Conic Section

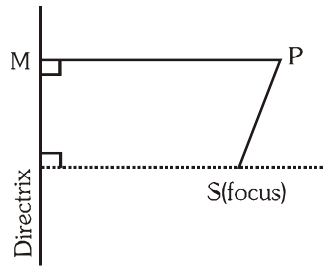
Contents

CONIC SECTION

This chapter focusses on parabolic curves, which constitutes one category of various curves obtained by slicing a cone by a plane, called conic sections. A cone (not necessarily right circular) can be out in various ways by a plane, and thus different types of conic sections are obtained.

Let us start with the definition of a conic section and then we will see how are they obtained by slicing a right circular cone.

1. DEFINITION OF CONIC SECTION :



It is locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always constant, is known as a conic section or a conic. The fixed point is called the **focus** of the conic and this fixed line is called the **directrix** of the conic. Also this constant ratio is called the **eccentricity** of the conic and is denoted by e.

If e = 1, the conic is called parabola.

If e < 1, the conic is called Ellipse.

If e > 1, the conic is called Hyperbola.

If e = 0, the conic is called Circle.

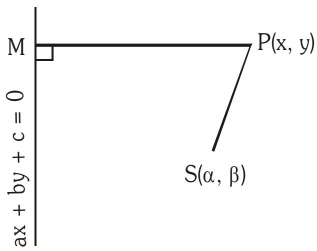
If e = ∞, the conic is called Pair of straight lines.

In the figure

$$\frac{SP}{PM} = \text{constant} = e \quad \text{or} \quad SP = ePM$$

- The line passing through the focus & perpendicular to the directrix is called the **Axis**.
- A point of intersection of a conic with its axis is called a **Vertex**.

1.1 EQUATION OF CONIC SECTION :



If the focus is (α, β) and the directrix is ax + by + c = 0 then the equation of the conic section whose eccentricity is e

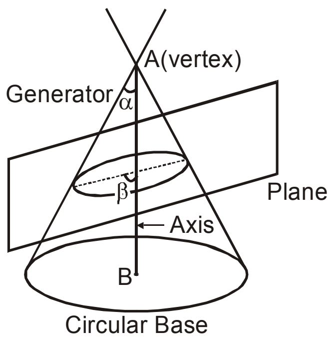
$$\sqrt{(x-\alpha)^2+(y-\beta)^2} = e.\frac{|ax+by+c|}{\sqrt{(a^2+b^2)}}$$

or $(x-\alpha)^2 + (y-\beta)^2 = e^2.\dfrac{(ax+by+c)^2}{(a^2+b^2)}$

Which is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

1.2 Section of right circular cone by different planes

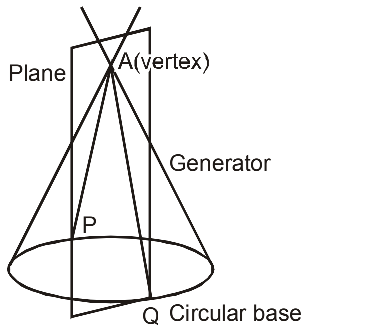
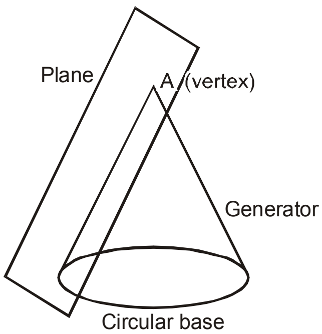
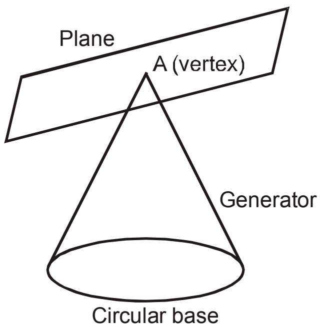
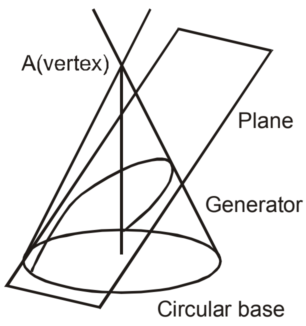
A right circular cone is as shown in the figure – 1



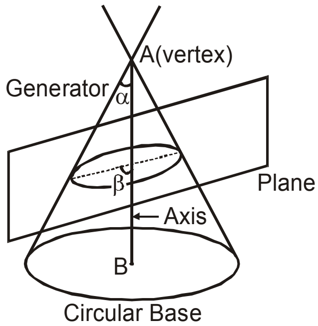
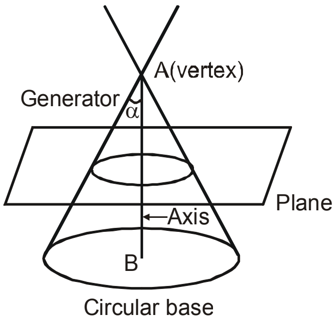
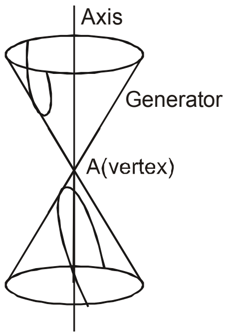
α is angle between generator and axis.

β is angle between plane and axis

Section of a right circular cone by a plane passing through its vertex is a pair of straight lines.
Section of a right circular cone by a plane not passing through vertex is either circle or parabola or ellipse or hyperbola which is shown in table below :

Type of conic section	3-D view of section of right circular cone with plane	Condition of conic in definition of conic	condition of conic in $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$
Two distinct real lines	 Plane passes through vertex A and $0 \le \beta < \alpha$	e > 1, focus lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, h^2 > ab$
Two real same lines	 Plane passes through vertex A and $\beta = \alpha$	e =1, focus lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, h^2 = ab$ (either $g^2 = ac$ of $f^2 = bc$)
Two imaginary lines/point	 Plane passes through vertex A and $\beta > \alpha$	0 < e < 1, focus lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, h^2 < ab$
Parabola		e = 1, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \ne 0, h^2 = ab$

lane passes through vertex A and $\beta > \alpha$

Ellipse	 Plane does not passes through vertex A and $\alpha < \beta$ < 90°	0 < e < 1, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0, h^2 < ab$ (either $a \neq b$ or $h \neq 0$)
Circle	 Plane does not passes through vertex A and β = 90°	$e = 0$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0, a = b, h = 0$
Hyperbola	 Plane does not passes through vertex A and β = 90°	$e > 1$, focus does not lies on directrix	$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0, h^2 > ab$

Note : (i) Pair of real parallel lines is not the part of conic but it is part of general two degree equation.

For it $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$, $h^2 = ab$, (either $g^2 > ac$ or $f^2 > bc$)

$\Rightarrow$ General two degree equation can represent real curve other than conic section.

(ii) For rectangular hyperbola $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$, $h^2 > ab$, $a + b = 0$

SOLVED EXAMPLE

Example 1: Find the locus of a point, which moves such that its distance from the point (0, –1) is twice its distance from the line 3x + 4y + 1 = 0

Solution : Let $P(x_1, y_1)$ be the point, whose locus is required.

Its distance from (0, –1) = 2 × its distance from the line 3x + 4y + 1 = 0

$$\Rightarrow \sqrt{(x_1-0)^2+(y_1+1)^2} = 2 \times \frac{|3x_1+4y_1+1|}{\sqrt{(3^2+4^2)}}$$

$$5\sqrt{x_1^2+(y_1+1)^2} = 2|3x_1 + 4y_1 + 1|$$

Squaring and simplifying, we have

$$25(x_1^2+y_1^2+2y_1+1) = 4(9x_1^2+16y_1^2+24x_1y_1+6x_1+8y_1+1)$$

Hence the locus of (x_1, y_1) is

$$11x^2 + 39y^2 + 96xy + 24x – 18y – 21 = 0$$

Example 2 : What conic does $13x^2 – 18xy + 37y^2 + 2x + 14y – 2 = 0$ represent ?

Solution : Compare the given equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore$ a = 13, h = –9, b = 37, g = 1, f = 7, c = –2

then $\Delta = abc + 2fgh – af^2 – bg^2 – ch^2$

$= (13)(37)(–2) + 2(7)(1)(–9) – 13(7)^2 – 37(1)^2 + 2(–9)^2$

$= – 962 – 126 – 637 – 37 + 162 = – 1600 \neq 0$

and also $h^2 = (–9)^2 = 81$ and ab = 14 × 37 = 481

Here $ab – h^2 = 400 > 0$

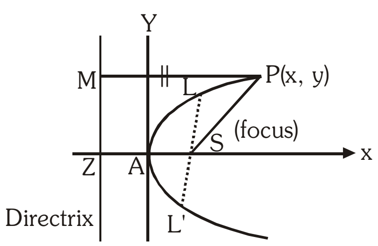
So we have $ab – h^2 > 0$ and $\Delta \neq 0$.

Hence the given equation represents an ellipse.

2. ELEMENTARY CONCEPTS OF PARABOLA :

2.1 Definition :

A parabola is the locus of a point which moves in a plane such a way that its distance from a fixed point is equal to its perpendicular distance from a fixed straight line.



From the diagram PS = PM

2.1.1 TERMS RELATED TO PARABOLA :

(a) Vertex : It is point of intersection of parabola with axis.

(b) Focal Length (Focal distance) : The distance of any point P(x, y) on the parabola from the focus is called the focal length i.e. the focal distance of P = the perpendicular distance of the point P from the directrix

(c) Double ordinate : The chord which is perpendicular to the axis of Parabola or parallel to Directrix is called double ordinate of the Parabola.

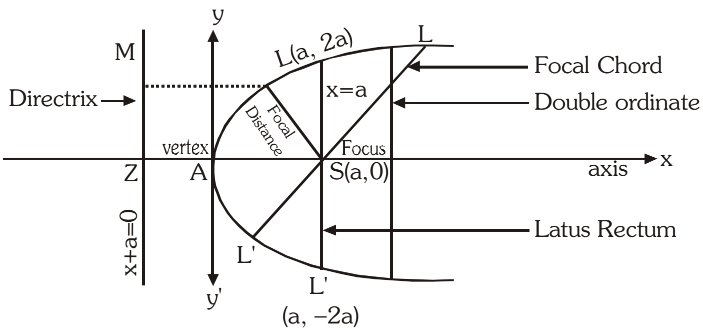
(d) Focal Chord : Any chord of the parabola passing through the focus is called focal chord.

(e) Latus Rectum : If a double ordinate passes through the focus of parabola then it is called as latus rectum.

2.2 STANDARD FORM OF EQUATION OF PARABOLA :

If we take vertex as the origin, axis as x-axis and distance between vertex and focus as 'a' then equation of the parabola in the simplest form will be-

$$y^2 = 4ax$$



From the diagram -

(a) Vertex A $\Rightarrow$ (0, 0)

(b) Focus S $\Rightarrow$ (a, 0)

(c) Directrix $\Rightarrow x + a = 0$

(d) Axis $\Rightarrow y = 0$ or x-axis

(e) Equation of Latus Rectum $\Rightarrow x = a$

(f) Length of L.R. $\Rightarrow 4a$

(g) Ends of L.R. $\Rightarrow$ (a, 2a), (a, –2a)

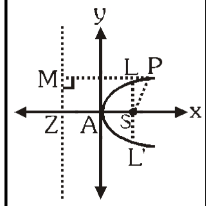
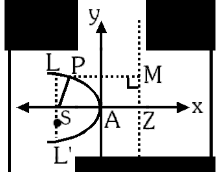
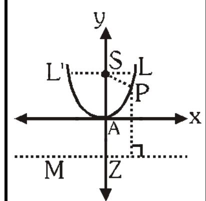
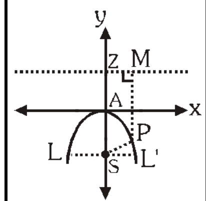
NOTE :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

All standard parabolas are given below :

Equation of Parabolas	vertex	Focus	Equation of the axis	Equation of directrix	Length of latus rectum	Extremities of latus rectum	Equation of the latus rectum	Equation of tangent at vertex	Parametrix Co-ordinates	Eccen-tricity	Figures
$y^2 = 4ax$	(0, 0)	(a, 0)	y = 0	x + a = 0	4a	(a, ± 2a)	x – a = 0	x = 0	$(at^2, 2at)$	1	
$y^2 = -4ax$	(0, 0)	(–a, 0)	y = 0	x – a = 0	4a	(–a, ±2a)	x + a = 0	x = 0	$(-at^2, 2at)$	1	[redacted]  [redacted]
$x^2 = 4ay$	(0, 0)	(0, a)	x = 0	y + a = 0	4a	(±2a, a)	y – a = 0	y = 0	$(2at, at^2)$	1	
$x^2 = -4ay$	(0, 0)	(0, –a)	x = 0	y – a = 0	4a	(±2a, –a)	y + a = 0	y = 0	$(2at, -at^2)$	1	

2.2.1 REDUCTION TO STANDARD EQUATION :

If the equation of a parabola is not in standard form and if it contains second degree term either in y or in x (but not in both) then it can be reduced into standard form. For this we change the given equation into the following forms-

$(y - k)^2 = 4a(x - h)$ or $(x - p)^2 = 4b(y - q)$

And then we compare from the following table for the results related to parabola.

Equation of parabola	Vertex	Axis	Focus	Directrix	Equation of L.R.	Length of LR.
$(y - k)^2 = 4a(x - h)$	(h, k)	y = k	(h + a, k)	x + a – h = 0	x = a + h	4a
$(x - p)^2 = 4b(y - q)$	(p, q)	x = p	(p, b + q)	y + b – q = 0	y = b + q	4b

SOLVED EXAMPLE

Example 3 : Find the equation of the parabola whose focus is (1, 1) and the directrix is x + y + 1 = 0.

Sol. Let P(x, y) be any point on the parabola.

Then the distance of P(x, y) from the focus (1, 1) = distance of P(x, y) from the directrix x + y + 1 = 0

$$\therefore \quad \sqrt{(x-1)^2 + (y-1)^2} = \frac{|x+y+1|}{\sqrt{(1)^2 + (1)^2}} \quad .....(1)$$

Squaring (1), we get

$$(x - 1)^2 + (y - 1)^2 = \left(\frac{x+y+1}{\sqrt{2}}\right)^2$$

or $2[x^2 + 1 - 2x + y^2 + 1 - 2y] = x^2 + y^2 + 2xy + 2y + 2x + 1$ or $x^2 - 2xy + y^2 - 6x - 6y + 3 = 0$

This is the required equation of the parabola.

Example 4 : Find the vertex, axis, focus, directrix, latusrectum of the parabola $x^2 + 2y - 3x + 5 = 0$.

Solution : Given equation

$x^2 + 2y - 3x + 5 = 0 \quad \Rightarrow \quad x^2 - 3x + 5 = -2y$

$x^2 - 3x = \frac{9}{4} + 5 - \frac{9}{4} = -2y \quad \Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = -2y - \frac{11}{4}$

$\left(x - \frac{3}{2}\right)^2 = -2\left(y + \frac{11}{8}\right) \quad \Rightarrow \quad X^2 = -4AY$

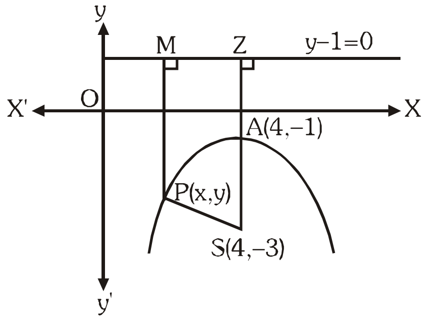
Hence vertex $\equiv \left(\frac{3}{2}, -\frac{11}{8}\right)$

Axis $x - \frac{3}{2} = 0 \quad \Rightarrow \quad x = \frac{3}{2}$

directirx $y = -\frac{7}{8} \quad \Rightarrow \quad$ latus rectum = 2.

Example 5 : Find the equation of the parabola whose focus is (4, –3) and vertex is (4, –1)

Solution : Let A(4, –1) be the vertex and S(4, –3) be the focus.



$\therefore$ Slope of AS = $\dfrac{-3+1}{4-4} = \infty$

which is parallel to y-axis

$\therefore$ Directrix parallel to x-axis.

Let $Z(x_1, y_1)$ be any point on the directrix, then A is the mid point of SZ

$\therefore$ $4 = \dfrac{x_1+4}{2} \Rightarrow x_1 = 4$

and $-1 = \dfrac{y_1-3}{2} \Rightarrow y_1 = 1$

$\therefore$ Z = (4, 1)

Also directrix is parallel to x-axis and passes through Z(4, 1), so equation of directrix is

$y = 1$ or $y - 1 = 0$

Now let P(x, y) be any point on the parabola. Join SP and drawn PM perpendicular to the directrix. Then by definition

SP = PM

$\Rightarrow$ $(SP)^2 = (PM)^2 \Rightarrow (x-4)^2 + (y+3)^2 = \left(\dfrac{|y-1|}{\sqrt{12}}\right)^2 \Rightarrow (x-4)^2 + (y+3)^2 = (y-1)^2$

or $x^2 - 8x + 8y + 24 = 0$

Example 6 : If focus of a parabola is (3, –4) and directrix is x + y – 2 = 0, then find its vertex and length of latusrectum

Solution : First we find the equation of axis of parabola, which is perpendicular to directrix. So its equations is x – y + k = 0. It passes through focus S(3, –4)

$\Rightarrow$ $3 - (-4) + k = 0 \Rightarrow k = -7$

Let Z is the point of intersection of axis and directrix

Solving equation x + y – 2 = 0 and x – y – 7 = 0 gives Z (9/2, –5/2)

Vertex A is the mid point of Z and S

$\Rightarrow$ $A\left(\dfrac{3+\frac{9}{2}}{2}, \dfrac{-4-\frac{5}{2}}{2}\right) = A\left(\dfrac{15}{4}, -\dfrac{13}{4}\right)$

length of latusrectum = $2 \times \left|\dfrac{3-4-2}{\sqrt{2}}\right| = 3\sqrt{2}$

Example 7 : The equation of the directrix of the parabola $x^2 - 4x - 3y + 10 = 0$ is :

(A) y = –5/4 (B) y = 5/4 (C) y = –3/4 (D) y = 4/5

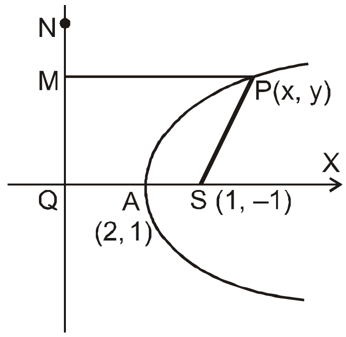
Solution : The given equation can be written as $(x-2)^2 = 3(y-2) \Rightarrow X^2 = 3Y$,

where x = X + 2, y = Y + 2. This directrix of this parabola with reference to new axes is

$Y = -a$, where $a = \dfrac{3}{4} \Rightarrow y - 2 = -\dfrac{3}{4} \Rightarrow y = 5/4$

Example 8 : Find the equation to the parabola whose focus is (1, –1) and vertex is (2, 1).

Sol. In this problem, equation of the directrix is not given.
Therefore, first of all we will determine the equation of the directrix.
Vertex A is (2, 1) and focus S is (1, –1).
Let the axis meet the directrix in Q.
Let the co-ordinates of Q be (α, β)
Since A is the mid-point of SQ



$\therefore \quad \frac{\alpha+1}{2} = 2 \qquad \therefore \quad \alpha = 3$

and $\quad \frac{\beta-1}{2} = 1 \qquad \therefore \quad \beta = 3$

$\therefore \quad Q \equiv (3, 3)$

Slope of axis QS = $\frac{1-(-1)}{2-1} = 2$

$\therefore$ Slope of directrix which is perpendicular to axis = $-\frac{1}{2}$

Directrix is line though Q(3, 3) which slope $-\frac{1}{2}$

$\therefore$ Equation of directrix is $y - 3 = -\frac{1}{2}(x - 3)$ or $x + 2y - 9 = 0$

Let P(x, y) be any point on the parabola. Draw PM $\perp$ QN. Then by definition of parabola PS = PM.

$\therefore \quad \sqrt{(x-1)^2+(y+1)^2} = \frac{|x+2y-9|}{\sqrt{(1)^2+(2)^2}} = \frac{|x+2y-9|}{\sqrt{5}}$

Squarring, we get $5[(x - 1)^2 + (y + 1)^2] = (x + 2y - 9)^2$ or $4x^2 - 4xy + y^2 + 8x + 46y - 71 = 0$.
This is the required equation of the parabola.

Example 9 : Find the equation of the parabola the extremities of whose latus rectum are (1, 2) and (1, –4).

Solution : Let $L \equiv (1, 2)$ and $L' \equiv (1, -4)$

Slope of L′L = $\frac{2+4}{1-1}$ (undefined)

Hence latus rectum L′L is perpendicular to x-axis. Therefore, axis of the parabola will be parallel to x-axis and tangent at the vertex will be parallel to y-axis.

Let the equation of the parabola be $(y - \beta)^2 = 4a(x - \alpha)$ ......(1)

Now $\quad L'L = |4a| \quad \therefore \quad 6 = |4a| \Rightarrow \quad 4a = \pm 6$

Hence from (1), equation of parabola becomes

$(y - \beta)^2 = \pm 6(x - \alpha)$ ......(2)

Since L and L′ lie on (2), therefore

$(2 - \beta)^2 = \pm 6(1 - \alpha)$ ......(3)

and $\quad (4 + \beta)^2 = \pm 6(1 - \alpha)$ ......(4)

From (3) and (4), $(2 - \beta)^2 = (4 + \beta)^2$

$\Rightarrow \quad 12\beta = -12 \quad \Rightarrow \quad \beta = -1$

$\therefore$ from (3), $9 = \pm 6(1 - \alpha) \quad \Rightarrow \quad \alpha = -\frac{1}{2}, \frac{5}{2}$

From (2), required parabolas are

$(y + 1)^2 = 6\left(x+\frac{1}{2}\right)$ or $(y + 1)^2 = 3(2x + 1)$

and $\quad (y + 1)^2 = -6\left(x-\frac{5}{2}\right)$ or $(y + 1)^2 = -3(2x - 5)$

Problems for Self Practice-1 :

(1) Name the conic represented by the equation $\sqrt{ax}+\sqrt{by}=1$, where a, b $\in$ R, a, b, > 0.

(2) Find the equation of the parabola whose focus is the point (0, 0)and whose directrix is the straight line 3x – 4y + 2 = 0.

(3) Find the vertex, axis, directrix, focus, latus rectum and the tangent at vertex for the parabola $9y^2 - 16x - 12y - 57 = 0$.

(4) Find the equation of the parabola whose focus is (1, –1) and whose vertex is (2, 1). Also find its axis and latus rectum.

(5) Find the equation of the parabola whose latus rectum is 4 units, axis is the line 3x + 4y = 4 and the tangent at the vertex is the line 4x – 3y + 7 = 0.

Answers :

(1) Parabola

(2) $16x^2 + 9y^2 + 24xy - 12x + 16y - 4 = 0$

(3) axis : y = $\frac{2}{3}$, directrix : $x=-\frac{613}{144}$, focus : $\left(-\frac{485}{144},\frac{2}{3}\right)$,

Length of the latus rectum = $\frac{16}{9}$, tangent at the vertex : $x=-\frac{61}{16}$.

(4) $4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$; Axis : 2x – y = 3; LR = $4\sqrt{5}$ unit

(5) $(3x + 4y - 4)^2 = 20(4x - 3y + 7)$

2.3 PARAMETRIC EQUATION OF PARABOLA :

The parametric equation Parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$

Hence any point on this parabola is $(at^2, 2at)$ which is called as 't' point.

Note :

(i) Any point on Parabola $y^2 = 4ax$ may also be written as $(a/t^2, 2a/t)$

(i) Parametric equations of the parabola $(y - h)^2 = 4a(x - k)^2$ is $x - k = at^2$ and $y - h = 2at$

Parametric form for : $y^2 = -4ax$ $(-at^2, 2at)$

$x^2 = 4ay$ $(2at, at^2)$

$x^2 = -4ay$ $(2at, -at^2)$

SOLVED EXAMPLE

Example 10 : The parametric equation of the parabola $y^2 = 8x$ are-

(A) $x = 2t, y = 4t^2$ (B) $x = 2t^2, y = 4t$ (C) $x = t^2, y = 2t$ (D) None of these

Solution : Here a = 2; y = 2at $\Rightarrow$ y = 2.2 . t = 4t

$x = at^2 \Rightarrow x = 2t^2$

Example 11 The parametric equation of the parabola $(y - 2)^2 = 12(x - 4)$ are-

(A) $6t, 3t^2$ (B) $2 + 3t, 4 + t^2$ (C) $4 + 3t^2, 2 + 6t$ (D) None of these

Solution : Here a = 3

$y - 2 = 2at \Rightarrow y = 2 + 2.3t = 2 + 6t$

$x - 4 = at^2 \Rightarrow x = 4 + 3.t^2 = 4 + 3t^2$

Problems for Self Practice-2 :

(1) Find the parametric equation of the parabola $(x - 1)^2 = -12(y - 2)$

(2) Parameter t of a point (4, –6) of the parabola $y^2 = 9x$ is-

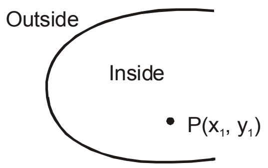
(A) 4/3 (B) –4/3 (C) –3/4 (D) –4/5

Answers :

(1) $x = 1 - 6t,\ y = 2 - 3t^2$ (2) (B)

2.4 Position of a point relative to a parabola:

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.



$S_1 : y_1^2 - 4ax_1$

$S_1 < 0 \rightarrow$ Inside

$S_1 > 0 \rightarrow$ Outside

SOLVED EXAMPLE

Example 12: Check whether the point (3, 4) lies inside or outside the parabola $y^2 = 4x$.

Solution : $y^2 - 4x = 0$

$\because$ $S_1 \equiv y_1^2 - 4x_1 = 16 - 12 = 4 > 0$

$\therefore$ (3, 4) lies outside the parabola.

Example 13 : Find the value of α for which point $(\alpha, 2\alpha + 1)$ doesn't lie outside the parabola $y = x^2 + x + 1$.

Solution : $x^2 + x + 1 - y = 0$

$(\alpha, 2\alpha + 1)$

$\alpha^2 + \alpha + 1 - (2\alpha + 1) \leq 0$

$\Rightarrow \alpha^2 - \alpha \leq 0 \Rightarrow \alpha(\alpha - 1) \leq 0 \Rightarrow \alpha \in [0, 1]$

Problems for Self Practice - 3 :

(1) Find the set of value's of α for which $(\alpha, -2-\alpha)$ lies inside the parabola $y^2 + 4x = 0$.

(2) Find the value of 'a' for which the point $(a^2 - 1, a)$ lies inside the parabola $y^2 = 8x$.

Answer :

(1) $\alpha \in (-4 - 2\sqrt{3}, -4 + 2\sqrt{3})$

(2) $\left(-\infty, -\sqrt{\frac{8}{7}}\right) \cup \left(\sqrt{\frac{8}{7}}, \infty\right)$

3. ELEMENTARY CONCEPTS OF ELLIPSE :

3.1 Definitions :

It is locus of a point which moves in such a way that the ratio of its distance from a fixed point and a fixed line (not passes through fixed point and all points and line lies in same plane) is constant (e), which is less than one.

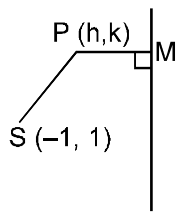
$$\frac{PS}{PM} = e\ (0 < e < 1)$$

Note : The general equation of a conic with focus (p, q) & directrix $\ell x + my + n = 0$ is:
$(\ell^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2(\ell x + my + n)^2$
$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
represent ellipse if $0 < e < 1$; $\Delta \neq 0$, $h^2 < ab$

SOLVED EXAMPLE

Example 14 : Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line $x - y + 3 = 0$ and eccentricity is $\frac{1}{2}$.

Solution : Let $P \equiv (h, k)$ be moving point,



$$e = \frac{PS}{PM} = \frac{1}{2}$$

$\Rightarrow$ $(h + 1)^2 + (k - 1)^2 = \frac{1}{4}\left(\frac{h - k + 3}{\sqrt{2}}\right)^2$

$\Rightarrow$ locus of P(h, k) is
$8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$
$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0.$

Problems for Self Practice-4 :

(1) Find the equation to the ellipse whose focus is (0, 0) directrix is $x + y - 1 = 0$ and $e = \frac{1}{\sqrt{2}}$.

