Question Set 10

INDEFINITE INTEGRATION

(Marks with option : 10)

Remember :

1. If $\frac{d}{dx}[f(x)] = g(x)$, then $\int g(x)\,dx = f(x) + c$, where $g(x)$ is integrand, $f(x)$ is integral of $g(x)$ with respect to x and c is an arbitrary constant.

2. $\int f(ax+b)\,dx = g(ax+b) \times \frac{1}{a} + c, \ (a \neq 0)$

3. (1) $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c, \ (n \neq -1)$

(2) $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$

(3) $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$

4. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \ (n \neq -1)$

5. $\int 1\,dx = x + c$

6. $\int \sin x\,dx = -\cos x + c$

7. $\int \cos x\,dx = \sin x + c$

8. $\int \sec^2 x\,dx = \tan x + c$

9. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

10. $\int \sec x \tan x\,dx = \sec x + c$

11. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

12. $\int \tan x\,dx = \log|\sec x| + c = -\log|\cos x| + c$

13. $\int \cot x\,dx = \log|\sin x| + c = -\log|\operatorname{cosec} x| + c$

14. $\int \sec x\,dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + c$

15. $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c = \log\left|\tan\left(\frac{x}{2}\right)\right| + c$

16. $\int a^x\,dx = \frac{a^x}{\log a} + c$

17. $\int e^x\,dx = e^x + c$

18. $\int \frac{1}{x}\, dx = \log|x| + c$

19. $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c = -\cos^{-1} x + c$

20. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c = -\cot^{-1} x + c$

21. $\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$

22. $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

23. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

24. $\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

25. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

26. $\int \frac{1}{\sqrt{a^2+x^2}}\, dx = \log\left|x + \sqrt{a^2+x^2}\right| + c$

27. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \log\left|x + \sqrt{x^2-a^2}\right| + c$

28. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$

29. $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \log|x + \sqrt{x^2+a^2}| + c$

30. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2-a^2}| + c$

31. $\int uv dx = u \int v dx - \int \left(\frac{du}{dx}\int v dx\right) dx$

The order in which u and v are to be chosen is according to the serial order of the letters of the word ***LIATE.***

L : Logarithmic, ***I*** : Inverse Trigonometric,

A : Algebraic, ***T*** : Trigonometric and

E : Exponential functions.

32. $\int e^x [f(x) + f'(x)] dx = e^x \cdot f(x) + c$

10.1 ELEMENTARY INTEGRATION

Solved Examples | **2 marks each**

Ex. 1. Integrate the following w.r.t. x :

(1) $(x-2)^2\sqrt{x}$ (2) $\dfrac{3x^3-2x+5}{x\sqrt{x}}$

Solution :

(1) Let $I=\int (x-2)^2\sqrt{x}\,dx$

$$=\int (x^2-4x+4)\sqrt{x}\,dx$$

$$=\int \left(x^{\frac{5}{2}}-4x^{\frac{3}{2}}+4x^{\frac{1}{2}}\right)dx$$

$$=\int x^{\frac{5}{2}}\,dx-4\int x^{\frac{3}{2}}\,dx+4\int x^{\frac{1}{2}}\,dx$$

$$=\frac{x^{\frac{7}{2}}}{\left(\frac{7}{2}\right)}-4\cdot\frac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}+4\cdot\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+c$$

$$=\frac{2}{7}x^{\frac{7}{2}}-\frac{8}{5}x^{\frac{5}{2}}+\frac{8}{3}x^{\frac{3}{2}}+c.$$

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(2) $\displaystyle\int \frac{3x^3-2x+5}{x\sqrt{x}}\,dx=\int x^{-\frac{3}{2}}(3x^3-2x+5)\,dx$

$$=\int\left(3x^{\frac{3}{2}}-2x^{-\frac{1}{2}}+5x^{-\frac{3}{2}}\right)dx$$

$$=3\int x^{\frac{3}{2}}\,dx-2\int x^{-\frac{1}{2}}\,dx+5\int x^{-\frac{3}{2}}\,dx$$

$$=3\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right)-2\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right)+5\left(\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right)+c$$

$$=\frac{6}{5}x^2\sqrt{x}-4\sqrt{x}-\frac{10}{\sqrt{x}}+c.$$

Ex. 2. Integrate the following w.r.t. x :

(1) $\dfrac{x+1}{(x+2)(x+3)}$ (2) $\dfrac{4x+3}{2x+1}$ (3) $\dfrac{2x-7}{\sqrt{3x-2}}$.

