

$=5\times 1-4=1,$

$(\text{RHL at } x=1)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 4x^3-3x$

$[\because f(x)=4x^3-3x, x>1]$

$=4(1)^3-3(1)=1,$

and, $f(1)=5\times 1-4=1$

$[\because f(x)=5x-4, \text{ where } x\le 1]$

$\therefore \quad \lim_{x\to 1^-} f(x)=f(1)=\lim_{x\to 1^+} f(x)$

So, f(x) is continuous at $x = 1$.

Example – 3

Test the continuity of the function f (x) at the origin :

$$f(x)=\begin{cases} \frac{|x|}{x}; & x\neq 0 \\ 1\ ; & x=0 \end{cases}$$

Sol. We have,

$(\text{LHL at } x=0)=\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$

$$=\lim_{h\to 0}\frac{|-h|}{-h}=\lim_{h\to 0}\frac{h}{-h}=\lim_{h\to 0}-1=-1$$

and, $(\text{RHL at } x=0)=\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$

$$=\lim_{h\to 0}\frac{|h|}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$$

Thus, f(x) is not continuous at the origin.

Example – 4

Discuss the continuity of the function of given by

$f(x)=|x-1|+|x-2|$ at $x=1$ and $x=2$.

Sol. We have,

$f(x)=|x-1|+|x-2|$

$$\Rightarrow \quad f(x)=\begin{cases} -(x-1)-(x-2), & \text{if } x<1 \\ (x-1)-(x-2), & \text{if } 1\le x<2 \\ (x-1)+(x-2), & \text{if } x\ge 2 \end{cases}$$

$$\Rightarrow \quad f(x)=\begin{cases} -2x+3, & \text{if } x<1 \\ 1, & \text{if } 1\le x<2 \\ 2x-3, & x\ge 2 \end{cases}$$

Continuity at $x=1$:

We have,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(-2x+3)=-2\times 1+3=1$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 1=1$$

and, $f(1)=1$.

$$\therefore \quad \lim_{x\to 1^-} f(x)=f(1)=\lim_{x\to 1^+} f(x)$$

So, $f(x)$ is continuous at $x=1$.

Continuity at $x=2$

We have,

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 1=1$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2}(2x-3)=2\times 2-3=1$$

and, $f(2)=2\times 2-3=1$.

$$\therefore \quad \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$

So, $f(x)$ is continuous at $x=2$.

Example – 5

Examine the function f(t) given by

$$f(t)=\begin{cases} \dfrac{\cos t}{\pi/2-t}; & t\neq \pi/2 \\ 1 \quad ; & t=\pi/2 \end{cases}$$

for continuity at $t=\pi/2$

Sol. We have,

$$(\text{LHL at } t=\pi/2)=\lim_{t\to \pi/2^-} f(t)$$

$$=\lim_{h\to 0} f(\pi/2-h)=\lim_{h\to 0}\frac{\cos(\pi/2-h)}{\pi/2-(\pi/2-h)}=\lim_{h\to 0}\frac{\sin h}{h}=1$$

and, $(\text{RHL at } t=\pi/2)=\lim_{t\to \pi/2^+} f(t)$

$$=\lim_{h\to 0} f(\pi/2+h)=\lim_{h\to 0}\frac{\cos(\pi/2+h)}{\pi/2-(\pi/2+h)}$$

$$=\lim_{h\to 0}\frac{-\sin h}{-h}=\lim_{h\to 0}\frac{\sin h}{h}=1$$

Also, $f(\pi/2)=1$.

$$\therefore \quad \lim_{t\to \pi/2^-} f(t)=\lim_{t\to \pi/2^+}=f(\pi/2)$$

So, f(t) is continuous at $t=\pi/2$.

Example – 6

Show that the function f(x) given by

$$f(x)=\begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x\neq 0 \\ 0 \quad , & \text{when } x=0 \end{cases}$$ is discontinuous at x=0.

Sol. We have,

$$(\text{LHL at } x=0)=\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$$

$$=\lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1}=\lim_{h\to 0}\frac{\dfrac{1}{e^{1/h}}-1}{\dfrac{1}{e^{1/h}}+1}=\frac{0-1}{0+1}=-1$$

$$\left[\because \lim_{h\to 0}\frac{1}{e^{1/h}}=0\right]$$

and, $(\text{RHL at } x=0)=\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$

$$=\lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1}=\lim_{h\to 0}\frac{1-1/e^{1/h}}{1+1/e^{1/h}}=\frac{1-0}{1+0}=1$$

So, f(x) is not continuous at $x=0$ as LHL $\neq$ RHL and has a discontinuity of first kind at $x=0$.

Example – 7

Show that the function $f(x)=|\sin x+\cos x|$ is continuous at $x=\pi$.

Sol. Let $g(x)=\sin x+\cos x$ and $h(x)=|x|$. Then, $f(x)=hog(x)$. In order to prove that $f(x)$ is continuous at $x=\pi$. It is sufficient to prove that $g(x)$ is continuous at $x=\pi$ and $h(x)$ is continuous at $y=g(\pi)=\sin\pi+\cos\pi=-1$.

Now,

$$\lim_{x\to\pi} g(x) = \lim_{x\to\pi} (\sin x + \cos x) = \sin\pi + \cos\pi = -1$$

and $g(\pi) = -1$

$\therefore \quad \lim_{x\to\pi} g(x) = g(\pi)$

LHL at $x = \pi$

$$\lim_{x\to\pi} f(x) = \lim_{x\to\pi} |\sin\pi + \cos\pi| = +1$$

RHL at $x = \pi$

$$\lim_{x\to\pi} |\sin\pi + \cos\pi| = +1$$

Hence continuous

Example – 8

Let $f(x+y) = f(x) + f(y)$ for all $x, y \in R$. If $f(x)$ is continuous at $x = 0$, show that $f(x)$ is continuous at all x.

Sol. Since $f(x)$ is continuous at $x = 0$. Therefore,

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{h\to0} f(0-h) = \lim_{h\to0} f(0+h) = f(0)$$

$$\Rightarrow \lim_{h\to0} f(0+(-h)) = \lim_{h\to0} f(0+h) = f(0)$$

$$\Rightarrow \lim_{h\to0} [f(0)+f(-h)] = \lim_{h\to0} [f(0)+f(h)] = f(0)$$

[Using : $f(x+y) = f(x) + f(y)$]

$$\Rightarrow f(0) + \lim_{h\to0} f(-h) = f(0) + \lim_{h\to0} f(h) = f(0)$$

$$\Rightarrow \lim_{h\to0} f(-h) = \lim_{h\to0} f(h) = 0 \qquad \text{... (i)}$$

Let a be any real number. Then,

$$\lim_{x\to a^-} f(x) = \lim_{h\to0} f(a-h) = \lim_{h\to0} f(a+(-h))$$

$$\Rightarrow \lim_{x\to a^-} f(x) = \lim_{h\to0} [f(a)+f(-h)]$$

[$\because f(x+y) = f(x) + f(y)$]

$$\Rightarrow \lim_{x\to a^-} f(x) = f(a) + \lim_{h\to0} f(-h)$$

$$\Rightarrow \lim_{x\to a^-} f(x) = f(a) + 0 \qquad \text{[Using (i)]}$$

$$\Rightarrow \lim_{x\to a^-} f(x) = f(a). \qquad \text{and,}$$

$$\lim_{x\to a^+} f(x) = \lim_{h\to0} f(a+h)$$

$$\Rightarrow \lim_{x\to a^+} f(x) = \lim_{h\to0} [f(a)+f(h)]$$

[$\because f(x+y) = f(x) + f(y)$]

$$\Rightarrow \lim_{x\to a^+} f(x) = f(a) + \lim_{h\to0} f(h)$$

$$\Rightarrow \lim_{x\to a^+} f(x) = f(a) + 0 = f(a) \qquad \text{[Using (i)]}$$

Thus, we have

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

$\therefore$ $f(x)$ is continuous at $x = a$.

Since a is an arbitrary real number. So, $f(x)$ is continuous at all $x \in R$.

Example – 9

If f is a real-valued function defined for all $x \neq 0, 1$ and satisfying the relation

$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x} - \frac{2}{1-x}$. Then $\lim_{x\to2} f(x)$ is______.

Sol. Given relation is

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x} - \frac{2}{1-x} \qquad \text{... (1)}$$

Replacing x with $\frac{1}{1-x}$ in above equation Eq. we have

$$f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = 2(1-x) + \frac{2(1-x)}{x} \qquad \text{... (2)}$$

Again replacing x with $\frac{1}{1-x}$ in Eq. (1), we get

$$f\left(\frac{x-1}{x}\right) + f(x) = -2x - \frac{2x}{1-x} \qquad \text{... (3)}$$

Now adding Eqs. (1) and (3) and subtracting Eq. (2) gives

$$2f(x) = \left(\frac{2}{x} - \frac{2}{1-x} - 2x - \frac{2x}{1-x}\right) - 2(1-x) - \frac{2(1-x)}{x}$$

$$=\left(\frac{2}{x}-\frac{2(1-x)}{x}\right)+\left(\frac{-2}{1-x}-\frac{2x}{1-x}\right)-2x-2(1-x)$$

$$=\frac{2x}{x}-\frac{2(1+x)}{(1-x)}-2$$

$$=2+\frac{2\,(x+1)}{x-1}-2$$

$$=\frac{2(x+1)}{x-1}$$

Therefore

$$f(x)=\frac{x+1}{x-1}$$

Taking limit we get

$$\lim_{x\to 2}\left(\frac{x+1}{x-1}\right)=\frac{2+1}{2-1}=3$$

Example – 10

Prove that the greatest integer function [x] is continuous at all points except at integer points.

Sol. Let f(x) = [x] be the greatest integer function. Let k be any integer. Then,

$$f(x)=[x]=\begin{cases}k-1, & \text{if } k-1\le x<k\\ k\ , & \text{if } k\le x<k+1\end{cases}\quad \text{[By def.]}$$

Now (LHL at x = k)

$$=\lim_{x\to k^-}f(x)=\lim_{h\to 0}f(k-h)=\lim_{h\to 0}[k-h]$$

$$=\lim_{h\to 0}(k-1)=k-1$$

$[\because\ k–1\le k-h<k\ \therefore\ [k–h]=k–1]$

and, (RHL at x = k)

$$=\lim_{x\to k^+}f(x)=\lim_{h\to 0}f(k+h)=\lim_{h\to 0}[k+h]$$

$$=\lim_{h\to 0}k=k$$

$[\because k\le k+h<k+1\ \therefore\ [k+h]=k]$

$$\therefore\quad \lim_{x\to k^-}f(x)\ne\lim_{x\to k^+}f(x)$$

So, f(x) is not continuous at x = k.

Since k is an arbitrary integer. Therefore, f(x) is not continuous at integer points.

Let a be any real number other than an integer. Then, there exists an integer k such that $k-1<a<k$.

Now, (LHL at x = a)

$$\lim_{x\to a^-}f(x)=\lim_{h\to 0}f(a-h)=\lim_{h\to 0}[a-h]$$

$$=\lim_{h\to 0}k-1=k-1$$

$[\because k-1<a-h<k\ \therefore\ [a–h]=k-1]$

(RHL at x = a)

$$=\lim_{x\to a^+}f(x)=\lim_{h\to 0}f(a+h)$$

$$=\lim_{h\to 0}[a+h]=\lim_{h\to 0}(k-1)=k-1$$

$[\because k-1<a+h<k\ \therefore\ [a+h]=k-1]$

and, $f(a)=k-1$

$[\because k-1<a<k\ \therefore\ [a]=k-1]$

Thus,

$$\lim_{x-a^-}f(x)=\lim_{x\to a^+}f(x)=f(a)$$

So, f(x) is continuous at x = a. Since a is an arbitrary real number, other than an integer. Therefore, f(x) is continuous at all real points except integer points.

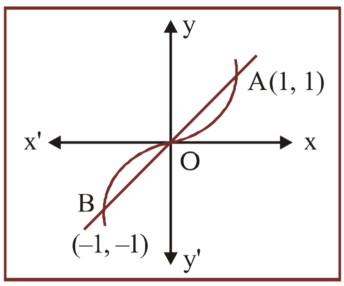
Example – 11

Let $f: R\to R$ be a function defined by $f(x)=\max\{x, x^3\}$. Show that the set of points $\{-1, 0, 1\}$, $f(x)$ is not differentiable.

Sol. $f(x)=\max\{x, x^3\}$ considering the graph separately, $y=x^3$ and $y=x$.

$$\text{Now, } f(x)=\begin{cases}x \text{ in } (-\infty, -1]\\ x^3 \text{ in } [-1, 0]\\ x \text{ in } [0, 1]\\ x^3 \text{ in } [1, \infty)\end{cases}$$

The point of consideration are



$f'(-1^-) = 1$ and $f'(-1^+) = 3$

$f'(-0^-) = 0$ and $f'(0^+) = 1$

$f'(1^-) = 1$ and $f'(1^+) = 3$

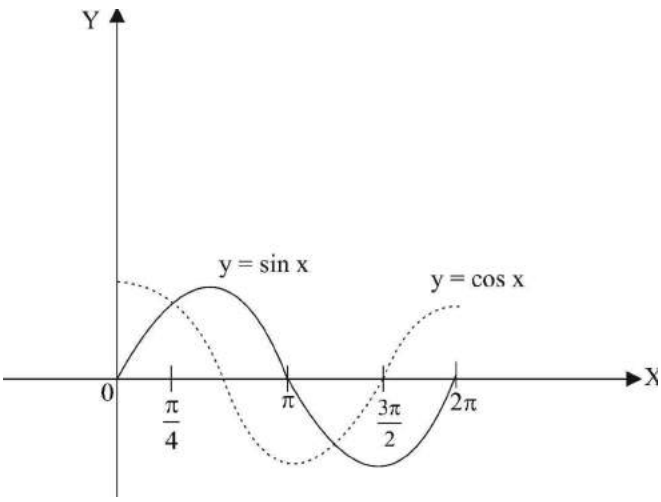
Hence, f is not differentiable at $-1, 0, 1$.

Example – 12

Find the number of points of non-differentiability of $f(x) = \max.\{\sin x, \cos x, 0\}$ in $(0, 2n\pi)$. Where $n \in N$

Sol. Here, we know sin x and cos x are periodic with period 2π. Thus we could sketch the curve as; (In the interval 0 to 2π)

Which shows



$y = \max.\{\sin x, \cos x, 0\}$

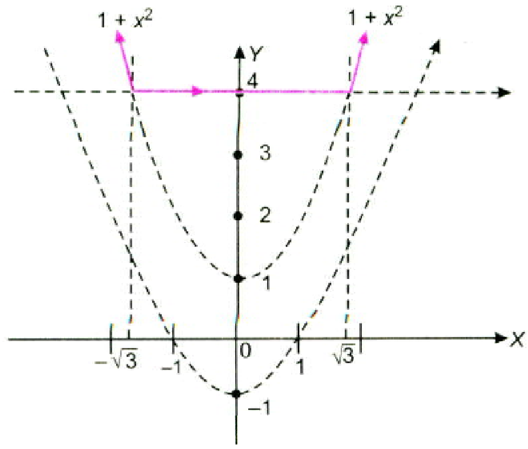
$$= \begin{cases} \cos x, 0 < x < \dfrac{\pi}{4} \text{ or } \dfrac{3\pi}{2} < x < 2\pi \\ 0, \pi < x < \dfrac{3\pi}{2} \\ \sin x, \dfrac{\pi}{4} < x < \pi \end{cases}$$

Clearly, $y = \max.\{\sin x, \cos x, 0\}$ is not differentiable at 3 points when $x = (0, 2\pi)$.

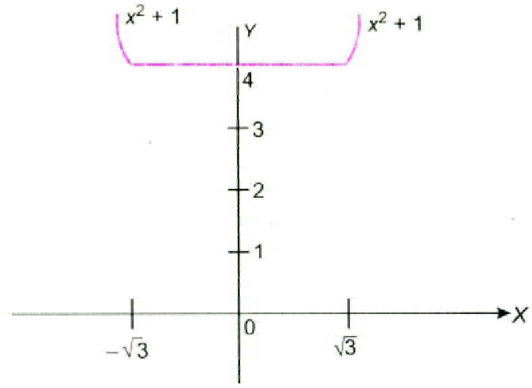
Example – 13

Let $f(x) = \text{maximum}\{4, 1 + x^2, x^2 - 1\}$ $\forall x \in R$. Then find the total number of points, where $f(x)$ is not differentiable.