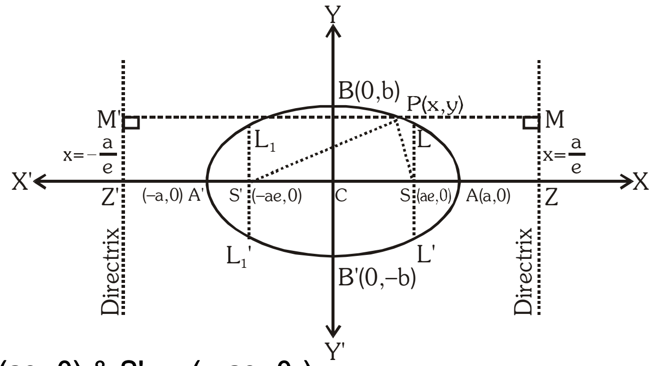
Answer :

(1) $3x^2 + 3y^2 - 2xy + 2x + 2y - 1 = 0.$

3.2 STANDARD EQUATION & DEFINITION :

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. where

a > b & b^2 = a^2 (1 – e^2) $\Rightarrow$ $a^2 - b^2 = a^2 e^2$, where e = eccentricity **(0 < e < 1)**.



(a) Foci : **S ≡ (ae, 0) & S' ≡ (– ae, 0).**

(b) **Equation of directrices : x =** $\frac{a}{e}$ **&** $x = -\frac{a}{e}$.

(c) **Vertices : A' ≡ (–a, 0) & A ≡ (a, 0).**

(d) **Major axis :** The line segment A' A in which the foci S' & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (z)** $\left(\pm\frac{a}{e}, 0\right)$.

(e) **Minor Axis :** The y-axis intersects the ellipse in the points B' ≡ (0,- b) & B ≡ (0, b). The line segment B'B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

(f) **Principal Axes :** The major & minor axis together are called **Principal Axes** of the ellipse.

(g) **Centre :** The point which bisects every chord of the conic drawn through it is called the **centre** of the conic. C ≡ (0,0) the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(h) **Focal Chord :** A chord which passes through a focus is called a **focal chord.**

(i) **Double Ordinate :** A chord perpendicular to the major axis is called a **double ordinate.**

(j) **Latus Rectum :** The focal chord perpendicular to the major axis is called the **latus rectum**.

(i) Length of latus rectum (LL') = $\frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

(ii) Equation of latus rectum : **x = ± ae**.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right), L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

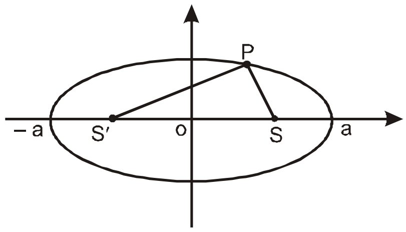
(k) **Focal radii : SP = a –ex & S'P = a + ex** $\Rightarrow$ **SP + S 'P = 2a = Major axis.**

(l) **Eccentricity :** $e = \sqrt{1 - \frac{b^2}{a^2}}$

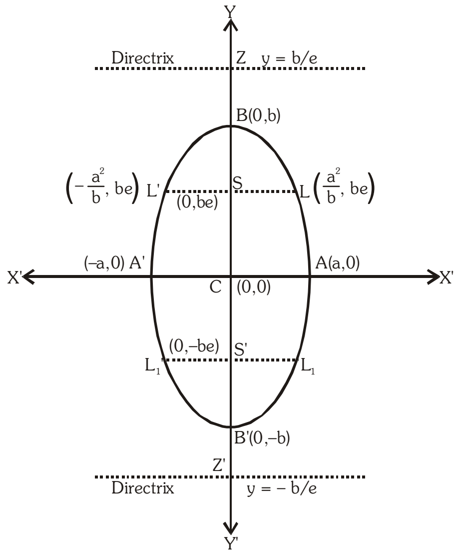
Note :

(i) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned, then the rule is to assume that a > b.

(ii) If P be any point on the ellipse with S & S′ as its foci then ℓ(SP) + ℓ(S′P) = 2a.



3.2.1 ANOTHER FORM OF ELLIPSE : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, **(a <b)**



(a) **AA' = Minor axis = 2a (x-axis)**

(b) **BB' = Major axis = 2b (y-axis)**

(c) **$a^2 = b^2(1 - e^2)$**

(d) **Latus rectum LL' = L_1L_1'** $=\frac{2a^2}{b}$, **equation y = ± be**

(e) **Ends of the latus rectum are :**

$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) **Equation of directrix y = ± b/e**

(g) **Eccentricity** : $e=\sqrt{1-\frac{a^2}{b^2}}$

SOLVED EXAMPLE

Example 15 : Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus - rectum and eccentricity of an ellipse $\frac{(x-3)^2}{25}+\frac{(y-2)^2}{16}=1$.

Solution : Let x – 3 = X, y – 2 = Y, so equation of ellipse becomes as $\frac{X^2}{5^2}+\frac{Y^2}{4^2}=1$.

equation of major axis is Y = 0 ⇒ y = 2.
equation of minor axis is X = 0 ⇒ x = 3.
centre (X = 0, Y = 0) ⇒ x = 3, y = 2
C ≡ (3, 2)

Length of semi-major axis a = 5
Length of major axis 2a = 10

Length of semi-minor axis b = 4
Length of minor axis = 2b = 8.
Let 'e' be eccentricity

$\therefore \quad b^2 = a^2 (1 - e^2)$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25-16}{25}} = \frac{3}{5}.$$

Length of latus rectum = $LL' = \frac{2b^2}{a} = \frac{2 \times 16}{5} = \frac{32}{5}$

Co-ordinates focii are X = ± ae, Y = 0

$\Rightarrow \quad S \equiv (X = 3, Y = 0) \quad \& \quad S' \equiv (X = -3, Y = 0)$

$\Rightarrow \quad S \equiv (6, 2) \quad \& \quad S' \equiv (0, 2)$

Example 16 : Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi minor axis is of length $\sqrt{5}$.

Solution : Here S is (2, 3) & S' is (–2, 3) and b = $\sqrt{5} \Rightarrow$ SS' = 4 = 2ae $\Rightarrow$ ae = 2

but $b^2 = a^2 (1 - e^2) \Rightarrow 5 = a^2 - 4 \Rightarrow a = 3.$

Hence the equation to major axis is y = 3

Centre of ellipse is midpoint of SS' i.e. (0, 3)

$\therefore$ Equation to ellipse is $\frac{x^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$ or $\frac{x^2}{9} + \frac{(y-3)^2}{5} = 1$

Example 17 : Find the equation of the ellipse having centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).

Solution : With centre at (1, 2), the equation of the ellipse is $\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$. It passes through the point (4, 6)

$\Rightarrow \quad \frac{9}{a^2} + \frac{16}{b^2} = 1 \qquad$ .... (i)

Distance between the focus and the centre = (6 – 1) = 5 = ae

$\Rightarrow \quad b^2 = a^2 - a^2e^2 = a^2 - 25 \qquad$ .... (ii)

Solving for a^2 and b^2 from the equations (i) and (ii), we get $a^2 = 45$ and $b^2 = 20$.

Hence the equation of the ellipse is $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$

Example 18 : If LR of an ellipse is half of its minor axis, then its eccentricity is -

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{2}}{3}$

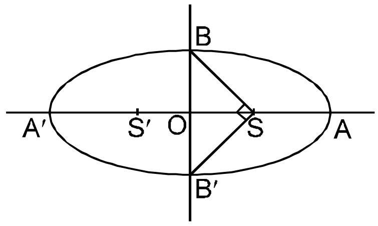
Solution : As given $\frac{2b^2}{a} = b \Rightarrow 2b = a \Rightarrow 4b^2 = a^2$

$\Rightarrow \quad 4a^2(1 - e^2) = a^2 \Rightarrow 1 - e^2 = 1/4$

$\therefore \quad e = \sqrt{3}/2$

Example 19 : If minor-axis of ellipse subtend a right angle at its focus then find the eccentricity of ellipse.

Solution : Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)



$\because$ $\angle BSB' = \frac{\pi}{2}$

and OB = OB′

$\therefore$ $\angle BSO = \frac{\pi}{4}$

$\Rightarrow$ OS = OB $\Rightarrow$ ae = b

$\Rightarrow$ $e^2 = \frac{b^2}{a^2} = 1 - e^2$ $\Rightarrow$ $e = \frac{1}{\sqrt{2}}$

Problems for Self Practice-5 :

(1) Find the eccentricity, foci and the length of the latus-rectum of the ellipse $x^2 + 4y^2 + 8y - 2x + 1 = 0$.

(2) Find the equation of the ellipse whose foci are (4, 6) & (16, 6) and whose semi-minor axis is 4.

(3) The foci of an ellipse are (0, ±2) and its eccentricity is $\frac{1}{\sqrt{2}}$. Find its equation

(4) If LR of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a < b) is half of its major axis, then find its eccentricity.

(5) The equation $\frac{x^2}{8-t} + \frac{y^2}{t-4} = 1$, will represent an ellipse if

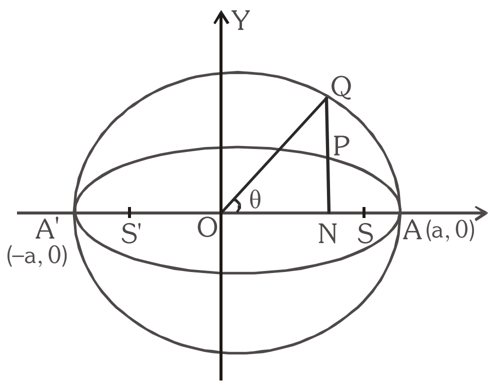
(A) $t \in (1, 5)$ (B) $t \in (2, 8)$ (C) $t \in (4, 8) - \{6\}$ (D) $t \in (4, 10) - \{6\}$

Answers :

(1) $e = \frac{\sqrt{3}}{2}$; foci $= (1 \pm \sqrt{3}, -1)$; LR $= 1$ (2) $\frac{(x-10)^2}{52} + \frac{(y-6)^2}{16} = 1$

(3) $e = \frac{1}{\sqrt{2}}$ (4) $\frac{x^2}{4} + \frac{y^2}{8} = 1$ (5) (C)

3.3 AUXILLIARY CIRCLE/ECCENTRIC ANGLE :



A circle described on major axis as diameter is called the **auxiliary circle.** Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively.'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse ($0 \le \theta < 2\pi$).

Note that $\frac{l(PN)}{l(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

3.4 PARAMETRIC REPRESENTATION :

The equations x = a cosθ & y = b sinθ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where θ is a parameter (eccentric angle).

Note that if P(θ) $\equiv$ (a cosθ, b sinθ) is on the ellipse then ; **Q(θ) $\equiv$ (a cosθ, a sinθ)** is on the auxiliary circle.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

SOLVED EXAMPLE

Example 20 : Find the focal distance of a point P(θ) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b)

Solution : Let 'e' be the eccentricity of ellipse.

$\therefore$ PS = e . PM

$= e\left(\frac{a}{e}-a\cos\theta\right)$

PS = (a – a e cosθ)

and PS′ = e. PM′

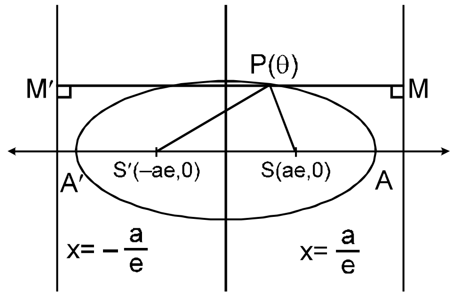
$= e\left(a\cos\theta+\frac{a}{e}\right)$

PS′ = a + ae cosθ

$\therefore$ focal distance are (a ± ae cosθ)

Note : PS + PS′ = 2a

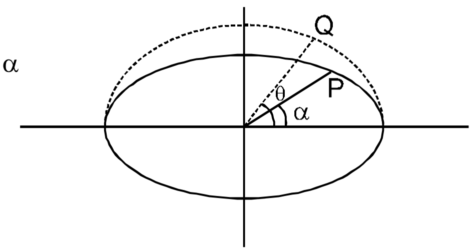
PS + PS′ = AA′



Example 21 : Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose radius makes angle α with x-axis.

Solution : Let P $\equiv$ (a cosθ, b sinθ)

$\therefore$ $m_{(op)}=\frac{b}{a}\tan\theta=\tan\alpha \Rightarrow \tan\theta=\frac{a}{b}\tan\alpha$



$OP=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}=\sqrt{\frac{a^2+b^2\tan^2\theta}{\sec^2\theta}}$

$=\sqrt{\frac{a^2+b^2\tan^2\theta}{1+\tan^2\theta}}=\sqrt{\frac{a^2+b^2\times\frac{a^2}{b^2}\tan^2\alpha}{1+\frac{a^2}{b^2}\tan^2\alpha}} \Rightarrow OP=\frac{ab}{\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}}$

Example 22 : Write the equation of chord of an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ joining two points P $\left(\frac{\pi}{4}\right)$ and Q $\left(\frac{5\pi}{4}\right)$.

Solution : Equation of chord is $\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} + \frac{y}{4}\cdot\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} = \cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$

$$\frac{x}{5}\cdot\cos\left(\frac{3\pi}{4}\right) + \frac{y}{4}\cdot\sin\left(\frac{3\pi}{4}\right) = 0$$

$$-\frac{x}{5} + \frac{y}{4} = 0 \quad \Rightarrow \quad 4x = 5y$$

Example 23 : If α, β are eccentric angles of end points of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then tan $\alpha/2$. tan $\beta/2$ is equal to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{e+1}{e-1}$ (D) $\frac{e-1}{e+1}$

Solution : Equation of line joining points 'α' and 'β' is $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

If it is a focal chord, then it passes through focus (ae, 0), so e cos $\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

$$\Rightarrow \frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}=\frac{e}{1} \quad \Rightarrow \quad \frac{\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}}=\frac{e-1}{e+1}$$

$$\Rightarrow \frac{2\sin\alpha/2\ \sin\beta/2}{2\cos\alpha/2\ \cos\beta/2}=\frac{e-1}{e+1} \quad \Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

using (–ae, 0) , we get $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e+1}{e-1}$

<u>Problems for Self Practice</u>-6 :

(1) Find the eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ whose distance from the centre is 2.

(2) Show that the area of triangle inscribed in an ellipse bears a constant ratio to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the first triangle.

(3) Find the locus of the foot of the perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the chord joining two points whose eccentric angles differ by $\frac{\pi}{2}$.

Answers :

(1) $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ (3) $2(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2$.

3.5 POSITION OF A POINT W.R.T. AN ELLIPSE :

The point **P(x_1, y_1)** lies **outside**, **inside** or **on** the ellipse according as ; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ **> < or = 0**.

Example 24 : Check whether the point P(3, 2) lies inside or outside of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

Solution : $S_1 \equiv \frac{9}{25}+\frac{4}{16}-1=\frac{9}{25}+\frac{1}{4}-1<0$

$\therefore$ Point P $\equiv$ (3, 2) lies inside the ellipse.

Example 25 : Find the set of those value(s) of 'α' for which the point $\left(7-\frac{5}{4}\alpha, \alpha\right)$ lies inside the ellipse

$\frac{x^2}{25}+\frac{y^2}{16}=1$.

Solution : For any point to lies inside the ellipse, $S_1 < 0$

$\Rightarrow \frac{\left(7-\frac{5}{4}\alpha\right)^2}{25}+\frac{\alpha^2}{16}-1<0 \Rightarrow \frac{(28-5\alpha)^2}{400}+\frac{\alpha^2}{16}-1<0$

$\Rightarrow (28-5\alpha)^2+25\alpha^2-400<0$

$\Rightarrow 50\alpha^2-280\alpha-384<0 \Rightarrow 25\alpha^2-140\alpha-192<0 \Rightarrow (5\alpha-12)(5\alpha-16)<0$

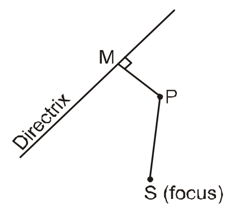
$\Rightarrow \frac{12}{5}<\alpha<\frac{16}{5}$

4. ELEMENTARY CONCEPTS OF HYPERBOLA :

Hyperbolic curves are of special importance in the field of science and technology especially astronomy and space studies. In this chapter we are going to study the characteristics of such curves.

4.1 Definition :

A hyperbola is defined as the locus of a point moving in a plane in such a way that the ratio of its distance from a fixed point to that from a fixed line (the point does not lie on the line) is a fixed constant greater than 1.

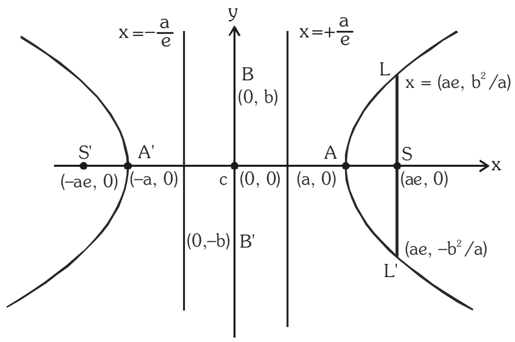


$\frac{PS}{PM}=e>1$, e – eccentricity

4.2 STANDARD EQUATION & DEFINITION(S) :

Standard equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(e^2 - 1)$$



or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(a) **Foci :**

S ≡ (ae, 0) & S′ ≡ (− ae, 0).

(b) **Equations of directrices :**

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(c) **Vertices :**

A ≡ (a , 0) & A′ ≡ (− a , 0).

(d) **Latus rectum :**

(i) Equation : **x = ± ae**

(ii) Length $= \frac{2b^2}{a} = \frac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})}$ **= 2a (e² − 1) = 2e (distance from focus to directrix)**

(iii) Ends : $\left(ae, \frac{b^2}{a}\right), \left(ae, \frac{-b^2}{a}\right)$; $\left(-ae, \frac{b^2}{a}\right), \left(-ae, \frac{-b^2}{a}\right)$

(e) (i) **Transverse Axis :**

The line segment A′A of length 2a in which the foci S′ & S both lie is called the **Transverse Axis of the Hyperbola.**

(ii) **Conjugate Axis :**

The line segment B′B between the two points B′ ≡ (0, − b) & B ≡ (0, b) is called as the **Conjugate Axis of the Hyperbola.**

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the **Principal axes of the hyperbola.**

(f) **Focal Property :**

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left| |PS| - |PS'| \right| = 2a$. The distance SS′ = focal length.

(g) **Focal distance :**

Distance of any point P(x, y) on Hyperbola from foci **PS = ex – a & PS′ = ex + a.**

General Note :

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having **–b²** instead of **b²** it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of **b²** .

4.3 CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **Conjugate Hyperbolas** of each other. eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.

Note that :

(i) If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(ii) The foci of a **hyperbola** and its **conjugate** are **concyclic and form the vertices of a square.**

(iii) Two hyperbolas are said to be **similar** if they have the **same eccentricity**.

(iv) Two similar hyperbolas are said to be equal if they have same latus rectum.

4.4 RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and **the length of it's latus rectum is equal to it's transverse or conjugate axis.**

SOLVED EXAMPLE

Example 26 : Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Solution : Let $P(x, y)$ be any point on the hyperbola and PM is perpendicular from P on the directrix.

Then by definition SP = e PM

$\Rightarrow \quad (SP)^2 = e^2 (PM)^2 \quad \Rightarrow \quad (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow \quad 5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Example 27 : Find the centre, eccentricity, foci and directrices of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

Solution : Here $16x^2 + 32x + 16 - (9y^2 - 36y + 36) - 144 = 0$ or $16(x + 1)^2 - 9(y - 2)^2 = 144$

$\therefore \quad \frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$

Putting $x + 1 = X$ and $y - 2 = Y$, the equation becomes $\frac{X^2}{9} - \frac{Y^2}{16} = 1$ which is in the standard form.

Here $a^2 = 9$ and $b^2 = 16$

$\because \quad b^2 = a^2(e^2 - 1)$, we get $16 = 9(e^2 - 1)$

$\therefore \quad e^2 - 1 = \frac{16}{9}$

$\therefore \quad e^2 = \frac{25}{9}$, i.e. $e = \frac{5}{3}$

Now, centre = $(0, 0)_{X, Y} = (-1, 2)$

{$\because$ when X = 0, x + 1 = X gives x = –1 and when Y = 0, y – 2 = Y gives y = 2}

foci = $(\pm ae, 0)_{X, Y} = \left(\pm 3 \cdot \frac{5}{3}, 0\right)_{X, Y} = (\pm 5, 0)_{X, Y} = (-1 \pm 5, 2) = (4, 2), (-6, 2)$

Directrices in X, Y coordinates have the equations

$$X \pm \frac{a}{e} = 0 \text{ or } x + 1 \pm \frac{3}{(5/3)} = 0 \text{ i.e. } x + 1 \pm \frac{9}{5} = 0$$

$\therefore \quad x = -\frac{14}{5}$ and $x = \frac{4}{5}$

Example 28 : Find the lengths of transverse axis and conjugate axis, eccentricity, the co-ordinates of foci, vertices, length of the latus-rectum and equations of the directrices of the following hyperbola $16x^2 - 9y^2 = -144$.

Solution : The equation $16x^2 - 9y^2 = -144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = -1$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore \quad a^2 = 9, b^2 = 16 \quad \Rightarrow \quad a = 3, b = 4$

Length of transverse axis : The length of transverse axis = 2b = 8

Length of conjugate axis : The length of conjugate axis = 2a = 6

Eccentricity : $e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$

Foci : The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$

Vertices : The co–ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$

Length of latus–rectum : The length of latus–rectum = $\frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$

Equation of directrices : The equation of directrices are

$$y = \pm \frac{b}{e} \quad \Rightarrow \quad y = \pm \frac{4}{(5/4)} \quad \Rightarrow \quad y = \pm \frac{16}{5}$$

Example 29 : Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.

Solution : Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then transverse axis = 2a and latus–rectum = $\frac{2b^2}{a}$. According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow \quad 2b^2 = a^2 \quad (\because \quad b^2 = a^2(e^2 - 1))$

$\Rightarrow \quad 2a^2(e^2 - 1) = a^2 \quad \Rightarrow \quad 2e^2 - 2 = 1 \quad \Rightarrow \quad e^2 = \frac{3}{2}$

$\therefore \quad e = \sqrt{\frac{3}{2}}$ Hence the required eccentricity is $\sqrt{\frac{3}{2}}$.

Example 30 : The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is-

(A) 2 (B) $2/\sqrt{3}$ (C) 4 (D) 4/3

Solution : Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

$$-x^2 + 3y^2 = 1 \quad \Rightarrow \quad -\frac{x^2}{1} + \frac{y^2}{1/3} = 1$$

Here $a^2 = 1$, $b^2 = 1/3$

$\therefore$ eccentricity $e = \sqrt{1 + a^2/b^2} = \sqrt{1+3} = 2$

Problems for Self Practice-7 :

(1) Find the equation to the hyperbola, whose eccentricity is $\frac{5}{4}$, focus is (a, 0) and whose directrix is $4x - 3y = a$.

(2) In the hyperbola $4x^2 - 9y^2 = 36$, find length of the axes, the co-ordinates of the foci, the eccentricity, and the latus rectum.

(3) Find the equation to the hyperbola, the distance between whose foci is 16 and whose eccentricity is $\sqrt{2}$.

(4) Find the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which passes through (4, 0) & $(3\sqrt{2}, 2)$

(5) The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of the hyperbola if its eccentricity is 2.

(6) Find eccentricity of conjugate hyperbola of hyperbola $4x^2 - 16y^2 = 64$, also find area of quadrilateral formed by foci of hyperbola & its conjugate hyperbola

Answers :

(1) $7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0$

(2) 6, 4; $(\pm\sqrt{13}, 0)$; $\sqrt{13}/3$; 8/3

(3) $x^2 - y^2 = 32$

(4) $\sqrt{3}$

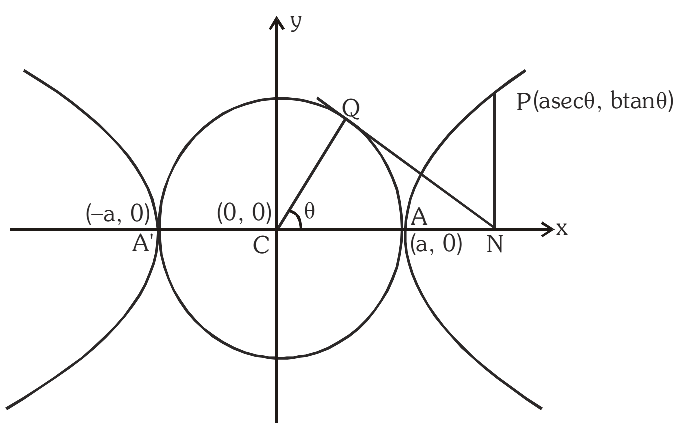
(5) $3x^2 - y^2 - 12 = 0$.

(6) $\sqrt{5}$ & 40 sq. units

4.5 AUXILIARY CIRCLE :

A circle drawn with centre C & T.A. as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola & the auxiliary circle. 'θ' is called the **eccentric angle** of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.



4.6 Parametric representation:

The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

The equation to the chord of the hyperbola joining the two points $P(\alpha)$ & $Q(\beta)$ is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}.$$

4.7 POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is **positive , zero** or **negative** according as the point (x_1, y_1) lies within , upon or outside the curve.

SOLVED EXAMPLE

Example 31 : Find the position of the point (2, 5) relative to the hyperbola $9x^2 - y^2 = 1$.

Solution : $\because$ $(9x^2 - y^2 - 1) > 0$ at (2,5)

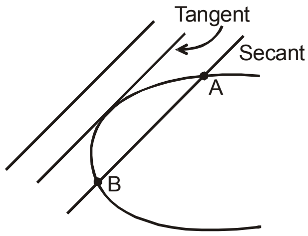
$\therefore$ point (2,5) lies inside.

5. LINE AND A PARABOLA :

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtreqless cm$ $\Rightarrow$ condition of tangency is, $c = a/m$.

Length of the chord intercepted by the parabola on the line $y = mx + c$ is :

$$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}.$$



Note : Length of the focal chord making an angle α with the x - axis is $4a\,\text{cosec}^2\,\alpha$.

5.1 CHORD JOINING TWO POINTS :

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is

$$y(t_1 + t_2) = 2x + 2at_1t_2$$

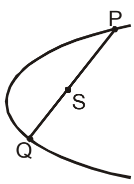
Note :

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(iii) length of focal chord at point P(t) to parabola $y^2 = 4ax$ is $a\left(t+\frac{1}{t}\right)^2$

(iv) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola.



$$\frac{2(PS)(SQ)}{PS+SQ} = 2a$$

SOLVED EXAMPLE

Example 32 : Discuss the position of line $y = x + 1$ with respect to parabola $y^2 = 4x$.

Solution : Solving we get $(x + 1)^2 = 4x \Rightarrow (x - 1)^2 = 0$

so $y = x + 1$ is tangent to the parabola.

Example 33 : If the endpoint t_1, t_2 of a chord satisfy the relation $t_1 t_2 = k$ (const.) then prove that the chord always passes through a fixed point. Find the point?

Solution : Equation of chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$(t_1 + t_2) y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$$

$$y = \frac{2}{t_1 + t_2}(x + ak) \qquad (\because t_1t_2 = k)$$

$\therefore$ This line passes through a fixed point $(-ak, 0)$.

Example 34 : If the line $2x - 3y = k$ touches the parabola $y^2 = 6x$, then the value of k is-

(A) 27/4 (B) –81/4 (C) –7 (D) –27/4

Solution : Given $x = \frac{3y+k}{2}$ ..... (1)

and $y^2 = 6x$ ..... (2)

$\Rightarrow$ $y^2 = 6\left(\frac{3y+k}{2}\right) \Rightarrow y^2 = 3(3y + k)$

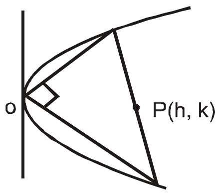
$\Rightarrow$ $y^2 - 9y - 3k = 0$ ..... (3)

If line (1) touches parabola (2) then roots of quadratic equation (3) is equal

$\therefore$ $(-9)^2 = 4 \times 1 \times (-3k) \Rightarrow k = -\frac{27}{4}$

Example 35 : Find the locus of the mid-points of the chords of the parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola.

