Session 3

Total Probability Theorem, Baye's Theorem or Inverse Probability

Total Probability Theorem

Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events i.e., $E_i \cap E_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{n} E_i = S$.

Suppose that, $P(E_i) > 0, \forall\, 1 \leq i \leq n$

Then for any event E

$$P(E) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)$$

Proof Since, $E_1, E_2, \ldots, E_n$ are disjoint

$\therefore E \cap E_1, E \cap E_2, \ldots, E \cap E_n$ are also disjoint.

Now, $$E = E \cap S = E \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (E \cap E_i)$$

$\therefore$ $$P(E) = \sum_{i=1}^{n} P(E \cap E_i) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)$$

Example 14. The probability that certain electronic component fails, when first used is 0.10. If it does not fail immediately, then the probability that it lasts for one year is 0.99. What is the probability that a new component will last for one year?

Sol. Given probability of electronic component fails, when first used = 0.10

i.e., $P(F) = 0.10$

$\therefore$ $P(\overline{F}) = 1 - P(F) = 0.90$

and let $P(Y)$ = Probability of new component to last for one year

$\because$ $P(F) + P(\overline{F}) = 1$

Obviously, the two events are mutually exclusive and exhaustive

$\therefore$ $P\left(\frac{Y}{F}\right) = 0$ and $P\left(\frac{Y}{\overline{F}}\right) = 0.99$

$\therefore$ $$P(Y) = P(F) \cdot P\left(\frac{Y}{F}\right) + P(\overline{F}).P\left(\frac{Y}{\overline{F}}\right)$$

$$= 0.10 \times 0 + 0.90 \times 0.99$$

$$= 0 + (0.9)(0.99) = 0.891$$

Example 15. Three groups A, B and C are contesting for positions on the Board of Directors of a company. The probabilities of their winning are 0.5, 0.3 and 0.2, respectively. If the group A wins, then the probability of introducing a new product is 0.7 and the corresponding probabilities for groups B and C are 0.6 and 0.5, respectively. Find the probability that the new product will be introduced.

Sol. Given, $P(A) = 0.5$, $P(B) = 0.3$ and $P(C) = 0.2$

$\therefore$ $P(A) + P(B) + P(C) = 1$

Then, events A, B, C are exhaustive.

If $P(E)$ = Probability of introducing a new product, then as given

$$P\left(\frac{E}{A}\right) = 0.7,\ P\left(\frac{E}{B}\right) = 0.6 \text{ and } P\left(\frac{E}{C}\right) = 0.5$$

$$P(E) = P(A) \cdot P\left(\frac{E}{A}\right) + P(B) \cdot P\left(\frac{E}{B}\right) + P(C) \cdot P\left(\frac{E}{C}\right)$$

$$= 0.5 \times 0.7 + 0.3 \times 0.6 + 0.2 \times 0.5$$

$$= 0.35 + 0.18 + 0.10 = 0.63$$

Example 16. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replace into urn, otherwise it is replaced along with another ball of the same colour. The process is repeated, find the probability that the third ball drawn is black.

Sol. For the first two draw, the balls taken out may be

Let E_1 = White and White; E_2 = White and Black

E_3 = Black and White; E_4 = Black and Black

$\therefore$ $$P(E_1) = P(W) \cdot P\left(\frac{W}{W}\right) = \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{6}$$

$$P(E_2) = P(W) \cdot P\left(\frac{B}{W}\right) = \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{3}$$

$$P(E_3) = P(B) \cdot P\left(\frac{W}{B}\right) = \frac{2}{4} \cdot \frac{2}{5} = \frac{1}{5}$$

and $$P(E_4) = P(B) \cdot P\left(\frac{B}{B}\right) = \frac{2}{4} \cdot \frac{3}{5} = \frac{3}{10}$$

$$\therefore\ P(E_1)+P(E_2)+P(E_3)+P(E_4)=\frac{1}{6}+\frac{1}{3}+\frac{1}{5}+\frac{3}{10}$$
$$=\frac{10+20+12+18}{60}=1$$