Sol. Using functions for sketching maximum $\{4, 1 + x^2, x^2 - 1\}$ as



OR



Thus, from above graph we can simply say,

$f(x)$ is not differentiable at $x = \pm\sqrt{3}$.

And it could be defined as :

$$f(x) = \begin{cases} 4, -\sqrt{3} \le x \le \sqrt{3} \\ x^2 + 1, x \le -\sqrt{3} \text{ or } x \ge \sqrt{3} \end{cases}$$

Example – 14

Discuss the differentiability of $f(x) = |x-1| + |x-2|$.

Sol. We have,

$f(x) = |x-1| + |x-2|$

$$\Rightarrow \quad f(x) = \begin{cases} -(x-1) - (x-2) \text{ for } x < 1 \\ x - 1 - (x-2) \text{ for } 1 \le x < 2 \\ (x-1) + (x-2) \text{ for } x \ge 2 \end{cases}$$

$$\Rightarrow \quad f(x)=\begin{cases}-2x+3 & , \quad x<1 \\ 1 & , \quad 1\le x<2 \\ 2x-3 & , \quad x\ge 2\end{cases}$$

When $x < 1$, we have

$f(x) = -2x + 3$ which, being a polynomial function is continuous and differentiable.

When $1 \le x < 2$, we have

$f(x) = 1$ which, being a constant function, is differentiable on $(1, 2)$.

When $x \ge 2$, we have

$f(x) = 2x - 3$ which, being a polynomial function, is differentiable for all $x > 2$. Thus, the possible points of non-differentiability of $f(x)$ are $x = 1$ and $x = 2$.

Now,

$$(\text{LHD at } x=1)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \quad (\text{LHD at } x=1)=\lim_{x\to 1}\frac{(-2x+3)-1}{x-1}$$

$[\because f(x)=-2x+3 \text{ for } x<1]$

$$\Rightarrow \quad (\text{LHD at } x=1)=\lim_{x\to 1}\frac{-2(x-1)}{x-1}=-2$$

$$(\text{RHD at } x=1)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \quad (\text{RHD at } x=1)=\lim_{x\to 1}\frac{1-1}{x-1}=0$$

$[\because f(x)=1 \text{ for } 1\le x<2]$

$\therefore \quad (\text{LHD at } x=1)\ne(\text{RHD at } x=1)$

So, $f(x)$ is not differentiable at $x = 1$.

$$(\text{LHD at } x=2)=\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$$

$$\Rightarrow \quad (\text{LHD at } x=2)=\lim_{x\to 2^-}\frac{1-(2\times 2-3)}{x-2}$$

$$\left[\begin{array}{l}\because f(x)=1 \text{ for } 1\le x<2 \\ \quad \text{and } f(2)=2\times 2-3\end{array}\right]$$

$$\Rightarrow \quad (\text{LHD at } x=2)=\lim_{x\to 2^-}\frac{1-1}{x-2}=0.$$

$$(\text{RHD at } x=2)=\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$

$$\Rightarrow \quad (\text{RHD at } x=2)=\lim_{x\to 2^+}\frac{(2x-3)-(2\times 2-3)}{x-2}$$

$[\because f(x)=2x-3 \text{ for } x\ge 2]$

$$\Rightarrow \quad (\text{RHD at } x=2)=\lim_{x\to 2^+}\frac{2x-4}{x-2}=\lim_{x\to 2^+}\frac{2(x-2)}{x-2}=2$$

$\therefore \quad (\text{LHD at } x=2)\ne(\text{RHD at } x=2)$

So, $f(x)$ not differentiable at $x = 2$.

Remark It should be noted that the function f(x) given by

$f(x)=|x-a_1|+|x-a_2|+|x-a_3|+\dots+|x-a_n|$

is not differentiable at $x = a_1, a_2, a_3, \dots, a_n$.

Example – 15

If $\sqrt{(1-x^{2n})}+\sqrt{(1-y^{2n})}=a(x^n-y^n)$ then prove that

$$\frac{dy}{dx}=\frac{x^{n-1}}{y^{n-1}}\sqrt{\left(\frac{1-y^{2n}}{1-x^{2n}}\right)}.$$

Sol. We have $\sqrt{(1-x^{2n})}+\sqrt{(1-y^{2n})}=a(x^n-y^n)$ ...(1)

$$\left.\begin{array}{l}\text{Putting } x^n=\sin\theta \Rightarrow \theta=\sin^{-1}x^n \\ \text{and} \quad y^n=\sin\phi \Rightarrow \phi=\sin^{-1}y^n\end{array}\right\} \quad ...(2)$$

then (1), becomes $\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$$\Rightarrow \quad 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)=a.2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow \quad \cot\left(\frac{\theta+\phi}{2}\right)=a \Rightarrow \theta-\phi=2\cot^{-1}a$$

$\Rightarrow \quad \sin^{-1}x^n-\sin^{-1}y^n=2\cot^{-1}a \qquad$ {from (2)}

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{(1-x^{2n})}}.nx^{n-1}-\frac{1}{\sqrt{(1-y^{2n})}}.ny^{n-1}\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x^{n-1}}{y^{n-1}}\sqrt{\left(\frac{1-y^{2n}}{1-x^{2n}}\right)}$$

Example – 16

If $y = \dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}$, show that

$\dfrac{dy}{dx} = \dfrac{-2a^2}{x^3}\left\{1+\dfrac{a^2}{\sqrt{a^4-x^4}}\right\}$.

Sol. We have,

$$y = \frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}$$

$$= \frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}} \cdot \frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}$$

$$\Rightarrow \quad y = \frac{\left[\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right]^2}{\left(a^2+x^2\right)-\left(a^2-x^2\right)}$$

$$= \frac{a^2+x^2+a^2-x^2+2\sqrt{a^2+x^2}\sqrt{a^2-x^2}}{2x^2}$$

$$\Rightarrow \quad y = \frac{2a^2+2\sqrt{a^4-x^4}}{2x^2}$$

$$\Rightarrow \quad y = \frac{a^2}{x^2}+\frac{\sqrt{a^4-x^4}}{x^2}$$

$$\Rightarrow \quad y = a^2x^{-2}+\sqrt{a^4-x^4}\,x^{-2}$$

$$\Rightarrow \quad \frac{dy}{dx} = a^2\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}\left\{\sqrt{a^4-x^4}\,x^{-2}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = -2a^2x^{-3}+(-2)x^{-3}\sqrt{a^4-x^4}$$

$$+\left(x^{-2}\right)\frac{1}{2}\left(a^4-x^4\right)^{-1/2}\frac{d}{dx}\left(a^4-x^4\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a^2}{x^3}-\frac{2}{x^4}\sqrt{a^4-x^4}+\frac{1}{2x^2\sqrt{a^4-x^4}}\left(-4x^3\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a^2}{x^3}-\frac{2}{x^3}\sqrt{a^4-x^4}-\frac{2x}{\sqrt{a^4-x^4}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a^2}{x^3}-2\left\{\frac{\sqrt{a^4-x^4}}{x^3}+\frac{x}{\sqrt{a^4-x^4}}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a^2}{x^3}-2\left\{\frac{a^4-x^4+x^4}{x^3\sqrt{a^4-x^4}}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a^2}{x^3}-\frac{2a^4}{x^3\sqrt{a^4-x^4}} = \frac{-2a^2}{x^3}\left\{1+\frac{a^2}{\sqrt{a^4-x^4}}\right\}$$

Example – 17

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, prove that $\dfrac{dy}{dx} = -\dfrac{1}{(x+1)^2}$

Sol. We have,

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$

$\Rightarrow \quad x^2(1+y) = y^2(1+x)$ [On squaring both sides]

$\Rightarrow \quad x^2-y^2 = y^2x-x^2y$

$\Rightarrow \quad (x+y)(x-y) = -xy(x-y)$

$\Rightarrow \quad x+y = -xy$ [$\because x-y \neq 0$. as $y = x$ does not satisfy the given equation]

$\Rightarrow \quad x = -y-xy$

$\Rightarrow \quad y(1+x) = -x$

$$\Rightarrow \quad y = -\frac{x}{1+x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left\{\frac{(1+x).1-x(0+1)}{(1+x)^2}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Example – 18

If $y = a^{x^{a^{x^{\cdots\infty}}}}$, prove that $\dfrac{dy}{dx} = \dfrac{y^2\log y}{x(1-y\log x\,.\,\log y)}$.

Sol. The given series may be written as

$$y = a^{(x^y)}$$

$\Rightarrow \quad \log y = x^y \log a$ [Taking log of both sides]

$\Rightarrow \quad \log(\log y) = y\log x+\log(\log a)$

$$\Rightarrow \quad \frac{1}{\log y}\frac{d}{dx}(\log y) = \frac{dy}{dx}\,.\,\log x+y\,.\frac{d}{dx}(\log x)+0$$

[Differentiating both sides with respect to x]

$$\Rightarrow \quad \frac{1}{\log y}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}.\log x+y.\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{1}{y\log y}-\log x\right\}=\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{1-y\log y.\log x}{y\log y}\right\}=\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2\log y}{x\{1-y\log y.\log x\}}.$$

Example – 19

If $y=e^{x+e^{x+e^{x+\dots\text{to }\infty}}}$, show that $\frac{dy}{dx}=\frac{y}{1-y}$

Sol. The given function may be written as

$y=e^{x+y}$

$\Rightarrow$ $\log y=(x+y).\log e$ [Taking log of both sides]

$\Rightarrow$ $\log y=x+y[\because \log e=1]$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx} \qquad \text{[Differentiating with respect to x]}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y}-1\right)=1\Rightarrow\frac{dy}{dx}=\frac{y}{1-y}$$

Example – 20

Show that the function $f(x)=\begin{cases}x-1, & \text{if } x<2\\ 2x-3, & \text{if } x\ge 2\end{cases}$ is not differentiable at $x=2$.

Sol. We have,

$$\text{(LHD at } x=2)=\lim_{x\to2^-}\frac{f(x)-f(2)}{x-2}$$

$$\Rightarrow \text{(LHD at } x=2)=\lim_{x\to2^-}\frac{(x-1)-(4-3)}{x-2}$$

$[\because f(x)=x-1 \text{ for } x<2]$

$$\Rightarrow \text{(LHD at } x=2)=\lim_{x\to2^-}\frac{x-2}{x-2}=\lim_{x\to2^-}1=1$$

and, $$\text{(RHD at } x=2)=\lim_{x\to2^+}\frac{f(x)-f(2)}{x-2}$$

$$\Rightarrow \text{(RHD at } x=2)=\lim_{x\to2^+}\frac{(2x-3)-(4-3)}{x-2}$$

$[\because f(x)=2x-3 \text{ for } x\ge2]$

$$\Rightarrow \text{(RHD at } x=2)=\lim_{x\to2^+}\frac{2x-4}{x-2}=\lim_{x\to2^+}2=2$$

$\Rightarrow$ (LHD at $x=2$) $\ne$ (RHD at $x=2$).

So, $f(x)$ is not differentiable at $x=2$.

Example – 21

Discuss the differentiability of

$$f(x)=\begin{cases}xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , \quad x\ne0\\ 0 & , \quad x=0\end{cases} \text{ at } x=0.$$

Sol. We have,

$$f(x)=\begin{cases}xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)}=xe^{-2/x}, & x\ge0\\ xe^{-\left(\frac{-1}{x}+\frac{1}{x}\right)}=x \quad , & x<0\\ 0 \quad , & x=0\end{cases}$$

Now,

$$\text{(LHD at } x=0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \text{(LHD at } x=0)=\lim_{x\to0^-}\frac{x-0}{x-0}=1$$

$[\because f(x)=x \text{ for } x<0 \text{ and } f(0)=0]$

and,

$$\text{(RHD at } x=0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \text{(RHD at } x=0)=\lim_{x\to0^+}\frac{xe^{-2/x}-0}{x}$$

$$\left[\begin{array}{r}\because f(x)=xe^{-2/x}\text{for } x>0\\ \text{and } f(0)=0\end{array}\right]$$

$$\Rightarrow \text{(RHD at } x=0)=\lim_{x\to0^+}e^{-2/x}=0.$$

$\frac{-2}{x}$ tends to $-\infty$ and $e^{-\infty}=0$

$\Rightarrow$ (LHD at $x=0$) $\ne$ (RHD at $x=0$)

So, $f(x)$ is not differentiable at $x=0$.

Example – 22

Show that the function $f(x)$ is continuous at $x = 0$ but its derivative does not exists at $x = 0$

$$\text{if } f(x) = \begin{cases} x\sin\left(\log x^2\right); & x \neq 0 \\ 0 \quad ; & x = 0 \end{cases}$$

Sol. $\text{LHL} = \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} f(-h)\sin\log(-h)^2$

$$= -\lim_{h\to 0} h\sin\log h^2$$

As $h \to 0$, $\log h^2 \to -\infty$.

Hence $\sin\log h^2$ oscillates between -1 and $+1$.

$$\text{L.H.L} = -\lim_{h\to 0}(h) \times \lim_{h\to 0}\left(\sin\log h^2\right)$$

$$= -0 \times (\text{number between } -1 \text{ and } +1) = 0$$

$$\text{R.H.L} = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} h\sin\log h^2 = \lim_{h\to 0} h \lim_{h\to 0}\sin\log h^2$$

$$= 0 \times (\text{oscillating between } -1 \text{ and } +1) = 0$$

$f(0) = 0$ (Given)

$\Rightarrow$ L.H.L. = R.H.L. = $f(0)$

Hence $f(x)$ is continuous at $x = 0$.

Test for differentiability :

$$Lf'(0) = \lim_{h\to 0}\frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{-h\sin\log(-h)^2 - 0}{-h}$$

$$= \lim_{h\to 0}\sin\left(\log h^2\right)$$

As the expression oscillates between -1 and $+1$, the limit does not exists.

$\Rightarrow$ Left hand derivative is not defined.

Hence the function is not differentiable at $x = 0$

Example – 23

Find the value of the constant λ so that the function given below is continuous at $x = -1$.

$$f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1}, & x \neq -1 \\ \lambda \quad , & x = -1 \end{cases}$$

Sol. Since $f(x)$ is continuous at $x = -1$. Therefore,

$$\lim_{x\to -1} f(x) = f(-1)$$

$$\Rightarrow \lim_{x\to -1}\frac{x^2 - 2x - 3}{x+1} = \lambda \qquad [\because f(-1) = \lambda]$$

$$\Rightarrow \lim_{x\to -1}\frac{(x-3)(x+1)}{x+1} = \lambda \Rightarrow \lim_{x\to -1}(x-3) = \lambda \Rightarrow -4 = \lambda$$

So, $f(x)$ is continuous at $x = -1$, if $\lambda = -4$.

Example – 24

If the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{\log(1+ax) - \log(1-bx)}{x}, & \text{if } x \neq 0 \\ k \quad , & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$, find k.

Sol. Since $f(x)$ is continuous at $x = 0$. Therefore,

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0}\frac{\log(1+ax) - \log(1-bx)}{x} = k \qquad [\because f(0) = k]$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{\log(1+ax)}{x} - \frac{\log(1-bx)}{x}\right] = k$$

$$\Rightarrow \lim_{x\to 0}\frac{\log(1+ax)}{x} - \lim_{x\to 0}\frac{\log(1-bx)}{x} = k$$

$$\Rightarrow a\lim_{x\to 0}\frac{\log(1+ax)}{ax} - (-b)\lim_{x\to 0}\frac{\log(1-bx)}{(-b)x} = k$$

$$\Rightarrow a(1) - (-b)(1) = k$$

$$\left[\text{Using} : \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

$\Rightarrow$ $a + b = k$

Thus, $f(x)$ is continuous at $x = 0$, if $k = a + b$.

Example – 25

Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & \text{if } x<0\\ a, & \text{if } x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{if } x>0\end{cases}$

Determine the value of a so that f (x) is continuous at x = 0.