Solution :



$$\frac{2at_1}{at_1} = \frac{2}{t_1}$$

$m_1m_2 = -1 \qquad \Rightarrow \qquad t_1t_2 = -4$

$2h = a(t_1^2 + t_2^2)$

$k = a(t_1 + t_2)$

eliminating t_1 & t_2

we get $y^2 - 2ax + 8a^2 = 0$

Alternative :

Equation of chord is $T = S_1$

$ky - 2a(x + h) = k^2 - 4ah$

$ky - 2ax + 2ah - k^2 = 0$

$\frac{ky - 2ax}{k^2 - 2ah} = 1$

$y^2 = 4ax$ (1)

$y^2 = 4ax\left(\frac{ky - 2ax}{k^2 - 2ah}\right)$

coeff. of x^2 + coeff of $y^2 = 0$

$1 + \frac{8a^2}{k^2 - 2ah} = 0$

$y^2 - 2ax + 8a^2 = 0$

Problems for Self Practice-8 :

(1) If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct point's then set of value's of 'λ' is

(2) If one end of focal chord of parabola $y^2 = 16x$ is (16, 16) then coordinate of other end is.

(3) Find the midpoint of the chord $x + y = 2$ of the parabola $y^2 = 4x$.

(4) If PSQ is focal chord of parabola $y^2 = 4ax$ $(a > 0)$, where S is focus then prove that

$\frac{1}{PS} + \frac{1}{SQ} = \frac{1}{a}$.

Answers : (1) $(-\infty, 1/3)$ (2) $(1, -4)$ (3) $(4, -2)$

5.2 TANGENT TO THE PARABOLA $y^2 = 4ax$:

5.2.1 Point form :

Equation of tangent to the given parabola at its point (x_1, y_1) is

$$yy_1 = 2a(x + x_1)$$

5.2.2 Slope form :

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

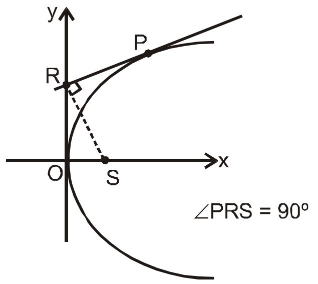
5.2.3 Parametric form :

Equation of tangent to the given parabola at its point P(t), is $ty = x + at^2$

Note :

(i) Point of intersection of the tangents at the point t_1 & t_2 is $[at_1t_2, a(t_1 + t_2)]$.

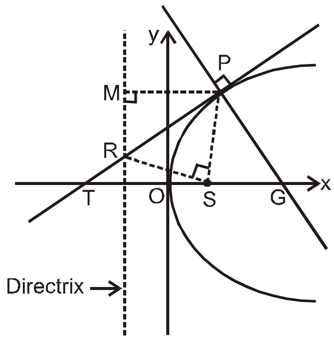
(ii) Perpendicular drawn from focus upon any tangent of a parabola lies on the tangent at the vertex.



(iii) Image of focus in any tangent to parabola lies on its directrix.

(iv) The area of triangle formed by three tangents to the parabola $y^2 = 4ax$ is half the area of triangle formed by their points of contacts.

(v) The portion of a tangent to a parabola cut off between the directrix and the parabola subtends a right angle at the focus.



(vi) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix.

SOLVED EXAMPLE

Example 36 : Find the equation of that tangent to the parabola $y^2 = 7x$ which is parallel to the straight line $4y - x + 3 = 0$. Find also its point of contact.

Solution : $y = mx + \frac{a}{m}$

$y = \frac{1}{4}x + \frac{7.4}{4.1}$

$y = \frac{x}{4} + 7 \Rightarrow x - 4y + 28 = 0$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = (28, 14)$

Example 37 : Find the equation of the tangents to the parabola $y^2 = 9x$ which go through the point (4, 10).

Solution : Equation of tangent to parabola $y^2 = 9x$ is

$y = mx + \frac{9}{4m}$

Since it passes through (4, 10)

$\therefore \quad 10 = 4m + \dfrac{9}{4m} \Rightarrow \quad 16m^2 - 40\,m + 9 = 0$

$m = \dfrac{1}{4}, \dfrac{9}{4}$

$\therefore$ equation of tangent's are $y = \dfrac{x}{4} + 9$ & $y = \dfrac{9}{4}x + 1$

Example 38 : Find the equations to the common tangents of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Solution : Equation of tangent to $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \qquad \text{........(i)}$$

Equation of tangent to $x^2 = 4by$ is

$$x = m_1y + \frac{b}{m_1} \quad \Rightarrow \quad y = \frac{1}{m_1}x - \frac{b}{(m_1)^2} \qquad \text{........(ii)}$$

for common tangent, (i) & (ii) must represent same line.

$$\therefore \quad \frac{1}{m_1} = m \quad \& \quad \frac{a}{m} = -\frac{b}{m_1^2}$$

$$\Rightarrow \quad \frac{a}{m} = -bm^2 \quad \Rightarrow \quad m = \left(-\frac{a}{b}\right)^{1/3}$$

$\therefore$ equation of common tangent is $y = \left(-\dfrac{a}{b}\right)^{1/3} x + a\left(-\dfrac{b}{a}\right)^{1/3}$.

Problems for Self Practice-9 :

(1) Find the equation of the tangent to the parabola $y^2 = 12x$, which passes through the point (2, 5). Find also the co-ordinates of their points of contact.

(2) Find the equation of the tangents to the parabola $y^2 = 16x$, which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$. Find also the co-ordinates of their points of contact.

(3) Prove that the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle θ is $(x + a)^2 \tan^2\theta = y^2 - 4ax$.

(4) If the tangents to parabola $y^2 = 4ax$ at P and Q meet in T, then prove that :

(i) TP and TQ subtend equal angles at the focus S.

(ii) $ST^2 = SP.\,SQ$

Answers : (1) $x - y + 3 = 0$, (3, 6); $3x - 2y + 4 = 0$, $\left(\dfrac{4}{3}, 4\right)$

(2) $2x - y + 2 = 0$, (1, 4) ; $x + 2y + 16 = 0$, (16, –16)

6. LINE AND AN ELLIPSE :

The line y = mx + c meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two points real, coincident or imaginary according as **c^2 is < = or > $a^2m^2 + b^2$**.

Hence y = mx + c is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if **$c^2 = a^2m^2 + b^2$**.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

SOLVED EXAMPLE

Example 39 : Find the set of value(s) of 'λ' for which the line 3x – 4y + λ = 0 intersect the ellipse $\frac{x^2}{16}$ + $\frac{y^2}{9}$ = 1 at two distinct points.

Solution : Solving given line with ellipse, we get $\frac{(4y-\lambda)^2}{9\times16}$ + $\frac{y^2}{9}$ = 1

$\frac{2y^2}{9}$ – $\frac{y\lambda}{18}$ + $\frac{\lambda^2}{144}$ –1 = 0

Since, line intersect the parabola at two distinct points,

$\therefore$ roots of above equation are real & distinct

$\therefore$ D > 0

$\Rightarrow$ $\frac{\lambda^2}{(18)^2}$ – $\frac{8}{9}$. $\left(\frac{\lambda^2}{144}-1\right)$ > 0 $\Rightarrow$ $-12\sqrt{2} < \lambda < 12\sqrt{2}$

Example 40 : If the line y = 2x + c be a tangent to the ellipse $\frac{x^2}{8}$ + $\frac{y^2}{4}$ = 1, then c is equal to

(A) ± 4 (B*) ± 6 (C) ± 1 (D) ± 8

Solution : C = $\pm\sqrt{a^2m^2+b^2}$

C = $\pm\sqrt{8\times4+4}$ = ± 6

<u>Problems for Self Practice</u>-10 :

(1) Find the value of 'λ' for which 2x – y + λ = 0 touches the ellipse $\frac{x^2}{25}$ + $\frac{y^2}{9}$ = 1

(2) Find the condition for the line x cosθ + ysinθ = P to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Answers : (1) $\lambda = \pm\sqrt{109}$ (2) $P^2 = a^2\cos^2\theta + b^2\sin^2\theta$

6.1 TANGENT TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

6.1.1 Point form : Equation of tangent to the given ellipse at its point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

6.1.2 Slope form : Equation of tangent to the given ellipse whose slope is 'm', is **y = mx ±** $\sqrt{a^2m^2+b^2}$

Point of contact are $\left(\frac{\mp a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2+b^2}}\right)$

Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

6.1.3 Parametric form : Equation of tangent to the given ellipse at its point (a cos θ , b sin θ), is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Note :

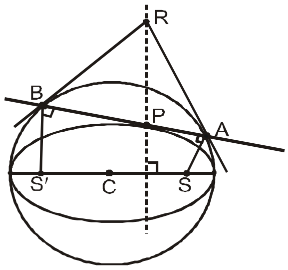
(i) The eccentric angles of point of contact of two parallel tangents differ by π.

(ii) Point of intersection of the tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(iii) Tangents at the extremities of any focal chord of an ellipse meet at directrix corresponding to the focus.

(iv) The portion of the tangent to an ellipse intercepted between the ellipse and the directrix subtends a right angle at the corresponding focus.

(v) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similiar ellipse as that of the original one.



SOLVED EXAMPLE

Example 41 : Find the equation of tangents to the ellipse $\frac{x^2}{50}+\frac{y^2}{32}=1$ which passes through a point (15, – 4).

Solution : y + 4 = m (x – 15)

y = mx – (15m + 4) .... (i)

for tangent

$(15m + 4)^2 = 50m^2 + 32$

$175m^2 + 120m - 16 = 0$

$(35m - 4)(5m + 4) = 0$

$m = \frac{4}{35}, \frac{-4}{5}$

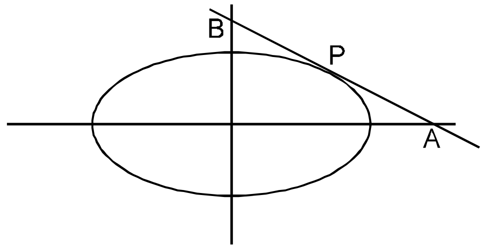
putting in (i)

$4x + 5y = 40$

$4x - 35y = 200$

Example 42 : A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant and meets the co-ordinate axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent.

Solution : Let $P \equiv (a\cos\theta, b\sin\theta)$



$\therefore$ equation of tangent is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$A \equiv (a\sec\theta, 0)$

$B \equiv (0, b\,\text{cosec}\theta)$

$\because$ P divide AB internally in the ratio 3 : 1

$\therefore \quad a\cos\theta = \frac{a\sec\theta}{4} \quad \Rightarrow \quad \cos^2\theta = \frac{1}{4} \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$

and $\quad b\sin\theta = \frac{3b\,\text{cosec}\theta}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ tangent is $\frac{x}{2a} + \frac{\sqrt{3}y}{2b} = 1 \quad \Rightarrow \quad bx + \sqrt{3}\,ay = 2ab$

Example 43 : Find the locus of foot of perpendicular drawn from centre to any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

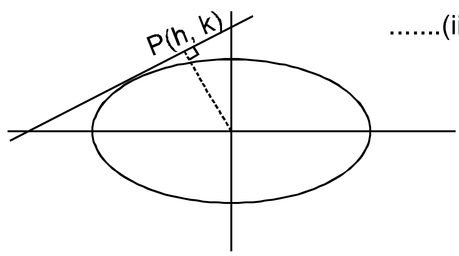
Solution : Let P(h, k) be the foot of perpendicular to a tangent $y = mx + \sqrt{a^2m^2 + b^2}$ ....(i)

from centre

$\therefore \quad \frac{k}{h} \cdot m = -1 \quad \Rightarrow \quad m = -\frac{h}{k}$ ....(ii)

$\because$ P(h, k) lies on tangent

$\therefore \quad k = mh + \sqrt{a^2m^2 + b^2}$ ....(iii)



from equation (ii) & (iii), we get

$\left(k + \frac{h^2}{k}\right)^2 = \frac{a^2h^2}{k^2} + b^2$

$\Rightarrow$ locus is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

Example 44 : The tangent at a point P on an ellipse intersects the major axis in T and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

Solution : Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P(acosθ, bsinθ) be a point on the ellipse. The equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. It meets the major axis at $T \equiv (a\sec\theta, 0)$.

The coordinates of N are $(a\cos\theta, 0)$. The equation of the circle with NT as its diameter is $(x - a\sec\theta)(x - a\cos\theta) + y^2 = 0$.

$\Rightarrow \quad x^2 + y^2 - ax(\sec\theta + \cos\theta) + a^2 = 0$

It cuts the auxiliary circle $x^2 + y^2 - a^2 = 0$ orthogonally if

$2g \cdot 0 + 2f \cdot 0 = a^2 - a^2 = 0$, which is true.

Problems for Self Practice-11 :

(1) Find the equation of the tangent to the ellipse $7x^2 + 8y^2 = 100$ at the point (2, –3).

(2) Find the equation of the tangents to the ellipse $9x^2 + 16y^2 = 144$ which are parallel to the line $x + 3y + k = 0$.

(3) Find the area of parallelogram formed by tangents at the extremities of latera recta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(4) Find the eccentric angle of the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tangent at which, is equally inclined to the axes.

(5) If y_1 is ordinate of a point P on the ellipse then show that the angle between its focal radius and tangent at it, is $\tan^{-1}\left(\frac{b^2}{aey_1}\right)$.

(6) Prove that the circle on any focal distance as diameter to an ellipse touches the auxiliary circle. Also prove that perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

(7) If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then prove that,

(i) $Tt \cdot PY = a^2 - b^2$

(ii) least value of Tt is a + b.

Answers :

(1) $7x - 12y = 50$

(2) $3y + x \pm \sqrt{97} = 0$

(3) $\frac{2a^3}{\sqrt{a^2-b^2}}$

(4) $\theta = \pm \tan^{-1}\left(\frac{b}{a}\right), \pi - \tan^{-1}\left(\frac{b}{a}\right), -\pi + \tan^{-1}\left(\frac{b}{a}\right)$

7. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > = < a^2m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $P(\alpha)$ & $Q(\beta)$ is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

SOLVED EXAMPLE

Example 45 : Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Solution : The given line is $x\cos\alpha + y\sin\alpha = p \Rightarrow y\sin\alpha = -x\cos\alpha + p$

$\Rightarrow y = -x\cot\alpha + p\,\text{cosec}\,\alpha$

Comparing this line with $y = mx + c$

$m = -\cot\alpha$, $c = p\,\text{cosec}\,\alpha$

Since the given line touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$c^2 = a^2m^2 - b^2 \Rightarrow p^2\,\text{cosec}^2\alpha = a^2\cot^2\alpha - b^2$ or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

Problems for Self Practice-12 :

(1) Find the condition for the line $\ell x + my + n = 0$ to touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(2) If the line $y = 5x + 1$ touch the hyperbola $\frac{x^2}{4} - \frac{y^2}{b^2} = 1$ $\{b > 4\}$, then -

(A) $b^2 = \frac{1}{5}$ (B) $b^2 = 99$ (C) $b^2 = 4$ (D) $b^2 = 100$

Answers :

(1) $n^2 = a^2\ell^2 - b^2m^2$ (2) (B)

7.1 TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

7.1.1 **Point form :** Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Note : In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $y - y_1 = m_1 (x - x_1)$ & $y - y_1 = m_2 (x - x_1)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2) m^2 - 2 x_1 y_1 m + y_1^2 + b^2 = 0$. If $D < 0$, then **no tangent** can be drawn from $(x_1 y_1)$ to **the hyperbola.**

7.1.2 **Slope form :** The equation of tangents of slope m to the given hyperbola is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

Point of contact are $\left(\mp\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2-b^2}}\right)$

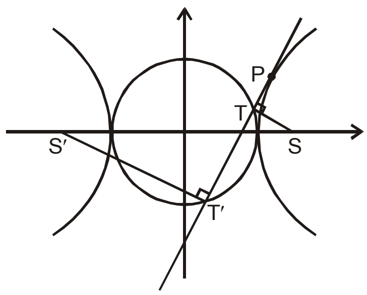
Note that there **are two parallel tangents having the same slope m.**

7.1.3 **Parametric form :** Equation of the tangent to the given hyperbola at the point **(a sec θ , b tan θ)** is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

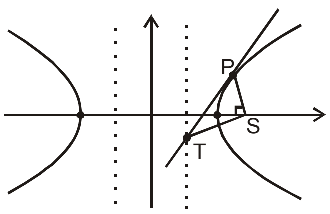
Note :

(i) Point of intersection of the tangents at θ_1 & θ_2 is $x = a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$, $y = b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$

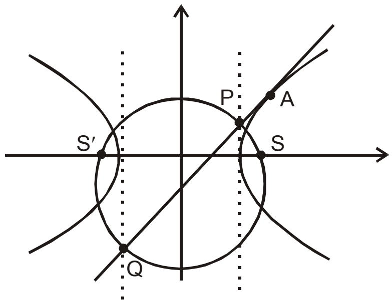
(ii) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of these perpendiculars is b^2.



(iii) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.



(iv) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



SOLVED EXAMPLE

Example 46 : Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Solution : Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore \quad m \times 1 = -1 \quad \Rightarrow \quad m = -1$

Since $x^2 - 4y^2 = 36$ or $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$

Comparing this with $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\therefore \quad a^2 = 36$ and $b^2 = 9$

So the equation of tangents are $y = (-1)x \pm \sqrt{36 \times (-1)^2 - 9}$

$y = -x \pm \sqrt{27} \quad \Rightarrow \quad x + y \pm 3\sqrt{3} = 0$

Example 47 : Find the equation and the length of the common tangents to hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.

Solution : Tangent at $(a \sec\phi, b \tan\phi)$ on the 1st hyperbola is

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1 \quad .....(1)$$

Similarly tangent at any point $(b \tan\theta, a \sec\theta)$ on 2nd hyperbola is

$$\frac{y}{a}\sec\theta - \frac{x}{b}\tan\theta = 1 \quad .....(2)$$

If (1) and (2) are common tangents then they should be identical. Comparing the co–effecients of x and y

$$\Rightarrow \quad \frac{\sec\theta}{a} = -\frac{\tan\phi}{b} \quad .....(3)$$

and $\quad -\dfrac{\tan\theta}{b} = \dfrac{\sec\phi}{a}$

or $\sec\theta = -\frac{a}{b}\tan\phi$ ......(4)

$\because$ $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow$ $\frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}\sec^2\phi = 1$ {from (3) and (4)}

or $\frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}(1+\tan^2\phi) = 1$ or $\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\tan^2\phi = 1 + \frac{b^2}{a^2}$

$\tan^2\phi = \frac{b^2}{a^2 - b^2}$

and $\sec^2\phi = 1 + \tan^2\phi = \frac{a^2}{a^2 - b^2}$

Hence the point of contact are

$\left\{\pm\frac{a^2}{\sqrt{(a^2-b^2)}}, \pm\frac{b^2}{\sqrt{(a^2-b^2)}}\right\}$ and $\left\{\pm\frac{b^2}{\sqrt{(a^2-b^2)}}, \pm\frac{a^2}{\sqrt{(a^2-b^2)}}\right\}$ {from (3) and (4)}

Length of common tangent i.e., the distance between the above points is $\sqrt{2}\frac{(a^2+b^2)}{\sqrt{(a^2-b^2)}}$ and equation of common tangent on putting the values of $\sec\phi$ and $\tan\phi$ in (1) is

$\pm\frac{x}{\sqrt{(a^2-b^2)}} \mp \frac{y}{\sqrt{(a^2-b^2)}} = 1$ or $x \mp y = \pm\sqrt{(a^2-b^2)}$

Example 48 : The locus of the point of intersection of two tangents of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if the product of their slopes is c^2, will be -

(A) $y^2 - b^2 = c^2(x^2 + a^2)$
(B) $y^2 + b^2 = c^2(x^2 - a^2)$
(C) $y^2 + a^2 = c^2(x^2 - b^2)$
(D) $y^2 - a^2 = c^2(x^2 + b^2)$

Solution : Equation of any tangent of the hyperbola with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

If it passes through (x_1, y_1) then

$(y_1 - mx_1)^2 = a^2m^2 - b^2 \Rightarrow (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$

If $m = m_1, m_2$ then as given $m_1m_2 = c^2 \Rightarrow \frac{y_1^2 + b^2}{x_1^2 - a^2} = c^2$

Hence required locus will be : $y^2 + b^2 = c^2(x^2 - a^2)$

Problems for Self Practice-13 :

(1) Find the equation of the tangent to the hyperbola $16x^2 - 9y^2 = 144$ at $\left(5, \frac{16}{3}\right)$.

(2) Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$, which is parallel to the line $4y = 5x + 7$.

(3) Find the common tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and an ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

Answers : (1) $5x - 3y = 9$ (2) $24y = 30x \pm \sqrt{161}$ (3) $y = \pm x \pm \sqrt{7}$

8. PAIR OF TANGENTS :

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the curve S = 0 is $SS_1 = T^2$

Curve (S=0)	T for point (x_1, y_1) & S = 0	S_1 for point (x_1, y_1) & S = 0	Combined equation of tangents from external point (x_1, y_1) to S = 0
Parabola $(y^2 - 4ax = 0)$	$T \equiv yy_1 - 2a(x + x_1)$	$S_1 = y_1^2 - 4ax_1$	$SS_1 = T^2$
Ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\right)$	$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$	$SS_1 = T^2$
Hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0\right)$	$T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$	$SS_1 = T^2$

SOLVED EXAMPLE

Example 49 : If two tangents to the parabola $y^2 = 4ax$ from a point P make angles θ_1 and θ_2 with the axis of the parabola, then find the locus of P in each of the following cases.

(i) $\theta_1 + \theta_2 = \alpha$ (a constant) (ii) $\theta_1 + \theta_2 = \frac{\pi}{2}$ (iii) $\tan\theta_1 + \tan\theta_2 = \lambda$ (is constant)

Solution: Let the equation of tangent and $p \equiv (h, k)$ $y = mx + \frac{a}{m}$

$p \equiv (h, k)$ $\quad k = mh + \frac{a}{m}$

$m^2h - mk + a = 0$

$m_1 + m_2 = \frac{k}{h}$ ...... (i)

$m_1 . m_2 = \frac{a}{h}$ ..... (ii)

(i) $\theta_1 + \theta_2 = \alpha$

$\tan(\theta_1 + \theta_2) = \tan\alpha$

$\frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \cdot \tan\theta_2} = \tan\alpha \Rightarrow \frac{\frac{k}{h}}{1 - \frac{a}{h}} = \tan\alpha$

$\frac{k}{h - a} = \tan\alpha$

$y = (x - a)\tan\alpha$

(ii) $\theta_1 + \theta_2 = \frac{\pi}{2} \Rightarrow \tan\theta_1 . \tan\theta_2 = 1$

$\frac{a}{h} = 1$ $\quad x = a$

(iii) $m_1 + m_2 = \lambda \Rightarrow \frac{k}{h} = \lambda$

$y = \lambda x$

Example 50 : Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2) . Find the area of the triangle that these tangents form with their chord of contact.

Solution: Let tangent $y = mx + \sqrt{9m^2 - 1}$

It passes through (3, 2)

$\therefore \quad (2 - 3m)^2 = (9m^2 - 1)$

$\therefore \quad$ m is not defined or $m = \dfrac{5}{12}$

$\therefore \quad$ tangents are

$x = 3 \qquad$ .... (i)

$5x - 12y + 9 = 0 \qquad$ .... (ii)

Chord of contact is $\quad 3x - 18y = 9 \qquad$ .... (iii)

by (i) & (iii) $\qquad$ by (ii) & (iii)

$x = 3,\ y = 0 \qquad x = -5,\ y = -\dfrac{4}{3}$

$$\text{Area of triangle} = \left|\frac{1}{2}\begin{vmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ -5 & -\frac{4}{3} & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2} \times 2 \times (3 + 5) = 8 \text{ sq. unit.}$$

Example 51 : Find the locus of point of intersection of perpendicular tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Solution : Let P(h, k) be the point of intersection of two perpendicular tangents equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) = \left(\frac{hx}{a^2} + \frac{ky}{b^2} - 1\right)^2$$

$$\Rightarrow \quad \frac{x^2}{a^2}\left(\frac{k^2}{b^2} - 1\right) + \frac{y^2}{b^2}\left(\frac{h^2}{a^2} - 1\right) + \ldots\ldots = 0 \qquad \ldots\ldots\ldots(i)$$

Since equation (i) represents two perpendicular lines

$$\therefore \quad \frac{1}{a^2}\left(\frac{k^2}{b^2} - 1\right) + \frac{1}{b^2}\left(\frac{h^2}{a^2} - 1\right) = 0$$

$\Rightarrow \quad k^2 - b^2 + h^2 - a^2 = 0 \qquad \Rightarrow \quad$ locus is $x^2 + y^2 = a^2 + b^2$

Example 52 : How many real tangents can be drawn from the point (4, 3) to the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$. Find the equation of these tangents & angle between them.

Solution : Given point $\quad P \equiv (4, 3) \quad$ Hyperbola $\quad S \equiv \dfrac{x^2}{16} - \dfrac{y^2}{9} - 1 = 0$

$\because \quad S_1 \equiv \dfrac{16}{16} - \dfrac{9}{9} - 1 = -1 < 0 \quad \Rightarrow \quad$ Point $P \equiv (4, 3)$ lies outside the hyperbola.

$\therefore \quad$ Two tangents can be drawn from the point P(4, 3).

Equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \quad \left(\frac{x^2}{16}-\frac{y^2}{9}-1\right).(-1) = \left(\frac{4x}{16}-\frac{3y}{9}-1\right)^2$$

$$\Rightarrow \quad -\frac{x^2}{16}+\frac{y^2}{9}+1 = \frac{x^2}{16}+\frac{y^2}{9}+1-\frac{xy}{6}-\frac{x}{2}+\frac{2y}{3}$$

$$\Rightarrow \quad 3x^2 - 4xy - 12x + 16y = 0 \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

Example 53 : Find the locus of point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Solution : Let P(h, k) be the point of intersection of two perpendicular tangents. Equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \quad \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}-1\right) = \left(\frac{hx}{a^2}-\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \quad \frac{x^2}{a^2}\left(-\frac{k^2}{b^2}-1\right)-\frac{y^2}{b^2}\left(\frac{h^2}{a^2}-1\right)+........=0 \qquad ....(i)$$

Since equation (i) represents two perpendicular lines

$$\therefore \quad \frac{1}{a^2}\left(-\frac{k^2}{b^2}-1\right)-\frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right)=0$$

$$\Rightarrow \quad -k^2 - b^2 - h^2 + a^2 = 0 \qquad \Rightarrow \qquad \text{locus is } x^2 + y^2 = a^2 - b^2$$

Problems for Self Practice-14 :

(1) If two tangents to the parabola $y^2 = 4ax$ from a point P make angles θ_1 and θ_2 with positve x-axis, then find the locus of P if $\cos\theta_1 \cos\theta_2 = \lambda$ (a constant)

Answers :

(1) $x^2 = \lambda^2 \{(x - a)^2 + y^2\}$

9. DIRECTOR CIRCLE :

Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle.

Curve(S = 0)	Locus of Director Circle of (S = 0)
Parabola $(y^2 - 4ax = 0)$	$x + a = 0$
Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$x^2 + y^2 = a^2 + b^2$
Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$x^2 + y^2 = a^2 - b^2$

Note: For hyperbola, if $b^2 < a^2$, then the director circle is real.

If $b^2 = a^2$ (i.e. rectangular hyperbola), then the radius of the director circle is zero and it reduces to a point circle at the origin. In this case centre is the only point from which two perpendicular tangents can be drawn on the curve.

If $b^2 > a^2$, then the radius of the director circle is imaginary, so that there is no such circle and so no pair of tangents at right angle can be drawn to the curve.

SOLVED EXAMPLE

Example 54 : A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Solution : Given ellipse are $\frac{x^2}{4}+\frac{y^2}{1}=1$ .... (i)

and, $\frac{x^2}{6}+\frac{y^2}{3}=1$ .... (ii)

any tangent to (i) is $\frac{x\cos\theta}{2}+\frac{y\sin\theta}{1}=1$ .... (iii)

It cuts (ii) at P and Q, and suppose tangent at P and Q meet at (h, k) Then equation of chord of contact of (h, k) with respect to ellipse (ii) is $\frac{hx}{6}+\frac{ky}{3}=1$ .... (iv)

comparing (iii) and (iv), we get $\frac{\cos\theta}{h/3}=\frac{\sin\theta}{k/3}=1$

$\Rightarrow \quad \cos\theta=\frac{h}{3}$ and $\sin\theta=\frac{k}{3} \quad \Rightarrow \; h^2+k^2=9$

locus of the point (h, k) is $x^2 + y^2 = 9 \quad \Rightarrow \quad x^2 + y^2 = 6 + 3 = a^2 + b^2$

i.e. director circle of second ellipse. Hence the tangents are at right angles.