Then, events E_1, E_2, E_3 and E_4 are exhaustive. Obviously, these events are mutually exclusive, then

$$P\left(\frac{B}{E_1}\right)=\frac{2}{2}=1;\ P\left(\frac{B}{E_2}\right)=\frac{3}{4}$$
$$P\left(\frac{B}{E_3}\right)=\frac{3}{4}\text{ and }P\left(\frac{B}{E_4}\right)=\frac{4}{6}=\frac{2}{3}$$

$\therefore$ Required probability,

$$P(B)=P(E_1)\cdot P\left(\frac{B}{E_1}\right)+P(E_2)\cdot P\left(\frac{B}{E_2}\right)+P(E_3)\cdot P\left(\frac{B}{E_3}\right)+P(E_4)\cdot P\left(\frac{B}{E_4}\right)$$
$$=\frac{1}{6}\times 1+\frac{1}{3}\times\frac{3}{4}+\frac{1}{5}\times\frac{3}{4}+\frac{3}{10}\times\frac{2}{3}$$
$$=\frac{1}{6}+\frac{1}{4}+\frac{3}{20}+\frac{1}{5}$$
$$=\frac{10+15+9+12}{60}=\frac{46}{60}=\frac{23}{30}$$

Baye's Theorem or Inverse Probability

If an event E can occur only with one of the n mutually exclusive and exhaustive events $E_1, E_2, E_3, \ldots, E_n$ and the probabilities $P(E/E_1), P(E/E_2), \ldots, P(E/E_n)$ are known, then

$$P\left(\frac{E_k}{E}\right)=\frac{P(E_k)\cdot P\left(\frac{E}{E_k}\right)}{\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)}$$

Proof The event E occurs with one of the n mutually exclusive and exhaustive events $E_1, E_2, E_3, \ldots, E_n$, then

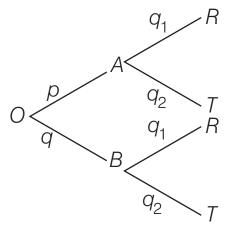
$$E=EE_1+EE_2+EE_3+\ldots+EE_n$$
$$\Rightarrow\quad P(E)=P(EE_1)+P(EE_2)+P(EE_3)+\ldots+P(EE_n)$$
$$=\sum_{k=1}^{n}P(EE_k)=\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)$$
$$\therefore\ P\left(\frac{E_k}{E}\right)=\frac{P(E_k\ E)}{P(E)}=\frac{P(E_k)\cdot P\left(\frac{E}{E_k}\right)}{\sum_{k=1}^{n}P(E_k)\cdot P\left(\frac{E}{E_k}\right)}$$

Remark

The probabilities $P(E_k)$ and $P\left(\frac{E_k}{E}\right)$ are known as **priori** and **posteriori** probabilities, respectively.

Remarks We can visualise a tree structure here

$$P(A)=p,\ P(B)=q$$
$$P\left(\frac{R}{A}\right)=p_1,\ P\left(\frac{T}{A}\right)=q_1$$
$$P\left(\frac{R}{B}\right)=p_2,\ P\left(\frac{T}{B}\right)=q_2$$



If we are to find $P\left(\frac{A}{R}\right)$, we go

$$\therefore\quad P\left(\frac{A}{R}\right)=\frac{P(A)\cdot P\left(\frac{R}{A}\right)}{P(A)\cdot P\left(\frac{R}{A}\right)+P(B)\cdot P\left(\frac{R}{B}\right)}$$

Example 17. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Then, find the probability that it was drawn from the bag B.

Sol. Let $E_1 \equiv$ The event of ball being drawn from bag A.

$E_2 \equiv$ The event of ball being drawn from bag B.

and $E \equiv$ The event of ball being red.