Sol. For f (x) to be continuous at x = 0, we must have

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

$$\Rightarrow \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=a \quad \text{... (i)}$$

Now, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\dfrac{1-\cos 4x}{x^2}$

$$\left[\because f(x)=\frac{1-\cos 4x}{x^2} \text{ for } x<0\right]$$

$$\Rightarrow \lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{2\sin^2 2x}{x^2}$$

$$\Rightarrow \lim_{x\to 0^-} f(x)=2\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^2$$

$$\Rightarrow \lim_{x\to 0^-} f(x)=2\times 4\,.\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2=8(1)^2=8 \quad \text{... (ii)}$$

and, $\lim_{x\to 0^+} f(x)=\lim_{x\to 0}\dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$

$$\left[\because f(x)=\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} \text{ for } x>0\right]$$

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{x\to 0}\frac{\sqrt{x}}{16+\sqrt{x}-16}.(\sqrt{16+\sqrt{x}}+4)$$

Rationalizing Denominator

$$\Rightarrow \lim_{x\to 0^+} f(x)=\lim_{x\to 0}(\sqrt{16+\sqrt{x}}+4)=4+4=8 \quad \text{... (iii)}$$

From (i), (ii) and (iii), we get a = 8.

Example – 26

Determine the value of the constant m so that the function

$f(x)=\begin{cases} m(x^2-2x), & \text{if } x<0\\ \cos x, & \text{if } x\ge 0\end{cases}$ is continuous

Sol. When x < 0, we have

$f(x)=m(x^2-2x)$, which being a polynomial is continuous at each x < 0.

When x > 0, we have

f (x) = cos x, which being a cosine function is continuous at each x > 0.

So, consider the point x = 0.

We have,

(LHL at x = 0) = $\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(x^2-2x)=0$, for all values of m and (RHL at x = 0) = $\lim_{x\to 0^+} f(x)=\lim_{x\to 0}\cos x=1$

Clearly, $\lim_{x\to 0^-} f(x)\ne \lim_{x\to 0^+} f(x)$ for all values of m.

So, f (x) cannot be made continuous for any value of m.

In other words, the value of m does not exist for which f(x) can be made continuous.

Example – 27

If $f(x)=\begin{cases}1, & \text{if } x\le 3\\ ax+b, & \text{if } 3<x<5\\ 7, & \text{if } 5\le x\end{cases}$

Determine the values of a and b so that f(x) is continuous.

Sol. The given function is a constant function for all x < 3 and for all x > 5 so it is continuous for all x < 3 and for all x > 5. We know that a polynomial function is continuous. So, the given function is continuous for all x ∈ (3, 5). Thus, f(x) is continuous at each x ∈ R except possibly at x = 3 and x = 5.

At, x = 3, we have

$\lim_{x\to 3^-} f(x)=\lim_{x\to 3} 1=1$, $\lim_{x\to 3^+} f(x)=\lim_{x\to 3} ax+b=3a+b$, and, f(3) = 1

For f(x) be continuous at x = 3, we must have

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3)$$

$\Rightarrow 1=3a+b$ .... (i)

At x = 5, we have

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5} ax + b = 5a + b;$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5} 7 = 7, \text{ and, } f(5) = 7$$

For f(x) to be continuous at x = 5, we must have

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$$

$\Rightarrow$ $5a + b = 7$ .... (ii)

Solving (i) and (ii), we get a = 3, b = – 8

Example – 28

Find the values of a & b so that the function is continuous for $0 \le x \le \pi$

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \dfrac{\pi}{4} \\ 2x\cot x + b, & \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \\ a\cos 2x - b\sin x, & \dfrac{\pi}{2} < x \le \pi \end{cases}$$

Sol. Since, $f(x)$ is continuous for $0 \le x \le \pi$

$$\therefore \text{ RHL}\left(\text{at } x = \frac{\pi}{4}\right) = \text{LHL}\left(\text{at } x = \frac{\pi}{4}\right)$$

$$\Rightarrow \left(2.\frac{\pi}{4}\cot\frac{\pi}{4} + b\right) = \left(\frac{\pi}{4} + a\sqrt{2}.\sin\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\pi}{2} + b = \frac{\pi}{4} + a$$

$$\Rightarrow a - b = \frac{\pi}{4} \qquad \text{....(i)}$$

$$\text{Also, RHL}\left(\text{at } x = \frac{\pi}{2}\right) = \text{LHL}\left(\text{at } x = \frac{\pi}{2}\right)$$

$$\Rightarrow \left(a\cos\frac{2\pi}{2} - b\sin\frac{\pi}{2}\right) = \left(2.\frac{\pi}{2}.\cot\frac{\pi}{2} + b\right)$$

$$\Rightarrow -a - b = b$$

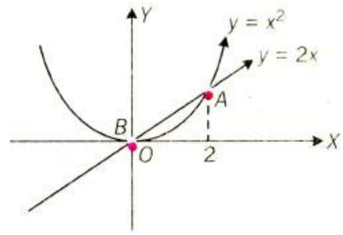
$$\Rightarrow a + 2b = 0 \qquad \text{...(ii)}$$

From eqs. (i) and (ii), $a = \dfrac{\pi}{6}$ and $b = \dfrac{-\pi}{12}$

Example – 29

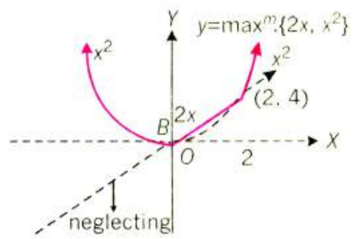
Draw graph for y = max. $\{2x, x^2\}$ and discuss the continuity and differentiability.

Sol. Here, to draw, y = max. $\{2x, x^2\}$



Firstly plot y = 2x and y = x^2 on graph and put $2x = x^2 \Rightarrow x =$ 0, 2 (i.e., their point of intersection).

Now, since y = max. $\{2x, x^2\}$ we have to neglect the curve below point of intersections thus, the required graph is, as shown.



Thus, from the given graph y = max. $\{2x, x^2\}$ we can say y = max. $\{2x, x^2\}$ is continuous for all $x \in R$.

But y = max. $\{2x, x^2\}$ is differentiable for all $x \in R - \{0, 2\}$

NOTES :

One must remember the formula we can write,

$$\max.\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \left|\frac{f(x)-g(x)}{2}\right|$$

$$\min.\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} - \left|\frac{f(x)-g(x)}{2}\right|$$

Example – 30

Differentiate the following functions with respect to x :

(i) x^{x^x} (ii) $(x^x)^x$

Sol. (i) Let $y = x^{x^x}$. Then,

$$y = e^{x^x.\log x}$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx}=e^{x^{x}.\log x}\frac{d}{dx}\left(x^{x}.\log x\right)$$

$$\Rightarrow \frac{dy}{dx}=x^{x^{x}}\frac{d}{dx}\left(e^{x\log x}.\log x\right)$$

$$\Rightarrow \frac{dy}{dx}=x^{x^{x}}\left\{\log x.\frac{d}{dx}\left(e^{x\log x}\right)+e^{x\log x}.\frac{d}{dx}\left(\log x\right)\right\}$$

$$\Rightarrow \frac{dy}{dx}=x^{x^{x}}\left\{\log x.e^{x}\log x\frac{d}{dx}(x\log x)+e^{x\log x}.\frac{1}{x}\right\}$$

$$\Rightarrow \frac{dy}{dx}=x^{x^{x}}\left\{\log x.x^{x}\left(x.\frac{1}{x}+\log x\right)+x^{x}.\frac{1}{x}\right\}$$

$$\Rightarrow \frac{dy}{dt}=x^{x^{x}}\left\{x^{x}\left(1+\log x\right).\log x+\frac{x^{x}}{x}\right\}$$

$$\Rightarrow \frac{dy}{dt}=x^{x^{x}}.\ x^{x}\left\{\left(1+\log x\right).\log x+\frac{1}{x}\right\}$$

(ii) Let $y=(x^{x})^{x}$. Then,

$$y=x^{x.x}=x^{x^{2}}$$

$$\log y=x^{2}\log x$$

$$\Rightarrow y=e^{x^{2}.\log x}$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx}=e^{x^{2}.\log x}\frac{d}{dx}\left(x^{2}.\log x\right)$$

$$\Rightarrow \frac{dy}{dx}=e^{x^{2}.\log x}\left(\log x.\frac{d}{dx}\left(x^{2}\right)+x^{2}.\frac{d}{dx}\left(\log x\right)\right)$$

$$\Rightarrow \frac{dy}{dx}=x^{x^{2}}\left(\log x.2x+x^{2}.\frac{1}{x}\right)$$

$$\left[\because e^{x^{2}}.\log x=x^{x^{2}}\right]$$

$$\Rightarrow \frac{dy}{dx}=x^{x^{2}}\left(2x.\log x+x\right)$$

$$\Rightarrow \frac{dy}{dx}=x.x^{x^{2}}\left(2\log x+1\right).$$

Example – 31

Differentiate : $(\log x)^{x}+x^{\log x}$ with respect to x.

Sol. Let $y=(\log x)^{x}+x\log x$. Then,

$$y=e^{\log(\log x)^{x}}+e^{\log(x^{\log x})}$$

$$\Rightarrow y=e^{x\log(\log x)}+e^{\log x\,.\,\log x}$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx}=e^{x\log(\log x)}.\frac{d}{dx}\{x\log(\log x)\}+e^{(\log x)^{2}}\frac{d}{dx}(\log x)^{2}$$

$$\Rightarrow \frac{dy}{dx}=(\log x)^{x}\left\{\log(\log x).\frac{d}{dx}(x)+x.\frac{d}{dx}(\log(\log x))\right\}+x^{\log x}$$

$$\left\{2(\log x.\frac{d}{dx}(\log x))\right\}$$

$$\Rightarrow \frac{dy}{dx}=(\log x)^{x}\left\{\log(\log x)+x.\frac{1}{\log x}.\frac{1}{x}\right\}+x^{\log x}\left\{2\log x.\frac{1}{x}\right\}$$

$$\Rightarrow \frac{dy}{dx}=(\log x)^{x}\left\{\log(\log x)+\frac{1}{\log x}\right\}+x^{\log x}\left\{\frac{2\log x}{x}\right\}.$$

Example – 32

Differentiate the following functions with respect to x :

$$x^{\cot x}+\frac{2x^{2}-3}{x^{2}+x+2}$$

Sol. Let $y=x^{\cot x}+\frac{2x^{2}-3}{x^{2}+x+2}$. Then,

$$y=e^{\cot x.\log x}+\frac{2x^{2}-3}{x^{2}+x+2}$$

$$\left[\because x^{\cot x}=e^{\log x^{\cot x}}=e^{\cot x.\log x}\right]$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\cot x\,.\,\log x}\right)+\frac{d}{dx}\left(\frac{2x^2-3}{x^2+x+2}\right)$$

$$\frac{dy}{dx}=e^{\cot x.\log x}\frac{d}{dx}(\cot x.\,\log x)$$

$$+\frac{(x^2+x+2)\frac{d}{dx}(2x^2-3)-(2x^2-3)\frac{d}{dx}(x^2+x+2)}{(x^2+x+2)^2}$$

$$\frac{dy}{dx}=x^{\cot x}\left\{\log x\,.\,\frac{d}{dx}(\cot x)+\cot x\,.\,\frac{d}{dx}(\log x)\right\}$$

$$+\frac{(x^2+x+2)(4x)-(2x^2-3)(2x+1)}{(x^2+x+2)^2}$$

$$\frac{dy}{dx}=x^{\cot x}\left\{-\operatorname{cosec}^2 x.\log x+\frac{\cot x}{x}\right\}+\frac{2x^2+14x+3}{(x^2+x+2)^2}$$

Example – 33

Given that $\cos\frac{x}{2}.\ \cos\frac{x}{4}.\ \cos\frac{x}{8}...=\frac{\sin x}{x}$, prove that

$$\frac{1}{2^2}\sec^2\frac{x}{2}+\frac{1}{2^4}\sec^2\frac{x}{4}+...=\operatorname{cosec}^2 x-\frac{1}{x^2}$$

Sol. We have,

$$\cos\frac{x}{2}.\ \cos\frac{x}{4}.\ \cos\frac{x}{8}...=\frac{\sin x}{x}$$

Taking log on both sides, we get

$$\log\cos\frac{x}{2}+\log\cos\frac{x}{4}+\log\cos\frac{x}{8}...=\log\sin x-\log x$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}-\frac{1}{4}\frac{\sin\frac{x}{4}}{\cos\frac{x}{4}}-\frac{1}{8}\frac{\sin\frac{x}{8}}{\cos\frac{x}{8}}...=\frac{\cos x}{\sin x}-\frac{1}{x}$$

$$\Rightarrow\ -\frac{1}{2}\tan\frac{x}{2}-\frac{1}{4}\tan\frac{x}{4}-\frac{1}{8}\tan\frac{x}{8}...=\cot x-\frac{1}{x}$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2^2}\sec^2\frac{x}{2}-\frac{1}{4^2}\sec^2\frac{x}{4}-\frac{1}{8^2}\sec^2\frac{x}{8}...=-\operatorname{cosec}^2 x+\frac{1}{x^2}$$

$$\frac{1}{2^2}\sec^2\frac{x}{2}+\frac{1}{4^2}\sec^2\frac{x}{4}+\frac{1}{8^2}\sec^2\frac{x}{8}...=\operatorname{cosec}^2 x-\frac{1}{x^2}$$

Example – 34

If $y=f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x)=\sin x^2$, find $\frac{dy}{dx}$.

Sol. Let $z=\frac{2x-1}{x^2+1}$. Then,

$y=f(z)$

$$\Rightarrow\ \frac{dy}{dx}=\frac{d}{dx}\{f(z)\}=\frac{d}{dz}\{f(z)\}\,.\,\frac{dz}{dx}$$

$$\Rightarrow\ \frac{dy}{dx}=f'(z)\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$\Rightarrow\ \frac{dy}{dx}=f'(z)\left\{\frac{2(x^2+1)-(2x-1)2x}{(x^2+1)^2}\right\}$$

$$\Rightarrow\ \frac{dy}{dx}=\sin z^2\ \frac{2(x^2+1)-(4x^2-2x)}{(x^2+1)^2}$$

$$\left[\begin{array}{l}\because\ f'(x)=\sin x^2\\ \therefore\ f'(z)=\sin z^2\end{array}\right]$$

$$\Rightarrow\ \frac{dy}{dx}=2\sin\left(\frac{2x-1}{x^2+1}\right)\left\{\frac{1+x-x^2}{\left(x^2+1\right)^2}\right\}$$

Example – 35

If $x=\sqrt{a^{\sin^{-1}t}}$, $y=\sqrt{a^{\cos^{-1}t}}$, $a>0$ and $-1<t<1$, show that $\frac{dy}{dx}=-\frac{y}{x}$

Sol. We have,

$$x=\sqrt{a^{\sin^{-1}t}} \text{ and } y=\sqrt{a^{\cos^{-1}t}}$$

$$\Rightarrow \frac{dx}{dt}=\frac{1}{2}\left(a^{\sin^{-1}t}\right)^{-1/2}\frac{d}{dt}\left(a^{\sin^{-1}t}\right) \text{ and } \frac{dy}{dx}=\frac{1}{2}\left(a^{\cos^{-1}t}\right)^{-1/2}\frac{d}{dt}\left(a^{\cos^{-1}t}\right)$$

$$\Rightarrow \frac{dx}{dt}=\frac{1}{2}\left(a^{\sin^{-1}t}\right)^{-1/2}\left(a^{\sin^{-1}t}\log_e a\right).\frac{d}{dt}\left(\sin^{-1}t\right) \text{ and ,}$$

$$\frac{dy}{dt}=\frac{1}{2}\left(a^{\cos^{-1}t}\right)^{-1/2}\left(a^{\cos^{-1}t}\log_e a\right).\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dx}{dt}=\frac{1}{2}\left(a^{\sin^{-1}t}\right)^{1/2}(\log_e a)\times\frac{1}{\sqrt{1-t^2}}=\frac{x\log_e a}{2\sqrt{1-t^2}} \text{ and ,}$$

$$\frac{dy}{dt}=\frac{1}{2}\left(a^{\cos^{-1}t}\right)^{1/2}(\log_e a)\times\frac{-1}{\sqrt{1-t^2}}=\frac{-y\log_e a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-y\log_e a}{2\sqrt{1-t^2}}\times\frac{2\sqrt{1-t^2}}{x\log_e a}=\frac{-y}{x}$$