10. CHORD OF CONTACT :

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the curve S = 0 is T = 0

Curve(S=0)	T for point (x_1, y_1) & $S = 0$	Equation of chord of contact from external point $(x_1 y_1)$ to $S = 0$ is $T = 0$
Parabola $(y^2 - 4ax = 0)$	$T \equiv yy_1 - 2a(x + x_1)$	$yy_1 - 2a(x + x_1) = 0$
Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0\right)$	$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$	$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$
Hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\right)$	$T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$	$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=0$

Note : The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$

SOLVED EXAMPLE

Example 55 : If tangents are drawn to the points of intersection of the line $7y - 4x = 10$ and parabola $y^2 = 4x$, then find the point of intersection of these tangents.

Solution: Here let the point be (x_1, y_1)

$\therefore$ chord of contact $yy_1 = 2(x + x_1)$

compare with the given line $\frac{2x_1}{-10} = \frac{-y_1}{7} = \frac{2}{-4}$

$\Rightarrow$ $x_1 = 5/2$, $y_1 = 7/2$

$\therefore$ Point reqd. $\left(\frac{5}{2}, \frac{7}{2}\right)$

Example 56 : Find the locus of point whose chord of contact w.r.t to the parabola $y^2 = 4bx$ is the tangents of the parabola $y^2 = 4ax$.

Solution : Equation of tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ ......(i)

Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is $yk = 2b(x + h)$

$y = \frac{2b}{k}x + \frac{2bh}{k}$ .....(ii)

From (i) & (ii)

$m = \frac{2b}{k}, \frac{a}{m} = \frac{2bh}{k}$ $\Rightarrow$ $a = \frac{4b^2h}{k^2}$

locus of P is $y^2 = \frac{4b^2}{a}x$.

Example 57 : If tangents to the parabola $y^2 = 4ax$ intersect the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution : Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

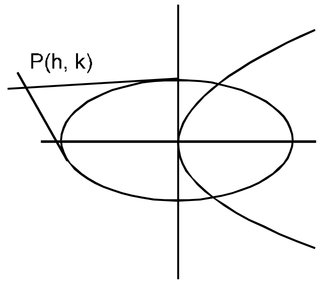
$\therefore$ equation of chord of contact AB is $\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ ....(i)

which touches the parabola equation of tangent to parabola $y^2 = 4ax$

$y = mx + \frac{a}{m}$

$\Rightarrow$ $mx - y = -\frac{a}{m}$ ....(ii)

equation (i) & (ii) as must be same



$\therefore \quad \dfrac{m}{\left(\dfrac{h}{a^2}\right)} = \dfrac{-1}{\left(\dfrac{k}{b^2}\right)} = \dfrac{-\dfrac{a}{m}}{1}$

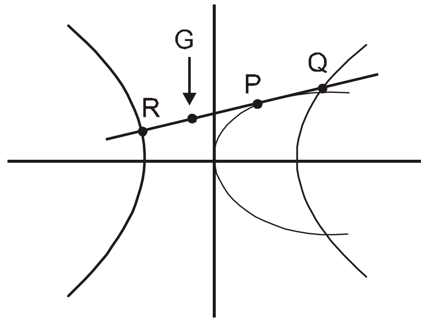
$\Rightarrow \quad m = -\dfrac{h}{k}\dfrac{b^2}{a^2}$ & $m = \dfrac{ak}{b^2}$

$\therefore \quad -\dfrac{hb^2}{ka^2} = \dfrac{ak}{b^2} \quad \Rightarrow \quad$ locus of P is $y^2 = -\dfrac{b^4}{a^3} \cdot x$

Example 58 : Chords of the hyperbola, $x^2 - y^2 = a^2$ touch the parabola, $y^2 = 4ax$. Prove that the locus of their middle points is the curve, $y^2(x - a) = x^3$.

Solution : If G(h,k) is mid–point of chord then chord is

$xh - yk = h^2 - k^2$ ....(i)



It must be in form of $y = mx + \dfrac{a}{m}$ ....(ii)

to touch $y^2 = 4ax$

On comparing (i), (ii) and eleminating m we get,

$k^2(h - a) = h^3 \quad \Rightarrow \quad y^2(x - a) = x^3$ H.P.

<u>Problems for Self Practice</u>-15 :

(1) Find the equation of the chord of contacts of tangents drawn from a point (2, 1) to the parabola $x^2 = 2y$.

(2) If from a variable point 'P' on the line $x - 2y + 1 = 0$ pair of tangent's are drawn to the parabola $y^2 = 8x$ then prove that chord of contact passes through a fixed point, also find that point.

(3) If a line $3x - y = 2$ intersects ellipse $\dfrac{x^2}{8} + \dfrac{y^2}{4} = 1$ at points A & B, then find co-ordinates of point of intersection of tangents at points A & B.

(4) If the chord of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are at right angles, then find $\dfrac{x_1x_2}{y_1y_2}$.

(5) Find the locus of point of intersection of tangents at the extremities of the chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ subtending a right angle at its centre.

Answers :

(1) $2x = y + 1$ (2) (1, 8) (3) (12, –2)

(4) $-\dfrac{a^4}{b^4}$ (5) $\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$

11. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the curve S = 0 whose middle point is (x_1, y_1) is $T = S_1$.

Curve(S = 0)	T for point (x_1, y_1) & S = 0	S_1 for point (x_1, y_1) & S = 0	Chord with middle point (x_1, y_1) for S = 0 is $T = S_1$
Parabola $(y^2 - 4ax = 0)$	$T = yy_1 - 2a(x + x_1)$	$S_1 = y_1^2 - 4ax_1$	$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$
Ellips $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right) = 0$	$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{xx_1}{a^2} + \frac{yy_1}{b^2}$
Hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0\right)$	$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$	$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

SOLVED EXAMPLE

Example 59 : Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given (p, q).

Solution : Let P(h, k) be the mid point of chord of the parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

Since it passes through (p, q)

$\therefore$ $qk - 2a(p + h) = k^2 - 4ah$

$\therefore$ Required locus is $y^2 - 2ax - qy + 2ap = 0$.

Example 60 : Find the locus of the mid - point of focal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : Let $P \equiv (h, k)$ be the mid-point

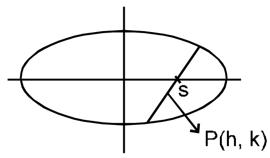
$\therefore$ equation of chord whose mid-point is given $\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ it passes through focus, either (ae, 0) or (–ae, 0)

If it passes through (ae, 0)

$\therefore$ locus is $\frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



If it passes through (–ae, 0)

$\therefore$ locus is $-\frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Example 61 : Find the condition on 'a' and 'b' for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through (a, – b) are bisected by the line x + y = b.

Solution : Let the line x + y = b bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ equation of chord whose mid-point is $P(\alpha, b - \alpha)$

$$\frac{x\alpha}{2a^2} + \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} + \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, –b)

$$\therefore \quad \frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} + \frac{(b-\alpha)^2}{2b^2}$$

$$\Rightarrow \quad \left(\frac{1}{a}+\frac{1}{b}\right)\alpha - 1 = \alpha^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) - \frac{2}{b}\alpha + 1$$

$$\Rightarrow \quad \alpha^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) - \left(\frac{3}{b}+\frac{1}{a}\right)\alpha + 2 = 0$$

since line bisect two chord

$\therefore$ above quadratic equation in α must have two distinct real roots

$$\therefore \quad \left(\frac{3}{b}+\frac{1}{a}\right)^2 - 4\left(\frac{1}{a^2}+\frac{1}{b^2}\right).2 > 0$$

$$\Rightarrow \quad \frac{9}{b^2} + \frac{1}{a^2} + \frac{6}{ab} - \frac{8}{a^2} - \frac{8}{b^2} > 0 \qquad \Rightarrow \quad \frac{1}{b^2} - \frac{7}{a^2} + \frac{6}{ab} > 0$$

$\Rightarrow$ $a^2 - 7b^2 + 6ab > 0$

$\Rightarrow$ $a^2 > 7b^2 - 6ab$ which is the required condition.

Example 62 : Find the locus of the mid point of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtend a right angle at the origin.

Solution : let (h,k) be the mid–point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{b^2} - \frac{k^2}{b^2} - 1 \qquad \text{or} \qquad \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad \text{........(1)}$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (1) is obtained by making homogeneous hyperbola with the help of (1)

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left(\frac{hx}{a^2}-\frac{ky}{b^2}\right)^2}{\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 y^2 = \frac{h^2}{a^4}x^2 + \frac{k^2}{b^4}y^2 - \frac{2hk}{a^2b^2}xy \quad \text{.......(2)}$$

The lines represented by (2) will be at right angle if coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 - \frac{k^2}{b^4} = 0$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right) = \frac{h^2}{a^4} + \frac{k^2}{b^4}$$

hence, the locus of (h,k) is $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right) = \frac{x^2}{a^4} + \frac{y^2}{b^4}$

Problems for Self Practice-16 :

(1) Find the equation of chord of parabola $y^2 = 4x$ whose mid point is (4, 2).

(2) Find the locus of mid - point of chord of parabola $y^2 = 4ax$ which touches the parabola $x^2 = 4by$.

(3) Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ whose slope is 'm'.

(4) Find the length of the chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ whose middle point is $\left(\frac{1}{2}, \frac{2}{5}\right)$

(5) Find the equation of the chord $\frac{x^2}{36} - \frac{y^2}{9} = 1$ which is bisected at (2, 1).

(6) Find the point 'P' from which pair of tangents PA & PB are drawn to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ in such a way that (5, 2) bisect AB

(7) From the points on the circle $x^2 + y^2 = a^2$, tangent are drawn to the hyperbola $x^2 - y^2 = a^2$, prove that the locus of the middle points of the chords of contact is the curve $(x^2 - y^2)^2 = a^2 (x^2 + y^2)$.

Answers : (1) $x - y - 2 = 0$ (2) $y (2ax - y^2) = 4a^2b$ (3) $y = \frac{2a}{m}$ (4) $\frac{7}{5}\sqrt{41}$ (5) $x = 2y$

(6) $\left(\frac{20}{3}, \frac{8}{3}\right)$

12. NORMAL :

12.1 Normal To The Parabola $y^2 = 4ax$:

12.1.1 Point form :

Equation of normal to the given parabola at its point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

12.1.2 Slope form :

Equation of normal to the given parabola whose slope is 'm', is

$\mathbf{y = mx - 2am - am^3}$

foot of the normal is $\mathbf{(am^2, -2am)}$

12.1.3 Parametric form :

Equation of normal to the given parabola at its point P(t), is

$\mathbf{y + tx = 2at + at^3}$

Note :

(i) Point of intersection of normals at t_1 & t_2 is $\mathbf{(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))}$.

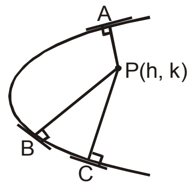
(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $\mathbf{t_1t_2 = 2}$; $\mathbf{t_3 = -(t_1 + t_2)}$ and the line joining t_1 & t_2 passes through a fixed point (–2a, 0).

(iv) If normal drawn to a parabola passes through a point P(h,k) then $k = mh - 2am - am^3$, i.e. $\mathbf{am^3 + m(2a - h) + k = 0}$.

This gives $m_1 + m_2 + m_3 = 0$; $m_1m_2+m_2m_3+m_3m_1 = \frac{2a-h}{a}$; $m_1m_2m_3 = \frac{-k}{a}$

where m_1, m_2, & m_3 are the slopes of the three concurrent normals :

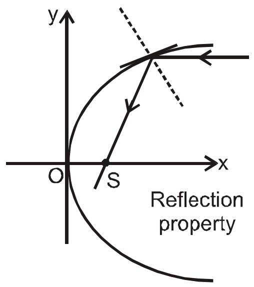


A, B, C $\rightarrow$ Conormal points

- Algebraic sum of slopes of the three concurrent normals is zero.
- Algebraic sum of ordinates of the three co-normal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the axis of parabola (x-axis).
- Condition for three real and distinct normals to be drawn froma point P (h, k) is

  $h > 2a$ & $k^2 < \frac{4}{27a}(h - 2a)^3$.

(vi) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of theparabola after reflection.

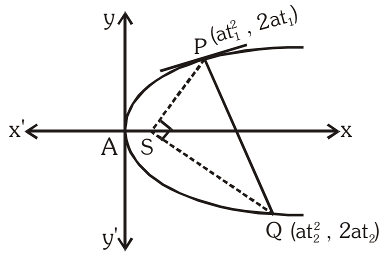


Example 63 : Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.

Solution : Let the normal at $P\left(at_1^2, 2at_1\right)$ meet the curve at $Q\left(at_2^2, 2at_2\right)$

$\therefore$ PQ is a normal chord.

and $t_2 = -t_1 - \frac{2}{t_1}$ ....(i)



By given condition $2at_1 = at_1^2$

$\therefore$ $t_1 = 2$ from equation (i), $t_2 = -3$

then P(4a, 4a) and Q(9a, –6a)

but focus S(a, 0)

$\therefore$ Slope of SP = $\frac{4a-0}{4a-a} = \frac{4a}{3a} = \frac{4}{3}$

and Slope of SQ = $\frac{-6a-0}{9a-a} = \frac{-6a}{8a} = -\frac{3}{4}$

$\therefore$ Slope of SP × Slope of SQ $= \frac{4}{3} \times -\frac{3}{4} = -1$

$\therefore$ $\angle PSQ = \pi/2$

i.e. PQ subtends a right angle at the focus S.

Example 64 : If ax + by = 1 is a normal to the parabola $y^2 = 4Px$, then prove that $Pa^3 + 2aPb^2 = b^2$.

Solution. Equation of normal to $y^2 = 4Px$ at (x_1, y_1) is $y - y_1 = \frac{-y_1}{2p}(x - x_1)$; $y - y_1 = \frac{-y_1}{2p}\left(x - \frac{y_1^2}{4p}\right)$

$\Rightarrow$ $4py_1x + 8p^2y = y_1^3 + 8p^2y_1$

Compair with ax + by = 1

$\Rightarrow$ $\frac{4py_1}{a} = \frac{8p^2}{b} = y_1^3 + 8p^2y_1$

$\Rightarrow$ $y_1 = \frac{2ap}{b}$ & $\frac{4p}{a} = y_1^2 + 8p^2$

$\Rightarrow$ $Pa^3 + 2aPb^2 = b^2$.

Example 65 : Find the locus of the point N from which 3 normals are drawn to the parabola $y^2 = 4ax$ are such that

(i) Two of them are equally inclined to x-axis

(ii) Two of them are perpendicular to each other

Solution : Equation of normal to $y^2 = 4ax$ is

$y = mx - 2am - am^3$

Let the normal passes through N(h, k)

$\therefore$ $k = mh - 2am - am^3$ $\Rightarrow$ $am^3 + (2a - h)m + k = 0$

For given value's of (h, k) it is cubic in 'm'.

Let m_1, m_2 & m_3 are root's of above equation

$\therefore$ $m_1 + m_2 + m_3 = 0$ ......(i)

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$ ......(ii)

$m_1m_2m_3 = -\frac{k}{a}$ ......(iii)

(i) If two normal are equally inclined to x-axis, then $m_1 + m_2 = 0$

$\therefore$ $m_3 = 0$ $\Rightarrow$ $y = 0$

(ii) If two normal's are perpendicular

$\therefore$ $m_1 m_2 = -1$

from (3) $m_3 = \frac{k}{a}$ .....(iv)

from (2) $-1 + \frac{k}{a}(m_1 + m_2) = \frac{2a-h}{a}$ .....(v)

from (1) $m_1 + m_2 = -\frac{k}{a}$ .....(vi)

from (5) & (6), we get

$-1 - \frac{k^2}{a} = 2 - \frac{h}{a}$

$y^2 = a(x - 3a)$

Problems for Self Practice-17 :

(1) If three distinct and real normals can be drawn to $y^2 = 8x$ from the point (a, 0), then -

(A) $a > 2$ (B) $a \in (2, 4)$ (C) $a > 4$ (D) none of these

(2) If $2x + y + k = 0$ is a normal to the parabola $y^2 = -16x$, then find the value of k.

(3) If the normal at point P(1, 2) on the parabola $y^2 = 4x$ cuts it again at point Q then Q = ?

(4) If normal chord at a point 't' on the parabola $y^2 = 4ax$ subtends a right angle at the vertex then prove that $t^2 = 2$

(5) Prove that the chord of the parabola $y^2 = 4ax$, whose equation is $y - x\sqrt{2} + 4a\sqrt{2} = 0$, is a normal to the curve and that its length is $6\sqrt{3}a$.

Answers : (1) (C) (2) 48 (3) (9, – 6)

12.2 NORMAL TO THE ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

12.2.1 Point form : Equation of the normal to the given ellipse at (x_1, y_1) is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2.$$

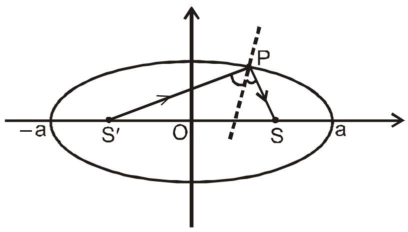
12.2.2 Slope form : Equation of a normal to the given ellipse whose slope is 'm' is

$$y = mx \mp \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}.$$

12.2.3 **Parametric form :** Equation of the normal to the given ellipse at the point **($a\cos\theta$, $b\sin\theta$)** is **$ax\sec\theta - by\,\mathrm{cosec}\,\theta = (a^2 - b^2)$.**

Note :

The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at



SOLVED EXAMPLE

Example 66 : Find the condition that the line $\ell x + my = n$ may be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : Equation of normal to the given ellipse at ($a\cos\theta$, $b\sin\theta$) is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ ...(i)

If the line $\ell x + my = n$ is also normal to the ellipse then there must be a value of θ for which line (i) and line $\ell x + my = n$ are identical. For that value of θ we have

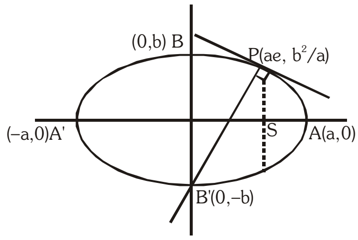
$$\frac{\ell}{\left(\frac{a}{\cos\theta}\right)} = \frac{m}{-\left(\frac{b}{\sin\theta}\right)} = \frac{n}{(a^2 - b^2)} \text{ or } \cos\theta = \frac{an}{\ell(a^2 - b^2)} \quad \text{...... (iii)}$$

and $\sin\theta = \frac{-bn}{m(a^2 - b^2)}$ ...... (iv)

Squaring and adding (iii) and (iv), we get $1 = \frac{n^2}{(a^2 - b^2)^2}\left(\frac{a^2}{\ell^2} + \frac{b^2}{m^2}\right)$ which is the required condition.

Example 67 : If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e=\sqrt{\frac{\sqrt{5}-1}{2}}$

Solution : The co-ordinates of an end of the latus-rectum are (ae, b^2/a).



The equation of normal at P(ae, b^2/a) is

$$\frac{a^2x}{ae}-\frac{b^2(y)}{b^2/a}=a^2-b^2 \quad \text{or} \quad \frac{ax}{e}-ay=a^2-b^2$$

It passes through one extremity of the minor axis whose co-ordinates are (0, –b)

$\therefore$ $0 + ab = a^2 - b^2$ $\Rightarrow$ $(a^2b^2) = (a^2 - b^2)^2$

$\Rightarrow$ $a^2.a^2(1 - e^2) = (a^2 e^2)^2$ $\Rightarrow$ $1 - e^2 = e^4$

$\Rightarrow$ $e^4 + e^2 - 1 = 0$ $\Rightarrow$ $(e^2)^2 + e^2 - 1 = 0$

$\therefore$ $e^2=\frac{-1\pm\sqrt{1+4}}{2}$ $\Rightarrow$ $e=\sqrt{\frac{\sqrt{5}-1}{2}}$ (taking positive sign)

Problems for Self Practice-18 :

(1) Find the equation of the normal to the ellipse $9x^2 + 16y^2 = 288$ at the point (4, 3)

(2) Find the value(s) of 'k' for which the line x + y = k is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(3) If the normal at the point P(θ) to the ellipse $\frac{x^2}{3}+\frac{y^2}{2}=1$ intersects it again at the point Q(2θ), then find cosθ.

(4) Find the locus of the mid-points of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(5) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then prove that

(i) PF. PG = b^2 (ii) PF. Pg = a^2

(iii) PG. Pg = SP. S′P (iv) CG. CT = CS^2

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

Answers : (1) 4x – 3y = 7 (2) $k=\pm\sqrt{\frac{(a^2-b^2)^2}{a^2+b^2}}$ (3) –1 (4) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2$

12.3 NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

12.3.1 Point form : The equation of the normal to the given hyperbola at the point P (x_1, y_1) on it is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 = a^2e^2.$$

12.3.2 Slope form : The equation of normal of slope m to the given hyperbola is $\mathbf{y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}}$ foot of

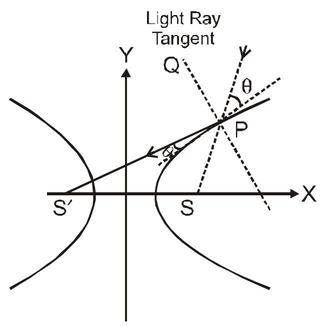
normal are $\left(\pm\frac{\mathbf{a^2}}{\sqrt{(\mathbf{a^2 - m^2b^2})}}, \mp\frac{\mathbf{mb^2}}{\sqrt{(\mathbf{a^2 - m^2b^2})}}\right)$

12.3.3 Parametric form : The equation of the normal at the point **P (a secθ , b tanθ)** to the given hyperbola

is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

Note :

The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This explains the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ (a > k > b > 0) are confocal and therefore orthogonal.

SOLVED EXAMPLE

Example 68 : Line x cos α + y sin α = p is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if -

(A) $a^2 \sec^2\alpha - b^2 \operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$ (C) $a^2 \sec^2\alpha + b^2 \operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$

(C) $a^2 \cos^2\alpha - b^2 \sin^2\alpha = \frac{(a^2+b^2)^2}{p^2}$ (D) $a^2 \cos^2\alpha + b^2 \sin^2\alpha = \frac{(a^2+b^2)^2}{p^2}$

Solution : Equation of a normal to the hyperbola is $ax\cos\theta + by\cot\theta = a^2 + b^2$ comparing it with the given line equation

$$\frac{a\cos\theta}{\cos\alpha} = \frac{b\cot\theta}{\sin\alpha} = \frac{a^2+b^2}{p} \Rightarrow \sec\theta = \frac{ap}{\cos\alpha(a^2+b^2)},\ \tan\theta = \frac{bp}{\sin\alpha(a^2+b^2)}$$

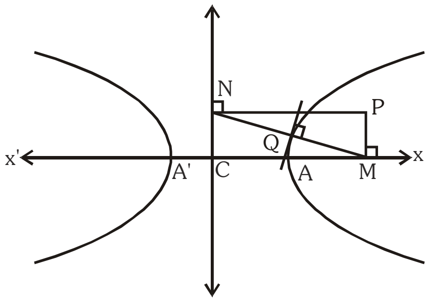
Eliminating θ, we get

$$\frac{a^2p^2}{\cos^2\alpha(a^2+b^2)^2} - \frac{b^2p^2}{\sin^2\alpha(a^2+b^2)^2} = 1 \Rightarrow a^2\sec^2\alpha - b^2\operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$$

Example 69 : The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is hyperbola $(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

Solution : Equation of normal at any point Q is $ax\cos\theta + by\cot\theta = a^2+b^2$

$\therefore$ $$M \equiv \left(\frac{a^2+b^2}{a}\sec\theta,\ 0\right),\ N \equiv \left(0,\ \frac{a^2+b^2}{b}\tan\theta\right)$$



$\therefore$ Let $P \equiv (h, k)$

$\Rightarrow$ $$h = \frac{a^2+b^2}{a}\sec\theta, \qquad k = \frac{a^2+b^2}{b}\tan\theta$$

$\Rightarrow$ $$\frac{a^2h^2}{(a^2+b^2)} - \frac{b^2k^2}{(a^2+b^2)^2} = \sec^2\theta - \tan^2\theta = 1$$

$\therefore$ locus of P is $(a^2x^2 - b^2y^2) = (a^2 + b^2)$.

Problems for Self Practice-19 :

(1) Find the equation of normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point $\left(6,\ \frac{3}{2}\sqrt{5}\right)$.

(2) Find the condition for the line $\ell x + my + n = 0$ is normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

(3) Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Answers : (1) $8\sqrt{5}x + 18y = 75\sqrt{5}$ (2) $\frac{a^2}{\ell^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$

(3) $(x^2 + y^2)^2 (a^2y^2 - b^2x^2) = x^2y^2 (a^2 + b^2)^2$

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Elementary concepts of Parabola

A-1. Find the value of λ for which the equation $\lambda x^2 + 4xy + y^2 + \lambda x + 3y + 2 = 0$ represents a parabola

A-2. **(i)** Find the equation of the parabola whose focus is at (–1, –2) and the directrix is the straight line $x - 2y + 3 = 0$

(ii) Find equation of the parabola whose vertex is (–3, 0) and directrix is $x + 5 = 0$ -

(iii) The extreme points of the latus rectum of a parabola are (7, 5) and (7, 3). Find the equation of the parabola.

A-3. Find the vertex, axis, focus, directrix, latusrectum of the parabola, also draw their rough sketches. $4y^2 + 12x - 20y + 67 = 0$

A-4. Find the equation of the parabola whose focus is (–6, –6) and vertex (–2, 2).

A-5. Find distance between the focus and directrix of the conic $\left(\sqrt{3}x - y\right)^2 = 48\left(x + \sqrt{3}y\right)$

A-6. Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

A-7. Find the set of values of α in the interval $[\pi/2, 3\pi/2]$, for which the point $(\sin\alpha, \cos\alpha)$ does not lie outside the parabola $2y^2 + x - 2 = 0$.

A-8. If a circle be drawn so as always to touch a given straight line and also a given circle externally then prove that the locus of its centre is a parabola.(given line and given circle are non intersecting)

Section (B) : Elementary concepts of Ellipse & Hyperbola

B-1. Find equation of the ellipse whose focus is (1, –1), directrix is the line $x - y - 3 = 0$ and the eccentricity is $\frac{1}{2}$.

B-2. Find equation of the ellipse with respect to coordinate axes whose minor axis is equal to the distance between its foci and whose LR = 10.

B-3. Find the centre, the length of the axes, eccentricity and the foci of ellipse

$12x^2 + 4y^2 + 24x - 16y + 25 = 0$

B-4 If focus and corresponding directrix of an ellipse are (3, 4) and $x + y - 1 = 0$ respectively and eccentricity is $\frac{1}{2}$ then find the co-ordinates of extremities of major axis.

B-5 Find the set of value(s) of 'α' for which the point $P(\alpha, -\alpha)$ lies inside the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

B-6. Find the equation of the ellipse having its centre at the point (2, –3), one focus at (3, –3) and one vertex at (4, –3).

B-7. If $\left(0, 3 + \sqrt{5}\right)$ is a point on the ellipse whose foci are (2, 3), (–2, 3) then find the length of semimajor axis.

B-8. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of the ellipse whose foci are (4, 0) and (–4, 0) and whose eccentricity is 1/3, then find its equation.

B-9. Write the parametric equation of ellipse $\frac{(x-3)^2}{25}+\frac{(y+2)^2}{16}=1$.

B-10. Find the coordinates of foci, the eccentricity and latus-rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

B-11. Find eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci.

B-12. Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

B-13. Find the equation of auxiliary circle of conic which passes through (1, 1) & is having foci (4, 5) & (2, 3).

B-14. Given the base of a triangle and the ratio of the tangent of half the base angles. Show that the vertex moves on a hyperbola whose foci are the extremities of the base.

B-15. Find the equation of a hyperbola which passes through the points of intersection of the line x+ y – 5 = 0 & hyperbola $2x^2 - 3y^2 - 6 = 0$ and is also passes through (3, 4).

B-16 Find equation of locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}\ \ k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different values of k.

B-17. If e and e′ are the eccentricities of the ellipse $5x^2 + 9y^2 = 45$ and the hyperbola $5x^2 - 4y^2 = 45$ respectively then find value of ee′.

B-18. An ellipse and a hyperbola have the same centre origin, the same foci and the minor-axis of the one is the same as the conjugate axis of the other. If e_1, e_2 be their eccentricities respectively, then prove that

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=2$$

B-19. For the hyperbola $x^2/100 - y^2/25 = 1$, prove that

(i) eccentricity = $\sqrt{5}/2$

(ii) SA . S′A = 25, where S & S′ are the foci & A is the vertex .

Section (C) : Position of line, Equation of chord and various forms of tangents of parabola

C-1. A line y = x + 7 intersect the parabola $(y - 4)^2 = 8(x + 3)$ at A & B. Find the length of chord AB.

C-2. Chord joining two distinct points $P(\alpha^2, k_1)$ and $Q\left(k_2, -\frac{54}{\alpha}\right)$ on the parabola $y^2 = 36x$ always passes through a fixed point. Find the co-ordinate of fixed point.