Solution :

(1) $\displaystyle\int\frac{x+1}{(x+2)(x+3)}\,dx=\int\frac{2(x+2)-(x+3)}{(x+2)(x+3)}\,dx$

$$= \int \left(\frac{2}{x+3} - \frac{1}{x+2} \right) dx$$

$$= 2\int \frac{1}{x+3}\, dx - \int \frac{1}{x+2}\, dx$$

$$= 2 \log |x+3| - \log |x+2| + c.$$

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(2) $$\int \frac{4x+3}{2x+1}\, dx = \int \frac{2(2x+1)+1}{2x+1}\, dx$$

$$= \int \left(\frac{2(2x+1)}{2x+1} + \frac{1}{2x+1} \right) dx$$

$$= 2 \int 1\, dx + \int \frac{1}{2x+1}\, dx$$

$$= 2x + \frac{1}{2} \log |2x+1| + c.$$

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(3) $$\int \frac{2x-7}{\sqrt{3x-2}}\, dx = \frac{1}{3} \int \frac{6x-21}{\sqrt{3x-2}}\, dx$$

$$= \frac{1}{3} \int \frac{2\,(3x-2) - 17}{\sqrt{3x-2}}\, dx$$

$$= \frac{1}{3} \int \left[\frac{2\,(3x-2)}{\sqrt{3x-2}} - \frac{17}{\sqrt{3x-2}} \right] dx$$

$$= \frac{2}{3} \int (3x-2)^{\frac{1}{2}}\, dx - \frac{17}{3} \int (3x-2)^{-\frac{1}{2}}\, dx$$

$$= \frac{2}{3} \cdot \frac{(3x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \times \frac{1}{3} - \frac{17}{3} \cdot \frac{(3x-2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} \times \frac{1}{3} + c$$

$$= \frac{4}{27} (3x-2)^{\frac{3}{2}} - \frac{34}{9} \sqrt{3x-2} + c.$$

Ex. 3. Integrate the following w.r.t. x :

(1) $\dfrac{\cos x - \cos 2x}{1 - \cos x}$ (2) $\sin^4 x$ (3) $\sin 4x \cdot \cos 3x$

(4) $\sqrt{1 + \sin 2x}$ (5) $\dfrac{3^x - 4^x}{5^x}$.

Solution :

(1) $$\int \frac{\cos x - \cos 2x}{1 - \cos x}\, dx = \int \left[\frac{\cos x - (\cos^2 x - \sin^2 x)}{1 - \cos x} \right] dx$$

$$= \int \left[\frac{\cos x - \cos^2 x + \sin^2 x}{1 - \cos x} \right] dx$$

$$= \int \left[\frac{\cos x\,(1 - \cos x) + (1 - \cos^2 x)}{1 - \cos x} \right] dx$$

$$= \int \left[\frac{\cos x\,(1 - \cos x) + (1 - \cos x)\,(1 + \cos x)}{1 - \cos x} \right] dx$$

$$= \int \left[\frac{(1 - \cos x)(\cos x + 1 + \cos x)}{1 - \cos x} \right] dx$$

$$= \int (1 + 2 \cos x)\, dx$$

$$= \int 1\, dx + 2 \int \cos x\, dx$$

$$= x + 2 \sin x + c.$$

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(2) $\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx$

$$= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4} \int \left(1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) dx$$

$$= \frac{1}{4} \int \left(\frac{3}{2} - 2 \cos 2x + \frac{1}{2} \cos 4x \right) dx$$

$$= \frac{3}{8} \int 1\, dx - \frac{2}{4} \int \cos 2x\, dx + \frac{1}{8} \int \cos 4x\, dx$$

$$= \frac{3}{8} x - \frac{1}{2} \times \frac{\sin 2x}{2} + \frac{1}{8} \times \frac{\sin 4x}{4} + c$$

$$= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c.$$

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(3) $\int \sin 4x \cos 3x\, dx = \frac{1}{2} \int \sin 4x \cos 3x\, dx$

$$= \frac{1}{2} \int [\sin (4x + 3x) + \sin (4x - 3x)]\, dx$$

$$= \frac{1}{2} \int \sin 7x\, dx + \frac{1}{2} \int \sin x\, dx$$

$$= \frac{1}{2} \left(\frac{-\cos 7x}{7} \right) - \frac{1}{2} \cos x + c$$

$$= -\frac{1}{14} \cos 7x - \frac{1}{2} \cos x + c.$$

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(4) $\int \quad 1 + \sin 2x\, dx = \int \quad \cos^2 x + \sin^2 x + 2 \sin x \cos x\, dx$

$$= \int \sqrt{(\cos x + \sin x)^2}\, dx$$

$$= \int (\cos x + \sin x)\, dx$$

$$= \int \cos x\, dx + \int \sin x\, dx$$

$$= \sin x - \cos x + c.$$

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(5) $\displaystyle\int \frac{3^x - 4^x}{5^x}\, dx = \int \left(\frac{3^x}{5^x} - \frac{4^x}{5^x}\right) dx$

$$= \int \left(\frac{3}{5}\right)^x dx - \int \left(\frac{4}{5}\right)^x dx$$

$$= \frac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)} - \frac{\left(\frac{4}{5}\right)^x}{\log\left(\frac{4}{5}\right)} + c.$$

Ex. 4. If $f'(x) = k(\cos x - \sin x)$, $f'(0) = 3$, $f\left(\frac{\pi}{2}\right) = 15$, find $f(x)$.