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

Continuity of the function

1. The function $f(x)=\begin{cases}\dfrac{1}{4^x-1}; & x\neq 0\\ 0 & x=0\end{cases}$ is continuous

(a) everywhere except at $x=0$ and $x=1$

(b) nowhere

(c) everywhere

(d) everywhere except at $x=0$

2. The function $f(x)=(1+x)^{\cot x}$ is not defined at $x=0$. The value of $f(0)$ so that $f(x)$ becomes continuous at $x=0$ is

(a) 1 (b) 0

(c) e (d) none of these

3. If $f(x)=\begin{cases}x, \text{ when } x \text{ is rational}\\ 1-x, \text{ when } x \text{ is irrational, then}\end{cases}$

(a) $f(x)$ is continuous for all real x

(b) $f(x)$ is discontinuous for all real x

(c) $f(x)$ is continuous only at $x=1/2$

(d) $f(x)$ is discontinuous only at $x=1/2$

Differentiability of the function

4. Let $f(x)=[n+p\sin x]$, $x\in(0,\pi)$, $n\in I$, p is a prime number and $[x]$ denotes the greatest integer less than or equal to x. The number of points at which $f(x)$ is not differentiable is

(a) $p-1$ (b) p

(c) $2p+1$ (d) $2p-1$

5. The set of points where the function

$f(x)=[x]+|1-x|, -1\le x\le 3$

where [.] denotes the greatest integer function, is not differentiable, is

(a) $\{-1,0,1,2,3\}$ (b) $\{-1,0,2\}$

(c) $\{0,1,2,3\}$ (d) $\{-1,0,1,2\}$

6. The set of points where the function $f(x)=|x-1|\,e^x$ is differentiable is

(a) R (b) $R-\{1\}$

(c) $R-\{-1\}$ (d) $R-\{0\}$

7. The function $f(x)$ = maximum $\left\{\sqrt{x(2-x)},\, 2-x\right\}$ is non-differentiable at x equal to :

(a) 1 (b) 0, 2

(c) 0,1 (d) 1, 2

Continuity/ Differentiability of the function

8. The function $f(x)=\sin^{-1}(\cos x)$ is

(a) discontinuous at $x=0$

(b) continuous at $x=0$

(c) differentiable at $x=0$

(d) None of these

9. If $f(x)=\begin{cases}\dfrac{|x+2|}{\tan^{-1}(x+2)}; & x\neq -2\\ =2; & x=-2\end{cases}$ then $f(x)$ is

(a) continuous at $x=-2$

(b) not continuous at $x=-2$

(c) differentiable at $x=-2$

(d) continuous but not diff. at $x=-2$

10. If $f(x)=\begin{cases}\dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}; & x\neq 0\\ 0; & x=0\end{cases}$ then

(a) $\lim\limits_{x\to 0} f(x)=1$

(b) $f(x)$ is continuous at $x=0$

(c) $f(x)$ is differentiable at $x=0$

(d) None of these

11. If $f(x)=x+\dfrac{x}{1+x}+\dfrac{x}{(1+x)^2}+\ldots$ to ∞, then at $x=0$, $f(x)$

(a) $\lim\limits_{x\to 0} f(x)$ does not exist

(b) is discontinuous

(c) is continuous but not differentiable

(d) is differentiable

12. If $f(x) = x^2|x| \operatorname{sgn} x$, then

(a) f is derivable at $x = 0$

(b) f is continuous but not derivable at $x = 0$

(c) LHD at $x = 0$ is 1

(d) RHD at $x = 0$ is 1

13. The function $f(x) = 1 + |\sin x|$ is

(a) continuous no where

(b) continuous every where and no differentiable at $x = 0$

(c) differentiable no where

(d) differentiable at $x = 0$

14. For the function $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$ which of the following is incorrect ?

(a) continuous at $x = 1$ (b) continuous at $x = 3$

(c) derivable at $x = 1$ (d) derivable at $x = 3$

15. If $f(x) = \begin{cases} 3^x, & -1 \le x \le 1 \\ 4 - x, & 1 < x \le 4 \end{cases}$, then $f(x)$ is

(a) continuous as well as differentiable at $x = 1$

(b) continuous but not differentiable at $x = 1$

(c) differentiable but not continuous at $x = 1$

(d) none of the above

16. A function is defined as follows :

$f(x) = \begin{cases} x^3; & x^2 < 1 \\ x; & x^2 \ge 1 \end{cases}$ The function is

(a) dis continuous at $x = 1$

(b) differentiable at $x = 1$

(c) continuous but not differentiable at $x = 1$

(d) none of these

17. If $f(x) = \begin{cases} 4, -3 < x < -1 \\ 5 + x, -1 \le x < 0 \\ 5 - x, 0 \le x < 2 \\ x^2 + x - 3, 2 \le x < 3 \end{cases}$, then $f\,|x|$ is

(a) differentiable but not continuous in (–3, 3)

(b) continuous but not differentiable in (–3, 3)

(c) continuous as well as differentiable in (–3, 3)

(d) neither continuous nor differentiable in (–3, 3)

Defferentiation of logarithmic functions

18. Derivative of $x^6 + 6^x$ with respect to x is

(a) $12x$ (b) $x + 4$

(c) $6x^5 + 6^x \log 6$ (d) $6x^5 + x6^{x-1}$

19. If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is equal to

(a) $(1 + \log x)^{-1}$ (b) $(1 + \log x)^{-2}$

(c) $\log x . (1 + \log x)^{-2}$ (d) None of these

20. If $y = \log_a x + \log_x a + \log_x x + \log_a a$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{x} + x \log a$ (b) $\frac{\log a}{x} + \frac{x}{\log a}$

(c) $\frac{1}{x \log a} + x \log a$ (d) $\frac{1}{x \log a} - \frac{\log a}{x(\log x)^2}$

21. If $f(x) = \log|x|, x \ne 0$ then $f'(x)$ equals

(a) $\frac{1}{|x|}$ (b) $\frac{1}{x}$

(c) $-\frac{1}{x}$ (d) None of these

22. The derivative of $y = x^{\ln x}$ is

(a) $x^{\ln x} \ln x$ (b) $x^{\ln x - 1} \ln x$

(c) $2x^{\ln x - 1} \ln x$ (d) $x^{\ln x - 2}$

23. If $y = \{f(x)\}^{\phi(x)}$, then $\frac{dy}{dx}$ is

(a) $e^{\phi(x) \log f(x)} \left\{ \frac{\phi(x) d f(x)}{f(x) dx} + \log f(x) . \frac{d \phi(x)}{dx} \right\}$

(b) $\frac{\phi(x)}{f(x)} \left(\frac{df(x)}{dx} \right) + \frac{d\phi(x)}{dx} \log f(x)$

(c) $e^{\phi(x) \log f(x)} \left\{ \phi(x) \frac{f'(x)}{f(x)} + \phi'(x) \log f'(x) \right\}$

(d) None of these

Parametric defferentiation

24. If $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $-\frac{y}{x}$ (b) $\frac{y}{x}$

(c) $-\frac{x}{y}$ (d) $\frac{x}{y}$

25. If $y = e^{\sin^{-1}x}$ and $u = \log x$, then $\frac{dy}{du}$ is

(a) $\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ (b) $x\,e^{\sin^{-1}x}$

(c) $\frac{x\,e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ (d) $\frac{e^{\sin^{-1}x}}{x}$

26. If $x = a\cos^4\theta$, $y = a\sin^4\theta$, then $\frac{dy}{dx}$ at $\theta = \frac{3\pi}{4}$ is

(a) -1 (b) 1

(c) $-a^2$ (d) a^2

Differentiation of function w.r.t functions

27. The derivative of e^{x^3} with respect to $\log x$ is

(a) e^{x^3} (b) $3x^2\,2e^{x^3}$

(c) $3x^3\,e^{x^3}$ (d) $3x^2\,e^{x^3} + 3x^2$

28. The derivative of $\log_{10} x$ with respect to x^2 is

(a) $\frac{1}{2x^2}\log_e 10$ (b) $\frac{2}{x^2}\log_{10} e$

(c) $\frac{1}{2x^2}\log_{10} e$ (d) None of these

29. The derivative of $\sin^2 x$ with respect to $\cos^2 x$ is

(a) $\tan^2 x$ (b) $\tan x$

(c) $-\tan x$ (d) None of these

Various rule of solving differentiation

30. If $y = \log_{\cos x} \sin x$, then $\frac{dy}{dx}$ is equal to

(a) $(\cot x \log\cos x + \tan x \log\sin x)/(\log\cos x)^2$

(b) $(\tan x \log\cos x + \cot x \log\sin x)/(\log\cos x)^2$

(c) $(\cot x \log\cos x + \tan x \log\sin x)/(\log\sin x)^2$

(d) None of these

31. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx}$ is equal to

(a) $\sin\left(\frac{2x-1}{x^2+1}\right)^2 \left\{\frac{x^2+2x+2}{(x^2+1)^2}\right\}$

(b) $\sin\left(\frac{2x-1}{x^2+1}\right)^2 \left\{\frac{2+2x-2x^2}{(x^2+1)^2}\right\}$

(c) $\sin\left(\frac{2x-1}{x^2+1}\right)^2 \left\{\frac{2+2x-x^2}{(x^2+1)}\right\}$

(d) None of these

32. If $y = 2^x \cdot 3^{2x-1}$, then $\frac{dy}{dx}$ is equal to

(a) $(\log 2)(\log 3)$ (b) $(\log 18)$

(c) $(\log 18^2)\,y^2$ (d) $(\log 18)\,y$

33. If $y = (1 + x^2)\tan^{-1}x - x$, then $\frac{dy}{dx}$ is equal to

(a) $\tan^{-1}x$ (b) $2x\tan^{-1}x$

(c) $2x\tan^{-1}x - 1$ (d) $\frac{2x}{\tan^{-1}x}$

34. If $y = (1+x)(1+x^2)(1+x^4)\ldots\left(1+x^{2^n}\right)$, then $\frac{dy}{dx}$ at $x = 0$ is

(a) -1 (b) 1

(c) 0 (d) None of these

Derivatives of inverse trigonometric functions

35. If $f(x)=2\tan^{-1}x+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then

(a) $f'(-2)=\frac{4}{5}$

(b) $f'(-1)=-1$

(c) $f'(x)=0$ for all $x<0$

(d) None of these

36. If $y=\left[\tan^{-1}\frac{1}{1+x+x^2}+\tan^{-1}\frac{1}{x^2+3x+3}+\right.$

$\left.+\tan^{-1}\frac{1}{x^2+5x+7}+\text{....upto n terms}\right]$ then $y'(0)$ equals

(a) $\frac{-1}{n^2+1}$ (b) $\frac{-n^2}{n^2+1}$

(c) $\frac{n^2}{n^2+1}$ (d) None of these

37. If $y=\cos^{-1}\left(\frac{2\cos x-3\sin x}{\sqrt{13}}\right)$; $0<x<\frac{\pi}{2}$, then $\frac{dy}{dx}$ is

(a) zero (b) constant = 1

(c) constant ≠ 1 (d) none of these

Differentiation of infinite series

38. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{-\cos x}{2y-1}$

(c) $\frac{\sin x}{1-2y}$ (d) $\frac{-\sin x}{1-2y}$

39. If $y=x^{x^{x^{x^{\cdots\infty}}}}$, then $x(1-y\log x)\frac{dy}{dx}$

(a) x^2 (b) y^2

(c) xy^2 (d) xy

40. If $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\ldots\ldots\infty}}}$, then find $\frac{dy}{dx}$

(a) $\frac{y}{2y-x}$ (b) $\frac{x}{2x-y}$

(c) $\frac{y}{y-x}$ (d) $\frac{x}{x-y}$

41. For $|x|<1$, let $y=1+x+x^2+\ldots$ to ∞, then $\frac{dy}{dx}$ equal to

(a) $\frac{x}{y}$ (b) $\frac{x^2}{y^2}$

(c) $\frac{x}{y^2}$ (d) xy^2+y

Higher order Derivative problems

42. If $x=a\sin\theta$ and $y=b\cos\theta$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{a}{b^2}\sec^2\theta$ (b) $-\frac{b}{a}\sec^2\theta$

(c) $\frac{b}{a^2}\sec^3\theta$ (d) $-\frac{b}{a^2}\sec^3\theta$

43. If f be a polynomial then, the second derivative of $f(e^x)$ is

(a) $f'(e^x)$ (b) $f''(e^x)e^x+f'(e^x)$

(c) $f''(e^x)e^{2x}+f''(e^x)$ (d) $f''(e^x)e^{2x}+f'(e^x)e^x$

44. If $x^2+y^2=1$, then

(a) $yy''-2(y')^2+1=0$ (b) $yy''+(y')^2+1=0$

(c) $yy''+(y')^2-1=0$ (d) $yy''+2(y')^2+1=0$

45. If $\sqrt{x+y}+\sqrt{y-x}=c$ then $\frac{d^2y}{dx^2}$ is

(a) $\frac{2x}{c^2}$ (b) $\frac{2}{c^3}$

(c) $-\frac{2}{c^2}$ (d) $\frac{2}{c^2}$

46. If $y^2 = ax^2 + bx + c$ where a, b, c are constants, then $y^3 \frac{d^2y}{dx^2}$, is equal to

(a) a constant

(b) a function of x

(c) a function of y

(d) a function of x and y both

47. Let $x = \sin(l\,nt)$ and $y = \cos(l\,nt)$ then $\frac{d^2y}{dx^2}$ is

(a) $-\frac{1}{y^2}$ (b) $-\frac{1}{y^3}$

(c) $\frac{1}{y^2}$ (d) $\frac{1}{y^3}$

48. If $y = a\cos(\log x) + b\sin(\log x)$ where a, b are parameters, the $x^2y'' + xy'$ is equal to

(a) y (b) –y

(c) 2y (d) –2y

49. If $y = A\cos nx + B\sin nx$, then $\frac{d^2y}{dx^2} =$

(a) $-n^2y$ (b) $-y$

(c) n^2y (d) none of these

Numerical Value Type Questions

50. Let $f(x) = \frac{1-\tan x}{4x-\pi}$, $x \neq \frac{\pi}{4}$, $x \in \left[0, \frac{\pi}{2}\right]$.

If $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is

51. If $y = \frac{a+bx^{3/2}}{x^{5/4}}$ and $y' = 0$ at $x = 5$, then the ratio a : b is equal to

52. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at $x = y = 1$, is

53. Let $f(x) = \frac{(x+1)^2(x-1)}{(x-2)^3}$, then $f'(0)$ is

54. If $t \in (0, ½)$ and $x = \sin^{-1}(3t - 4t^3)$ and $y = \cos^{-1}\left(\sqrt{1-t^2}\right)$, then $\frac{dy}{dx}$ is equal to

55. If $u = f(x^2)$, $v = g(x^3)$, $f'(x) = \sin x$ and $g'(x) = \cos x$ then $\frac{du}{dx}$ =k $\sin(x^2)/x\cos(x^3)$, then find the value of k

56. Find the derivative of f (tan x) with respect to g (sec x) at $x = \pi/4$, if $f'(1) = 2$, $g'\left(\sqrt{2}\right) = 4$.