C-3. Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

C-4. Two perpendicular chords are drawn from the origin 'O' to the parabola $y = x^2$, which meet the parabola at P and Q Rectangle POQR is completed. Find the locus of vertex R.

C-5. If y = 2x – 3 is a tangent to the parabola $y^2 = 4a\left(x-\frac{1}{3}\right)$, then find the value of 'a', where $a \neq 0$:

C-6. A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

C-7. Points A, B & C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B & C, taken in pairs, intersect at points P, Q & R, then find the ratio of the areas of the triangles ABC & PQR.

C-8. A parabola $y = ax^2 + bx + c$ crosses the x-axis at $(\alpha, 0)$ $(\beta, 0)$ both to the right of the origin. A circle also passes through these two points. Find the length of a tangent from the origin to the circle.

C-9. A tangent at any point P(1, 7) the parabola $y = x^2 + 6$, which is touching to the circle $x^2 + y^2 + 16x + 12y + c = 0$ at point Q, then find co-ordinate of point Q

C-10. Find equation of a tangent common to the parabolas $y^2 = 4x$ and $x^2 = 4y$ is

Section (D) : Position of line, Equation of chord and various forms of tangents of Ellipse & Hyperbola

D-1. Find the length of chord $x - 2y - 2 = 0$ of the ellipse $4x^2 + 16y^2 = 64$.

D-2. If $\tan\theta_1 \tan\theta_2 = -\frac{a^2}{b^2}$, then prove that chord of two point θ_1 and θ_2 on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will subtend a right angle at centre.

D-3. Check whether the line $4x + 5y = 40$ touches the ellipse $\frac{x^2}{50} + \frac{y^2}{32} = 1$ or not. If yes, then also find its point of contact.

D-4. Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

D-5. Find equation of the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ included between the co-ordinate axes.

D-6. An ellipse passes through the point $(4, -1)$ and touches the line $x + 4y - 10 = 0$. Find its equation if its axes coincide with co–ordinate axes.

D-7. Any tangent to an ellipse is cut by the tangents at the ends of major axis in the points T and T′. Prove that the circle, whose diameter is T T′ will pass through the focii of the ellipse.

D-8. If 'P' be a moving point on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ in such a way that tangent at 'P' intersect $x = \frac{25}{3}$ at Q then circle on PQ as diameter passes through a fixed point. Find that fixed point.

D-9. A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ having slope -1 intersects the axis of x & y in point A & B respectively. If O is the origin then find the area of triangle OAB.

D-10. For what value of λ, does the line $y = 3x + \lambda$ touch the hyperbola $9x^2 - 5y^2 = 45$?

D-11. Find the area of a triangle formed by the lines $x - y = 0$, $x + y = 0$ and any tangent to the hyperbola $x^2 - y^2 = a^2$.

D-12. Find equation of common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$.

D-13. If m_1 & m_2 are the slopes of the tangents to the hyperbola $x^2/25 - y^2/16 = 1$ which passes through the point (4, 2), find the value of (i) $m_1 + m_2$ & (ii) $m_1 m_2$.

Section (E) : Pair of tangents, Director circle, chord of contact and chord with given middle point of Parabola

E-1. Find the locus of the point P from which tangents are drawn to the parabola $y^2 = 4ax$ having slopes m_1 and m_2 such that -

(i) $m_1^2 + m_2^2 = \lambda$ (constant) (ii) $\theta_1 - \theta_2 = \theta_0$ (constant)

where θ_1 and θ_2 are the inclinations of the tangents from positive x-axis.

E-2. If tangent at P and Q to the parabola $y^2 = 4ax$ intersect at R then prove that mid point of R and M lies on the parabola, where M is the mid point of P and Q.

E-3. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. Find the point on this line from which the other tangents to the parabola is perpendicular to the given tangent.

E-4 If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

E-5. Find locus of the middle points of the focal chords of the parabola, $y^2 = 4x$.

Section (F) : Pair of tangents, Director circle, chord of contact and chord with given middle point of Ellipse & Hyperbola

F-1. Find equation of the chord of the ellipse $2x^2 + 5y^2 = 20$ which is bisected at the point (2, 1)

F-2. How many real tangents can be drawn from the point (4, 3) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the equation of these tangents & angle between them.

F-3. A chord PQ of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ subtends right angle at its centre. Find locus of the point of intersection of tangents drawn at P and Q.

F-4. Find the locus of the middle points of chords of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are drawn through the positive end of the minor axis.

F-5. Find equation to the chord of the hyperbola $x^2 - y^2 = 9$ which is bisected at (5, – 3)

F-6. Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line $x + y = b$.

F-7. The chords passing through L(2, 1) intersects the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at P and Q. If the tangents at P and Q intersects at R then find locus of R.

F-8. Prove that the locus of the middle points of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which pass through a fixed point (α, β) is a hyperbola whose centre is $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

F-9. If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Section (G) : Equation of normal, co-normal points of parabola

G-1. Find the equation of a normal at the parabola $y^2 = 4x$ which is parallel to the line $y = 3x + 4$

G-2. If the line $x + y = 1$ is a normal to the parabola $y^2 = kx$, then the value of k is-

G-3. The normal at the point $P(ap^2, 2ap)$ meets the parabola $y^2 = 4ax$ again at $Q(aq^2, 2aq)$ such that the lines joining the origin to P and Q are at right angle. Then prove that $p^2 = 2$.

G-4. If a line $x + y = 1$ cut the parabola $y^2 = 4ax$ in points A and B and normals drawn at A and B meet at C (C does not lies on parabola). The normal to the parabola from C other, than above two meet the parabola in D, then find D

G-5. If normal of circle $x^2 + y^2 + 6x + 8y + 9 = 0$ intersect the parabola $y^2 = 4x$ at P and Q then find the locus of point of intersection of tangent's at P and Q.

G-6. If two normals drawn from any point to the parabola $y^2 = 4ax$ make angle α and β with the axis such that $\tan\alpha . \tan\beta = 2$, then find the locus of this point.

Section (H) : Equation of normal, co-normal points of Ellipse & Hyperbola

H-1 If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e^4 + e^2 - 1 = 0$

H-2. Find condition for which line $x\cos\alpha + y\sin\alpha = p$ be normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

H-3. P and Q are corresponding points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b

H-4 The normal at P to a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M respectively. Show that the locus of the middle point of LM is a hyperbola of eccentricity $\frac{e}{\sqrt{(e^2-1)}}$.

H-5. A ray originating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lying in first quadrant.

PART-II : OBJECTIVE QUESTIONS

Section (A) : Elementary concepts of Parabola

A-1. The directrix and axis of the parabola $4y^2 - 6x - 4y = 5$ are respectively-

(A) $8x + 11 = 0$; $y - 1 = 0$ (B) $8x - 11 = 0$; $2y - 1 = 0$

(C) $8x + 11 = 0$; $2y - 1 = 0$ (D) None of these

A-2. The length of latus rectum of a parabola, whose focus is (2, 3) and directrix is the line $x - 4y + 3 = 0$ is -

(A) $\frac{7}{\sqrt{17}}$ (B) $\frac{14}{\sqrt{21}}$ (C) $\frac{7}{\sqrt{21}}$ (D) $\frac{14}{\sqrt{17}}$

A-3. Vertex, focus, latus rectum, length of the latus rectum and equation of directrix of the parabola $y^2 = 4x + 4y$ are-

(A) (1, 2), (0, 2), y = 0, 4, x = –2 (B) (–1, 2), (0, 2), x = 0, 4, x = –2
(C) (–1, 2), (1, 2), x = 0, 4, x = 2 (D) (–1, 2) (0, 2), y = 0, 2, y = –2

A-4. Length of the latus rectum of the parabola $25[(x-2)^2 + (y-3)^2] = (3x - 4y + 7)^2$ is:

(A) 4 (B) 2 (C) 1/5 (D) 2/5

A-5. A parabola is drawn with its focus at (3, 4) and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is:

(A) $x^2 - 6x - 8y + 25 = 0$ (B) $y^2 - 8x - 6y + 25 = 0$
(C) $x^2 - 6x + 8y - 25 = 0$ (D) $x^2 + 6x - 8y - 25 = 0$

A-6. The vertex of a parabola is the point (a, b) and latus rectum is of length 1. If the axis of the parabola is along the positive direction of y-axis, then its equation is :

(A) $(x + a)^2 = \frac{1}{2}(2y - 2b)$ (B) $(x - a)^2 = \frac{1}{2}(2y - 2b)$

(C) $(x + a)^2 = \frac{1}{4}(2y - 2b)$ (D) $(x - a)^2 = \frac{1}{8}(2y - 2b)$

A-7. For the parabola $y^2 + 8x - 12y + 20 = 0$ which of the following is not correct-

(A) vertex (2, 6) (B) focus (0, 6)
(C) length of the latus rectum = 4 (D) axis is y = 6

A-8. The area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum is-

(A) 16 sq. units (B) 12 sq. units (C) 18 sq. units (D) 24 sq. units

A-9. The length of L.R. of the parabola $y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$ is-

(A) $\frac{2u^2\cos^2\alpha}{g}$ (B) $\frac{u^2\sin^2 2\alpha}{g}$ (C) $\frac{u^2\cos^2 2\alpha}{g}$ (D) None of these

A-10. Which one of the following equations parametrically represents equation to a parabolic profile?

(A) $x = 3\cos t;\ y = 4\sin t$ (B) $x^2 - 2 = -2\cos t;\ y = 4\cos^2\frac{t}{2}$

(C) $\sqrt{x} = \tan t;\ \sqrt{y} = \sec t$ (D) $x = \sqrt{1 - \sin t}\,;\ y = \sin\frac{t}{2} + \cos\frac{t}{2}$

A-11. The points on the parabola $y^2 = 12x$ whose focal distance is 4, are

(A) $(2, \sqrt{3}), (2, -\sqrt{3})$ (B) $(1, 2\sqrt{3}), (1, -2\sqrt{3})$ (C) (1, 2), (2, 1) (D) $(2, 2\sqrt{3}), (3, -2\sqrt{3})$

A-12. If on a given base, a triangle be described such that the sum of the tangents of the base angles is a constant, then the locus of the vertex is:

(A) a circle (B) a parabola (C) an ellipse (D) a hyperbola

A-13. The angle made by a double ordinate of length 8a at the vertex of the parabola $y^2 = 4ax$ is :

(A) $\pi/3$ (B) $\pi/2$ (C) $\pi/4$ (D) $\pi/6$

Section (B) : Elementary concepts of Ellipse & Hyperbola

B-1. Equation of the ellipse whose focus is (6, 7) directrix is x + y + 2 = 0 and e = $1/\sqrt{3}$ is-

(A) $5x^2 + 2xy + 5y^2 - 76x - 88y + 506 = 0$ (B) $5x^2 - 2xy + 5y^2 - 76x - 88y + 506 = 0$
(C) $5x^2 - 2xy + 5y^2 + 76x - 88y + 506 = 0$ (D) None of these

B-2. Eccentricity of an ellipse of which distance between the focii is 10 and that of focus and corresponding directrix is 15, is.

(A) $\frac{1}{2}$ (B) $\frac{2}{5}$ (C) $\frac{3}{4}$ (D) $\frac{6}{7}$

B-3. If distance between the directrices be thrice the distance between the focii, then eccentricity of ellipse is

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{4}{5}$

B-4. The equation $\frac{x^2}{10-a} + \frac{y^2}{4-a} = 1$ represents an ellipse if-

(A) a < 4 (B) a > 4 (C) 4 < a < 10 (D) a > 10

B-5. The equation of the ellipse which passes through origin and has its foci at the points (1, 0) and (3, 0) is-

(A) $3x^2 + 4y^2 = x$ (B) $3x^2 + y^2 = 12x$ (C) $x^2 + 4y^2 = 12x$ (D) $3x^2 + 4y^2 = 12x$

B-6. An arc of a bridge is semi-elliptical with major axis horizontal. The length of the base is 9 meter and the highest part of the bridge is 3 meter from the horizontal. The best approximation of the Pillar 2 meter from the centre of the base is :

(A) 11/4 m (B) 8/3 m (C) 7/2 m (D) 2 m

B-7. The position of the point (1, 3) with respect to the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$

(A) outside the ellipse (B) on the ellipse (C) on the major axis (D) on the minor axis

B-8. **Statement-1 :** Eccentricity of ellipse whose length of latus rectum is same as distance between foci is 2sin18°.

Statement-2 : For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, eccentricity e = $\sqrt{1-\frac{b^2}{a^2}}$.

(A)STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

(E) Both STATEMENTS are false

B-9 The conic represented by x = 2 (cos t + sin t), y = 5 (cos t – sin t) is

(A) a circle (B) a parabola (C) an ellipse (D) a hyperbola

B-10. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) 1/2

B-11. The foci of the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$ are

(A) (2, 3), (5, 7) (B) (4, 1), (–6, 1) (C) (0, 0), (5, 3) (D) (–4, 1), (6, 1)

B-12. If vertex and focus of hyperbola are (2, 3) and (6, 3) respectively and eccentricity e of the hyperbola is 2, then equation of the hyperbola is :

(A) $\frac{(x+1)^2}{16} - \frac{(y-3)^2}{48} = 1$ (B) $\frac{(x+2)^2}{9} - \frac{(y-3)^2}{27} = 1$

(C) $\frac{(x+2)^2}{16} - \frac{(y-3)^2}{48} = 1$ (D) $\frac{(x+2)^2}{6} - \frac{(y-3)^2}{27} = 1$

B-13. If e and e′ are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, then the point $\left(\frac{1}{e}, \frac{1}{e'}\right)$ lies on the circle :

(A) $x^2 + y^2 = 1$ (B) $x^2 + y^2 = 2$ (C) $x^2 + y^2 = 3$ (D) $x^2 + y^2 = 4$

B-14. Which of the following pair, may represent the eccentricities of two conjugate hyperbolas, for all $\alpha \in (0, \pi/2)$?

(A) $\sin \alpha$, $\cos \alpha$ (B) $\tan \alpha$, $\cot \alpha$ (C) $\sec \alpha$, $\operatorname{cosec} \alpha$ (D) $1 + \sin \alpha$, $1 + \cos \alpha$

B-15. **Statement-1 :** If $\sec\theta$, $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ represent eccentricity of a hyperbola then eccentricity of its conjugate hyperbola is given by $\operatorname{cosec}\theta$.

Statement-2 : If e_1, e_2 are eccentricities of two hyperbolas which are conjugate to each other then $e_1^{-2} + e_2^{-2} = 1$

(A)STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

(E) Both STATEMENTS are false

B-16. The equation $\frac{x^2}{12-\lambda} + \frac{y^2}{8-\lambda} = 1$ represents -

(A) a hyperbola if $\lambda < 8$ (B) an ellipse if $\lambda > 0$

(C) a hyperbola if $8 < \lambda < 12$ (D) none of these

B-17. If foci of a hyperbola are foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If the eccentricity of the hyperbola be 2, then its equation is -

(A) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (B) $\frac{x^2}{12} - \frac{y^2}{4} = 1$

(C) $\frac{x^2}{12} + \frac{y^2}{4} = 1$ (D) $\frac{x^2}{12} - \frac{y^2}{8} = 1$

B-18. The parametric representation of a point on the ellipse whose foci are (–1, 0) and (7, 0) and eccentricity 1/2 is-

(A) $(3 + 8\cos\theta, 4\sqrt{3}\cos\theta)$ (B) $(3+ 8\cos\theta, 4\sqrt{3}\sin\theta)$

(C) $(3 + 4\sqrt{3}\cos\theta, 8\sin\theta)$ (D) None of these

B-19. If P $(\sqrt{2}\sec\theta, \sqrt{2}\tan\theta)$ is a point on the hyperbola whose distance from the origin is $\sqrt{6}$ where P is in the first quadrant then θ =

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{15}$

B-20. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is-

(A) 9 (B) 1 (C) 5 (D) 7

B-21. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α varies ?

(A) Abscissae of vertices (B) Abscissae of foci

(C) Eccentricity (D) Directrix

Section (C) : Position of line, Equation of chord and various forms of tangents of parabola

C-1. The locus of point of trisections of the focal chords of the parabola, $y^2 = 4x$ is:

(A) $y^2 = x - 1$ (B) $9y^2 = 4.(3x - 4)$ (C) $y^2 = 2(1 - x)$ (D) None of these

C-2 In the parabola $y^2 = 6x$, the equation of the chord through vertex and negative end of latus rectum, is-

(A) $x = 2y$ (B) $y + 2x = 0$ (C) $y = 2x$ (D) $x + 2y = 0$

C-3 If PSQ is the focal chord of the parabola $y^2 = 8x$ such that SP = 6. Then the length SQ is-

(A) 4 (B) 6 (C) 3 (D) None of these

C-4. If focal chord of $y^2 = 16x$ touches $(x - 6)^2 + y^2 = 2$ then slope of such chord is-

(A) 1, –1 (B) $2, -\frac{1}{2}$ (C) $\frac{1}{2}, -2$ (D) 2, –2

C-5 If a tangent to the parabola $4y^2 = x$ makes an angle of 60° with the x-axis, then its point of contact is-

(A) $\left(\frac{1}{48}, \frac{1}{8\sqrt{3}}\right)$ (B) $\left(\frac{3}{16}, \frac{\sqrt{3}}{8}\right)$

(C) $\left(\frac{1}{48}, -\frac{1}{8\sqrt{3}}\right)$ (D) $\left(\frac{3}{16}, -\frac{\sqrt{3}}{8}\right)$

C-6 A tangent to the parabola $y^2 = 4ax$ at P(p, q) is perpendicular to the tangent at the other point Q, then coordinates of Q are-

(A) $(a^2/p, -4a^2/q)$ (B) $(-a^2/p, -4a^2/q)$ (C) $(-a^2/p, 4a^2/q)$ (D) $(a^2/p, 4a^2/q)$

C-7. The feet of the perpendicular drawn from focus upon any tangent to the parabola, $y = x^2 - 2x - 3$ lies on

(A) $y + 4 = 0$ (B) $y = 0$ (C) $y = -2$ (D) $y + 1 = 0$

C-8. The common tangent of the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$ is-

(A) $y = x + a$ (B) $y = x - a$ (C) $y = x - 2a$ (D) $y = x + 2a$

C-9. Identify following statements for true/false (T/F) in order

S1 : The circles on focal radii of a parabola as diameter touch the tangent at the vertex

S2 : The circles on focal radii of a parabola as diameter touch the axis

S3 : A circle described on any focal chord of the parabola as its diameter will touch the directrix of the parabola

S4 : A circle described on any focal chord of the parabola as its diameter will touch the axis of the parabola

(A) TTFF (B) TFTF (C) FFTT (D) FTFT

C-10. Identify the following statements for true/false (T/F) in order

S1 : The tangents at the extremities of a focal chord of a parabola are perpendicular

S2 : The tangents at the extremities of a focal chord of a parabola are parallel

S3 : The tangents at the extremities of a focal chord of a parabola intersect on the directrix

S4 : The tangents at the extremities of a focal chord of a parabola intersect at the vertex

(A) TFTF (B) TTFF (C) TTTT (D) FFFF

C-11. M is the foot of the perpendicular from a point P on the parabola $y^2 = 8(x - 3)$ to its directrix and S is the focus of the parabola, if SPM is an equilateral triangle, the length of each side of the triangle is-

(A) 2 (B) 3 (C) 4 (D) 8

C-12. Range of c for which the line $y = mx + c$ touches the parabola $y^2 = 8(x + 2)$, is

(A) $(-4, 4]$ (B) $(-\infty, -4] \cup [4, \infty)$ (C) $(-\infty, -2] \cup [2, \infty)$ (D) $[4, \infty)$

C-13. The circle drawn with variable chord $x + ay - 5 = 0$ (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line -

(A) $x + 5 = 0$ (B) $y + 5 = 0$ (C) $x + y + 5 = 0$ (D) $x - y + 5 = 0$

Section (D) : Position of line, Equation of chord and various forms of tangents of Ellipse & Hyperbola

D-1. For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

(A) ± 5 (B) ± 4 (C) ± 3 (D) ± 2

D-2. If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point P, then eccentric angle of P is

(A) 0 (B) 45° (C) 60° (D) 90°

D-3. If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepts lengths h and k on the axes then-

(A) $\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$ (B) $\frac{h^2}{a^2} + \frac{k^2}{b^2} = 2$ (C) $\frac{a^2}{h^2} + \frac{b^2}{k^2} = 1$ (D) $\frac{a^2}{h^2} + \frac{b^2}{k^2} = 2$

D-4. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its corresponding point Q on the auxiliary circle, lies on the line :

(A) $x = a/e$ (B) $x = 0$ (C) $y = 0$ (D) none of these

D-5. At which of the following points of the hyperbola $x^2 - y^2 = 3$, tangent is parallel to the line $2x + y + 8 = 0$?

(A) (2, 1) (B) (2, –1) (C) (–2, –1) (D) (1, 3)

D-6. The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$ is

(A) $\sqrt{\frac{17}{20}}$ (B) $\sqrt{\frac{20}{17}}$ (C) $\sqrt{\frac{3}{20}}$ (D) $\sqrt{\frac{3}{20}}$

D-7. If $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equal to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{1+e}{1-e}$ (D) $\frac{e+1}{e-1}$

D-8. If the straight line $2x + \sqrt{2}y + n = 0$ touches the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, then the value of n is equal to

(A) ± 0.5 (B) ± 1 (C) ± 2 (D) ± 4

D-9. If F_1 & F_2 are the feet of the perpendiculars from the focii S_1 & S_2 of an ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$ on the tangent at any point P on the ellipse, then $(S_1F_1).(S_2F_2)$ is equal to :

(A) 2 (B) 3 (C) 4 (D) 5

Section (E) : Pair of tangents, Director circle, chord of contact and chord with given middle point of Parabola

E-1. The area of triangle made by the chord of contact and tangents drawn from point (4, 6) to the parabola $y^2 = 8x$ is-

(A) $\frac{1}{2}$ (B) 2 (C) $\frac{1}{2}\sqrt{3}$ (D) 1

E-2 Locus of perpendicular tangents of the parabola $x^2 - 4x + 2y + 3 = 0$ is :

(A) x = 1 (B) y = 1 (C) x = 2 (D) y = 2

E-3. Angle between the tangents drawn from (1, 4) to the parabola $y^2 = 4x$ is-

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

Section (F) : Pair of tangents, Director circle, chord of contact and chord with given middle point of Ellipse & Hyperbola

F-1. The angle between the pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2) is-

(A) $\tan^{-1}\left(\frac{12}{5}\right)$ (B) $\tan^{-1}(6\sqrt{5})$ (C) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (D) $\tan^{-1}(12\sqrt{5})$

F-2. If 3x + 4y = 12 intersect the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at P and Q, then point of intersection of tangents at P and Q is-

(A) $\left(6, \frac{16}{3}\right)$ (B) $\left(\frac{25}{4}, 4\right)$ (C) $\left(\frac{25}{4}, \frac{16}{3}\right)$ (D) $\left(\frac{25}{3}, 4\right)$

F-3. Equation of chord of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ whose mid point is (3, 1) is.

(A) 48x + 25y – 169 = 0 (B) 25x + 48y – 169 = 0
(C) 48x – 25y – 119 = 0 (D) 25x + 48y – 123 = 0

F-4. The number of points from where a pair of perpendicular tangents can be drawn to the hyperbola, $x^2 \sec^2 \alpha - y^2 \operatorname{cosec}^2 \alpha = 1$, $\alpha \in (0, \pi/4)$, is :

(A) 0 (B) 1 (C) 2 (D) infinite

F-5. What will be the equation of the chord of hyperbola $25x^2 - 16y^2 = 400$, whose mid point is (5, 3) -

(A) 115x – 117y = 17 (B) 125x – 48y = 481 (C) 127x + 33y = 341 (D) 15x + 121y = 105

F-6. Locus of the mid - point of focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is-

(A) $\frac{x^2}{a^2} - \frac{y^2}{b^2} \pm \frac{ex}{a} = 0$ (B) $\frac{x^2}{a^2} - \frac{y^2}{b^2} \pm \frac{ey}{a} = 0$

(C) $\frac{x^2}{b^2} - \frac{y^2}{a^2} \pm \frac{ex}{a} = 0$ (D) $\frac{x^2}{b^2} - \frac{y^2}{a^2} \pm \frac{ey}{a} = 0$

F-7. The tangents from $(1, 2\sqrt{2})$ to the hyperbola $16x^2 - 25y^2 = 400$ include between them an angle equal to:

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

F-8. The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to y = 2x is

(A) 3x – 4y = 4 (B) 3y – 4x + 4 = 0 (C) 4x – 4y = 3 (D) 3x – 4y = 2

Section (G) : Equation of normal, co-normal points of parabola

G-1. If the normal of the parabola $y^2 = 4ax$ drawn at (a, 2a) meets the parabola again at point $(at^2, 2at)$ then t is equal to-

(A) 3 (B) 1 (C) –1 (D) –3

G-2. Which of the following lines, is a normal to the parabola $y^2 = 16x$

(A) $y = x - 11 \cos\theta - 3\cos3\theta$ (B) $y = x - 11 \cos\theta - \cos3\theta$
(C) $y = (x - 11) \cos\theta + \cos3\theta$ (D) $y = (x - 11) \cos\theta - \cos3\theta$

G-3 If two of the normal of the parabola $y^2 = 4x$, that pass through (15, 12) are 4x + y = 72, and 3x – y = 33, then the third normal is-

(A) y = x – 3 (B) x + y = 3 (C) y = x + 3 (D) y = x + 4

G-4. At what point on the parabola $y^2 = 4x$ the normal makes equal angles with the axes?

(A) (4, 4) (B) (9, 6) (C) (4, – 1) (D) (1, 2)

G-5. The equation of the other normal to the parabola $y^2 = 4ax$ which passes through the intersection of those at (4a, –4a) & (9a, –6a) is:

(A) $5x - y + 115a = 0$ (B) $5x + y - 135a = 0$

(C) $5x - y - 115a = 0$ (D) $5x + y + 115 = 0$

G-6. The area of triangle formed by tangent and the normal to the parabola $y^2 = 4ax$. both drawn at the same end of the latus rectum, and the axis of the parabola is-

(A) $2\sqrt{2}\,a^2$ (B) $2a^2$ (C) $4a^2$ (D) None of these

G-7 If m is the slope of tangent drawn at one extremity (–3, 2) of focal chord of the parabola $y^2 + 4x + 4y = 0$, then the slope of normal at the other extremity is-

(A) m (B) –1/m (C) 1/m (D) None of these

G-8. The equation of the normal having slope m of the parabola $y^2 = x + a$ is-

(A) $y = mx - am - am^3$ (B) $y = mx - 2am - am^3$

(C) $4y = 4mx + 4am - 2m - m^3$ (D) $4y = 4mx + 2am - am^3$

Section (H) : Equation of normal, co-normal points of Ellipse & Hyperbola

H-1. P & Q are corresponding points on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and the auxiliary circle respectively. The normal at P to the ellipse meets CQ in R where C is centre of the ellipse. Then ℓ (CR) is

(A) 5 units (B) 6 units (C) 7 units (D) 8 units

H-2. A ray emanating from the point (– 4, 0) is incident on the ellipse $9x^2 + 25y^2 = 225$ at the point P with abscissa 3. The equation of the reflected ray after first reflection is-

(A) $5x \pm 12y = 48$ (B) $6x \pm 13y = 48$ (C) $13x \pm 6y = 48$ (D) $12x \pm 5y = 48$

H-3. The tangent & normal at a point on $x^2/a^2 - y^2/b^2 = 1$ cut the y – axis respectively at A & B. A circle on AB as diameter passes through -

(A) Foci (B) Vertices

(C) End points of conjugate axis (D) None of these

H-4 The locus of the foot of perpendicular drawn from the centre of the hyperbola $x^2 - y^2 = 25$ to its normal is-

(A) $100x^2y^2 = (x^2 + y^2)^2 (y^2 - x^2)$ (B) $10x^2y^2 = (x^2 + y^2)^2 (y^2 - x^2)$

(C) $200x^2y^2 = (x^2 - y^2)^2 (y^2 + x^2)$ (D) $100x^2y^2 = (x^2 - y^2)^2 (y^2 + x^2)$

H-5 The value of $|\lambda|$, for which the line $2x - \frac{8}{3}\lambda y = -3$ is a normal to the conic $x^2 + \frac{y^2}{4} = 1$ is –

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{3}{8}$

PART-III : MATCH THE COLUMN

1. AB is a chord of the parabola $y^2 = 4ax$ joining $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$. Match the following

Column – I		Column – II	
(A)	AB is a normal chord	(p)	$t_2 = -t_1 + 2$
(B)	AB is a focal chord	(q)	$t_2 = -\frac{4}{t_1}$
(C)	AB subtends 90º at (0, 0)	(r)	$t_2 = -\frac{1}{t_1}$
(D)	AB is inclined at 45º to the axis of parabola	(s)	$t_2 = -t_1 - \frac{2}{t_1}$

2. Normals are drawn at point P, Q and R lying on the parabola $y^2 = 4x$ which intersect at (3, 0). Then

Column - I		Column - I	
(A)	Area of ΔPQR	(p)	2
(B)	Radius of circumcircle of ΔPQR	(q)	5/2
(C)	Centroid of ΔPQR	(r)	(5/2, 0)
(D)	Circumcentre of ΔPQR	(s)	(2/3, 0)

3.