Solution : By the definition of integration,

$$f(x) = \int f'(x)\, dx = \int k(\cos x - \sin x)\, dx$$

$$= k\left[\int \cos x\, dx - \int \sin x\, dx\right]$$

$$= k[\sin x - (-\cos x)] + c$$

$\therefore\ f(x) = k(\sin x + \cos x) + c$ ... (1)

Now, $f'(0) = 3$ gives

$f'(0) = k(\cos 0 - \sin 0) = 3$

$\therefore\ k(1 - 0) = 3 \qquad \therefore\ k = 3$

$\therefore$ from (1), $f(x) = 3(\sin x + \cos x) + c$ ... (2)

Further, $f\left(\frac{\pi}{2}\right) = 15$ gives $f\left(\frac{\pi}{2}\right) = 3\left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) + c = 15$

$\therefore\ 3(1 + 0) + c = 15$

$\therefore\ 3 + c = 15$

$\therefore\ c = 12$

$\therefore$ from (2), $f(x) = 3(\sin x + \cos x) + 12.$

Ex. 5. If $f'(x) = x^{-1}$, then find $f(x)$. *(March '22) (1 mark)*

Solution : By the definition of integration,

$$f(x) = \int f'(x)\, dx = \int x^{-1}\, dx$$

$$= \int \left(\frac{1}{x}\right) dx = \log|x| + c.$$

Examples for Practice	**2 marks each**

Integrate the following w.r.t. x :

1. $\dfrac{2}{\sqrt{x} - \sqrt{x+3}}$

2. $\dfrac{\sqrt{x}+1}{x+\sqrt{x}}$

3. $x^2\left(1 - \dfrac{2}{x}\right)^2$

4. $\dfrac{2x+3}{5x-1}$

5. $\dfrac{2x-7}{\sqrt{4x-1}}$

6. $\dfrac{\cos 2x}{\sin^2 x}$

7. $\dfrac{\sin x}{1+\sin x}$

8. $\sin 5x \cdot \cos 7x$

9. $\dfrac{\sin 2x}{\cos x}$ *(Sept. '21) (1 mark)*

10. $\cos^3 x$

11. $\dfrac{\sin^3 x - \cos^3 x}{\sin^2 x \cdot \cos^2 x}$

12. $\dfrac{\cos x}{1 - \cos x}$

13. $x^3 + 3^x$

14. $\dfrac{e^{4\log x} - e^{5\log x}}{x^5}$.

ANSWERS

1. $\dfrac{4}{9}\left[x^{\frac{3}{2}} + (x+3)^{\frac{3}{2}}\right] + c$

2. $2\sqrt{x} + c$

3. $\dfrac{x^3}{3} - 2x^2 + 4x + c$

4. $\dfrac{2}{5}x + \dfrac{17}{25}\log|5x - 1| + c$

5. $\dfrac{1}{12}(4x-1)^{\frac{3}{2}} - \dfrac{13}{4}\sqrt{4x-1} + c$

6. $-\cot x - 2x + c$

7. $\sec x - \tan x + x + c$

8. $-\dfrac{1}{24}\cos 12x + \dfrac{1}{4}\cos 2x + c$

9. $-2\cos x + c$

10. $\dfrac{1}{12}\sin 3x + \dfrac{3}{4}\sin x + c$

11. $\sec x + \operatorname{cosec} x + c$

12. $-\operatorname{cosec} x - \cot x - x + c$

13. $\dfrac{x^4}{4} + \dfrac{3^x}{\log 3} + c$

14. $\log|x| - x + c$.

10.2 METHOD OF SUBSTITUTION

Theory Question	3 or 4 marks

Q. 1. If $x = \phi(t)$ is a differentiable function of t, then
$$\int f(x)\,dx = \int f[\phi(t)]\cdot\phi'(t)\,dt.$$

Ans. $x = \phi(t)$ is differentiable function of t.