57. If $f(x) = \sqrt{1+\cos^2(x^2)}$, then the value of $f'\left(\frac{\sqrt{\pi}}{2}\right)$ is

58. If $y = |\cos x| + |\sin x|$ then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is

59. If Differentiation of $\log(\sin x^2)$ with respect to x is $k.x\cot(x^2)$, then the value of k is

60. If $f(x) = \log_x(\log_e x)$, then $f'(x)$ at $x = e$ is

61. If $y = \tan^{-1}\left[\frac{\sin x + \cos x}{\cos x - \sin x}\right]$, then $\frac{dy}{dx}$ is equal to

62. Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, find $f'(1/2)$

63. Find the derivative of $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ at $x = 0$

64. If $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+\cfrac{\cos x}{1+\sin x}\ldots\ldots\infty}}}$ then $y'(0)$ is

65. If $e^y + xy = e$, then the value of $\frac{d^2y}{dx^2}$ for $x = 0$, is

EXERCISE - 2 : PREVIOUS YEAR JEE MAINS QUESTIONS

1. If the function.

$g(x)=\begin{cases} k\sqrt{x+1} & , \quad 0\le x\le 3 \\ mx+2 & , \quad 3<x\le 5 \end{cases}$ is differentiable, then the value of $k+m$ is: **(2015)**

(a) $\frac{10}{3}$ (b) 4

(c) 2 (d) $\frac{16}{5}$

2. For $x\in R$, $f(x)=|\log 2-\sin x|$ and $g(x)=f(f(x))$, then : **(2016)**

(a) $g'(0)=\cos(\log 2)$

(b) $g'(0)=-\cos(\log 2)$

(c) g is differentiable at $x=0$ and $g'(0)=-\sin(\log 2)$

(d) g is not differentiable at $x=0$

3. If $y=\left[x+\sqrt{x^2-1}\right]^{15}+\left[x-\sqrt{x^2-1}\right]^{15}$,

then $(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}$ is equal to : **(2017/Online Set–1)**

(a) $125y$ (b) $224y^2$

(c) $225y^2$ (d) $225y$

4. If $2x=y^{\frac{1}{5}}+y^{-\frac{1}{5}}$ and $(x^2-1)\frac{d^2y}{dx^2}+\lambda x\frac{dy}{dx}+ky=0$, then $\lambda+k$ is equal to : **(2017/Online Set–2)**

(a) –23 (b) –24

(c) 26 (d) –26

5. Let f be a polynomial function such that $f(3x)=f'(x)\cdot f''(x)$, for all $x\in R$. Then : **(2017/Online Set–2)**

(a) $f(2)+f'(2)=28$ (b) $f''(2)-f'(2)=0$

(c) $f''(2)-f(2)=4$ (d) $f(2)-f'(2)+f''(2)=10$

6. The value of k for which the function

$$f(x)=\begin{cases} \left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}}, & 0<x<\frac{\pi}{2} \\ k+\frac{2}{5}, & x=\frac{\pi}{2} \end{cases}$$

is continuous at $x=\frac{\pi}{2}$, is : **(2017/Online Set–2)**

(a) $\frac{17}{20}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $-\frac{2}{5}$

7. Let $S=\{t\in R: f(x)=|x-\pi|.\left(e^{|x|}-1\right)\sin|x|$ is not differentiable at $t\}$. Then the set S is equal to: **(2018)**

(a) $\{0,\pi\}$ (b) ϕ (an empty set)

(c) $\{0\}$ (d) $\{\pi\}$

8. Let $f(x)=\begin{cases} (x-1)^{\frac{1}{2-x}} & ,x>1, x\ne 2 \\ k & ,x=2 \end{cases}$

The value of k for which f is continuous at x = 2 is: **(2018/Online Set–2)**

(a) 1 (b) e

(c) e^{-1} (d) e^{-2}

9. If $2y=\left(\cot^{-1}\left(\frac{\sqrt{3}\cos x+\sin x}{\cos x-\sqrt{3}\sin x}\right)\right)^2, x\in\left(0,\frac{\pi}{2}\right)$

then $\frac{dy}{dx}$ is equal to: **(2019-04-08/Shift-1)**

(a) $\frac{\pi}{6}-x$ (b) $x-\frac{\pi}{6}$

(c) $\frac{\pi}{3}-x$ (d) $2x-\frac{\pi}{3}$

10. Let f: $[-1, 3] \to R$ be defined as

$$f(x)=\begin{cases}|x|+[x], & -1\le x<1\\ x+|x|, & 1\le x<2\\ x+[x], & 2\le x\le 3,\end{cases}$$

where [t] denotes the greatest integer less than or equal to t. Then f is discontinuous at: **(2019-04-08/Shift-2)**

(a) only one point (b) only two points

(c) only three points (d) four or morepoints

11. If $f(1)=1$, $f'(1)=3$, then the derivative of $f(f(f(x)))+(f(x))^2$ at x = 1 is: **(2019-04-08/Shift-2)**

(a) 33 (b) 12

(d) 9 (c) 15

12. If the function f defined on $\left(\frac{\pi}{6},\frac{\pi}{3}\right)$ by

$$f(x)=\begin{cases}\dfrac{\sqrt{2}\cos x-1}{\cot x-1}, & x\ne\dfrac{\pi}{4}\\ k, & x=\dfrac{\pi}{4}\end{cases}$$

is continuous, then k is equal to: **(2019-04-09/Shift-1)**

(a) 2 (b) $\frac{1}{2}$

(c) 1 (d) $\frac{1}{\sqrt{2}}$

13. If $f(x)=[x]-\left[\frac{x}{4}\right], x\in R$, where $[x]$ denotes the greatest integer function, then: **(2019-04-09/Shift-2)**

(a) f is continuous at $x = 4$.

(b) $\lim\limits_{x\to4^+} f(x)$ exists but $\lim\limits_{x\to4^-} f(x)$ does not exist

(c) Both $\lim\limits_{x\to4^-} f(x)$ and $\lim\limits_{x\to4^+} f(x)$ exist but are not equal.

(d) $\lim\limits_{x\to4^-} f(x)$ exists but $\lim\limits_{x\to4^+} f(x)$ does not exist

14. If the function

$$f(x)=\begin{cases}a|\pi-x|+1, x\le5\\ b|x-\pi|+3, x>5\end{cases}$$

is continuous at x = 5, then the value of a-b is: **(2019-04-09/Shift-2)**

(a) $\frac{2}{\pi+5}$ (b) $\frac{-2}{\pi+5}$

(c) $\frac{2}{\pi-5}$ (d) $\frac{2}{5-\pi}$

15. Let f: R $\to$ R be differentiable at c $\in$ R and f(c) = 0. If g $(x)=|f(x)|$, then at x = c, g is : **(2019-04-10/Shift-1)**

(a) not differentiable if $f'(c)=0$

(b) differentiable if $f'(c)\ne0$

(c) differentiable if $f'(c)=0$

(d) not differentiable

16. If $f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x}, & x<0\\ q, & x=0\\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}, & x>0\end{cases}$

is continuous at x = 0, then the ordered pair (p, q) is equal to: **(2019-04-10/Shift-1)**

(a) $\left(-\frac{3}{2},-\frac{1}{2}\right)$ (b) $\left(-\frac{1}{2},\frac{3}{2}\right)$

(c) $\left(-\frac{3}{2},\frac{1}{2}\right)$ (d) $\left(\frac{5}{2},\frac{1}{2}\right)$

17. If $e^y+xy=e$, the ordered pair $\left(\frac{dy}{dx},\frac{d^2y}{dx^2}\right)$ at $x = 0$ is equal to : **(2019-04-12/Shift-1)**

(a) $\left(\frac{1}{e},-\frac{1}{e^2}\right)$ (b) $\left(-\frac{1}{e},\frac{1}{e^2}\right)$

(c) $\left(\frac{1}{e},\frac{1}{e^2}\right)$ (d) $\left(-\frac{1}{e},-\frac{1}{e^2}\right)$

18. The derivative of $\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$ with respect to $\frac{x}{2}$ where $\left(x\in\left(0,\frac{\pi}{2}\right)\right)$ is _____. **(2019-04-12/Shift-2)**

(a) 1 (b) $\frac{2}{3}$

(c) $\frac{1}{2}$ (d) 2

19. Let f : R → R be a function defined as

$$f(x)=\begin{cases}5, & if \quad x\le 1\\ a+bx, & if \quad 1<x<3\\ b+5x, & if \quad 3\le x<5\\ 30, & if \quad x\ge 5\end{cases}$$

Then, f is: **(2019-01-09/Shift-1)**

(a) continuous if a = 5 and b = 5

(b) continuous if a = - 5 and b = 10

(c) continuous if a= 0 and b = 5

(d) not continuous for any values of a and b

20. Let f be a differentiable function from R to R such that $|f(x)-f(y)|\le 2|x-y|^{3/2}$, for all x, y, ∈ R.

If $f(0)=1$ then $\int_0^1 (f(x))^2\,dx$ is equal to

(2019-01-09/Shift-2)

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) 0

21. If x = 3 tan t and y = 3 sec t, then the value of $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{4}$, is: **(2019-01-09/Shift-2)**

(a) $\frac{1}{3\sqrt{2}}$ (b) $\frac{1}{6\sqrt{2}}$

(c) $\frac{3}{2\sqrt{2}}$ (d) $\frac{1}{6}$

22. Let $f(x)=\begin{cases}\max\{|x|,x^2\} & |x|\le 2\\ 8-2|x| & 2<|x|\le 4\end{cases}$

Let S be the set of points in the interval (- 4, 4) at which f is not differentiable. Then S: **(2019-01-10/Shift-1)**

(a) is an empty set (b) equals {-2, -1, 0, 1, 2}

(c) equals {-2, -1, 1, 2} (d) equals {-2, 2}

23. Let $f:R\to R$ be a function such that

$f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3),\ x\in$ R.

then f(2) equals: **(2019-01-10/Shift-1)**

(a) -4 (b) 30

(c) -2 (d) 8

24. Let $f:(-1,1)\to$ R be a function defined by $f(x)=\max\{-|x|,-\sqrt{1-x^2}\}$. If K be the set of all points at which f is not differentiable, then K has exactly: **(2019-01-10/Shift-2)**

(a) five elements (b) one element

(c) three elements (d) two elements

25. If $x\log_e(\log_e x)-x^2+y^2=4(y>0)$, then $\frac{dy}{dx}$ at $x=e$ is equal to **(2019-01-11/Shift-1)**

(a) $\frac{(1+2e)}{2\sqrt{4+e^2}}$ (b) $\frac{(2e-1)}{2\sqrt{4+e^2}}$

(c) $\frac{(1+2e)}{\sqrt{4+e^2}}$ (d) $\frac{e}{\sqrt{4+e^2}}$

26. Let k be the set of all real values of x where the function $f(x) = \sin|x| - |x| + 2(x-\pi)\cos|x|$ is not differentiable. Then the set k is equal to: **(2019-01-11/Shift-2)**

(a) ϕ {an empty set} (b) {π}

(c) {0} (d) {0, π}

27. For x > 1, If $(2x)^{2y}=4e^{2x-2y}$, then $(1+\log_e 2x)^2\frac{dy}{dx}$ is equal to ______. **(2019-01-12/Shift-1)**

(a) $\frac{x\log_e 2x-\log_e 2}{x}$ (b) $\log_e 2x$

(c) $\frac{x\log_e 2x+\log_e 2}{x}$ (d) $x\log_e 2x$

28. If a function $f(x)$ defined by

$$f(x)=\begin{cases}ae^x+be^{-x} & , \quad -1\le x<1\\ cx^2 & , \quad 1\le x\le 3\\ ax^2+2cx & , \quad 3<x\le 4\end{cases}$$ be continuous for some $a,b,c\in R$ and $f'(0)+f'(2)=$ e, then the value of a is : **(2020-09-02/Shift-1)**

(a) $\frac{1}{e^2-3e+13}$ (b) $\frac{e}{e^2-3e-13}$

(c) $\frac{e}{e^2+3e+13}$ (d) $\frac{e}{e^2-3e+13}$

29. If $y=\sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx-\frac{4}{5}\sin kx\right\}$,

then $\frac{dy}{dx}$ at $x=0$ is .... **(2020-09-02/Shift-2)**

30. If $y^2+\log_e(\cos^2 x)=y,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then

(2020-09-03/Shift-1)

(a) $|y'(0)|+|y''(0)|=1$

(b) $y''(0)=0$

(c) $|y'(0)|+|y''(0)|=3$

(d) $|y''(0)|=2$

31. If $(a+\sqrt{2}\,b\cos x)\ (a-\sqrt{2}\,b\cos y)=a^2-b^2$, where a>b>0, then $\frac{dx}{dy}$ at $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$ is: **(2020-09-04/Shift-1)**

(a) $\frac{a+b}{a-b}$ (b) $\frac{a-2b}{a+2b}$

(c) $\frac{a-b}{a+b}$ (d) $\frac{2a+b}{2a-b}$

32. Suppose a differentiable function $f(x)$ satisfies the identity $f(x+y)=f(x)+f(y)+xy^2+x^2y$, for all real x and y. If $\lim_{x\to 0}\frac{f(x)}{x}=1$, then $f'(3)$ is equal to ....

(2020-09-04/Shift-1)

33. The function $f(x)=\begin{cases}\frac{\pi}{4}+\tan^{-1}x, & |x|\le 1\\ \frac{1}{2}(|x|-1), & |x|>1\end{cases}$ is :

(2020-09-04/Shift-2)

(a) both continuous and differentiable on $R-\{-1\}$

(b) continuous on $R-\{-1\}$ and differentiable on $R-\{-1,1\}$

(c) continuous on $R-\{1\}$ and differentiable on $R-\{-1,1\}$

(d) both continuous and differentiable on $R-\{1\}$

34. If the function $f(x)=\begin{cases}k_1(x-\pi)^2-1, & x\le\pi\\ k_2\cos x, & x>\pi\end{cases}$ is twice differentiable, then the ordered pair (k_1,k_2) is equal to:

(2020-09-05/Shift-1)

(a) (1,1) (b) (1,0)

(c) $\left(\frac{1}{2},-1\right)$ (d) $\left(\frac{1}{2},1\right)$

35. Let $f(x)=x.\left[\frac{x}{2}\right]$, for $-10<x<10$, where [t] denotes the greatest integer function. Then the number of points of discontinuity of f is equal to ....

(2020-09-05/Shift-1)

36. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x=\frac{1}{2}$ is: **(2020-09-05/Shift-2)**

(a) $\frac{2\sqrt{3}}{3}$ (b) $\frac{2\sqrt{3}}{5}$

(c) $\frac{\sqrt{3}}{12}$ (d) $\frac{\sqrt{3}}{10}$

37. Let $f:R\to R$ be defined as

$$f(x)=\begin{cases}x^5\sin\left(\frac{1}{x}\right)+5x^2, & x<0\\ 0, & x=0\\ x^5\cos\left(\frac{1}{x}\right)+\lambda x^2, & x>0\end{cases}$$

The value of λ for which $f''(0)$ exists, is_______.

(2020-09-06/Shift-1)

38. Let $f:R\to R$ be a function defined by $f(x)=\max\{x,x^2\}$. Let S denote the set of all points in R, where f is not differentiable. Then **(2020-09-06/Shift-2)**

(a) {0, 1} (b) f(an empty set)

(c) {1} (d) {0}

39. If $y(\alpha)=\sqrt{2\left(\frac{\tan\alpha+\cot\alpha}{1+\tan^2\alpha}\right)+\frac{1}{\sin^2\alpha}}$

where $\alpha\in\left(\frac{3\pi}{4},\pi\right)$, then $\frac{dy}{d\alpha}$ at $\alpha=\frac{5\pi}{6}$ is

(2020-01-07/Shift-1)

(a) $-\frac{1}{4}$ (b) $\frac{4}{3}$

(c) 4 (d) -4

40. Let $x^k+y^k=a^k,(a,k>0)$ and $\frac{dy}{dx}+\left(\frac{y}{x}\right)^{\frac{1}{3}}=0$ then k is

(2020-01-07/Shift-1)

(a) $\frac{1}{3}$ (b) $\frac{3}{2}$

(c) $\frac{2}{3}$ (d) $\frac{4}{3}$

41. Let S be the set of points where the function, $f(x)=\left|2-\left|x-3\right|\right|, x\in$ R, is not differentiable. Then, the value of $\sum_{x\in s} f\left(f\left(x\right)\right)$ is equal to

(2020-01-07/Shift-1)

42. Let y = y(x) be a function of x satisfying $y\sqrt{1-x^2}=k-x\sqrt{1-y^2}$ where k is a constant and $y\left(\frac{1}{2}\right)=-\frac{1}{4}$. Then $\frac{dy}{dx}$ at $x=\frac{1}{2}$, is equal to :

(2020-01-07/Shift-2)

(a) $-\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{5}}{2}$

(c) $-\frac{\sqrt{5}}{4}$ (d) $\frac{2}{\sqrt{5}}$

43. If the function f defined on $\left(-\frac{1}{3},\frac{1}{3}\right)$ by

$$f(x)=\begin{cases}\left(\frac{1}{x}\right)\log_e\left(\frac{1+3x}{1-2x}\right), & when\ x\neq 0\\ k, & when\ x=0\end{cases}$$

is continuous, then k is equal to ________.