Column-I		Column-II	
(A)	End points of a stick 'AB' of length 10m slides on the coordinate axes (Point A on x-axis) then locus of the point M dividing this stick such that $\frac{BM}{AM} = \frac{3}{2}$ is a curve whose eccentricity is e, then 3e is	(p)	$\sqrt{6}$
(B)	AA' is the major axis of the ellipse $3x^2 + 2y^2 + 6x - 4y - 1 = 0$ and P is a variable point on ellipse, then greatest area of ΔAPA' is	(q)	$2\sqrt{7}$
(C)	The distance between the foci of the curve represented by the equation $x = 1 + 4\cos\theta$, $y = 2 + 3\sin\theta$ is	(r)	$\frac{128}{3}$
(D)	Tangents are drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ at the end points of the latusrectum, the area of the quadrilateral so formed is	(s)	$\sqrt{5}$

4. **Match the column**

Column – I	Column – II
(A) If the mid point of a chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is (0, 3), then length of the chord is $\frac{4k}{5}$, then k is	(p) 6
(B) Sum of distances of a point on the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ from the focii	(q) 8
(C) Number of positive integral values of b for which tangent parallel to line y = x + 1 can be drawn to hyperbola $\frac{x^2}{5}-\frac{y^2}{b^2}=1$ is	(r) 2
(D) The product of the lengths of the perpendiculars from the two focii on any tangent to the hyperbola $\frac{x^2}{25}-\frac{y^2}{3}=1$ is k, then 2k is	(s) 16

5. AB is a chord to the curve S and C is a point on line AB.

Column – I	Column – II
(A) If $S \equiv \frac{x^2}{9}+\frac{y^2}{16}-1=0$, A(0, 3) & AC : AB = 2 : 1 then locus of C is	(p) $x^2+y^2+12x+16y-125=0$
(B) If $S \equiv y^2-12x=0$, A(0, 0) & AC : AB = 1 : 3 then locus of C is	(q) $x^2-y^2-4x=0$
(C) If $S \equiv x^2-y^2-16=0$, A(4, 0) & AC : AB = 1 : 2 then locus of C is	(r) $4x-y^2=0$
(D) If $S \equiv x^2+y^2-25=0$, A(3, 4) & AC : AB = 3 : 1 then locus of C is	(s) $16x^2+9y^2+54y-495=0$

Exercise # 2

PART-I : OBJECTIVE QUESTIONS

1. A parabola has its vertex and focus in the first quadrant and axis along the line y = x. If the distances of the vertex and focus from the origin are respectively $\sqrt{2}$ and $2\sqrt{2}$, then an equation of the parabola is

(A) $(x + y)^2 = x - y + 2$ (B) $(x - y)^2 = x + y - 2$

(C) $(x - y)^2 = 8(x + y - 2)$ (D) $(x + y)^2 = 8(x - y + 2)$

2. Range of a for which the point (–2a, a + 1) will be an interior point of the smaller region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 4x$, is :

(A) $(-1, \infty)$ (B) $(-5 - 2\sqrt{6}, -5 + 2\sqrt{6})$

(C) $(-1, -5 + 2\sqrt{6})$ (D) $(1, 5 + 2\sqrt{6})$

3. The equation of locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 4ax$ is-

(A) $y^2 = 2a(x - a)$ (B) $y^2 = 2a(x - 2a)$ (C) $y^2 = 2a\left(x - \frac{a}{2}\right)$ (D) $(y - a)^2 = 2a\left(x - \frac{a}{2}\right)$

4. A variable chord of the parabola $y^2 = 24x$ which subtend 90° angle at its vertex always passes through a fixed point whose distance from the origin is equal to :

(A) 24 (B) 12 (C) 6 (D) 48

5. If P_1Q_1 and P_2Q_2 are two focal chords of the parabola $y^2 = 4ax$. then the chords P_1P_2 and Q_1Q_2 intersect on

(A) tangent at the vertex of the parabola (B) the directrix of the parabola

(C) at x = –2a (D) y = 2a and x = –2a

6. If P, Q, R are three points on a parabola $y^2 = 4ax$ whose ordinates are in geometrical progression, then the tangents at P and R meet on-

(A) the line through Q parallel to x-axis (B) the line through Q parallel to y-axis

(C) the line joining Q to the vertex (D) the line joining Q to the focus.

7. If two tangents to the parabola $y^2 = 8x$ meet the tangent at its vertex in the points P & Q. If PQ = 4 units, then the locus of the point of the intersection of the two tangents is -

(A) $y^2 = 8(x - 2)$ (B) $y^2 = 8(x + 2)$ (C) $y^2 = 8(x + 4)$ (D) $x^2 = 8(y + 2)$

8. Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then,

(A) C_1 and C_2 touch each other only at one point

(B) C_1 and C_2 touch each other exactly at two points

(C) C_1 and C_2 intersect (but do not touch) at exactly two points

(D) C_1 and C_2 neither intersect nor touch each other

9. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola at the point (1, 2) is

(A) $(x - 1)^2 = 4(y - 2)$ (B) $(x + 3)^2 = 4(y + 2)$

(C) $(x + 1)^2 = 4(y - 1)$ (D) $(x - 1)^2 = 4(y - 1)$

10. The centre of the circle which passes through the focus of the parabola $x^2 = 4y$ & touches it at the point (6, 9) is

(A) (–9, –14) (B) (–9, 14) (C) (8, 15) (D) (–8, 15)

11. If a parabola whose length of latus rectum is 4a touches both the coordinate axes then the locus of its focus is

(A) $xy = a^2(x^2 + y^2)$ (B) $x^2y^2 = a^2(x^2 + y^2)$

(C) $x^2 - y^2 = a^2(x^2 + y^2)$ (D) $x^2y^2 = a^2(x^2 - y^2)$

12. In the parabola $y^2 = 4ax$, the locus of middle points of all chords of constant length c is

(A) $(4ax - y^2)(y^2 - 4a^2) = a^2c^2$ (B) $(4ax + y^2)(y^2 + 4a^2) = a^2c^2$

(C) $(4ax + y^2)(y^2 - 4a^2) = a^2c^2$ (D) $(4ax - y^2)(y^2 + 4a^2) = a^2c^2$

13. Locus of a point P if the three normals drawn from it to the parabola $y^2 = 4ax$ are such that two of them make complementry angles with the axis of the parabola, is-

(A) $x^2 = a(y - a)$ (B) $y^2 = 4a(x - 2a)$ (C) $y^2 = a(x - a)$ (D) $y^2 = 4a(x - 3a)$

14. If two normals to a parabola $y^2 = 4ax$ intersect at right angles then the chord joining their feet passes through a fixed point whose co-ordinates are:

(A) (– 2a, 0) (B) (a, 0) (C) (2a, 0) (D) (–a, 0)

15. Locus of the middle points of normal chords of the parabola $y^2 = 4ax$ is-

(A) $y^4 - 2a(x - 2a).\, y^2 + 8a^4 = 0$ (B) $x^4 - 2a(y - 2a).\, x^2 + 8a^2 = 0$

(C) $y^2 - 2a(x - 2a).\, y + 8a^4 = 0$ (D) $x^2 - 2a(y - 2a).\, x + 8a^2 = 0$

16. A line of fixed length (a + b) moves so that its ends are always on two fixed perpendicular straight lines. The locus of the point which divided this line into portions of lengths a & b, is:

(A) an ellipse (B) an hyperbola (C) a circle (D) a straight line

17. Set of all possible value of α for which point $P(\alpha, 3\alpha)$ lies on the smaller region of the ellipse $9x^2 + 16y^2 = 144$ divided by the line $3x + 4y = 12$ is-

(A) $\frac{3}{5} < \alpha < \frac{3}{\sqrt{10}}$ (B) $\frac{4}{5} < \alpha < \frac{4}{\sqrt{17}}$

(C) $\frac{2}{5} < \alpha < \frac{4}{\sqrt{17}}$ (D) $0 < \alpha < 1$

18. Condition such that the line $px + qy = r$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points whose eccentric angles differ by $\frac{\pi}{4}$ is $a^2p^2 + b^2q^2 = k.r^2$ **then k is equal to**

(A) $(4 - 2\sqrt{2})$ (B) $(4 + 2\sqrt{2})$ (C) 4 (D) $2\sqrt{2}$

19. $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ & $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$. Then the value of 'b' and the other common tangent are given by :

(A) $b = \sqrt{3}$; $x + 2y + 4 = 0$ (B) $b = 3$; $x + 2y + 4 = 0$

(C) $b = \sqrt{3}$; $x + 2y - 4 = 0$ (D) $b = \sqrt{3}$; $x - 2y - 4 = 0$

20. An ellipse with major axis 4 and minor axis 2 touches both the coordinate axis, then Locus of its centre is

(A) $x^2 - y^2 = 5$ (B) $x^2.y^2 = 5$ (C) $\frac{x^2}{4} + y^2 = 5$ (D) $x^2 + y^2 = 5$

21. An ellipse with major axis 4 and minor axis 2 touches both the coordinate axes, then locus of its focus is

(A) $(x^2 - y^2)(1 + x^2y^2) = 16x^2y^2$ (B) $(x^2 - y^2)(1 - x^2y^2) = 16x^2y^2$.

(C) $(x^2 + y^2)(1 + x^2y^2) = 16x^2y^2$ (D) $(x^2 + y^2)(1 - x^2y^2) = 16x^2y^2$.

22. Find the locus of the point, the chord of contact of the tangents drawn from which to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the circle $x^2 + y^2 = c^2$, where $c < b < a$.

(A) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{c^2}$ (B) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^4}$ (C) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$ (D) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{c^4}$

23. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectum of the given ellipse at the points

(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$ (C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

24. Let two variable ellipse E_1 and E_1 touches each other externally at (0, 0). Their common tangent at (0, 0) is y = x. If one of the focus at E_1 & one of the focus of E_1 always lies or line y = 2x then find locus of other focus of E_1 & E_1.

(A) y = 4x (B) y = –2x (C) y = x/2 (D) y = –x/2

25. Find an equation of the hyperbola whose directrix is the normal to circle $x^2 + y^2 - 4x - 6y + 9 = 0$ having slope is 2 and eccentricity is equal to radius of given circle when focus of hyperbola is centre of the given circle.

(A) $11x^2 - y^2 + 4x - 8xy + 19y - 61 = 0$ (B) $7x^2 - y^2 + 2x - 16xy + 38y - 50 = 0$

(C) $11x^2 - y^2 + 4x - 16xy + 38y - 61 = 0$ (D) $11x^2 - y^2 + 4x - 16xy + 38y + 61 = 0$

26. If AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔOAB (O is the origin) is an equilateral triangle, then eccentricity 'e' of the hyperbola.

(A) is greater than $\frac{2}{\sqrt{3}}$ (B) is less than $\frac{2}{\sqrt{3}}$ (C) is equal to $\frac{2}{\sqrt{3}}$ (D) is less than $\frac{1}{\sqrt{3}}$

27. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is

(A) $x^2\,\text{cosec}^2\theta - y^2\sec^2\theta = 1$ (B) $x^2\sec^2\theta - y^2\,\text{cosec}^2\theta = 1$

(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$ (D) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

28. If two points P & Q on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ whose centre is C be such that CP is perpendicular to CQ & a < b, then

(A) $\frac{1}{CP^2} + \frac{1}{CQ^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right)$ (B) $\frac{1}{CP^2} + \frac{1}{CQ^2} = 2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$

(C) $\frac{1}{CP^2} + \frac{1}{CQ^2} = 3\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$ (D) $\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$

29. A point moves so that the sum of the squares of its distances from two intersecting straight lines whose equations are $y = x\tan\theta$, $y = -x\tan\theta$ is constant $(2\lambda^2)$, then equations of its locus is-

(A) $\frac{x^2}{(\lambda\sin\theta)^2}+\frac{y^2}{(\lambda\cos\theta)^2}=1$

(B) $\frac{x^2}{(\lambda\operatorname{cosec}\theta)^2}-\frac{y^2}{(\lambda\sec\theta)^2}=1$

(C) $\frac{x^2}{(\lambda\sin\theta)^2}-\frac{y^2}{(\lambda\cos\theta)^2}=1$

(D) $\frac{x^2}{(\lambda\operatorname{cosec}\theta)^2}+\frac{y^2}{(\lambda\sec\theta)^2}=1$

30. A point moves such that the sum of the squares of its distances from the two sides of length 'a' of a rectangle is twice the sum of the squares of its distances from the other two sides of length 'b' ($b^2 \neq 2a^2$). The locus of the point can be :

(A) a circle (B) an ellipse (C) a hyperbola (D) a pair of lines

31. Tangent at any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cut the axes at A and B respectively. If the rectangle OAPB (where O is origin) is completed then locus of point P is given by

(A) $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$ (B) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ (C) $\frac{a^2}{y^2}-\frac{b^2}{x^2}=1$ (D) none of these

32. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ to the circle $x^2+y^2=9$, then the locus of mid-point of the chord of contact is

(A) $\frac{x^2}{9}-\frac{y^2}{4}=\left(\frac{x^2+y^2}{9}\right)^2$

(B) $\frac{x^2}{9}+\frac{y^2}{4}=\left(\frac{x^2+y^2}{9}\right)^2$

(C) $\frac{x^2}{9}+\frac{y^2}{4}=\left(\frac{x^2-y^2}{9}\right)^2$

(D) $\frac{x^2}{9}-\frac{y^2}{4}=\left(\frac{x^2-y^2}{9}\right)^2$

33. A circle of variable radius with centre (h, k) cuts the rectangular hyperbola $x^2 - y^2 = 9a^2$ in points P, Q, R and S. Then equation of locus of the centroid of triangle PQR is

(A) $\left(x-\frac{h}{3}\right)^2-\left(y-\frac{k}{3}\right)^2=a^2$

(B) $\left(x-\frac{2h}{3}\right)^2-\left(y-\frac{2k}{3}\right)^2=a^2$

(C) $\left(x-\frac{2h}{3}\right)^2+\left(y-\frac{2k}{3}\right)^2=a^2$

(D) $\left(x-\frac{h}{3}\right)^2+\left(y-\frac{k}{3}\right)^2=a^2$

PART-II : NUMERICAL QUESTIONS

1. If ℓ is the distance between focus and directrix of the parabola $9x^2 - 24xy + 16y^2 - 20x - 15y - 60 = 0$ then value of ℓ is equal to
2. Length of the focal chord of the parabola $y^2 = 8x$ at a distance 5 units from the vertex is:
3. AB is a chord of the parabola $y^2 = 9x$ with vertex at A. BC is drawn perpendicular to AB meeting the axis at C. The projection of BC on the axis of the parabola is
4. Number of integral values of m for which a chord of slope m of the circle $x^2 + y^2 = 4$ touches parabola $y^2 = 4x$, which do not satisfy, is -
5. A variable chord PQ of the parabola, $y^2 = 4x$ is drawn parallel to the line $y = 8x$. If the parameters of the points P & Q on the parabola be p & q respectively, then (p + q) equal to.
6. Through the vertex O of the parabola $y^2 = 8x$, a perpendicular is drawn to any tangent meeting it at P & the parabola at Q, then the value of OP. OQ is
7. AB, AC are tangents to a parabola $y^2 = 4ax$. Lengths of the perpendiculars from A, B & C respectively one of tangent to the curve is k, 6, 24. then k is equal to:
8. Let tangent at point A, B and vertex (V) of parabola is $x - 2y + 1 = 0$, $3x + y + 4 = 0$ and $y = x$ respectively. If focus of parabola is $\left(\frac{a}{7}, \frac{b}{7}\right)$ then find the value of (a + 5b).
9. T is a point on the tangent to a parabola $y^2 = 4ax$ at its point P. TL and TN are the perpendiculars on the focal radius SP and the directrix of the parabola respectively. Then $\frac{SL}{TN}$ is equal to :
10. A normal is drawn to a parabola $y^2 = 4ax$ at any point other than the vertex and it cuts the parabola again at a point whose distance from the vertex is always greater than or equal to $\lambda\sqrt{6}\,a$, then the maximum value of λ is
11. If three normal are drawn through (c, 0) to $y^2 = 4x$ and two of which of perpendicular then the value of c is
12. If the distance between the focii of an ellipse is equal to the length of its latus rectum, then eccentricity of the ellipse is :
13. P & Q are the points with eccentric angles θ & $\theta + \pi/6$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, then the area of the triangle OPQ is :
14. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is
15. If P is a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose focii are S and S′ and e_1 is the eccentricity and the locus of the incentre of $\Delta PSS'$ is an ellipse whose eccentricity is e_2, then the value of $\left(1 + \frac{1}{e_1}\right)e_2^2$ is :
16. Minimum length of the intercept made by the axes on the tangent to the ellipse $\frac{x^2}{81} + \frac{4y^2}{9} = 1$ is-

17. Area of quadrilateral from by common tangents to the circle $x^2 + y^2 = 16$ and ellipse $\frac{x^2}{25}+\frac{y^2}{4} = 1$ is -

18. If common tangent of $x^2 + y^2 = r^2$ and $\frac{x^2}{16}+\frac{y^2}{9} = 1$ forms square then find its area.

19. Number of values of 'λ', for which the line $2x - \frac{8}{3}\lambda y = -3$ is a normal to the conic $x^2 + \frac{y^2}{4} = 1$ is

20. If $7x^2 + pxy + qy^2 + rx - sy + t = 0$ is the eqaution of the hyperbola whose one focus is (–1, 1), eccentricity = 3 and the equation of the corresponding directrix is $x - y + 3 = 0$, then the value of 't' is :

21. The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ passes through the point of intersection of the lines, $7x + 13y - 87 = 0$ & $5x - 8y + 7 = 0$ & the latus rectum is $32\sqrt{2}/5$. The value of $(a^2 + b^2)$ is :

22. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$ **[IIT-JEE-2008, Paper-2(3, –1)/81]** with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

23. If P is any point common to the hyperbola $\frac{x^2}{16}-\frac{y^2}{25}=1$ and the circle having line segment joining its focii as diameter then find the sum of focal distances of point P.

24. If $x \cos\alpha + y \sin\alpha = p$, a variable chord of the hyperbola $\frac{x^2}{8}-\frac{y^2}{16} = 1$ subtends a right angle at the centre of the hyperbola, then the chords touch a fixed circle, then radius of the circle is-

25. P is a point on the hyperbola $\frac{x^2}{81}-\frac{y^2}{16} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then OT. ON is equal to :

26. If m_1 and m_2 are slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16} = 1$ which passes through the point of contact of $3x - 4y = 5$ and $x^2 - 4y^2 = 5$ then value of $(m_1 + m_2 - m_1m_2)$ is equal to

27. Tangents are drawn from the point $(\alpha, 2)$ to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ & ϕ to the x –axis . If $\tan\theta . \tan\phi = 2$, then the value of $\alpha^2 - 7$ is

28. If radii of director circles of $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ are 2r and r respectively and e_e and e_h be the eccentricities of the ellipse and the hyperbola respectively then $4e_h^2 - 3e_e^2$ is equal to

PART - III : ONE OR MORE THAN ONE CORRECT QUESTION

1. Let A be the vertex and L the length of the latus rectum of the parabola, $y^2 - 2y - 4x - 7 = 0$. The equation of the parabola with A as vertex, 2 L the length of the latus rectum and the axis at right angles to that of the given curve is:

(A) $x^2 + 4x + 8y - 4 = 0$ (B) $x^2 + 4x - 8y + 12 = 0$

(C) $x^2 + 4x + 8y + 12 = 0$ (D) $x^2 + 8x - 4y + 8 = 0$

2. If $(a^2, a - 2)$ be a point interior to the region of the parabola $y^2 = 2x$ bounded by the chord joining the points (2, 2) and (8, – 4), then of all possible values of a lies in -

(A) (0, 3) (B) (1, 3) (C) (–2, 5) (D) (5, 7)

3. If the triangle PQR of area 'A' is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. Then :

(A) modulus of the difference of the ordinates of the points Q and R is $\frac{2A}{a}$

(B) modulus of the difference of the ordinates of the points Q and R is $\frac{4A}{a}$

(C) minimum value of area A is $2a^2$

(D) length of QR for which area is minimum is equal to 4a

4. If one end of a focal chord of the parabola $y^2 = 4x$ is (1, 2), the other end lies on

(A) $x^2 y + 2 = 0$ (B) $xy + 2 = 0$

(C) $xy - 2 = 0$ (D) $x^2 + xy - y - 1 = 0$

5. Let PQ be a variable focal chord of the parabola $y^2 = 4ax$ where vertex is A. If locus of, centroid of triangle APQ is a parabola 'P_1' then

(A) Latus rectum of parabola P_1 is $\frac{4a}{3}$ (B) Vertex of parabola P_1 is $\left(\frac{2a}{3}, 0\right)$

(C) Latus rectum of parabola P_1 is $\frac{2a}{3}$ (D) Vertex of parabola P_1 is $\left(\frac{4a}{3}, 0\right)$

6. P is a point on the parabola $y^2 = 4ax$ (a > 0) whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. If a circle is described on MD as a diameter then it intersects the x–axis at a point whose co–ordinates are:

(A) (– 3a, 0) (B) (– a, 0) (C) (– 2a, 0) (D) (a, 0)

7. If a point R that divides a chord of slope 2 of the parabola $y^2 = 4x$. internally in the ratio 1 : 2 then

(A) locus of point R is $\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}\left(x - \frac{2}{9}\right)$ (B) locus of point R is $\left(y - \frac{2}{9}\right)^2 = \frac{4}{9}\left(x - \frac{8}{9}\right)$

(C) focus of locus of point R is $\left(\frac{4}{9}, \frac{8}{9}\right)$ (D) focus of locus of point R is $\left(1, \frac{8}{9}\right)$

8. Let $y^2 = 4ax$ be a parabola and $x^2 + y^2 + 2bx = 0$ be a circle. If parabola and circle touch each other externally then:

(A) $a > 0, b > 0$ (B) $a > 0, b < 0$ (C) $a < 0, b > 0$ (D) $a < 0, b < 0$

9. The parabola whose axis is parallel to the y-axis and which passes through the points (0, 4), (1, 9) and (–2,6) then

(A) focus of parabola is $\left(\frac{-5}{8}, \frac{23}{8}\right)$

(B) length of latus-rectum is $\frac{1}{4}$

(C) if it passes through $(2, \alpha)$ then the value of α is 16

(D) if it passes through $(2, \alpha)$ then the value of α is 18

10. P is a point on the parabola $y^2 = 4x$ where abscissa and ordinate are equal. Equation of a circle passing through the focus and touching the parabola at P is:

(A) $x^2 + y^2 - 13x + 2y + 12 = 0$

(B) $x^2 + y^2 - 3x - 18y + 2 = 0$

(C) $x^2 + y^2 + 13x - 2y - 14 = 0$

(D) $x^2 + y^2 - x = 0$

11. Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. If tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S then

(A) The ratio of the areas of the triangles PQS and PQR is 1 : 4

(B) The ratio of the areas of the triangles PQS and PQR is 1 : 2

(C) The radius of the circumcircle of the triangle PRS is $3\sqrt{3}$

(D) The radius of the incircle of the triangle PQR is 2

12. If from the focus of the parabola, $y^2 = 8x$ as centre, a circle is described so that a common chord of the curves is equidistant from the vertex & focus of the parabola, then-

(A) Equation of the circle is $(x - 2)^2 + y^2 = 9$

(B) Equation of the circle is $(x - 2)^2 + y^2 = 4$

(C) length of tangent from foot of directrix to this circle is $\sqrt{7}$

(D) length of tangent from foot of directrix to this circle is $2\sqrt{3}$

13. If from the vertex of a parabola $y^2 = 8x$ a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, then which of the following points does not satisfy locus of the further angle of the rectangle

(A) (16, 0) (B) (4, 3) (C) (20, 4) (D) (24, –8)

14. If three normals can be drawn to the curve $y^2 = x$ from point (c, 0) then 'c' can be equal to

(A) 0 (B) 1/2 (C) 1 (D) 2

15*. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose

(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is x = 0 (C) latus rectum is $\frac{2a}{3}$ (D) focus is (a, 0)

16. The equation, $3x^2 + 4y^2 - 18x + 16y + 43 = C$.

(A) cannot represent a real pair of straight lines for any value of C

(B) represents an ellipse, if C > 0

(C) no locus, if C < 0

(D) a point, if C = 0

17*. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are

(A) $x^2 + 2\sqrt{3}\ y = 3 + \sqrt{3}$ (B) $x^2 - 2\sqrt{3}\ y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}\ y = 3 - \sqrt{3}$ (D) $x^2 - 2\sqrt{3}\ y = 3 - \sqrt{3}$

18. A circle has the same centre as an ellipse & passes through the focii F_1 & F_2 of the ellipse, such that the two curves intersect at 4 points. Let 'P' be any one of their points of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then :

(A) the distance between the focii is 13 (B) the distance between the focii is 17

(C) length of minor axis is $2\sqrt{30}$ (D) length of minor axis is $4\sqrt{15}$

19*. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then

(A) b + c = 4a (B) b + c = 2a

(C) locus of points A is an ellipse (D) locus of point A is a pair of straight lines

20. If coordinates of the vertices B and C of a triangle ABC are (2, 0) and (8, 0) respectively. The vertex A is varying in such a way that $4\tan\frac{B}{2} \cdot \tan\frac{C}{2} = 1$, then

(A) eccentricity of locus of A is $\frac{3}{5}$

(B) eccentricity of locus of A is $\frac{5}{3}$

(C) maximum area of triangle ABC is 12 square unit

(D) minimum area of triangle ABC is 10 square unit

21. Point 'O' (origin) is the centre of the ellipse with major axis AB & minor axis CD along coordinate axis. Point F is one focus of the ellipse. If OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, then equation of ellipse is

(A) $\frac{x^2}{169}+\frac{y^2}{25}=1$ (B) $\frac{x^2}{169}+\frac{y^2}{25}=\frac{1}{4}$ (C) $\frac{x^2}{25}+\frac{y^2}{169}=\frac{1}{4}$ (D) $\frac{x^2}{25}+\frac{y^2}{169}=1$

22. If P is a point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose focii are S and S′. Let $\angle PSS' = \alpha$ and $\angle PS'S = \beta$, then

(A) PS + PS′ = 2a, if a > b
(B) PS + PS′ = 2b, if a < b
(C) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}$
(D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{\sqrt{a^2-b^2}}{b^2}[a - \sqrt{a^2-b^2}]$ when a > b

23. A tangent having slope $-\frac{4}{3}$ touches the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$ at point P and intersects the major and minor axes at A & B respectively, O is the centre of the ellipse then

(A) Distance between the parallel tangents having slopes $-\frac{4}{3}$, is $\frac{48}{5}$

(B) Area of ΔAOB is 24

(C) If the tangent in first quadrant touches the ellipse at (h, k) then value of hk is 12

(D) If equation of the tangent intersecting positive axes is $\ell x + my = 1$, then $\ell + m$ is equal to $\frac{7}{24}$

24. Let A(α) and B(β) be the extremeties of a chord of an ellipse . If the slope of AB is equal to the slope of the tangent at a point C(θ) on the ellipse, then the value of θ, is

(A) $\frac{\alpha+\beta}{2}$
(B) $\frac{\alpha-\beta}{2}$
(C) $\frac{\alpha+\beta}{2} + \pi$
(D) $\frac{\alpha-\beta}{2} - \pi$

25. The tangent at any point 'P' on the standard ellipse with focii as S & S′ meets the tangents at the vertices A & A′ in the points V & V′, then :

(A) (AV) (A′ V′) = b^2
(B) (AV) (A′ V′) = a^2
(C) $\angle V'SV = 90°$
(D) V′ S′ SV is a cyclic quadrilateral

26. Point/points, from which tangents to the ellipse $5x^2 + 4y^2 = 20$ are perpendicular, is/are :

(A) $(1, 2\sqrt{2})$
(B) $(2\sqrt{2}, 1)$
(C) $(2, \sqrt{5})$
(D) $(\sqrt{5}, 2)$

27. Let F_1, F_2 be two focii of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P then

(A) PN bisects $\angle F_1PF_2$
(B) PT bisects $\angle F_1PF_2$
(C) PT bisects angle $(180° - \angle F_1PF_2)$
(D) None of these

28. The equation of a hyperbola with co-ordinate axes as principal axes, if the distances of one of its vertices from the foci are 3 & 1 can be :

(A) $3x^2 - y^2 = 3$ (B) $x^2 - 3y^2 + 3 = 0$ (C) $x^2 - 3y^2 - 3 = 0$ (D) $x^2 - 3y^2 - 6 = 0$

29*. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then

(A) Equation of ellipse is $x^2 + 2y^2 = 2$ (B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$ (D) The foci of ellipse are $(\pm \sqrt{2}, 0)$

30. If (5, 12) and (24, 7) are the foci of a conic, passing through the origin then the eccentricity of conic is

(A) $\sqrt{386}/12$ (B) $\sqrt{386}/13$ (C) $\sqrt{386}/25$ (D) $\sqrt{386}/38$

31. If $(3\sin\alpha, 2\cos\alpha)$ lies on the same side as that of origin w.r.t conic $2x^2 - 3y^2 = 6$, then $\cos\alpha$ may be

(A) $-\sqrt{\frac{4}{5}}$ (B) $\sqrt{\frac{2}{5}}$ (C) $\frac{1}{\sqrt{5}}$ (D) $\sqrt{\frac{3}{5}}$

32*. If a hyperbola passes through the focii of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Its transverse and conjugate axes coincide respectively with the major and minor axes of the ellipse and if the product of eccentricities of hyperbola and ellipse is 1, then

(A) the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (B) the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$

(C) focus of hyperbola is (5, 0) (D) focus of hyperbola is $(5\sqrt{3}, 0)$

33. Which of the following equations in parametric form can represent a hyperbolic profile, where 't' is a parameter.

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$ (B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$ (D) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

34. $x^2 + y^2 = 16$ is the auxilliary circle of

(A) $9x^2 - 16y^2 - 144 = 0$ (B) $16x^2 - 9y^2 + 144 = 0$

(C) $9(x - y)^2 - 16(x + y)^2 - 288 = 0$ (D) $16(x - y)^2 - 9(x + y)^2 + 288 = 0$

35. If the chord joining the points whose eccentric angles are 'α' and 'β' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a focal chord then

(A) $\pm e\cos\left(\frac{\alpha - \beta}{2}\right) = \cos\left(\frac{\alpha + \beta}{2}\right)$ (B) $\pm e\cos\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\alpha - \beta}{2}\right)$

(C) $\tan(\alpha/2)\tan(\beta/2) + \left(\frac{ke - 1}{ke + 1}\right) = 0$ where $k = \pm 1$

(D) $\tan(\alpha/2)\tan(\beta/2) + \left(\frac{ke + 1}{ke - 1}\right) = 0$ where $k = \pm 1$

36. The equations of the transverse and conjugate axes of a hyperbola are respectively $x + 2y - 3 = 0$, $2x - y + 4 = 0$, and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. Then .