$\therefore \dfrac{dx}{dt} = \phi'(t)$

Let $\int f(x)\,dx = F(x)$

$\therefore \dfrac{d}{dx}[F(x)] = f(x)$

$\therefore$ by the chain rule,

$$\frac{d}{dt}[F(x)] = \frac{d}{dx}[F(x)]\cdot\frac{dx}{dt}$$
$$= f(x)\cdot\frac{dx}{dt} = f[\phi(t)]\cdot\phi'(t)$$

$\therefore$ by the definition of integration,

$F(x) = \int f[\phi(t)]\cdot\phi'(t)\,dt$

$\therefore \int f(x)\,dx = \int f[\phi(t)]\cdot\phi'(t)\,dt.$

Solved Examples	2 marks each

Ex. 6. Integrate the following w.r.t. x :

(1) $\dfrac{x^{n-1}}{\sqrt{1+4x^n}}$ (2) $\dfrac{1}{x\cdot\log x\cdot\log(\log x)}$ (3) $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$.

Solution :

(1) Let $I = \displaystyle\int \frac{x^{n-1}}{\sqrt{1+4x^n}}\,dx$

Put $x^n = t$ $\therefore nx^{n-1}\,dx = dt$

$\therefore x^{n-1}\,dx = \dfrac{dt}{n}$

$$\therefore I = \int \frac{1}{\sqrt{1+4t}}\cdot\frac{dt}{n} = \frac{1}{n}\int (1+4t)^{-\frac{1}{2}}\,dt$$

$$=\frac{1}{n}\cdot\frac{(1+4t)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\times\frac{1}{4}+c$$

$$=\frac{1}{2n}\cdot\sqrt{1+4x^n}+c.$$

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(2) Let $I=\int\frac{1}{x\cdot\log x\cdot\log(\log x)}\,dx$

$$=\int\frac{1}{\log(\log x)}\cdot\frac{1}{x\cdot\log x}\,dx$$

Put $\log(\log x)=t$ $\quad\therefore\ \frac{1}{\log x}\cdot\frac{1}{x}\,dx=dt$

$\therefore\ \frac{1}{x\cdot\log x}\,dx=dt$

$\therefore\ I=\int\frac{1}{t}\,dt=\log|t|+c$

$=\log|\log(\log x)|+c.$

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(3) Let $I=\int\frac{\cos\sqrt{x}}{\sqrt{x}}\,dx=\int\cos\sqrt{x}\cdot\frac{1}{\sqrt{x}}\,dx$

Put $\sqrt{x}=t$ $\quad\therefore\ \frac{1}{2\sqrt{x}}\,dx=dt$

$\therefore\ \frac{1}{\sqrt{x}}\,dx=2\,dt$

$\therefore\ I=\int\cos t\cdot 2dt=2\int\cos t\,dt$

$=2\sin t+c=2\sin\sqrt{x}+c.$

Ex. 7. Integrate the following w.r.t. x :

(1) $\frac{x^2\cdot\tan^{-1}(x^3)}{1+x^6}$ **(2)** $\frac{\sec^8 x}{\operatorname{cosec} x}$ **(3)** $\frac{\sqrt{\tan x}}{\sin x\cdot\cos x}$ **(4)** $\frac{e^x(1+x)}{\cos(x\cdot e^x)}$.

Solution :

(1) Let $I=\int\frac{x^2\cdot\tan^{-1}(x^3)}{1+x^6}\,dx=\int\tan^{-1}(x^3)\cdot\frac{x^2}{1+x^6}\,dx$

Put $\tan^{-1}(x^3)=t$ $\quad\therefore\ \frac{1}{1+(x^3)^2}\cdot 3x^2\,dx=dt$

$\therefore\ \frac{x^2}{1+x^6}\,dx=\frac{dt}{3}$

$$\therefore\ I=\int t\cdot\frac{dt}{3}=\frac{1}{3}\int t\,dt$$

$$=\frac{1}{3}\cdot\frac{t^2}{2}+c=\frac{1}{6}[\tan^{-1}(x^3)]^2+c.$$

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(2) Let $I=\int\frac{\sec^8 x}{\operatorname{cosec} x}\,dx=\int\frac{\sin x}{\cos^8 x}\,dx$

Put $\cos x=t$ $\quad\therefore\ -\sin x\,dx=dt$

$\therefore\ \sin x\,dx=-dt$

$$\therefore\ I=\int\frac{1}{t^8}(-dt)=-\int t^{-8}\,dt$$

$$=-\frac{t^{-7}}{-7}+c=\frac{1}{7t^7}+c$$

$$=\frac{1}{7\cos^7 x}+c=\frac{1}{7}\sec^7 x+c.$$

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(3) Let $I=\int\frac{\sqrt{\tan x}}{\sin x\cdot\cos x}\,dx$

Dividing numerator and denominator by $\cos^2 x$, we get

$$I=\int\frac{\left(\frac{\sqrt{\tan x}}{\cos^2 x}\right)}{\left(\frac{\sin x}{\cos x}\right)}\,dx=\int\frac{\sqrt{\tan x}\cdot\sec^2 x}{\tan x}\,dx=\int\frac{\sec^2 x}{\sqrt{\tan x}}\,dx$$