(2020-01-07/Shift-2)

44. If $f(x)=\begin{cases}\frac{\sin(a+2)x+\sin x}{x}, & x<0\\ b & ,x=0\\ \frac{\left(x+3x^2\right)^{\frac{1}{3}}-x^{\frac{1}{3}}}{x^{\frac{4}{3}}} & ,x>0\end{cases}$

is continuous at $x=0$ then a + 2b is equal to:

(2020-01-09/Shift-1)

(a) –2 (b) 1

(c) 0 (d) –1

45. Let f and g be differentiable functions on R, such that fog is the identity function. If for some $a,b\in R, g'(a)=5$ and $g(a)=b$, then $f'(b)$ is equal to :

(2020-01-09/Shift-2)

(a) $\frac{2}{5}$ (b) 5

(c) 1 (d) $\frac{1}{5}$

46. Let $[t]$ denotes the greatest integer $\leq t$ and $\lim_{x\to 0} x\left[\frac{4}{x}\right]=A$.

Then the function, $f(x)=\left[x^2\right]\sin\pi x$ is discontinuous, when x is equal to **(2020-01-09/Shift-2)**

(a) $\sqrt{A+1}$ (b) $\sqrt{A}$

(c) $\sqrt{A+5}$ (d) $\sqrt{A+21}$

47. If $f:R\to R$ is a fugreatest integer functionnction defined by $f(x)=[x-1]\cos\left(\frac{2x-1}{2}\right)\pi$ where [.] denotes the greatest integer function, then f is

(24-02-2021/Shift-1)

(a) Discontinuous at all integral values of x except at

(b) Discontinuous only at $x=1$

(c) Continuous only at $x=1$

(d) Continuous for every real x

48. The number of points, at which the function

$f(x)=|2x+1|-3|x+2|+|x^2+x-2|, x\in R$

is not differentiable, is ______.

(25-02-2021/Shift-1)

49. A function f is defined on $[-3,3]$ as

$f(x)=\begin{cases}\min\{|x|,2-x^2\}, & -2\le x\le 2\\ [|x|] & ,2<|x|\le 3\end{cases}$ where $[x]$

denotes the greatest integer $\le x$. The number of points, where f is not differentiable $(-3,3)$ is ______.

(25-02-2021/Shift-2)

50. Let $f:R\to R$ be define as

$$f(x)=\begin{cases}2\sin\left(-\frac{\pi x}{2}\right), & \text{if } x<-1\\ |ax^2+x+b|, & \text{if } -1\le x\le 1\\ \sin(\pi x), & \text{if } x>1\end{cases}$$

If $f(x)$ is continuous on R, then $a+b$ equals:

(26-02-2021/Shift-2)

(a) -3 (b) 1

(c) 3 (d) -1

51. Let the function $f:R\to R$ and $g:R\to R$ be defined as:

$f(x)=\begin{cases}x+2, & x<0\\ x^2, & x\ge 0\end{cases}$ and $g(x)=\begin{cases}x^3, & x<1\\ 3x-2, & x\ge 1\end{cases}$

Then, the number of points in R where $(fog)(x)$ is NOT differentiable is equal to **(16-03-2021/Shift-1)**

(a) 0 (b) 2

(c) 3 (d) 1

52. Let $\alpha\in R$ be such that the function

$f(x)=\begin{cases}\dfrac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3}, & x\ne 0\\ \alpha, & x=0\end{cases}$ is

continuous at $x=0$, where $\{x\}=x-[x],[x]$ is the greatest integer less than or equal to x. Then

(16-03-2021/Shift-2)

(a) $\alpha=0$ (b) $\alpha=\frac{\pi}{\sqrt{2}}$

(c) no such α exists (d) $\alpha=\frac{\pi}{4}$

53. Let $f:S\to S$ where $S=(0,\infty)$ be a twice differentiable function such that $f(x+1)=xf(x)$. If $g:S\to R$ be defined as $g(x)=\log_e f(x)$, then the value of $|g''(5)-g''(1)|$ is equal to : **(16-03-2021/Shift-2)**

(a) 1 (b) $\frac{197}{144}$

(c) $\frac{187}{144}$ (d) $\frac{205}{144}$

54. Let $f:R\to R$ and $g:R\to R$ be defined as

$f(x)=\begin{cases}x+a, & x<0\\ |x-1|, & x\ge 0\end{cases}$ and $g(x)=\begin{cases}x+1, & x<0\\ (x-1)^2+b, & x\ge 0\end{cases}$

where a, b are non-negative real numbers. If $(gof)(x)$ is continuous for all $x\in R$, then $a+b$ is equal to.

(16-03-2021/Shift-2)

55. If the function $f(x)=\dfrac{\cos(\sin x)-\cos x}{x^4}$ is continuous at each point in its domain and $f(0)=\dfrac{1}{k}$, then k is ....

(17-03-2021/Shift-1)

56. If $f(x)=\begin{cases}\dfrac{1}{|x|}; & |x|\ge 1\\ ax^2+b; & |x|<1\end{cases}$ is differentiable at every point of the domain, then the value of a and b are respectively : **(18-03-2021/Shift-1)**

(a) $\dfrac{5}{2},-\dfrac{3}{2}$ (b) $\dfrac{1}{2},-\dfrac{3}{2}$

(c) $-\dfrac{1}{2},\dfrac{3}{2}$ (d) $\dfrac{1}{2},\dfrac{1}{2}$

57. Let $f:R\to R$ be a function defined as

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin 2x}{2x}, & \text{if } x<0\\ b, & \text{if } x=0\\ \dfrac{\sqrt{x+bx^3}-\sqrt{x}}{bx^{5/2}}, & \text{if } x>0\end{cases}$$

If f is continuous at $x=0$, then the value of $a+b$ is equal to **(18-03-2021/Shift-2)**

(a) $-\dfrac{5}{2}$ (b) -3

(c) $-\dfrac{3}{2}$ (d) -2

58. Let a function $f:R\to R$ be defined as

$$f(x)=\begin{cases}\sin x-e^x & \text{if } x\le 0\\ a+[-x] & \text{if } 0<x<1\\ 2x-b & \text{if } x\ge 1\end{cases}$$

where $[x]$ is the greatest integer less than or equal to x.

If f is continuous on R, then $(a+b)$ is equal to:

(20-07-2021/Shift-1)

(a) 5 (b) 3

(c) 2 (d) 4

59. Let a function $g:[0,4]\to R$ be defined as,

$$g(x)=\begin{cases}\max\{t^3-6t^2+9t-3\}, & 0\le t\le x, 0\le x\le 3\\ 4-x, & 3<x\le 4\end{cases},$$

then the number of points in the interval $(0,4)$ where $g(x)$ **(20-07-2021/Shift-2)**

60. Let $r:R\to R$ be function defined as

$$f(x)=\begin{cases}3\left(1-\dfrac{|x|}{2}\right) & \text{if } |x|\le 2\\ 0 & \text{if } |x|>2\end{cases}$$

Let $g:R\to R$ be given by $g(x)=f(x+2)-f(x-2)$

If If n and m denote the number of points in R where g is not continuous and not differentiable, respectively, then $n+m$ is equal to **(22-07-2021/Shift-2)**

61. Let $f:R\to R$ be defined as

$$f(x)=\begin{cases}\dfrac{\lambda|x^2-5x+6|}{\mu(5x-x^2-6)} & ; x<2\\ e^{\frac{\tan(x-2)}{x-[x]}} & ; x>2\\ \mu & ; x=2\end{cases}$$

Where $[x]$ is the greatest integer less than or equal to x.

If f is continuous at $x=2$, then $\lambda+\mu$ is equal to?

(25-07-2021/Shift-1)

(a) $e(e-2)$ (b) $2e-1$

(c) $e(-e+1)$ (d) 1

62. Conside the function $f(x)=\begin{cases}\dfrac{P(x)}{\sin(x-2)} & ; \quad x\neq 2\\ 7 & ; \quad x=2\end{cases}$

where $P(x)$ $P(x)$ is a polynomial such that $P''(x)$ is always a constant and $P(3)=9$. If $f(x)$ is continuous at $x=2$, then $P(5)$ is equal to **(25-07-2021/Shift-2)**

63. Let $f:[0,\infty)\to[0,3]$ be a function defined by

$$f(x)=\begin{cases}\max\{\sin t: 0\le t\le x\}, & 0\le x\le\pi\\ 2+\cos x, & x>\pi\end{cases}$$

Then which of the following is true?

(27-07-2021/Shift-2)

(a) f is differentiable everywhere in $(0,\infty)$

(b) f is continuous everywhere but not differentiable exactly at two points in $(0,\infty)$

(c) f is not continuous exactly at two points in $(0,\infty)$

(d) f is continuous everywhere but not differentiable exactly at one point in $(0,\infty)$

64. Let $f:\left(-\dfrac{\pi}{4},\dfrac{\pi}{4}\right)\to R$ be defined as

$$f(x)=\begin{cases}(1+|\sin x|)^{\frac{3a}{|\sin x|}}, & -\dfrac{\pi}{4}<x<0\\ b, & x=0\\ e^{\cot 4x/\cot 2x}, & 0<x<\dfrac{\pi}{4}\end{cases}$$

If f is continuous at $x=0$, the the value of $6a+b^2$ is equal to : **(27-07-2021/Shift-1)**

(a) e (b) $1+e$

(c) $1-e$ (d) $e-1$

65. Let $f:[0,3]\to R$ be defined by

$$f(x)=\min\{x-[x],1+[x]-x\}$$

where $[x]$ is the greatest integer less than or equal to x.

Let P denote the set containing all $x\in[0,3]$ where where f is discontinuous, and Q denote the set containing all $x\in(0,3)$ where f is not differentiable. Then the sum of number of elements in P and Q is equal to __________.

(27-07-2021/Shift-1)

66. Let $f(x)=\cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\dfrac{1-x}{x}}\right)\right)$, $0<x<1$.

Then **(26-08-2021/Shift-1)**

(a) $(1+x)^2 f'(x)+2(f(x))^2=0$

(b) $(1-x)^2 f'(x)+2(f(x))^2=0$

(c) $(1+x)^2 f'(x)-2(f(x))^2=0$

(d) $(1-x)^2 f'(x)-2(f(x))^2=0$

67. If $y=y(x)$ is an implicit function of x such that $\log_e(x+y)=4xy$, then $\dfrac{d^2y}{dx^2}$ at $x=0$ is equal to

(26-08-2021/Shift-1)

68. Let $a, b\in R$, $b\neq 0$. Define a function

$$f(x)=\begin{cases}a\sin\dfrac{\pi}{2}(x-1) & \text{for } \le 0\\ \dfrac{\tan 2x-\sin 2x}{bx^3} & \text{for } x>0\end{cases}$$

If f is continuous at $x=0$, then $10-ab$ is equal to _________. **(26-08-2021/Shift-1)**

69. Let [t] denote the greatest integer less than or equal to t.

Let $f(x)=x-[x], g(x)=1-x+[x]$ and

$h(x)=\min\{f(x), g(x)\}, x\in[-2, 2]$. Then h is:

(26-08-2021/Shift-2)

(a) not continuous at exactly four points in [-2, 2]

(b) not continuous at exactly three points in [-2, 2]

(c) continuous in [-2, 2] but not differentiable at exactly three point in (-2, 2)

(d) continuous in [-2, 2] but not differentiable at more than four points in (-2, 2)

70. If $y^{\frac{1}{4}}+y^{-\frac{1}{4}}=2x$, and $(x^2-1)\frac{d^2y}{dx^2}+\alpha x\frac{dy}{dx}+\beta y=0$,

then $|\alpha-\beta|$ is equal to ______. **(27-08-2021/Shift-1)**

71. If $y(x)=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), x\in\left(\frac{\pi}{2},\pi\right)$,

then $\frac{dy}{dx}$ at $x=\frac{5\pi}{6}$ is: **(27-08-2021/Shift-2)**

(a) 0 (b) -1

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

72. If the function $f(x)=\begin{cases}\frac{1}{x}\log_e\left(\frac{1+\frac{x}{a}}{1-\frac{x}{b}}\right), & x<0\\ k, & x=0\\ \frac{\cos^2 x-\sin^2 x-1}{\sqrt{x^2+1}-1}, & x>0\end{cases}$ is

continuous at $x=0$, then $\frac{1}{a}+\frac{1}{b}+\frac{4}{k}$ is equal to ?

(31-08-2021/Shift-1)

(a) -4 (b) -5

(c) 4 (d) 5

73. The function $f(x)=|x^2-2x-3|.e^{|9x^2-12x+4|}$ is not differentiable at exactly ? **(31-08-2021/Shift-1)**

(a) Two points (b) One points

(c) Four points (d) Three

74. Let $[t]$ denote the greatest integer $\le t$. The number of points where the function

$f(x)=[x]|x^2-1|+\sin\left(\frac{\pi}{[x]+3}\right)-[x+1], x\in(-2,2)$ is

not continuous is ________ **(01-09-2021/Shift-2)**

75. Let $f: R\to R$ be defined as

$$f(x)=\begin{cases}\frac{x^3}{(1-\cos 2x)^2}\log_e\left(\frac{1+2xe^{-2x}}{(1-xe^{-x})^2}\right) & x\neq 0\\ \alpha & x=0\end{cases}$$

If f is continuous at $x=0$, then α is equal to :

(22-07-2021/Shift-2)

(a) 1 (b) 0

(c) 3 (d) 2

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

Objective Questions I [Only one correct option]

1. Let U(x) and V(x) are differentiable functions such that $\frac{U(x)}{V(x)} = 7$. If $\frac{U'(x)}{V'(x)} = p$ and $\left(\frac{V(x)}{U(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to

(a) 1 (b) 0

(c) 7 (d) –7

2. Suppose the function f(x) – f(2x) has the derivative 5 at x = 1 and derivative 7 at x = 2. The derivative of the function f(x) – f(4x) at x = 1, has the value equal to

(a) 19 (b) 9

(c) 17 (d) 14

3. If f is a periodic function, then

(a) f' and f'' are also periodic

(b) f' is periodic but f'' is not periodic

(c) f'' is periodic but f' is not periodic

(d) None of these

4. If $y = f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ equals

(a) 4 (b) 2

(c) –2 (d) 0

5. If $f(x) = \log|2x|, x \neq 0$, then $f'(x)$ is equal to

(a) $\frac{1}{x}$ (b) $-\frac{1}{x}$

(c) $\frac{1}{|x|}$ (d) None of these

6. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in AP, then $f'(a), f'(b)$ and $f'(c)$ are in

(a) AP

(b) GP

(c) HP

(d) Arithmetico-Geometric progression

7. If $x^2 + y^2 = t + \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y}{x}$ (b) $-\frac{y}{x}$

(c) $\frac{x}{y}$ (d) $-\frac{x}{y}$

8. If $f(x) = |x-3|$ and $\phi(x) = (fof)(x)$, then for $x > 10$, $\phi'(x)$ is equal to

(a) 1 (b) 0

(c) –1 (d) None of these

9. Let $f(x) = \sin x$, $g(x) = 2x$ and $h(x) = \cos x$.

If $\phi(x) = [go(fh)](x)$, then $\phi''\left(\frac{\pi}{4}\right)$ is equal to

(a) 4 (b) 0

(c) –4 (d) None of these

10. If $f(x) = |x-1|$ and $g(x) = f[f\{f(x)\}]$, then for $x > 2$, $g'(x)$ is equal to

(a) –1 if $2 < x < 3$ (b) 1 if $2 \leq x < 3$

(c) 1 for all $x > 2$ (d) None of these

11. Let f (x) = 2/(x+1) and g(x) = 3x. It is given that $(fog)(x_0) = (gof)(x_0)$. Then $(gof)'(x_0)$ equals

(a) – 32 (b) $\frac{32}{3}$

(c) $\frac{-32}{9}$ (d) $\frac{-32}{3}$

12. Let $f(x)$ be a polynomial of degree 3 such that $f(3) = 1$, $f'(3) = -1$, $f''(3) = 0$ and $f'''(3) = 12$. Then the value of $f'(1)$ is

(a) 12 (b) 23

(c) –13 (d) None of these

13. If $f(x) = \sin\left(\frac{\pi}{2}[x]-x^5\right), 1<x<2$ and $[x]$ denotes the greatest integer less than or equal to x, then $f'\left(\sqrt[5]{\frac{\pi}{2}}\right)$ is equal to