(A) Equation of one of the directrix is $2x - y + 4 + \sqrt{\frac{3}{2}} = 0$

(B) Coordinates of one of possible focus of hyperbola is $\left(-1+\frac{2}{\sqrt{6}}, 2-\frac{1}{\sqrt{6}}\right)$

(C) Coordinates of one of possible focus of hyperbola is $\left(\left(-1+\frac{2}{\sqrt{5}}\right), \left(2-\frac{1}{\sqrt{5}}\right)\right)$

(D) Equation of one of the directrix is $2x - y + 4 + \sqrt{\frac{3}{2}} = \sqrt{3}$

37. If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then

(A) PG = PC (B) Pg = PC (C) PG = Pg (D) Gg = PC

PART - IV : COMPREHENSION

Comprehension # 1 (Q.1 to Q.3)

Let ABCD be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of the square ABCD. L is a line through A.

1 If P is a point on C_1 and Q in another point on C_2, $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to

(A) 0.75 (B) 1.25 (C) 1 (D) 0.5

2. A circle touch the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

(A) ellipse (B) hyperbola

(C) parabola (D) parts of straight line

3. A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is

(A) $\frac{1}{2}$ sq. units (B) $\frac{2}{3}$ sq. units (C) 1 sq. units (D) 2 sq. units

Comprehension # 2 (Q.4 to Q.6)

Consider three lines y axis, y = 2 and $\ell x + my = 1$ where (ℓ, m) lies on $y^2 = 4x$. answer the following :

4. Locus of circum centre of triangle formed by given three lines is a parabola whose vertex is

(A) (–2, 3/2) (B) (2, –3/2) (C) (–2, –3/2) (D) (2, –5/2)

5. Area of triangle formed by vertex and end points of latus rectum of parabola obtained in questions (1) is

(A) $\frac{1}{2^8}$ (unit)^2 (B) $\frac{1}{2^9}$ (unit)^2 (C) $\frac{1}{2^{10}}$ (unit)^2 (D) $\frac{1}{2^7}$ (unit)^2

6. Any point on the parabola obtained in question (1) can be represented as

(A) $\left(2+\frac{1}{32}t^2, \frac{3}{2}+\frac{t}{16}\right)$ (B) $\left(2+\frac{t^2}{32}, \frac{-3}{2}+\frac{1}{16}t^2\right)$ (C) $\left(-2+\frac{1}{32}t^2, \frac{3}{2}+\frac{t}{16}\right)$ (D) $\left(-2+\frac{1}{16}t^2, \frac{3}{2}+\frac{t}{5}\right)$

Comprehension # 3 (Q.7 to Q.9)

Two tangents PA and PB are drawn from a point P(h, k) to the ellipse E : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b). Angle of the tangents with the positive x - axis are θ_1 and θ_2. Normals at A and B are intersecting at Q point.

On the basis of above information answer the following questions.

7. Locus of P, if $\tan\theta_1 . \tan\theta_2 = 4$, is

(A) $\frac{y-b}{x-a} = 2(x + a)$ (B) $y^2 - b^2 = 2(x^2 + a^2)$ (C) $\frac{y+b}{x+a} = \frac{4(x-a)}{y-b}$ (D) $\frac{y+b}{x-b}=\frac{x+a}{y-b}$

8. Circumcentre of ΔQAB is

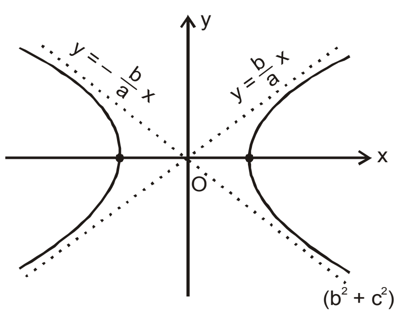
(A) mid point of AB (B) mid point of PQ (C) orthocentre of ΔPAB (D) can't say

9. Locus of P, if $\cot\theta_1 + \cot\theta_2 = \lambda$, is

(A) $2xy = \lambda(y^2 - b^2)$ (B) $2xy - \lambda(b^2 - y^2) = 0$ (C) $xy = \lambda$ (D) $x^2 + xy = \lambda$

Comprehension # 4 (Q.10 to Q.13)

Asymptotes are lines whose distance from the curve at infinity tends to zero Let y = mx + c is asymptote of $H = \frac{x^2}{a^2}-\frac{y^2}{b^2}-1 = 0$. Solving the two equations, we have $(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0$. Both roots of this equation must be infinite so $m = \pm\frac{b}{a}$ and c = 0 which implies that $y = \pm\frac{b}{a}x$ are asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$. Note that no real tangent can be drawn to the hyperbola from its centre and only one real tangent can be drawn from a point lying on its asymptote other than centre. Further combined equation of asymptotes is $A = \frac{x^2}{a^2}-\frac{y^2}{b^2} = 0$ and conjugate hyperbola $C = \frac{x^2}{a^2}-\frac{y^2}{b^2} + 1 = 0$.



Hence 2A = H + C, as we can see, equation of A, H and C vary only by a constant, for asymptotes which can be evaluated by applying condition of pair of lines.

10. The points of contact of tangents drawn to the hyperbola $\frac{x^2}{3}-\frac{y^2}{2}=1$ from point (2, 1) are

(A) (3, 2), (1, 5) (B) (3, 2), $\left(\frac{9}{5},\frac{2}{5}\right)$ (C) (1, 2), (3, 4) (D) (3, 2), (3, 4)

11. The number of real distinct tangents drawn to hyperbola $4x^2 - y^2 = 4$ from point (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

12. The number of real distinct tangents drawn from point (1, 2) to hyperbola $x^2 - y^2 - 2x + 4y - 4 = 0$ is

(A) 1 (B) 2 (C) 3 (D) None of these

13. The asymptotes of $xy - 3y - 2x = 0$ is

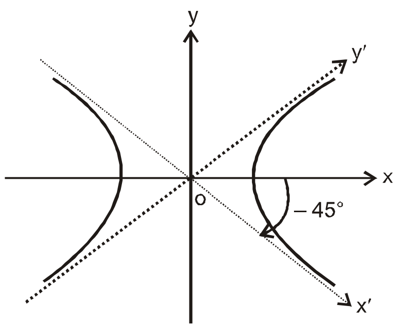
(A) $x + 2 = 0$ and $y + 3 = 0$ (B) $x - 2 = 0$ and $y - 3 = 0$

(C) $x - 3 = 0$ and $y - 2 = 0$ (D) $x + 3 = 0$ and $y + 2 = 0$

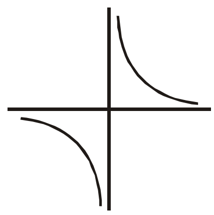
Comprehension # 5 (Q.14 to Q.17)

A particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral or rectangular Hyperbola. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$.

Since a = b. If axis of hyperbola are along coordinate axis then its equation becomes $x^2 - y^2 = a^2$



Rotation of this system through an angle of 45° in clockwise direction gives another form to the equation of rectangular hyperbola. Which is $xy = c^2$ where $c^2 = \frac{a^2}{2}$.



Rectangular hyperbola ($xy = c^2$) :

Vertices : (c, c) & (– c, – c);

Foci : $\left(\sqrt{2}\,c, \sqrt{2}\,c\right)$ & $\left(-\sqrt{2}\,c, -\sqrt{2}\,c\right)$,

Directrices : $x + y = \pm\sqrt{2}\,c$

Latus Rectum (l) : $\ell = 2\sqrt{2}\,c$ = T.A. = C.A.

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of a chord joining the points P (t_1) & Q(t_2) is $x + t_1 t_2 y = c(t_1 + t_2)$.

Equation of the tangent at P (x_1, y_1) is $\frac{x}{x_1}+\frac{y}{y_1}=2$ & at P (t) is $\frac{x}{t} + ty = 2c$.

Equation of the normal at P (t) is $xt^3 - yt = c(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Solve the following questions

14. A rectangular hyperbola circumscribe a triangle ABC, then it will always pass through its

(A) orthocenter (B) circum centre (C) centroid (D) incentre

15. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, the coordinates of orthocentre of the ΔPQR are

(A) (x_4, y_4) (B) $(x_4, - y_4)$ (C) $(-x_4, - x_4)$ (D) $(- x_4, - y_4)$

16. If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again at t', then

(A) $t' = -\frac{1}{t^3}$ (B) $t' = \frac{1}{t}$ (C) $t' = \frac{1}{t^2}$ (D) $t'^2 = -\frac{1}{t^2}$

17. If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 & t_4, then

(A) $t_1 t_2 t_3 t_4 = 1$

(B) The arthmetic mean of the four points bisects the distance between the centres of the two curves.

(C) The geometrical mean of the four points bisects the distance between the centres of the two curves.

(D) the centre of the circle through the points t_1, t_2 & t_3 is : $\left\{\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1t_2t_3\right)\right\}$

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1*. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be

[IIT-JEE-2010, Paper-1(3, 0)/84]

(A) $-\frac{1}{r}$ (B) $\frac{1}{r}$ (C) $\frac{2}{r}$ (D) $-\frac{2}{r}$

Comprehension (2 to 4)

Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4} = 1$ touching the ellipse at point A and B.

2. The coordinates of A and B are **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) (3, 0) and (0, 2) (B) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(C) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2) (D) (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

3. The orthocentre of the triangle PAB is **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) $\left(5, \frac{8}{7}\right)$ (B) $\left(\frac{7}{5}, \frac{25}{8}\right)$ (C) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (D) $\left(\frac{8}{25}, \frac{7}{5}\right)$

4. The equation of the locus of the point whose distances from the point P and the line AB are equal, is

[IIT-JEE 2010, Paper-2, (3, –1), 79]

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$ (B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$ (D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Comprehension (5 to 6)

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

5. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

[IIT-JEE-2010, Paper-1(3, –1)/84]

(A) $2x - \sqrt{5}\,y - 20 = 0$ (B) $2x - \sqrt{5}\,y + 4 = 0$

(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

6. Equation of the circle with AB as its diameter is [IIT-JEE-2010, Paper-1(3, –1)/84]

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

7. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then find the eccentricity of the hyperbola. [IIT-JEE-2010, Paper-1(3, 0)/84]

8. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is [IIT-JEE 2011, Paper-1, (4, 0), 80]

9. Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0, 0) to (x, y) in the ratio 1 : 3. Then the locus of P is [IIT-JEE 2011, Paper-2, (3, –1), 80]

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

10*. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by [IIT-JEE 2011, Paper-2, (4, 0), 80]

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

11*. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) a focus of the hyperbola is (2, 0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

12. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is [IIT-JEE 2011, Paper-2, (3, –1), 80]

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

13. The ellipse $E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

14. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line 2x – y = 1. The points of contacts of the tangents on the hyperbola are **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

(A) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ (C) $\left(3\sqrt{3},-2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3},2\sqrt{2}\right)$

15. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is. **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

Comprehension (16 to 17)

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line y = 2x + a, a > 0. **[IIT-JEE - 2013, Paper-2, (3,–1), 60]**

16. Length of chord PQ is

(A) 7a (B) 5a (C) 2a (D) 3a

17. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta$ =

(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$

18. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral PQRS is
[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) 3 (B) 6 (C) 9 (D) 15

Comprehension (19 to 20)

Let a, r, s, t be nonzero real numbers. Let P(at^2, 2at), Q, R (ar^2, 2ar) and (as^2, 2as) be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a, 0) **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

19. The value of r is

(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$

20. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(A) $\frac{(t^2+1)^2}{2t^3}$ (B) $\frac{a(t^2+1)^2}{2t^3}$ (C) $\frac{a(t^2+1)^2}{t^3}$ (D) $\frac{a(t^2+2)^2}{t^3}$

21. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is

[JEE (Advanced) 2015, P-1]

22. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is **[JEE (Advanced) 2015, P-1]**

23. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is(are) the coordinates of P ?

[JEE (Advanced) 2015, P-1 (4, –2)/ 80]

(A) $\left(4, 2\sqrt{2}\right)$ (B) $\left(9, 3\sqrt{2}\right)$ (C) $\left(\frac{1}{4}, -\frac{1}{\sqrt{2}}\right)$ (D) $\left(1, \sqrt{2}\right)$

24. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P,Q and R, respectively. Suppose that PQ = PR = $\frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are)

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A) $e_1^2 + e_2^2 = \frac{43}{40}$ (B) $e_1e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$ (C) $|e_1^2 - e_2^2| = \frac{5}{8}$ (D) $e_1e_2 = \frac{\sqrt{3}}{4}$

25. Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$ (B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$ (D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

26. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1,0)$ and $(f_2,0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0,0) and with foci at $(f_1,0)$ and $(2f_2,0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2,0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is **[JEE (Advanced) 2015, P-2]**

27. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then- **[JEE (Advanced) 2016, P-2]**

(A) $SP = 2\sqrt{5}$

(B) $SQ : QP = (\sqrt{5}+1) : 2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$

Comprehension (28 to 29)

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant. **[JEE (Advanced) 2016, P-2]**

28. The orthocentre of the triangle F_1MN is-

(A) $\left(-\frac{9}{10},0\right)$ (B) $\left(\frac{2}{3},0\right)$ (C) $\left(\frac{9}{10},0\right)$ (D) $\left(\frac{2}{3},\sqrt{6}\right)$

29. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is-

(A) 3 : 4 (B) 4 : 5 (C) 5 : 8 (D) 2 : 3

30. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k ? **[JEE (Advanced) 2017, P-1]**

(A) p = 5, h = 4, k = –3 (B) p = –1, h = 1, k = –3

(C) p = –2, h = 2, k = –4 (D) p = 2, h = 3, k = –4

31. If 2x – y + 1 = 0 is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{16}=1$, then which of the following CANNOT be sides of a right angled triangle ? **[JEE (Advanced) 2017, P-1]**

(A) 2a, 4, 1 (B) 2a, 8, 1 (C) a, 4, 1 (D) a, 4, 2

Column 1,2 and 3 contain conics, equation of tangents to the conics and points of contact, respectively. **[JEE (Advanced) 2017, P-1]**

	Column 1		Column 2		Column 3
(I)	$x^2 + y^2 = a^2$	(i)	$my = m^2x + a$	(P)	$\left(\frac{a}{m^2},\frac{2a}{m}\right)$
(II)	$x^2 + a^2y^2 = a^2$	(ii)	$y=mx+a\sqrt{m^2+1}$	(Q)	$\left(\frac{-ma}{\sqrt{m^2+1}},\frac{a}{\sqrt{m^2+1}}\right)$
(III)	$y^2 = 4ax$	(iii)	$y = mx + \sqrt{a^2m^2-1}$	(R)	$\left(\frac{-a^2m}{\sqrt{a^2m^2+1}},\frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV)	$x^2 - a^2y^2 = a^2$	(iv)	$y=mx+\sqrt{a^2m^2+1}$	(S)	$\left(\frac{-a^2m}{\sqrt{a^2m^2-1}},\frac{-1}{\sqrt{a^2m^2-1}}\right)$

32. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3},\frac{1}{2}\right)$ is found to be $\sqrt{3}x+2y=4$, then which of the following options is the only **CORRECT** combination ?

(A) (II) (iii) (R) (B) (IV) (iv) (S) (C) (IV) (iii) (S) (D) (II) (iv) (R)

33. If a tangent to a suitable conic (Column 1) is found to be y = x + 8 and its point of contact is (8,16), then which of the following options is the only **CORRECT** combination ?

(A) (III) (i) (P) (B) (III) (ii) (Q) (C) (II) (iv) (R) (D) (I) (ii) (Q)

34. For $a=\sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1,1), then which of the following options is the only **CORRECT** combination for obtaining its equation ?

(A) (II) (ii) (Q) (B) (III) (i) (P) (C) (I) (i) (P) (D) (I) (ii) (Q)

35. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ? **[JEE (Advanced) 2018, P-2]**

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and x = 1 is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and x = 1 is $\frac{1}{16}(\pi - 2)$

36. Let H : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

[JEE (Advanced) 2018, P-2]

LIST-I		**LIST-II**
P. The length of the conjugate axis of H is	**1.**	8
Q. The eccentricity of H is	**2.**	$\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.**	$\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.**	4

The correct option is :

(A) P → 4; Q → 2, R → 1; S → 3

(B) P → 4; Q → 3; R → 1; S → 2

(C) P → 4; Q → 1, R → 3; S → 2

(D) P → 3; Q → 4; R → 2; S → 1

37. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$;

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1}, $n > 1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.

Then which of the following options is/are correct ? **[JEE (Advanced) 2019, P-1]**

(1) The eccentricities of E_{18} and E_{19} are NOT equal

(2) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(3) The length of latus rectum of E_9 is $\frac{1}{6}$

(4) $\sum_{n=1}^{N}(\text{area of } R_n) < 24$, for each positive integer N

Comprehension (38 to 39)

Answer the following by appropriately matching the lists based on the information given in the paragraph **[JEE (Advanced) 2019, P-2]**

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below:

	List-I		**List-II**
(I)	$2h + k$	(P)	6
(II)	$\frac{\text{Length of ZW}}{\text{Length of XY}}$	(Q)	$\sqrt{6}$
(III)	$\frac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R)	$\frac{5}{4}$
(IV)	α	(S)	$\frac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\frac{10}{3}$

38. Which of the following is the only INCORRECT combination ?

(A) (IV), (S) (B) (IV), (U) (C) (III), (R) (D) (I), (P)

39. Which of the following is the only CORRECT combination ?

(A) (II), (T) (B) (I), (S) (C) (I), (U) (D) (II), (Q)

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1*. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{\frac{2}{5}}$ is : **[AIEEE 2011, I, (4, –1), 120]**

(1) $3x^2 + 5y^2 - 32 = 0$ (2) $5x^2 + 3y^2 - 48 = 0$
(3) $3x^2 + 5y^2 - 15 = 0$ (4) $5x^2 + 3y^2 - 32 = 0$

2. The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by :
[AIEEE 2011, II, (4, –1), 120]

(1) $x^2 - 3y^2 = 3$ (2) $3x^2 - y^2 = 3$ (3) $-x^2 + 3y^2 = 3$ (4) $-3x^2 + y^2 = 3$

3. **Statement-1** : An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}\,x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$. **[AIEEE - 2013, (4, – ¼) 120]**

Statement-2 : If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}\,x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$.

(1) Statement-1 is false, Statement-2 is true.
(2) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(3) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(4) Statement-1 is true, statement-2 is false.

4. An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$ is semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is : **[AIEEE-2012, (4, –1)/120]**

(1) $4x^2 + y^2 = 4$ (2) $x^2 + 4y^2 = 8$ (3) $4x^2 + y^2 = 8$ (4) $x^2 + 4y^2 = 16$

5. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having centre at (0, 3) is
[AIEEE - 2013, (4, – ¼)]

(1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$
(3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$

6. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is:
[JEE(Main) 2014, (4, – ¼), 120]

(1) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (2) $(x^2 + y^2)^2 = 6x^2 - 2y^2$
(3) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (4) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

7. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is : **[Conic Section]**
[JEE(Main) 2014, (4, – ¼), 120]

(1) $\frac{1}{8}$ (2) $\frac{2}{3}$ (3) $\frac{1}{2}$ (4) $\frac{3}{2}$

8. The area (in sq.units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$, is **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $\frac{27}{4}$ (2) 18 (3) $\frac{27}{2}$ (4) 27

[JEE(Main) 2015, (4, – ¼), 120]

9. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $x^2 = y$ (2) $y^2 = x$ (3) $y^2 = 2x$ (4) $x^2 = 2y$

10. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is : **[JEE(Main) 2016]**

(1) $\sqrt{3}$ (2) $\frac{4}{3}$ (3) $\frac{4}{\sqrt{3}}$ (4) $\frac{2}{\sqrt{3}}$

11. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the cente C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is : **[JEE(Main) 2016]**

(1) $x^2 + y^2 - 4x + 9y + 18 = 0$ (2) $x^2 + y^2 - 4x + 8y + 12 = 0$

(3) $x^2 + y^2 - x + 4y - 12 = 0$ (4) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

12. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directices is x= –4, then the equation of the normal to it at $\left(1,\frac{3}{2}\right)$ is :- **[JEE(Main) 2017]**

(1) $x + 2y = 4$ (2) $2y - x = 2$ (3) $4x - 2y = 1$ (4) $4x + 2y = 7$

13. A hyperbola passes through the point $P\left(\sqrt{2},\sqrt{3}\right)$ and has foci at (± 2, 0). Then the tangent to this hyperbola at P also passes through the point : **[JEE(Main) 2017]**

(1) $\left(-\sqrt{2},-\sqrt{3}\right)$ (2) $\left(3\sqrt{2},2\sqrt{3}\right)$ (3) $\left(2\sqrt{2},3\sqrt{3}\right)$ (4) $\left(\sqrt{3},\sqrt{2}\right)$

14. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the point P and Q. If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is - **[JEE(Main) 2018]**

(1) $54\sqrt{3}$ (2) $60\sqrt{3}$ (3) $36\sqrt{5}$ (4) $45\sqrt{5}$

15. Tangent and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is- **[JEE(Main) 2018]**

(1) 2 (2) 3 (3) $\frac{4}{3}$ (4) $\frac{1}{2}$

16. Equation of a common tangent to the circle, $x^2 + y^2 - 6x = 0$ and the parabola, $y^2 = 4x$, is:
[JEE(Main) Jan-2019]

(1) $2\sqrt{3}\,y = 12x + 1$ (2) $2\sqrt{3}\,y = -x - 12$ (3) $\sqrt{3}\,y = x + 3$ (4) $\sqrt{3}\,y = 3x + 1$

17. Axis of a parabola lies along x-axis. If its vertex and focus are at distances 2 and 4 respectively from the origin, on the positive x-axis then which of the following points does not lie on it ?
[JEE(Main) Jan-2019]

(1) (4, –4) (2) $(5, 2\sqrt{6})$ (3) (8, 6) (4) $6, 4\sqrt{2}$

18. Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval : **[JEE(Main) Jan-2019]**

(1) (2, 3] (2) $(3, \infty)$ (3) (3/2, 2] (4) (1, 3/2]

19. Let A(4,–4) and B(9,6) be points on the parabola, $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is:
[JEE(Main) Jan-2019]

(1) $31\frac{3}{4}$ (2) 32 (3) $30\frac{1}{2}$ (4) $31\frac{1}{4}$

20. A hyperbola has its centre at the origin, passes through the point (4,2) and has transverse axis of length 4 along the x-axis. Then the eccentricity of the hyperbola is : **[JEE(Main) Jan-2019]**

(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{3}{2}$ (3) $\sqrt{3}$ (4) 2

21. If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to :
[JEE(Main) Jan-2020]

(1) 128 (2) –64 (3) –128 (4) –32

22. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is : **[JEE(Main) Jan-2020]**

(1) $\sqrt{3}$ (2) $2\sqrt{3}$ (3) $3\sqrt{2}$ (4) $\frac{3}{\sqrt{2}}$

23. If $3x + 4y = 12\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$ for some $a \in R$, then the distance between the foci of the ellipse is : **[JEE(Main) Jan-2020]**

(1) 4 (2) $2\sqrt{7}$ (3) $2\sqrt{5}$ (4) $2\sqrt{2}$

Answers

Exercise # 1

PART - I

Section (A) :

A-1. 4

A-2. (i) $4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$
(ii) $y^2 = 8(x + 3)$
(iii) $(y - 4)^2 = 2(x - 6.5)$ or $(y - 4)^2 = -2(x - 7.5)$

A-3. Vertex - $\left(-\frac{7}{2}, \frac{5}{2}\right)$, Axis - $y = \frac{5}{2}$,
Focus- (–17/4, 5/2), Directrix - $X = \frac{3}{4}$,
Latusrectum = 3

A-4. $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$

A-5. 12 **A-6.** $\alpha \in \phi$

A-7. $\alpha \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$

Section (B) :

B-1. $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$

B-2. $2x^2 + y^2 = 100$, $x^2 + 2y^2 = 100$

B-3. $C \equiv (-1, 2)$, length of major axis = $2b = \sqrt{3}$, length of minor axis = 2a = 1; $e = \sqrt{\frac{2}{3}}$;
$f\left(-1,\ 2 \pm \frac{1}{\sqrt{2}}\right)$

B-4 ((2, 3) & (6, 7))

B-5 $\alpha \in \left(-\frac{12}{5}, \frac{12}{5}\right)$

B-6. $3x^2 + 4y^2 - 12x + 24y + 36 = 0$

B-7. 3

B-8. $\frac{x^2}{144} + \frac{y^2}{128} = 1$

B-9. $(x = 3 + 5\cos\theta,\ y = -2 + 4\sin\theta)$

B-11. $\frac{2}{\sqrt{3}}$ **B-12.** Inside

B-13. $(x - 3)^2 + (y - 4)^2 = 25$

B-15. $2x^2 - 3y^2 + 18x + 18y - 96 = 0$

B-16 $\frac{x^2}{16} - \frac{y^2}{48} = 1$ **B-17.** 1

Section (C) :

C-1. $8\sqrt{2}$ **C-2.** (9, 0) or (–9, 0)

C-4. $y = x^2 + 2$ **C-5.** $\frac{-14}{3}$

C-6. Equation of tangent is $y = -2x - 1$ and point of contact is $\left(\frac{1}{2}, -2\right)$
Equation of tangent is $y = \frac{1}{2}x + 4$ and point of contact is (8, 8)

C-7. 2 **C-8.** $\sqrt{\frac{c}{a}}$

C-9. (–6, –7) **C-10.** $x + y + 1 = 0$

Section (D) :

D-1. $\sqrt{35}$ **D-3.** (yes, (5, 4))

D-4. $x - 2y \pm 4 = 0$. **D-5.** $9x^2 + 16y^2 = 4x^2y^2$

D-6. $x^2 + 64y^2 = 80$ & $x^2 + 4y^2 = 20$

D-8. (3, 0) **D-9.** $\frac{1}{2}(a^2 + b^2)$

D-10. $\lambda = \pm 6$ **D-11.** a^2

D-12. $y = \pm 3\sqrt{\frac{2}{7}}x \pm \frac{15}{\sqrt{7}}$

D-13. **(i)** –16/9 **(ii)** –20/9

Section (E) :

E-1. (i) $y^2 - 2ax = \lambda x^2$
(ii) $y^2 - 4ax = (x + a)^2\tan^2\theta_0$

E-3. (–2, 0) **E-4** (–1, 4)

E-5. $y^2 = 2(x - 1)$

Section (F) :

F-1. $4x + 5y = 13$

F-.3 $\frac{x^2}{81} + \frac{y^2}{16} = \frac{13}{36}$

F-4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{y}{b}$

F-5. $5x + 3y = 16$

F-6. $a \neq \pm b$
F-7. $9x - 8y = 72$
F-9. $y^2 = -\frac{b^4}{a^3} \cdot x$

Section (G) :
G-1. $y = 3x - 33$
G-2. $k = 4/3$
G-4. $(4a, 4a)$
G-5. $x + 2y - 3 = 0$
G-6. $y^2 = 4ax$.