Put $\tan x=t$ $\quad\therefore\ \sec^2 x\,dx=dt$

$$\therefore\ I=\int\frac{1}{\sqrt{t}}\,dt$$

$$=\int t^{-\frac{1}{2}}\,dt=\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c=2\sqrt{t}+c=2\sqrt{\tan x}+c.$$

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(4) Let $I=\int\frac{e^x(1+x)}{\cos(x\cdot e^x)}\,dx$

Put $x\cdot e^x=t$ $\quad\therefore\ (xe^x+e^x\times 1)\,dx=dt$ $\quad\therefore\ e^x(1+x)\,dx=dt$

$$\therefore\ I=\int\frac{1}{\cos t}\,dt=\int\sec t\,dt$$

$$=\log|\sec t+\tan t|+c$$

$$=\log|\sec(x\cdot e^x)+\tan(x\cdot e^x)|+c.$$

Examples for Practice	2 marks each

Integrate the following w.r.t. x :

1. $\dfrac{\cos(\log x)}{x}$

2. $\dfrac{1+\log x}{\sin^2(x\log x)}$

3. $\dfrac{e^{x-1}+x^{e-1}}{e^x+x^e}$

4. $\dfrac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}$

5. $\dfrac{x\cdot\sec^2(x^2)}{\sqrt{\tan^3(x^2)}}$

6. $\dfrac{1}{\sqrt{x}+\sqrt{x^3}}$

7. $\dfrac{2\sin x\cos x}{3\cos^2 x+4\sin^2 x}$

8. $\dfrac{1}{\sqrt{1-x^2}\cdot(2+3\sin^{-1}x)}$

9. $\dfrac{1}{x+\sqrt{x}}$

10. $\dfrac{10x^9+10^x\cdot\log 10}{10^x+x^{10}}$

11. $e^{3\log x}\cdot(x^4+1)^{-1}$

12. $\dfrac{1}{3x+7x^{-n}}$

13. $\sin^4 x\cdot\cos^3 x$

14. $\dfrac{1}{1+e^{-x}}$.

ANSWERS

1. $\sin(\log x)+c$

2. $-\cot(x\log x)+c$

3. $\dfrac{1}{e}\log|e^x+x^e|+c$

4. $\dfrac{1}{4}[\sin^{-1}(x^2)]^2+c$

5. $\dfrac{-1}{\sqrt{\tan(x^2)}}+c$

6. $2\tan^{-1}(\sqrt{x})+c$

7. $\log|3\cos^2 x+4\sin^2 x|+c$

8. $\dfrac{1}{3}\log|2+3\sin^{-1}x|+c$

9. $2\log|\sqrt{x}+1|+c$

10. $\log|10^x+x^{10}|+c$

11. $\dfrac{1}{4}\log|x^4+1|+c$

12. $\dfrac{1}{3(n+1)}\log|3x^{n+1}+7|+c$

13. $\dfrac{1}{5}\sin^5 x-\dfrac{1}{7}\sin^7 x+c$

14. $\log|e^x+1|+c$.

Solved Examples	3 marks each

Ex. 8. Integrate the following w.r.t. x :

(1) $\dfrac{\log(x+2)-\log x}{x(x+2)}$

(2) $\dfrac{1}{\sin^4 x+\cos^4 x}$

(3) $(5-3x)(2-3x)^{-\frac{1}{2}}$

(4) $\dfrac{1}{\sqrt{\sin^3 x\cdot\sin(x+\alpha)}}$.

Solution :

(1) Let $I = \int \frac{\log(x+2) - \log x}{x(x+2)} dx$

$$= \int [\log(x+2) - \log x] \cdot \frac{dx}{x(x+2)}$$

Put $t = \log(x+2) - \log x$. Then

$$dt = \left(\frac{1}{x+2} - \frac{1}{x}\right) dx = \frac{x - x - 2}{x(x+2)} dx = \frac{-2}{x(x+2)} dx$$

$$\therefore \frac{dx}{x(x+2)} = -\frac{dt}{2}.$$

$$\therefore I = \int t\left(-\frac{dt}{2}\right) = -\frac{1}{2}\int t\,dt = -\frac{1}{2}\cdot\frac{t^2}{2} + c$$

$$= -\frac{1}{4}[\log(x+2) - \log x]^2 + c$$

$$= -\frac{1}{4}\left[\log\left(\frac{x+2}{x}\right)\right]^2 + c.$$

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(2) Let $I = \int \frac{1}{\sin^4 x + \cos^4 x} dx$