(a) $5\left(\frac{\pi}{2}\right)^{4/5}$ (b) $-5\left(\frac{\pi}{2}\right)^{4/5}$

(c) 0 (d) None of these

14. If $f(x) = \sqrt{x^2-10x+25}$, then the derivative of $f(x)$ on the interval [0, 7] is

(a) 1 (b) –1

(c) 0 (d) none of these

15. If $y = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, then $\frac{dy}{dx}$ is equal to

(a) 1 (b) –1

(c) 0 (d) None of these

16. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are given, then

(a) $\Delta_1 = 3(\Delta_2)^2$ (b) $\frac{d}{dx}\Delta_1 = 3\Delta_2$

(c) $\frac{d}{dx}\Delta_1 = 3(\Delta_2)^2$ (d) $\Delta_1 = 3(\Delta_2)^{3/2}$

17. If $f(x) = \cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) - \cos x \cos\left(x+\frac{\pi}{3}\right)$ then $f'(x)$ is

(a) 0 (b) 1

(c) 2 (d) 3

18. The function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \pi/4$. The value of $f(\pi/4)$ so that f is continuous at $x = \pi/4$ is

(a) $\sqrt{e}$ (b) $1/\sqrt{e}$

(c) 2 (d) None of these

19. Let $f(x) = a + b|x| + c|x|^4$, where a, b and c are real constants. Then $f(x)$ is differentiable at $x = 0$ if

(a) $a = 0$ (b) $b = 0$

(c) $c = 0$ (d) None of these

20. The derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'\left(\sqrt{2}\right) = 4$, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) 1 (d) None of these

21. Let $y = x^3 - 8x + 7$ and $x = f(t)$. If $\frac{dy}{dt} = 2$ and $x = 3$ at $t = 0$, then $\frac{dx}{dt}$ at $t = 0$ is given by

(a) 1 (b) $\frac{19}{2}$

(c) $\frac{2}{19}$ (d) None of these

22. $f(x)$, $g(x)$, $h(x)$ are functions having non-zero derivatives. The derivative of $f(x)$ w.r.t $g(x)$ is $\alpha(x)$ and derivative of $g(x)$ w.r.t $h(x)$ is $\beta(x)$. Then derivative of $h(x)$ w.r.t $f(x)$ =

(a) $\alpha(x) . \beta(x)$ (b) $\frac{\alpha(x)}{\beta(x)}$

(c) $\frac{1}{\alpha(x)\beta(x)}$ (d) $\frac{\beta(x)}{\alpha(x)}$

23. If f is an even function such that $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$ has some finite non-zero value, then

(a) f is continuous and derivable at $x = 0$

(b) f is continuous but not derivable at $x = 0$

(c) f may be discontinuous at $x = 0$

(d) None of these

24. If $\sin y = x \sin(a+y)$, then $\frac{dy}{dx}$ is

(a) $\frac{\sin a}{\sin^2(a+y)}$ (b) $\frac{\sin^2(a+y)}{\sin a}$

(c) $\sin a \sin^2(a+y)$ (d) $\frac{\sin^2(a-y)}{\sin a}$

25. Let $f(x) = \alpha(x)\beta(x)\gamma(x)$ for all real x, where $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are differentiable functions of x. If $f'(2) = 18f(2)$, $\alpha'(2) = 3\alpha(2)$, $\beta'(2) = -4\beta(2)$ and $\gamma'(2) = k\gamma(2)$, then the value of k is

(a) 14 (b) 16

(c) 19 (d) None of these

26. If $y = \frac{ax+b}{x^2+c}$, where a, b, c are constants then $(2xy'+y)y'''$ is equal to

(a) $3(xy''+y')y''$ (b) $3(xy'+y'')y''$

(c) $3(xy''+y')y'$ (d) None of these

27. Function $f : R \to R$ satisfies the functional equation

$$f(x-y) = \frac{f(x)}{f(y)}$$

If $f'(0) = p$ and $f'(a) = q$, then $f'(-a)$ is

(a) $\frac{p^2}{q}$ (b) $\frac{q}{p}$

(c) $\frac{p}{q}$ (d) q

28. Let $f(x) = x^n$, $n \in W$. The number of values of n for which $f'(p+q) = f'(p) + f'(q)$ is valid for all +ve p & q is

(a) 0 (b) 1

(c) 2 (d) None of these

29. Let $f : R \to R$ be a function such that

$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, $f(0) = 0$ and $f'(0) = 3$. Then

(a) $f(x)$ is a quadratic function

(b) $f(x)$ is continuous but not differentiable

(c) $f(x)$ is differentiable in R

(d) $f(x)$ is bounded in R

30. If $f_r(x), g_r(x), h_r(x)$, r = 1, 2, 3 are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, r = 1, 2, 3 and

$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$, then $F'(a)$ is equal to

(a) a (b) –a

(c) 0 (d) None of these.

31. Let f & g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ & $fog = I$ (Identity function). Then $f'(b)$ is equal to

(a) 2 (b) $\frac{2}{3}$

(c) $\frac{1}{2}$ (d) None

32. Let $f(x) = a[x] + b e^{|x|} + c|x|^2$, where a, b and c are real constants. where [x] denotes greatest integer $\le$ x. If $f(x)$ is differentiable at x = 0, then

(a) $b = 0, c = 0, a \in R$ (b) $a = 0, c = 0, b \in R$

(c) $a = 0, b = 0, c \in R$ (d) None of these

33. If $f(x) = \begin{cases} \frac{x\left(3e^{1/x}+4\right)}{2-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

(a) continuous as well as differentiable at x = 0

(b) continuous but not differentiable at x = 0

(c) differentiable but not continuous at x = 0

(d) None of these

34. If $y = e^{\tan x}$, then $\cos^2 x \dfrac{d^2y}{dx^2} =$

(a) $(1 - \sin 2x)\dfrac{dy}{dx}$ (b) $-(1 + \sin 2x)\dfrac{dy}{dx}$

(c) $(1 + \sin 2x)\dfrac{dy}{dx}$ (d) None of these

35. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$, and that $f''(x) - 2f'(x) - 15f(x) = 0$ for all x. Then the product ab is

(a) 25 (b) –15

(c) 9 (d) –9

36. If $(\sin y)^{\sin(\pi x/2)} + \dfrac{\sqrt{3}}{2}\operatorname{Sec}^{-1}(2x) + 2^x \tan(\log(x+2)) = 0$ then dy/dx at $x = -1$ is

(a) $\dfrac{3}{\sqrt{\pi^2 - 3}}$ (b) $\dfrac{1}{\pi\sqrt{\pi^2 - 3}}$

(c) $\dfrac{3}{\pi\sqrt{\pi^2 - 3}}$ (d) $\dfrac{3\pi}{\sqrt{\pi^2 - 3}}$

37. If $ax^2 + 2hxy + by^2 = 1$, then $\dfrac{d^2y}{dx^2}$ is equal to

(a) $\dfrac{ab - h^2}{(hx + by)^3}$ (b) $\dfrac{h^2 - ab}{(hx + by)^3}$

(c) $\dfrac{h^2 + ab}{(hx + by)^3}$ (d) None of these

38. If $y^2 = P(x)$, a polynomial of degree $n \geq 3$, then $2\dfrac{d}{dx}\left(y^3 \dfrac{d^2y}{dx^2}\right)$

(a) $-P(x) . P'''(x)$ (b) $P(x) . P'''(x)$

(c) $P(x) . P''(x)$ (d) None of these

39. Let $f(x) = e^x - e^{-x} - 2\sin x - \dfrac{2}{3}x^3$, then the least value of n for which $\left|\dfrac{d^n f(x)}{dx^n}\right|_{x=0}$ is non-zero

(a) 4 (b) 5

(c) 7 (d) 3

40. People living at Mars, instead of the usual definition of derivative D f(x), define a new kind of derivative, D*f(x) by the formula

$$D^* f(x) = \lim_{h \to 0} \frac{f^2(x+h) - f^2(x)}{h} \text{ where } f^2(x) \text{ means } [f(x)]^2.$$

If $f(x) = x \ln x$ then

$D^* f(x)\big|_{x=e}$ has the value

(a) e (b) 2e

(c) 4e (d) 8 e

41. If $y = k\sin px$, then the value of the determinant

$$\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$$ is equal to

(a) 1 (b) 0

(c) –1 (d) None of these.

where y_n denotes nth derivative of y w.r.t. x.

42. If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\dfrac{n\pi}{2} & 4 \\ \sin x & \sin\dfrac{n\pi}{2} & 8 \end{vmatrix}$ then the value of $\dfrac{d^n}{dx^n}[f(x)]_{x=0}$ is

(a) 0 (b) 1

(c) –1 (d) None of these

43. A non zero polynomial with real coefficients has the property that $f(x) = f'(x) . f''(x)$. The leading coefficient of f(x) is

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{9}$

(c) $\dfrac{1}{12}$ (d) $\dfrac{1}{18}$

44. If $0 < x < 1$, then

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+...\infty =$$

(a) $\frac{1}{1-x}$ (b) $\frac{x}{1-x}$

(c) $\frac{x}{1+x}$ (d) $\frac{1-x}{1+x}$

45. If $y=\left(\frac{ax+b}{cx+d}\right)$, then $2\frac{dy}{dx}.\frac{d^3y}{dx^3}$ is equal to

(a) $\left(\frac{d^2y}{dx^2}\right)^2$ (b) $3\frac{d^2y}{dx^2}$

(c) $3\left(\frac{d^2y}{dx^2}\right)^2$ (d) $3\frac{d^2x}{dy^2}$

Objective Questions II [One or more than one correct option]

46. The function $f(x)=\max.\{(1-x),(1+x),2\}, x\in(-\infty,\infty)$, is

(a) continuous at all points

(b) differentiable at all points

(c) differentiable at all points except at $x = 1$ and $x = -1$.

(d) continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous.

47. If $f(x)=\frac{1}{[\sin x]}$, where [.] denotes the greatest function, then

(a) Domain of $f(x)$ is $(2n\pi+\pi, 2n\pi+2\pi)\cup\left\{2n\pi+\frac{\pi}{2}\right\}$ where $n \in I$

(b) $f(x)$ is continuous when $x \in (2n\pi+\pi, 2n\pi+2\pi)$

(c) $f(x)$ is differentiable at $x = \pi/2$

(d) None of these

48. Let [x] denotes the greatest integer less than or equal to x. If $f(x)=[x\sin \pi x]$, then $f(x)$ is

(a) continuous at $x = 0$ (b) continuous in $(-1, 0)$

(c) differentiable at $x = 1$ (d) differentiable in $(-1, 1)$

49. Let $y=\sqrt{\begin{array}{l}(\sin x+\sin 2x+\sin 3x)^2\\+(\cos x+\cos 2x+\cos 3x)^2\end{array}}$ then which of the following is correct ?

(a) $\frac{dy}{dx}$ when $x=\frac{\pi}{2}$ is -2

(b) value of y when $x=\frac{\pi}{5}$ is $\frac{3+\sqrt{5}}{2}$

(c) value of y when $x=\frac{\pi}{12}$ is $\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}}{2}$

(d) y simplifies to $(1 + 2\cos x)$ in $[0, \pi]$

50. Let $f(x)=\frac{1-x^{n+1}}{1-x}$ and $g(x)=1-\frac{2}{x}+\frac{3}{x^2}-.........+(-1)^n\frac{n+1}{x^n}$.
Then the constant term in $f'(x)\times g(x)$ is equal to

(a) $\frac{n(n^2-1)}{6}$ when n is even

(b) $\frac{n(n+1)}{2}$ when n is odd

(c) $-\frac{n}{2}(n+1)$ when n is even

(d) $\frac{n(n-1)}{2}$ when n is odd

51. If $F(x)=f(x)\,g(x)$ and $f'(x)\,g'(x)=c$, then

(a) $F'=c\left[\frac{f}{f'}+\frac{g}{g'}\right]$ (b) $\frac{F''}{F}=\frac{f''}{f}+\frac{g''}{g}+\frac{2c}{fg}$

(c) $\frac{F'''}{F}=\frac{f'''}{f}+\frac{g'''}{g}$ (d) $\frac{F'''}{F''}=\frac{f'''}{f''}+\frac{g'''}{g''}$

52. Suppose f and g are functions having second derivatives f'' and g'' everywhere, if $f(x).g(x)=1$ for all x and f' and g' are never zero, then $\frac{f''(x)}{f'(x)}-\frac{g''(x)}{g'(x)}$ equals

(a) $-2\frac{f'(x)}{f(x)}$ (b) $-\frac{2g'(x)}{g(x)}$

(c) $-\frac{f'(x)}{f(x)}$ (d) $2\frac{f'(x)}{f(x)}$

Numerical Value Type Questions

53. If $f(x)=\begin{cases}\dfrac{8^x-4^x-2^x+1^x}{x^2}, & x>0\\ e^x\sin x+\pi x+\lambda\ln 4, & x\le 0\end{cases}$ is continuous at $x=0$, then the value of $1000\,e^{\lambda}$ must be

54. Let $f(x)=\dfrac{\cos^{-1}(1-\{x\}).\sin^{-1}(1-\{x\})}{\sqrt{2\{x\}}.(1-\{x\})}$, then the value of $\dfrac{2008\sqrt{2}}{\pi}\lim\limits_{x\to 0^-}f(x)$ must be (where $\{x\}$ denotes the fractional part of x).

55. If $f(x)=\begin{cases}\dfrac{36^x-9^x-4^x+1}{\sqrt{2}-\sqrt{(1+\cos x)}}, & x\ne 0\\ \lambda, & x=0\end{cases}$ is continuous at $x=0$, then $\lambda=\sqrt{\mu}\ \ln 2.\ln 3$ then the value of μ must be

56. The function given by

$$f(x)=\begin{cases}\dfrac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}}, & x\ne -1\\ \dfrac{1}{\sqrt{\lambda\pi}}, & x=-1\end{cases}$$

The value of λ for which the function $f(x)$ is continuous at $x=-1$ from the right, must be

57. Let P (x) be a polynomial of degree 4 such that $P(1)=P(3)=P(5)=P'(7)=0$. If the real number $x\ne 1,3,5$ is such that $P(x)=0$ can be expressed as $x=p/q$ where 'p' and 'q' are relatively prime, then find $(p+q)$.

58. If $f(x)=\dfrac{1}{\sin x-\sin a}-\dfrac{1}{(x-a)\cos x}$ then

$$\frac{d}{da}\lim_{x\to a}f(x)=\frac{-1}{k}\sec a-\sec a\tan^2 a.$$

The numerical quantity k should be equal to

59. If the third derivative of $\dfrac{x^4}{(x-1)(x-2)}$ is $\dfrac{-k}{(x-2)^4}+\dfrac{6}{(x-1)^4}$ then the numerical quantity k must be equal to

Match the Following

Each question has two columns. Four options are given representing matching of elements from Column-I and Column-II. Only one of these four options corresponds to a correct matching.For each question, choose the option corresponding to the correct matching.

60.

Column–I	**Column–II**
(A) If the function $f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & x\ne 0\\ \dfrac{K}{2}, & x=0\end{cases}$ is continuous at $x=0$, then $k=$	**(P)** 6
(B) If $f(x)=\dfrac{x^2-10x+25}{x^2-7x+10}$ for $x\ne 5$ & it is continuous at $x=5$ then $f(5)=$	**(Q)** $2\log\lvert a\rvert$
(C) If $f\ R\to R$ defined by $f(x)=\begin{cases}a^2\cos^2 x+b^2\sin^2 x\,(x\le 0)\\ e^{ax+b}\,(x>0)\end{cases}$ is continuous function then $b=$	**(R)** 3
	(S) $\log\lvert a\rvert$
	(T) 0

The correct matching is

(a) A - P; B - T; C - Q

(b) A - T; B - P; C - Q

(c) A - Q; B - T; C - P

(d) A - P; B - Q; C - T

61.

Column - I		Column - II	
(A)	If $f'(x)=\sqrt{3x^2+6}$ & $y=f(x^3)$ then at $x=1$, $\frac{dy}{dx}=$	**(P)**	-2
(B)	If f be a differential function such that $f(xy)=f(x)+f(y);\ \forall\ x, y \in R$ then $f(e)+f(1/e)=$	**(Q)**	-1
(C)	If f be a twicedifferential function such that $f''(x)=-f(x)$ & $f'(x)=g(x)$; If $h(x)=(f(x))^2+(g(x))^2$ & $h(5)=9$ then $h(10)=$	**(R)**	0
(D)	$y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$, $\frac{\pi}{2}<x<\pi$ then $\frac{dy}{dx}=$	**(S)**	9

The correct matching is

(a) A–S; B–R; C–S; D–P

(b) A–R; B–S; C–S; D–P

(c) A–S; B–S; C–R; D–P

(d) A–S; B–R; C–P; D–S

Paragraph Type Questions

Using the following passage, solve Q.62 to Q.64

Passage

A curve is represented parametrically by the equations $x=f(t)=a^{\ln(b^t)}$ and $y=g(t)=b^{-\ln(a^t)}$ a, b > 0 and $a \neq 1$, $b \neq 1$ where $t \in R$.