Since, both the bag are equally likely to be selected, therefore

$$P(E_1)=P(E_2)=\frac{1}{2}\text{ and }P\left(\frac{E}{E_1}\right)=\frac{3}{5}\text{ and }P\left(\frac{E}{E_2}\right)=\frac{5}{9}$$

$\therefore$ Required probability,

$$P\left(\frac{E_2}{E}\right)=\frac{P(E_2)\cdot P\left(\frac{E}{E_2}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$
$$=\frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}}=\frac{\frac{5}{9}}{\frac{3}{5}+\frac{5}{9}}=\frac{25}{52}$$

Example 18. A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Sol. Let E_1 be the event that the man reports that it is a six and E be the event that a six occurs.

Then, $P(E) = \frac{1}{6}$

$\therefore \quad P(\bar{E}) = 1 - P(E) = 1 - \frac{1}{6} = \frac{5}{6}$

$\therefore \quad P\left(\frac{E_1}{E}\right) = P$ (man speaking the truth) $= \frac{3}{4}$

and $P\left(\frac{E_1}{\bar{E}}\right) = P$ (man not speaking the truth) $= 1 - \frac{3}{4} = \frac{1}{4}$

Clearly, $\left(\frac{E}{E_1}\right)$ is the event that it is actually a six, when it is known that the man reports a six.

$$P\left(\frac{E}{E_1}\right) = \frac{P(E)\cdot P\left(\frac{E_1}{E}\right)}{P(E)\cdot P\left(\frac{E_1}{E}\right) + P(\bar{E})\cdot P\left(\frac{E_1}{\bar{E}}\right)}$$

$$= \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \frac{3}{8}$$

Example 19. In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{6}$. The probability that his answer is correct given that he copied it is $\frac{1}{8}$. Find the probability that he knew the answer to the question given that he correctly answered it.

Sol. Lest E_1 be the event that the answer is guessed, E_2 be the event that the answer is copied, E_3 be the event that the examinee knows the answer and E be the event that the examinee answers correctly.

Given, $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$

Assume that events E_1, E_2 and E_3 are exhaustive

$\therefore \quad P(E_1) + P(E_2) + P(E_3) = 1$

$\therefore \quad P(E_3) = 1 - P(E_1) - P(E_2) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$

Now, $P\left(\frac{E}{E_1}\right)$

$\equiv$ Probability of getting correct answer by guessing

$= \frac{1}{4}$ [since 4 alternatives]

$P\left(\frac{E}{E_2}\right) \equiv$ Probability of answering correctly by copying $= \frac{1}{8}$

and $P\left(\frac{E}{E_3}\right) \equiv$ Probability of answering correctly by knowing $= 1$

Clearly, $\left(\frac{E_3}{E}\right)$ is the event he knew the answer to the question, given that he correctly answered it.

$$\therefore \quad P\left(\frac{E_3}{E}\right) = \frac{P(E_3)\cdot P\left(\frac{E}{E_3}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right) + P(E_2)\cdot P\left(\frac{E}{E_3}\right)}$$

$$= \frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1} = \frac{24}{29}$$

Example 20. A and B are two independent witnesses (i.e., there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statements. Show that the probability that the statements is true, is $\frac{xy}{1 - x - y + 2xy}$.

Sol. Let E_1 be the event that both A and B speak the truth, E_2 be the event that both A and B tell a lie and E be the event that A and B agree in a certain statements.

And also, let C be the event that A speak the truth and D be the event that B speaks the truth.

$\therefore \quad E_1 = C \cap D$

[$\because C$ and D are independent events]

and $\quad E_2 = \bar{C} \cap \bar{D}$

then, $\quad P(E_1) = (C \cap D) = P(C)\cdot P(D) = xy$

and $\quad P(E_2) = P(\bar{C} \cap \bar{D}) = P(\bar{C})\,P(\bar{D})$

$= \{1 - P(C)\}\{1 - P(D)\} = (1 - x)(1 - y)$

$= 1 - x - y - xy$

Now, $P\left(\frac{E}{E_1}\right) \equiv$ Probability that A and B will agree, when both of them speak the truth $= 1$

and $P\left(\frac{E}{E_2}\right) =$ Probability that A and B will agree, when both of them tell a lie $= 1$