Dividing numerator and denominator by $\cos^4 x$, we get

$$I = \int \frac{\sec^4 x}{\tan^4 x + 1} dx$$

$$= \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^4 x + 1} dx$$

$$= \int \frac{(1 + \tan^2 x) \cdot \sec^2 x}{\tan^4 x + 1} dx$$

Put $\tan x = t$ $\quad \therefore \sec^2 x\, dx = dt$

$$\therefore I = \int \frac{1 + t^2}{t^4 + 1} dt$$

Dividing numerator and denominator by t^2, we get

$$I = \int \frac{\frac{1}{t^2} + 1}{t^2 + \frac{1}{t^2}} dt$$

$$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\, dt$$

Put $t-\frac{1}{t}=y$ $\quad\therefore\ \left(1+\frac{1}{t^2}\right)dt = dy$

$$\therefore\ I = \int \frac{1}{y^2+2}\, dy = \int \frac{1}{y^2+(\sqrt{2})^2}\, dy$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{1}{\sqrt{2}}\left(t-\frac{1}{t}\right)\right]+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{t^2-1}{\sqrt{2}\,t}\right]+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right]+c.$$

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(3) Let $I = \int (5-3x)(2-3x)^{-\frac{1}{2}}\, dx$

Put $2-3x=t$ $\quad\therefore\ -3dx = dt$

$$\therefore\ dx = \frac{-dt}{3}$$

Also, $x = \frac{2-t}{3}$

$$\therefore\ I = \int \left[5-3\left(\frac{2-t}{3}\right)\right] t^{-\frac{1}{2}} \cdot \left(\frac{-dt}{3}\right)$$

$$= -\frac{1}{3}\int (5-2+t)\, t^{-\frac{1}{2}}\, dt$$

$$= -\frac{1}{3}\int (3+t)t^{-\frac{1}{2}}\, dt$$

$$= -\frac{1}{3}\int (3t^{-\frac{1}{2}}+t^{\frac{1}{2}})\, dt$$

$$= -\frac{3}{3}\int t^{-\frac{1}{2}}\, dt - \frac{1}{3}\int t^{\frac{1}{2}}\, dt$$

$$= -\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} - \frac{1}{3}\cdot\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c$$

$$= -2\sqrt{2-3x} - \frac{2}{9}(2-3x)^{\frac{3}{2}} + c.$$

---

(4) Let $I = \int \frac{dx}{\sqrt{\sin^3 x \cdot \sin(x+\alpha)}}$

$$= \int \frac{dx}{\sqrt{\sin^3 x \cdot (\sin x \cos\alpha + \cos x \sin\alpha)}}$$

$$= \int \frac{dx}{\sqrt{\sin^4 x\ (\cos\alpha + \cot x \sin\alpha)}}$$

$$= \int \frac{\operatorname{cosec}^2 x\, dx}{\sqrt{\cos\alpha + \cot x \cdot \sin\alpha}}$$

Put $\cos\alpha + \cot x \cdot \sin\alpha = t$

$\therefore\ -\operatorname{cosec}^2 x \cdot \sin\alpha\, dx = dt$

$\therefore\ \operatorname{cosec}^2 x\, dx = -\frac{dt}{\sin\alpha}$

$\therefore\ I = \int \frac{1}{\sqrt{t}}\left(\frac{-dt}{\sin\alpha}\right)$

$$= \frac{-1}{\sin\alpha}\int t^{-\frac{1}{2}}\, dt$$

$$= \frac{-1}{\sin\alpha}\cdot\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$$

$$= \frac{-2}{\sin\alpha}\sqrt{t} + c$$

$$= -2\operatorname{cosec}\alpha\sqrt{\cos\alpha + \cot x \sin\alpha} + c.$$

Ex. 9. Integrate the following w.r.t. x :

(1) $\frac{\sqrt{a^2-x^2}}{x^2}$ **(2)** $\sqrt{\frac{a-x}{x}}$.

Solution : Let $I = \int \frac{\sqrt{a^2-x^2}}{x^2}\, dx$

Put $x = a\sin\theta$ $\therefore\ dx = a\cos\theta\, d\theta$ and $\sin\theta = \frac{x}{a}$

$$\therefore\ I=\int \frac{\sqrt{a^2-a^2\sin^2\theta}}{a^2\sin^2\theta}\cdot a\cos\theta\, d\theta$$

$$=\int \frac{\sqrt{a^2(1-\sin^2\theta)}}{a^2\sin^2\theta}\cdot a\cos\theta\, d\theta$$

$$=\int \frac{a\cos\theta}{a^2\sin^2\theta}\cdot a\cos\theta\, d\theta$$

$$=\int \cot^2\theta\, d\theta=\int(\operatorname{cosec}^2\theta-1)\, d\theta$$

$$=\int \operatorname{cosec}^2\theta\, d\theta-\int 1\, d\theta$$

$$=-\cot\theta-\theta+c \qquad \text{... (1)}$$

Now, $\sin\theta=\frac{x}{a}$ $\qquad\therefore\ \theta=\sin^{-1}\left(\frac{x}{a}\right)$

and $\cot\theta=\sqrt{\operatorname{cosec}^2\theta-1}=\sqrt{\frac{1}{\sin^2\theta}-1}$

$$=\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}=\frac{\sqrt{1-\frac{x^2}{a^2}}}{\left(\frac{x}{a}\right)}$$

$$=\frac{\sqrt{a^2-x^2}}{x}$$

$\therefore$ from (1), $I=-\frac{\sqrt{a^2-x^2}}{x}-\sin^{-1}\left(\frac{x}{a}\right)+c.$