62. Which of the following is not a correct expression for $\frac{dy}{dx}$?

(a) $\frac{-1}{f(t)^2}$ (b) $-(g(t))^2$

(c) $\frac{-g(t)}{f(t)}$ (d) $\frac{-f(t)}{g(t)}$

63. The value of $\frac{d^2y}{dx^2}$ at the point where f (t) = g (t) is

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) 2

64. The value of $\frac{f(t)}{f'(t)}\cdot\frac{f''(-t)}{f'(-t)}+\frac{f(-t)}{f'(-t)}\cdot\frac{f''(t)}{f'(t)}\ \forall\ t \in R$, is equal to

(a) –2 (b) 2

(c) –4 (d) 4

Text

65. Differentiate the following functions with respect to x :

(i) log (sec x + tan x) (ii) $e^{x \sin x}$

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

Objective Questions I [Only one correct option]

1. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x. Then g is : **(2000)**

(a) onto if f is onto

(b) one-one if f is one-one

(c) continuous if f is continuous

(d) differentiable if f is differentiable

2. Let $f: R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is : **(2001)**

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$

(c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

3. The left hand derivative of $f(x) = [x]\sin(\pi x)$ at $x = k$, k is an integer is **(2001)**

(a) $(-1)^k(k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$

(c) $(-1)^k k\pi$ (d) $(-1)^{k-1}k\pi$

4. Which of the following functions is differentiable at $x = 0$? **(2001)**

(a) $(\cos|x|) + |x|$ (b) $\cos(|x|) - |x|$

(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

5. The domain of the derivative of the functions

$$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1), & \text{if } |x| > 1 \end{cases} \text{ is}$$ **(2002)**

(a) $R - \{0\}$ (b) $R - \{1\}$

(c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

6. If y is a function of x and $\log(x + y) - 2xy = 0$, then the value of $y'(0)$ is equal to **(2004)**

(a) 1 (b) – 1

(c) 2 (d) 0

7. If $y = y(x)$ and it follows the relation

$x \cos y + y \cos x = \pi$, then $y''(0)$ **(2005)**

(a) -1 (b) π

(c) $-\pi$ (d) 1

8. Let $f(x) = ||x| - 1|$, then points where $f(x)$, is not differentiable is/(are) : **(2005)**

(a) 0 (b) 1

(c) ± 1 (d) $0, \pm 1$

9. $\frac{d^2x}{dy^2}$ equals **(2007)**

(a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

10. Let $g(x) = \log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x + 1) = x f(x)$. Then, for N = 1, 2, 3, ..., **(2008)**

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(a) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N+1)^2}\right\}$

11. Let f be a real-valued function defined on the interval (–1,1) such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to **(2010)**

(a) 1 (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{1}{e}$

12. Let $f(x)=\begin{cases} x^2\left|\cos\dfrac{\pi}{x}\right|, & x\neq 0,\ x\in R, \\ 0, & x=0 \end{cases}$ then f is **(2012)**

(a) differentiable both at $x=0$ and $x=2$

(b) differentiable at $x=0$ but not differentiable at $x=2$

(c) not differentiable at $x=0$ but differentiable at $x=2$

(d) differentiable neither at $x=0$ nor at $x=2$

13. Let $f_1: R\to R$, $f_2:[0,\infty)\to R$, $f_3: R\to R$ and $f_4: R\to[0,\infty)$ be defined by

$$f_1(x) = =\begin{cases} |x| & \text{if } x<0, \\ e^x & \text{if } x\geq 0; \end{cases}$$

$f_2(x)=x^2$;

$$f_3(x)=\begin{cases} \sin x & \text{if } x<0, \\ x & \text{if } x\geq 0 \end{cases}$$

and

$$f_4(x)=\begin{cases} f_2(f_1(x)) & \text{if } x<0, \\ f_2(f_1(x))-1 & \text{if } x\geq 0. \end{cases}$$ **(2014)**

	List I		List II
P.	f_4 is	1.	onto but not one-one
Q.	f_3 is	2.	neither continuous nor one-one
R.	f_2 of$_1$ is	3.	differentiable but not one-one
S.	f_2 is	4.	continuous and one-one

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

Objective Questions II [One or more than one correct option]

14. If $f(x)=\min\{1, x^2, x^3\}$, then **(2006)**

(a) $f(x)$ is continuous $\forall\ x\in R$

(b) $f(x)>0,\ \forall\ x>1$

(c) $f(x)$ is continuous but not differentiable $\forall\ x\in R$

(d) $f(x)$ is not differentiable at two points.

15. Let $f: R\to R$ be a function such that $f(x+y)=f(x)+f(y)$, $\forall\ x, y\in R$. If $f(x)$ is differentiable at $x=0$, then **(2011)**

(a) $f(x)$ is differentiable only in a finite interval containing zero

(b) $f(x)$ is continuous $\forall\ x\in R$

(c) $f'(x)$ is constant $\forall\ x\in R$

(d) $f(x)$ is differentiable except at finitely many points

16. If $f(x)=\begin{cases} -x-\dfrac{\pi}{2}, & x\leq -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2}<x\leq 0 \\ x-1, & 0<x\leq 1 \\ \ln x, & x>1 \end{cases}$, then **(2011)**

(a) $f(x)$ is continuous at $x=-\dfrac{\pi}{2}$

(b) $f(x)$ is not differentiable at $x=0$

(c) $f(x)$ is differentiable at $x=1$

(d) $f(x)$ is differentiable at $x=-\dfrac{3}{2}$

17. For every integer n, let a_n and b_n be real numbers. Let function $f: R\to R$ be given by

$f(x)=\begin{cases} a_n+\sin\pi x, & \text{for } x\in[2n, 2n+1] \\ b_n+\cos\pi x, & \text{for } x\in(2n-1, 2n) \end{cases}$, for all integers n.

If f is continuous, then which of the following hold(s) for all n? **(2012)**

(a) $a_{n-1}-b_{n-1}=0$ (b) $a_n-b_n=1$

(c) $a_n-b_{n+1}=1$ (d) $a_{n-1}-b_n=-1$

18. Let $f: [a, b]\to[1,\infty)$ be a continuous function and let $g: R\to R$ be defined as

$$g(x)=\begin{cases} 0 & \text{if } x<a, \\ \int_a^x f(t)\,dt & \text{if } a\leq x\leq b, \\ \int_a^b f(t)\,dt & \text{if } x>b \end{cases}$$

Then **(2014)**

(a) g (x) is continuous but not differentiable at a

(b) g (x) is differentiable on R

(c) g (x) is continuous but not differentiable at b

(d) g (x) is continuous and differentiable at either a or b but not both.

19. Let $g : R \to R$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$.

Let $f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $h(x) = e^{|x|}$ **(2015)**

(a) f is differentiable at $x = 0$
(b) h is differentiable at $x = 0$
(c) foh is differentiable at $x = 0$
(d) hof is differentiable at $x = 0$

20. Let a, $b \in R$ and $f : R \to R$, be defined by $f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$. Then f is **(2016)**

(a) differentiable at $x = 0$ if $a = 0$ and $b = 1$
(b) differentiable at $x = 1$ if $a = 1$ and $b = 0$
(c) **NOT** differentiable at $x = 0$ if $a = 1$ and $b = 0$
(d) **NOT** differentiable at $x = 1$ if $a = 1$ and $b = 1$

21. Let $f: R \to R$, $g: R \to R$, and $h : R \to R$, be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then **(2016)**

(a) $g'(2) = \frac{1}{15}$ (b) $h'(1) = 666$
(c) $h(0) = 16$ (d) $h(g(3)) = 36$

22. Let the function $f : R \to R$ be defined by $f(x) = x^3 - x^2 + (x - 1)\sin x$ and let $g : R \to R$ be an arbitrary function. Let $fg : R \to R$ be the product function defined by $(fg)(x)=f(x)g(x)$. Then which of the following statements is/are TRUE ? **(2020)**

(a) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$
(b) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$
(c) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$
(d) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

23. Let $f : R \to R$ and $g : R \to R$ be functions satisfying $f(x+y) = f(x) + f(y) + f(x) f(y)$ and $f(x) = xg(x)$ for all $x, y \in R$. If $\lim_{x\to 0} g(x) = 1$, then which of the following statements is/are TRUE? **(2020)**

(a) f is differentiable at every $x \in R$
(b) If $g(0)=1$, then g is differentiable at every $x \in R$
(c) The derivative $f'(1)$ is equal to 1
(d) The derivative $f'(0)$ is equal to 1

Numerical Value Type Questions

24. If two functions 'f' and 'g' satisfying given conditions for $\forall$ x, y $\in$ R, $f(x - y) = f(x)\, g(y) - f(y) \,.\, g(x)$ and $g(x - y) = g(x) \,.\, g(y) + f(x) \,.\, f(y)$. If right hand derivative at $x = 0$ exists for $f(x)$ then find the derivative of $g(x)$ at $x = 0$ **(2005)**

25. Let f: $R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$ by

$$h(x) = \begin{cases} \max \ \{f(x), g(x)\} & if\ x \leq 0, \\ \min \ \{f(x), g(x)\} & if\ x \geq 0. \end{cases}$$ **(2014)**

The number of points at which h(x) is not differentiable is

26. Let the functions $f : (-1,1) \to R$ and $g : (-1,1) \to (-1,1)$ be defined by $f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$, where [x] denotes the greatest integer less than or equal to x. Let $fog : (-1,1) \to R$ be the composite function defined by $(fog)(x) = f(g(x))$. Suppose c is the number of points in the interval (-1,1) at which fog is not continuous, and suppose d is the number of points in the interval (-1,1) at which fog is not differentiable. Then the value of c + d is ______ . **(2020)**

Match the Following

Each question has two columns. Four options are given representing matching of elements from Column-I and Column-II. For each question, choose the option corresponding to the correct matching.

27. In the following, [x] denotes the greatest integer less than or equal to x. **(2007)**

	Column I	Column II
(A)	$x\|x\|$	(P) continuous in (–1, 1)
(B)	$\sqrt{\|x\|}$	(Q) differentiable in (–1, 1)
(C)	$x + [x]$	(R) strictly increasing (–1, 1)
		(S) not differentiable at least at
(D)	$\|x - 1\| + \|x + 1\|$	one point in (–1, 1)

The correct matching is

(a) A–P,Q,R; B–P,S; C–R,S; D–P,Q
(b) A–P,S,R; B–P,Q,R; C–R,S; D–P,Q
(c) A–P,Q,R; B–P,S; C–P,Q; D–R,S
(d) A–P,Q; B–P,S; C–R,S; D–P,Q,R

Text

28. Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α if and only if there is a function $g : R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x-\alpha)$ for all $x \in R$. **(2001)**

29. $$f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right), & -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2}, & x = 0 \\ \dfrac{e^{ax/2}-1}{x}, & 0 < x < \dfrac{1}{2} \end{cases}$$

If $f(x)$ is differentiable at $x = 0$ and $|c| < \dfrac{1}{2}$, then find the value of a and prove that $64b^2 = (4 - c^2)$. **(2004)**

30. If $f : [-1, 1] \to R$ and $f(0) = 0$ then $f'(0) = \lim\limits_{n\to\infty} n f\left(\dfrac{1}{n}\right)$

Find the value of $\lim\limits_{n\to\infty} \dfrac{2}{\pi}(n+1)\cos^{-1}\left(\dfrac{1}{n}\right) - n$

Given that $0 < \left|\lim\limits_{n\to\infty} \cos^{-1}\left(\dfrac{1}{n}\right)\right| < \dfrac{\pi}{2}$ **(2004)**

Answer Key

CHAPTER -3 | CONTINUITY, DIFFERENTIABILITY & DIFFERENTIATION

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

1. (d)	**2.** (c)	**3.** (c)	**4.** (d)	**5.** (c)
6. (b)	**7.** (d)	**8.** (b)	**9.** (b)	**10.** (b)
11. (b)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (b)
16. (c)	**17.** (b)	**18.** (c)	**19.** (c)	**20.** (d)
21. (b)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (c)
26. (a)	**27.** (c)	**28.** (c)	**29.** (d)	**30.** (a)
31. (b)	**32.** (d)	**33.** (b)	**34.** (b)	**35.** (c)
36. (b)	**37.** (b)	**38.** (a)	**39.** (b)	**40.** (a)
41. (d)	**42.** (d)	**43.** (d)	**44.** (b)	**45.** (d)
46. (a)	**47.** (b)	**48.** (b)	**49.** (a)	**50.** (−0.5)
51. (2.24)	**52.** (-1)	**53.** (0.31)	**54.** (0.33)	**55.** (0.67)
56. (0.707)	**57.** (−0.72)	**58.** (0.37)	**59.** (2)	**60.** (0.37)
61. (1)	**62.** (1.6)	**63.** (2)	**64.** (0.5)	**65.** (0.13)

EXERCISE - 2 : PREVIOUS YEAR JEE MAIN QUESTIONS

1. (c)	**2.** (a)	**3.** (d)	**4.** (b)	**5.** (b)
6. (c)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (c)
11. (a)	**12.** (b)	**13.** (a)	**14.** (d)	**15.** (c)
16. (c)	**17.** (b)	**18.** (d)	**19.** (d)	**20.** (a)
21. (b)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (b)
26. (a)	**27.** (a)	**28.** (d)	**29.** (91.00)	**30.** (d)
31. (1)	**32.** (10.00)	**33.** (c)	**34.** (d)	**35.** (8.00)
36. (d)	**37.** (5)	**38.** (a)	**39.** (c)	**40.** (c)
41. (3.00)	**42.** (a)	**43.** (5.00)	**44.** (c)	**45.** (d)
46. (a)	**47.** (d)	**48.** (2.00)	**49.** (5.00)	**50.** (d)
51. (d)	**52.** (c)	**53.** (d)	**54.** (1.00)	**55.** (6.00)
56. (c)	**57.** (c)	**58.** (b)	**59.** (1.00)	**60.** (4.00)
61. (c)	**62.** (39.00)		**63.** (a)	**64.** (b)
65. (5.00)	**66.** (b)	**67.** (40.00)		
68. (14.00)	**69.** (d)	**70.** (17.00)	**71.** (d)	**72.** (b)
73. (a)	**74.** (2.00)	**75.** (a)		

ANSWER KEY

CHAPTER -3 | CONTINUITY, DIFFERENTIABILITY & DIFFERENTIATION

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

1. (a)	**2.** (a)	**3.** (a)	**4.** (c)	**5.** (c)
6. (a)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (a)
11. (d)	**12.** (b)	**13.** (b)	**14.** (d)	**15.** (a)
16. (b)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (a)
21. (c)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (c)
26. (a)	**27.** (a)	**28.** (c)	**29.** (c)	**30.** (c)
31. (c)	**32.** (c)	**33.** (b)	**34.** (c)	**35.** (b)
36. (c)	**37.** (b)	**38.** (b)	**39.** (c)	**40.** (c)
41. (b)	**42.** (a)	**43.** (d)	**44.** (a)	**45.** (c)
46. (a,c)	**47.** (a,b)	**48.** (a,b,d)		**49.** (a,b)
50. (b,c)	**51.** (a,b,c)	**52.** (b,d)	**53.** (2000)	**54.** (1004)
55. (512)	**56.** (2)	**57.** (100)	**58.** (2)	**59.** (96)
60. (a)	**61.** (a)	**62.** (d)	**63.** (d)	**64.** (a)

65. (i) sec x (ii) $e^{x \sin x}(x \cos x + \sin x)$

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

1. (c)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (d)
6. (a)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (a)
11. (b)	**12.** (b)	**13.** (d)	**14.** (a,b,c)	**15.** (b,c)
16. (a,b,c,d)		**17.** (b,d)	**18.** (a,c)	**19.** (a,d)
20. (a,b)	**21.** (b,c)	**22.** (a,c)	**23.** (a,b,d)	**24.** (0)
25. (3)	**26.** (4.00)	**27.** (a)	**29.** a = 1	**30.** $1-\frac{2}{\pi}$