Section (H) :
H-2. $p^2 (a^2 \sec^2 \alpha + b^2 \text{cosec}^2 \alpha) = (a^2 - b^2)^2$
H-5. $3\sqrt{3}\, x - 13y + 15\sqrt{3} = 0$.

PART - II

Section (A) :

A-1.	(C)	A-2.	(D)	A-3.	(B)
A-4.	(D)	A-5.	(A)	A-6.	(B)
A-7.	(C)	A-8.	(C)	A-9.	(A)
A-10.	(B)	A-11.	(B)	A-12.	(B)
A-13.	(B)				

Section (B) :

B-1.	(B)	B-2.	(A)	B-3.	(C)
B-4.	(A)	B-5.	(D)	B-6.	(B)
B-7.	(C)	B-8.	(C)	B-9	(C)
B-10.	(B)	B-11.	(B)	B-12.	(C)
B-13.	(A)	B-14.	(C)	B-15.	**(D)**
B-16.	(C)	B-17.	(A)	B-18.	(B)
B-19.	(A)	B-20.	(D)	B-21.	(B)

Section (C) :

C-1.	(D)	C-2	(B)	C-3	(C)
C-4.	(A)	C-5	(A)	C-6	(A)
C-7.	(A)	C-8.	(D)	C-9.	(B)
C-10.	(A)	C-11.	(D)	C-12.	(B)
C-13.	(A)				

Section (D) :

D-1.	(A)	D-2.	(B)	D-3.	(C)
D-4.	(C)	D-5.	(B)	D-6.	(A)
D-7.	(C)	D-8.	(C)	D-9.	(B)

Section (E) :

E-1.	(B)	E-2	(B)	E-3.	(B)

Section (F) :

F-1.	(C)	F-2.	(C)	F-3.	(A)
F-4.	(D)	F-5.	(B)	F-6.	(A)
F-7.	(D)	F-8.	(A)		

Section (G) :

G-1.	(D)	G-2.	(D)	G-3	(A)
G-4.	(D)	G-5.	(B)	G-6.	(C)
G-7	(A)	G-8.	(C)		

Section (H) :

H-1.	(C)	H-2.	(D)	H-3.	(A)
H-4	(A)	H-5	(A)		

PART - III

1. (A) → (s), (B) → (r), (C) → (q), (D) → (p)
2. (A) → (p), (B) → (q), (C) → (s), (D) → (r)
3. (A) → (s), (B) → (p), (C) → (q) (D) → (s)
4. (A) → (q), (B) → (q), (C) → (r), (D) → (p)
5. (A) → (s), (B) → (r), (C) → (q), (D) → (p)

Exercise # 2

PART - I

1.	(C)	2.	(C)	3.	(C)
4.	(A)	5.	(B)	6.	(B)
7.	(B)	8.	(B)	9.	(C)
10.	(B)	11.	(B)	12.	(D)
13.	(C)	14.	(B)	15.	(A)
16.	(A)	17.	(B)	18.	(A)
19.	(A)	20.	(D)	21.	(C)
22.	(C)	23.	(C)	24.	(C)
25.	(C)	26.	(A)	27.	(A)
28.	(A)	29.	(D)	30.	(C)
31.	(A)	32.	(A)	33.	(C)

PART - II

1.	0.5	2.	1.28	3.	9.00
4.	0	5.	0.25	6.	16
7.	12	8.	6	9.	1.00
10.	4	11.	3		
12.	0.61 or 0.62			13.	3.75
14.	2.70	15.	2	16.	10.5
17.	22.4	18.	50	19.	2.00
20.	77	21.	28.5	22.	0.22
23.	16.24 or 16.25			24.	4
25.	81.00	26.	0.68 or 0.69	27.	5.5
28.	5.20				

PART - III

1.	(A), (B)	2.	(A), (C)
3.	(A),(C), (D)	4.	(A), (B), (D)
5.	(A), (B)	6.	(A), (D)
7.	(A), (C)	8.	(A), (D)
9.	(A), (D)	10.	(A), (D)
11.	(A), (C), (D)	12.	(A), (C)
13.	(B), (C)	14.	(C), (D)
15.	(A), (D)	16.	(A), (B), (C), (D)
17.	(B), (C)	18.	(A), (C)
19.	(B), (C)	20.	(A), (C)
21.	(B), (C)	22.	(A), (B), (C)
23.	(A),(B),(C),(D)	24.	(A), (C)
25.	(A), (C), (D)	26.	(A), (B), (C), (D)
27.	(A), (C)	28.	(A), (B)
29*.	(A), (B)	30.	(A), (D)
31.	(A), (D)	32.	(A), (C)
33.	(A), (C), (D)	34.	(A), (B), (C), (D)
35.	(A), (C)	36.	(A), (B)
37.	(A), (B), (C)		

PART - IV

1	(A)	2.	(C)	3.	(C)
4.	(A)	5.	(B)	6.	(C)
7.	(C)	8.	(B)	9.	(A)
10.	(B)	11.	(A)	12.	(D)
13.	(C)	14.	(A)	15.	(D)
16.	(A)	17.	(A), (B), (D)		

Exercise # 3

PART - I

1.	(C), (D)	2.	(D)
3.	(C)	4.	(A)
5.	(B)	6.	(A)
7.	2	8.	2
9.	(C)	10.	(A), (B), (D)
11.	(B), (D)	12.	(B)
13.	(C)	14.	(A), (B)
15.	(4)	16.	(B)
17.	(D)	18.	(D)
19.	(D)	20.	(B)
21.	4	22.	2
23.	(A), (D)	24.	(A), (B)
25.	(A),(B),(D)	26.	4
27.	(A),(C),(D)	28.	(A)
29.	(C)	30.	(D)
31.	(B), (C), (D)	32.	(D)
33.	(A)	34.	(D)
35.	(A),(C)	36.	(B)
37.	(3), (4)	38.	(A)
39.	(D)		

PART - II

1.	(1), (2)	2.	(2)
3.	(2)	4.	(4)
5.	(1)	6.	(1)
7.	(3)	8.	(4)
9.	(4)	10.	(4)
11.	(2)	12.	(3)
13.	(3)	14.	(4)
15.	(1)	16.	(3)
17.	(3)	18.	(2)
19.	(4)	20.	(1)
21.	(3)	22.	(3)
23.	(2)		

SUBJECTIVE QUESTIONS

1. Prove that in a parabola the angle θ that the latus rectum subtends at the vertex of the parabola isindependent of the latus rectum and lies between $\frac{2\pi}{3}$ & $\frac{3\pi}{4}$

2. A parabola is drawn to pass through A and B, the ends of a diameter of a given circle of radius a, and to have as directrix a tangent to a concentric circle of radius b; then axes being AB and a perpendicular diameter, prove that the locus of the focus of the parabola is $\frac{x^2}{b^2} + \frac{y^2}{b^2 - a^2} = 1$

3. If r_1, r_2 be the length of the perpendicular chords of the parabola $y^2 = 4ax$ drawn through the vertex, then show that $(r_1 r_2)^{4/3} = 16a^2(r_1^{2/3} + r_2^{2/3})$.

4. If the parabola $y^2 = 4ax$ cuts the ellipse $\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$ in three distinct points then show that the eccentricity of the ellipse e belongs to $\left(\frac{1}{\sqrt{2}}, 1\right)$.

5. Prove that the bisector of the angle contain origin of the lines $y = m(x - 1)$ and $y = \frac{2 + 2\sqrt{1+m^2}}{m}$ (m is real parameter) touches a fixed parabola. Find the equation of the parabola.

6. Let PQ be the common chord of the parabola $y^2 = 4ax$ and the circle touching the parabola at P and Q. Tangent at P and the chord PQ are equally inclined to the axes of the parabola. Prove that locus of middle point of PQ is another parabola whose latus rectum is one fifth of the given parabola.

7. if a point P on the axis of the parabola $y^2 = 4x$ is taken such that the point P is at shortest distance from the circle $x^2 + y^2 + 2x - 2\sqrt{2} + 2 = 0$. common tangents are drawn to the circle and the parabola, then find the area of the triangle PAB where A and B are the points of contact on two different lines on circle and parabola respectively.

8. Consider a triangle ABC, equation of sides AB, BC and AC are $7x - y + 3 = 0$, $x + y - 3 = 0$ and $3x + y + 7 = 0$ respectively. Find the locus of the focus of a parabola having BC as chord of contact.

9. If AB and CD are two perpendicular focal chords of a parabola $y^2 = 28x$. If a quadrilateral ACBD is formed then evaluate the minimum area of the quadrilateral.

10. The sides of triangle ABC are tangents to the parabola $y^2 = 4ax$. Let D, E, F be the points of contact of side BC, CA and AB respectively. If lines AD, BE and CF are concurrent at focus of the parabola then prove that ABC is equilateral.

11. Let the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$ intersect at O and A(O being origin). Then show that the parabola whose directrix is the common tangent to the two parabolas and whose focus is the point which divides OA internally in the ratio $(1 + \sqrt{3}) : (7 - \sqrt{3})$ will pass through the foci of the given parabolas.

12. If l, m are variable real numbers such that $5l^2 + 6m^2 \quad 4lm + 3l = 0$. Prove that variable line $lx + my = 1$ always touches a fixed parabola, whose axes is parallel to x axis. Find the focus and directrix of the parabola.

13. A pair of tangents are drawn to the parabola which are equally inclined to a straight line whose inclination to the axis is α ; prove that the locus of their point of intersection is the straight line $y = (x - a) \tan 2\alpha$.

14. Prove that the equation to the circle, which passes through the focus and touches the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$, is $x^2 + y^2 - ax(3t^2 + 1) - ay(3t - t^3) + 3a^2t^2 = 0$.
Prove also that the locus of its centre is the curve $27ay^2 = (2x - a)(x - 5a)^2$.

15. Two parabolas $y^2 = 4a(x - \lambda_1)$ and $x^2 = 4a(y - \lambda_2)$ always touch one another, λ_1, λ_2 being parameters. Find the eccentricity of the locus of point of contact.

16. If tangent drawn at a point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is same as the normal drawn at a point $(\sqrt{5}\cos\phi, 2\sin\phi)$ on the ellipse $4x^2 + 5y^2 = 20$. Find the values of t & ϕ.

17. Prove that the normals at the points, where the straight line $\ell x + my = 1$ meets the parabola $y^2 = 4ax$, meet on the normal at the point $\left(\frac{4am^2}{\ell^2}, \frac{4am}{\ell}\right)$ on the parabola.

18. From an external point P, tangents are drawn to the parabola; find the equation of the locus of P when these tangents make angles θ_1 and θ_2 with the axis, such that $\cos\theta_1 \cos\theta_2 = \mu$, which is constant.

19. TP and TQ are tangents to the parabola and the normals at P and Q meet at a point R on the curve ; prove that the centre of the circle circumscribing the triangle TPQ lies on the parabola $2y^2 = a(x - a)$.

20. A chord is a normal to a parabola and is inclined at an angle θ to the axis; prove that the area of the triangle formed by it and the tangents at its extremities is $4a^2 \sec^3\theta \operatorname{cosec}^3\theta$

21. From a given point 'P' a variable straight line is drawn to cut a given straight line (not passing through the point 'P') at point 'Q'. Show that the line through 'Q' and perpendicular to the variable line, will always touch a fixed parabola. Also find focus of that parabola.

22. Find the locus of centre of a family of circles passing through the vertex of the parabola $y^2 = 4ax$, and cutting the parabola orthogonally at the other point of intersection.

23. If α, β are eccentric angles of the extremities of a focal chord of an ellipse, then eccentricity of the ellipse is

24. If circumcentre of an equilateral triangle inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with vertices having eccentric angles α, β, γ respectively is (x_1, y_1), then find $\Sigma \cos\alpha \cos\beta + \Sigma \sin\alpha \sin\beta$.

25. Find the locus of extremities of latus rectum of the family of ellipse $b^2x^2 + y^2 = a^2b^2$ where b is a parameter ($b^2 < 1$).

26. A series of concentric ellipses $E_1, E_2, ..., E_n$ are drawn such that E_n touches the extremities of the major axis of E_{n-1} and the foci of E_n coincide with the extremities of minor axis of E_{n-1}. If the eccentricity of the ellipses is independent of n, then find the value of the eccentricity.

27. A point moves such that the sum of the square of the distances from two fixed straight lines intersecting at angle 2α is a constant. Prove that the locus is an ellipse of eccentricity

$\frac{\sqrt{\cos 2\alpha}}{\cos\alpha}$ if $\alpha < \frac{\pi}{4}$ and $\frac{\sqrt{-\cos 2\alpha}}{\sin\alpha}$ if $\alpha > \frac{\pi}{4}$

28. A straight line PQ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1 and the circle $x^2 + y^2 = r^2 (b < r < a)$. RS is a focal chord of the ellipse. If RS is parallel to PQ and meets the circle at points R and S. Find the length of RS.

29. The vertex of a parabola lies on with one of the vertices of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its axis is along the major axis of the ellipse. For the area of the region bounded by the parabola and its common chord (not passing through the vertex) with the ellipse to be maximum, prove that the latus rectum of the ellipse is four times that of the parabola.

30. Find the maximum area of the parallelogram inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, one of whose diagonals is the line y = mx.

31. An ellipse has the points (1, –1) and (2, –1) as its focii and x + y – 5 = 0 as one of its tangent. Find the coordinates of the point where this line touches the ellipse.

32. Find slope of all the tangents to the parabola $y^2 = 4x$ which bisect two distinct chords of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ drawn from (-2, 0).

33. If l_1 and l_2 be the length of the focal chord other than latus rectum of ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ = 1 which are perpendicular to each other then prove that $\frac{1}{l_1}+\frac{1}{l_2}=\frac{7}{12}$.

34. Find the range of eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b) such that the line segment joining the foci does not subtend a right angle at any point on the ellipse.

35. A line of gradient m intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at A and B and its auxiliary circle at C and D. Tangents to ellipse at A and B intersect at P and tangents to circle at C and D intersects at Q. Find the locus of mid point of PQ.

36. From a variable point P on the line y = x – 3, tangents PQ and PR are drawn to the circle $x^2 + y^2 = 4$. Q' and R' are corresponding points of Q and R on the ellipse $\frac{x^2}{4}+\frac{y^2}{1} = 1$. Find the locus of mid point of the chord Q'R;' of the ellipse.;

37. A line through P(λ, 3) meet the ellipse $\frac{x^2}{16}+\frac{y^2}{9} = 1$ at A and D and meet the axes at B and C; so that PA . PD = PB . PC, then find the range of λ.

38. Consider the ellipse $\frac{x^2}{4}+y^2 = p$ and its given chord x – y = 1. For what values of p, the pair of tangents drawn from an external point to the ellipse meet the given chord at its extremities and the reflection of the external point about the given chord lie within the ellipse.

39. Show that the equation of the pair of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ at the points of intersection with the line, p x + q y + 1 = 0 is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$ $(p^2 a^2 + q^2 b^2 - 1) = (p x + q y + 1)^2$.

40. Prove that the sum of the eccentric angles of the extremities of a chord of an ellipse, which is drawn in a given direction is constant and is equal to twice the eccentric angle of the point at which the tangent is parallel to the given direction.

41. If the normals at α, β, γ , δ on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ are concurrent, prove that $(\Sigma\cos\alpha)(\Sigma\sec\alpha) = 4$

42. The transverse axis of a hyperbola is of length 2a and a vertex divides the segment of the axis between the centre and the corresponding focus in the ratio 2 : 1. Find the equation of the hyperbola.

43. If the distance between the centres of the hyperbolas :
$x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$ ..... (i)
$9x^2 - 16y^2 - 18x - 32y - 151 = 0$ ..... (ii) is d then $125\ d^2$ = ...

44. PQ is the chord joining the points whose eccentric angles are ϕ_1 and ϕ_2 on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$, If $\phi_1 - \phi_2 = 2\alpha$, where α is constant, prove that PQ touches the hyperbola $\frac{x^2}{a^2}\cos^2\alpha - \frac{y^2}{b^2} = 1$.

45. Chords of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ are tangents to the circle drawn on the line joining the foci as diameter . Find the locus of the point of intersection of tangents at the extremities of the chords .

46. If a chord of hyperbola $xy = c^2$ is a normal at point A, subtending an angle α at origin. Then prove that $\frac{\sin(\alpha - A)}{\sin(\alpha + A)} = 3$.

47 Let $S_1 \equiv x^2 - y^2 + 2y - \sqrt{2} - 1$ and $S_2 \equiv 4x^2 + 9y^2 - 18y - 27$ be such that $S_1 = 0$ and $S_2 = 0$ intersect in four real points Q, R, S and T and let P be the point $(2^{3/4}, 1)$. Show that PQ + PR + PS + PT = $4\sqrt{2}\sqrt{\frac{36+9\sqrt{2}}{13}}$.

48. P and Q two points on the hyperbola $x^2 - y^2 = 1$ such that PQ subtends an angle of $\pi/4$ at its centre. Find the angle between the tangent at P and the normal at Q.

49 A tangent is drawn at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, cutting x-axis and y-axis at Q and R respectively. From point A of rectangle OQAR (where O is origin) two lines are drawn to cut one branch of the hyperbola at point A1 and A2. Find the locus of the centroid of the triangle AA1A2, if A_1A_2 forms a focal chord of minimum length.

50 A tangent is drawn at a point (x_1, y_1) on the parabola $y^2 = 4ax$. Now tangents are drawn from points on this tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that all the chords of contact pass through the point (x_2, y_2). Show that x_1 and x_2 are of opposite signs.

51. Find the values of 'a' if there exists a tangent to the ellipse $a^2x^2 + \frac{y^2}{2} = 1$ such that chord intercepted between given tangent and hyperbola $\frac{x^2}{a^2} - 2y^2 = 1$ subtends a right angle at the centre of the curves.

52. Find the equation of parabola with focus as the vertex A of one segment of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre O and vertex as the focus S of that segment. Prove that if the parabola cuts positive y-axis at K then OK is geometric mean of OS and the latus rectum of parabola.

53 From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ tangents are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Prove that the chord of contact thus formed will touch the hyperbola at the point which is the reflection of the first point w.r.t. the x axis. Also show that the same thing happens if the point is on the ellipse and tangents are drawn to the hyperbola.

54. An equilateral triangle PQR is drawn on the circle $x^2 + y^2 = a^2$. Points P' , Q ' and R' the corresponding points of the points P, Q and R on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then find the minimum area of the triangle P' Q' R'.

Answers

5. $y^2 = 4x$.

7. $\sqrt{2}$

8. $(x + 1)^2 + (y - 4)^2 = \sqrt{\left[x^2+(y-3)^2\right]\left[(x+5)^2+(y-8)^2\right]}$

9. 1568

12. focus = $\left(\frac{1}{3},\frac{4}{3}\right)$, directrix - 3x + 11 = 0.

15. $\sqrt{2}$

16. $\phi = \pi - \tan^{-1} 2,\ t = -\frac{1}{\sqrt{5}}$; $\phi = \pi + \tan^{-1}2,\ t = \frac{1}{\sqrt{5}}$; $\phi = \pm\frac{\pi}{2}$, t = 0

18. $x^2 = \mu^2 \{(x - a)^2 + y^2\}$

21. (a, 0)

22. $2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$

23. $\frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$

24. $\frac{9x_1^2}{2a^2} + \frac{9y_1^2}{2b^2} - \frac{3}{2}$

25. $x^2 \pm ay = a^2$

26. $\frac{\sqrt{5}-1}{2}$

28. RS = 2b

30. 2ab

31. $\left(\frac{34}{9}, \frac{11}{9}\right)$

32. $\left(-\infty, \frac{-2}{3}\right)\cup\left(\frac{2}{3}, \infty\right)$

34. $e \in \left(0, \frac{1}{\sqrt{2}}\right)$

35. $X + \left(\frac{2a^2m}{a^2+b^2}\right)Y = 0$

36. $3x^2 + 12y^2 - 4x + 8y = 0$.

37. $\lambda \ge 8$ or $\lambda \le -8$.

38. $\frac{1}{5} < p < \frac{5}{13}$

42. $5x^2 - 4y^2 = 5a^2$

43. 0025

45. $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2+b^2}$

48. $\frac{\pi}{4}$

49 $\left(\frac{a}{3x-2\sqrt{a^2+b^2}}\right)^2 - \left(\frac{b}{3y}\right)^2 = 1$ or $\left(\frac{a}{3x+2\sqrt{a^2+b^2}}\right)^2 - \left(\frac{b}{3y}\right)^2 = 1$

51. $a \in \left[-\sqrt{\sqrt{2}-1}, -\sqrt{\frac{2}{5}}\right]\cup\left[\sqrt{\frac{2}{5}}, \sqrt{\sqrt{2}-1}\right]$

54. $\frac{3\sqrt{3}}{4}ab$

Self Assessment Paper (SAP)

JEE ADVANCED

Maximum Marks : 62 **Total Time : 1:00 Hr**

SECTION-1 : ONE OPTION CORRECT (Marks - 12)

1. A parabola 'C' whose focus is S(0, 0) and passes through P(3, 4), Equation of tangent at 'P' to parabola is $3x + 4y - 25 = 0$. A chord through 'S' parallel to tangennt at 'P' intersects the parabola at A & B. Then which of the following is/are correct

(A) Length of AB is 40 units (B) Length of AB is 20 units

(C) Length of latus rectum of parabola 40 units (D) Length of latus rectum of parabola 10 units

2. If a quadrilateral formed by four tangents to the ellipse $3x^2 + 4y^2 = 12$ is square, then

(A) The vertices of the square lie on $y = \pm x$ (B) The vertices of the square lie on $x^2 + y^2 = 25$

(C) The area of all such square is constant. (D) None of these

3. x and y are real numbers and $x^2 + 9y^2 - 4x + 6y + 4 = 0$ then the maximum value of $(4x - 9y)$ is

(A) 5 (B) 25 (C) $\sqrt{97}$ (D) 16

4. If PQ is a double ordinate of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ such that OPQ is equilateral triangle, 'O' being the centre of the hyperbola and 'e' is eccentricity of hyperbola, then

(A) $1 < e < \frac{2}{\sqrt{3}}$ (B) $e \geq \frac{2}{\sqrt{3}}$ (C) $e \geq \frac{4}{3}$ (D) $1 < e < \frac{4}{3}$

SECTION-2 : ONE OR MORE THAN ONE CORRECT (Marks - 32)

5. A triangle ABC with vertices A(2,5), B(5, 2) and C(–1, –1) is inscribed on the curve $xy - x - y - 3 = 0$ and $H(\alpha, \alpha)$ is orthocentre. Let D, E, F be feet of perpendicular from A, B & C on BC, CA & AB respectively and inradius of ΔDEF is 'r', then

(A) orthocentre of ΔABC is (3, 3) (B) $r = \frac{2}{5}\sqrt{2}$ units

(C) equation of DE is $x + y = \frac{26}{5}$ (D) $r = \frac{4}{5}\sqrt{2}$ units

6. Let A, B & C are three distinct point on $y^2 = 8x$ such that normal at these points are concurrent at P. The slope of AB is '2' and abscissa of centroid of ΔABC is $\frac{4}{3}$. Which of the following is/are correct ?

(A) Area of ΔABC is 8 square units (B) Coordinate of 'P' is (6, 0)

(C) Angle between normals are 45°, 45°, 90° (D) Angle between normals are 60°, 60°, 60°

7. If two tangent can be drawn to the different branches of hyperbola $x^2 - \frac{y^2}{4} = 1$ from the point (α, α^2), then

(A) $\alpha \in (-2, 0)$ (B) $\alpha \in (0, 2)$ (C) $\alpha \in (-\infty, -2)$ (D) $\alpha \in (2, \infty)$

8. If eccentricity of hyperbola $\left|\sqrt{(x-1)^2+(y-2)^2} - \sqrt{(x-5)^2+(y-5)^2}\right| = 3$ is e_1 and its conjugate hyperbola is e_2, then

(A) $\frac{1}{e_1} + \frac{1}{e_2} = 1$ (B) $\frac{e_1}{e_2} = \frac{4}{3}$ (C) $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ (D) $e_1 e_2 = \frac{25}{12}$

9. Suppose that the ellipse E has equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Suppose that 'C' is any circle concentric with E. Let 'A' is a point on E and B is a point on 'C' such that AB is tangent to both E & C of maximum length then which of the following is/are correct

(A) Length of AB is 2 (B) Length of AB is $2\sqrt{2}$

(C) Slope of tangent AB is $\pm\frac{1}{\sqrt{2}}$ (D) Slope of tangent AB is $2\sqrt{2}$

10. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let 'B' be one of the end point of the latus rectum and 'C' is the focus of the hyperbola nearest to the point 'A', then for ΔABC

(A) The area is $\sqrt{\frac{3}{2}} - 1$ (B) The area is $\sqrt{\frac{3}{2}} + 1$

(C) Circumradius is $\frac{1}{2}\sqrt{11 - 4\sqrt{6}}$ (D) Circumradius is $\frac{1}{2}\sqrt{11 + 4\sqrt{6}}$

11. Chord of the parabola $y^2 = 4ax$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the locus of the points of intersection of tangents of the parabola drawn at the extremities of such chords is

(A) ellipse with major axis is double the transverse axis of given hyperbola (if a = b)
(B) circle with radius 'a' (if b = 2a)
(C) hyperbola (if a = b)
(D) circle (if a = b)

12. For ellipse $2x^2 - 2xy + 4y^2 - \left(3 + \sqrt{2}\right) = 0$, then

(A) Length of major axis of ellipse is $\sqrt{\frac{3+\sqrt{2}}{3-\sqrt{2}}}$ (B) Length of minor axis of ellipse is 2

(C) Product of perpendicular from the foci to any tangent to given ellipse is 1.

(D) Length of major axis of ellipse is $\frac{6+2\sqrt{2}}{\sqrt{7}}$

SECTION-3 : NUMERICAL VALUE TYPE (Marks - 18)

13. If all chords of the parabola $y^2 = 4x + 4$ which subtends a right angle at point $(1, 2\sqrt{2})$ a fixed point (α, β). Then the value of $\beta - \alpha$ is

14. The tangent at a point P of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets its auxilliary circle in two points A & B. If chord AB subtends right angle at centre then least value of 'e' is

15. If a chord of hyperbola $xy = c^2$ is normal at point A subtending an angle 'α' at origin O, then the value of $\frac{\tan\alpha}{\tan A}$ (where $A = \angle OAB$) is equal to

16. The number of integral points on the hyperbola $x^2 - y^2 = 4000000$ are (an integral point is a point both of whose co-ordinates are integer)

17. A parabola which have directrix $x + y + 2 = 0$ and touches a line $= 2x + y - 5 = 0$ at (2, 1), then the length of latusrectum is

18. An ellipse has foci at $F_1(9, 20)$ and $F_2(49,55)$ in the xy plane and is tangent to the x-axis, then length of major axis is

Answers

Self Assessment Paper

1.	(B)	2.	(C)
3.	(D)	4.	(B)
5.	(A) (B) (C)	6.	(A) (B) (C)
7.	(C) (D)	8.	(B) (C) (D)
9.	(A) (C)	10.	(A) (C)
11.	(A) (B)	12.	(B) (C) (D)
13.	–7.82	14.	0.71
15.	–2	16.	98
17.	12.73	18.	85