Clearly, $\left(\frac{E_1}{E}\right)$ be the event that the statement is true

$$\therefore \quad P\left(\frac{E_1}{E}\right) = \frac{P(E_1)\cdot P\left(\frac{E}{E_1}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$

$$= \frac{xy\cdot 1}{xy\cdot 1 + (1 - x - y + xy)\cdot 1} = \frac{xy}{1 - x - y + 2xy}$$

Exercise for Session 3

1. A bag A contains 3 white and 2 black balls and another bag B contains 2 white and 4 black balls. A bag and a ball out of it are picked at random. The probability that the ball is white, is

(a) $\frac{2}{7}$ (b) $\frac{7}{9}$ (c) $\frac{4}{15}$ (d) $\frac{7}{15}$

2. There are two bags, one of which contains 3 black and 4 white balls, while the other contains 4 black and 3 white balls. A die is cast. If the face 1 or 3 turns up a ball is taken out from the first bag and if any other face turns up, a ball is taken from the second bag. The probability of choosing a black ball, is

(a) $\frac{7}{15}$ (b) $\frac{8}{15}$ (c) $\frac{10}{21}$ (d) $\frac{11}{21}$

3. There are two groups of subjects, one of which consists of 5 Science subjects and 3 Engineering subjects and the other consists of 3 Science and 5 Engineering subjects. An unbiased die is cast. If number 3 or 5 turns up, a subject from group I is selected, otherwise a subject is selected from group II. The probability that an Engineering subject is selected ultimately, is

(a) $\frac{7}{13}$ (b) $\frac{9}{17}$ (c) $\frac{13}{24}$ (d) $\frac{11}{20}$

4. Urn A contains 6 red and 4 white balls and urn B contains 4 red and 6 white balls. One ball is drawn at random from urn A and placed in urn B. Then a ball is drawn from urn B and placed in urn A. Now, if one ball is drawn from urn A, the probability that it is red, is

(a) $\frac{6}{11}$ (b) $\frac{17}{50}$ (c) $\frac{16}{55}$ (d) $\frac{32}{55}$

5. A box contains N coins, of which m are fair and the rest are biased. The probability of getting head when a fair coin is tossed is $\frac{1}{2}$, while it $\frac{2}{3}$ when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. The probability that the coin drawn is fair, is

(a) $\frac{5m}{m+8N}$ (b) $\frac{3m}{m+8N}$ (c) $\frac{7m}{m+8N}$ (d) $\frac{9m}{m+8N}$

6. A pack of playing cards was found to contain only 51 cards. If the first 13 cards which are examined are all red, then the probability that the missing card is black, is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{15}{26}$ (d) $\frac{16}{39}$

7. A purse contains n coins of unknown values. A coin is drawn from it at random and is found to be a rupee. Then the chance that it is the only rupee coin in the purse, is

(a) $\frac{1}{n}$ (b) $\frac{2}{n+1}$ (c) $\frac{1}{n(n+1)}$ (d) $\frac{2}{n(n+1)}$

8. A card is lost from a pack of 52 playing cards. From the remainder of the pack, one card is drawn and is found to be a spade. The probability that the missing card is a spade, is

(a) $\frac{2}{17}$ (b) $\frac{3}{17}$ (c) $\frac{4}{17}$ (d) $\frac{5}{17}$

9. A person is known to speak the truth 4 times out of 5. He throws a die and reports that it is an ace. The probability that it is actually an ace, is

(a) $\frac{1}{3}$ (b) $\frac{2}{9}$ (c) $\frac{4}{9}$ (d) $\frac{5}{9}$

10. Each of the n urns contains 4 white and 6 black balls, the $(n+1)$ th urn contains 5 white and 5 black balls. Out of $(n+1)$ urns an urn is chosen at random and two balls are drawn from it without replacement. Both the balls are found to be black. If the probability that the $(n+1)$th urn was chosen to drawn the balls is $\frac{1}{16}$, the value of n, is

(a) 10 (b) 11 (c) 12 (d) 13

Answers

Exercise for Session 3

1. (d) **2.** (d) **3.** (c) **4.** (d) **5.** (d) **6.** (b)
7. (d) **8.** (c) **9.** (c) **10.** (a)