---

(2) Let $I=\int\sqrt{\frac{a-x}{x}}\,dx$

Put $x=a\sin^2\theta$ $\qquad\therefore\ dx=a\times 2\sin\theta\cos\theta\, d\theta$

and $\sin^2\theta=\frac{x}{a}$

$$\therefore\ I=\int\sqrt{\frac{a-a\sin^2\theta}{a\sin^2\theta}}\cdot 2a\sin\theta\cos\theta\, d\theta$$

$$=\int\sqrt{\frac{a(1-\sin^2\theta)}{a\sin^2\theta}}\cdot 2a\sin\theta\cos\theta\, d\theta$$

$$=\int\frac{\cos\theta}{\sin\theta}\cdot 2a\sin\theta\cos\theta\, d\theta$$

$$= \int 2a \cos^2\theta \, d\theta = a \int 2 \cos^2\theta \, d\theta = a \int (1 + \cos 2\theta) \, d\theta$$

$$= a \int 1 \, d\theta + a \int \cos 2\theta \, d\theta$$

$$= a\theta + a \cdot \frac{\sin 2\theta}{2} + c$$

$$= a\theta + a \cdot \frac{2 \sin\theta \cos\theta}{2} + c$$

$$= a\,\theta + a \sin\theta \cos\theta + c \qquad \text{... (1)}$$

Now, $\sin^2\theta = \frac{x}{a}$ $\quad\therefore\ \sin\theta = \sqrt{\frac{x}{a}}$

$\therefore\ \theta = \sin^{-1}\sqrt{\frac{x}{a}}$

and $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{x}{a}} = \sqrt{\frac{a-x}{a}}$

$\therefore$ from (1),

$$I = a \sin^{-1}\sqrt{\frac{x}{a}} + a \cdot \sqrt{\frac{x}{a}} \cdot \sqrt{\frac{a-x}{a}} + c$$

$$= a \sin^{-1}\sqrt{\frac{x}{a}} + \sqrt{x(a-x)} + c.$$

Ex. 10. Integrate the following w.r.t. *x* :

(1) $\dfrac{\sin x + 2\cos x}{3\sin x + 4\cos x}$ **(2)** $\dfrac{4e^x - 25}{2e^x - 5}$ **(3)** $\dfrac{1}{1 - \tan x}$.

Solution :

(1) Let $I = \int \frac{\sin x + 2\cos x}{3\sin x + 4\cos x} dx$

Put Numerator $= A\,(\text{Denominator}) + B\left[\frac{d}{dx}(\text{Denominator})\right]$

$\therefore\ \sin x + 2\cos x = A\,(3\sin x + 4\cos x) + B\left[\frac{d}{dx}(3\sin x + 4\cos x)\right]$

$$= A(3\sin x + 4\cos x) + B\,(3\cos x - 4\sin x)$$

$\therefore\ \sin x + 2\cos x = (3A - 4B)\sin x + (4A + 3B)\cos x$

Equating the coefficients of $\sin x$ and $\cos x$ on both the sides, we get

$$3A - 4B = 1 \qquad \text{... (1)}$$

and $4A + 3B = 2$ ... (2)

Multiplying equation (1) by 3 and equation (2) by 4, we get

$9A - 12B = 3$

$16A + 12B = 8$

On adding, we get

$25A = 11 \qquad \therefore\ A = \dfrac{11}{25}$

$\therefore$ from (2), $4\left(\dfrac{11}{25}\right) + 3B = 2$

$\therefore\ 3B = 2 - \dfrac{44}{25} = \dfrac{6}{25} \qquad \therefore\ B = \dfrac{2}{25}$

$\therefore\ \sin x + 2\cos x = \dfrac{11}{25}(3\sin x + 4\cos x) + \dfrac{2}{25}(3\cos x - 4\sin x)$

$$\therefore\ I = \int\left[\frac{\frac{11}{25}(3\sin x + 4\cos x) + \frac{2}{25}(3\cos x - 4\sin x)}{3\sin x + 4\cos x}\right]dx$$

$$= \int\left[\frac{11}{25} + \frac{\frac{2}{25}(3\cos x - 4\sin x)}{(3\sin x + 4\cos x)}\right]dx$$

$$= \frac{11}{25}\int 1\,dx + \frac{2}{25}\int\frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\,dx$$

$$= \frac{11}{25}x + \frac{2}{25}\log|3\sin x + 4\cos x| + c \quad \ldots\left[\because \int\frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

---

(2) Let $I = \displaystyle\int\frac{4e^x - 25}{2e^x - 5}\,dx$

Put Numerator $= A$ (Denominator) $+ B\left[\dfrac{d}{dx}\text{(Denominator)}\right]$

$\therefore\ 4e^x - 25 = A(2e^x - 5) + B\left[\dfrac{d}{dx}(2e^x - 5)\right]$

$\qquad = A(2e^x - 5) + B(2e^x - 0)$

$\therefore\ 4e^x - 25 = (2A + 2B)e^x - 5A$

Equating the coefficient of e^x and constant on both sides, we get

$2A + 2B = 4 \qquad \ldots (1)$

and $5A = 25 \qquad \therefore\ A = 5$

$\therefore$ from (1), $2(5) + 2B = 4$

$\therefore\ 2B = -6 \qquad \therefore\ B = -3$

$\therefore\ 4e^x - 25 = 5(2e^x - 5) - 3(2e^x)$

$$\therefore\ I = \int \left[\frac{5(2e^x - 5) - 3(2e^x)}{2e^x - 5}\right] dx$$

$$= \int \left[5 - \frac{3(2e^x)}{2e^x - 5}\right] dx$$

$$= 5\int 1 dx - 3\int \frac{2e^x}{2e^x - 5}\, dx$$

$$= 5x - 3\log|2e^x - 5| + c \qquad \ldots\left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c\right]$$

---

(3) Let $I = \int \frac{1}{1 - \tan x}\, dx$

$$= \int \frac{1}{1 - \left(\frac{\sin x}{\cos x}\right)}\, dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2}\int \left[\frac{(\cos x - \sin x) - (-\sin x - \cos x)}{\cos x - \sin x}\right] dx$$

$$= \frac{1}{2}\int \left[1 - \left(\frac{-\sin x - \cos x}{\cos x - \sin x}\right)\right] dx$$

$$= \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \frac{-\sin x - \cos x}{\cos x - \sin x}\, dx$$

$$= \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + c$$

$$\ldots\left[\because \frac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x \text{ and } \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$$

Ex. 11. Integrate the following w.r.t. x :

(1) $\dfrac{\sin(x + a)}{\cos(x - b)}$ **(2)** $\dfrac{1}{\cos(x - a)\cdot\cos(x - b)}$

Solution :

(1) $\displaystyle\int \frac{\sin(x + a)}{\cos(x - b)}\, dx = \int \frac{\sin[(x - b) + (a + b)]}{\cos(x - b)}\, dx$

$$= \int \frac{\sin(x - b)\cos(a + b) + \cos(x - b)\sin(a + b)}{\cos(x - b)}\, dx$$

$$= \int \left[\frac{\sin(x-b)\cos(a+b)}{\cos(x-b)} + \frac{\cos(x-b)\sin(a+b)}{\cos(x-b)} \right] dx$$

$$= \int [\cos(a+b) \cdot \tan(x-b) + \sin(a+b)]\, dx$$

$$= \cos(a+b) \int \tan(x-b)\, dx + \sin(a+b) \int 1\, dx$$

$$= \cos(a+b) \cdot \log|\sec(x-b)| + x\sin(a+b) + c.$$

---

(2) $\displaystyle\int \frac{1}{\cos(x-a) \cdot \cos(x-b)}\, dx$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a) \cdot \cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a) \cdot \cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a) \cdot \cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)}{\cos(x-b)} - \frac{\sin(x-a)}{\cos(x-a)} \right] dx$$

$$= \frac{1}{\sin(a-b)} \cdot \left[\int \tan(x-b)\, dx - \int \tan(x-a)\, dx \right]$$

$$= \frac{1}{\sin(a-b)} \left[\log|\sec(x-b)| - \log|\sec(x-a)| \right] + c$$

$$= \frac{1}{\sin(a-b)} \cdot \log\left| \frac{\sec(x-b)}{\sec(x-a)} \right| + c$$

$$= \frac{1}{\sin(a-b)} \log\left| \frac{\cos(x-a)}{\cos(x-b)} \right| + c.$$

Remark : If the denominator contains one sine and one cosine, we adjust $\cos(a-b)$.

For example : $\displaystyle\int \frac{1}{\sin(x-a)\cos(x-b)}\, dx$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}\, dx$$

Now, we can solve this example as above.

Examples for Practice	3 marks each

Integrate the following w.r.t. x :

1. (1) $\dfrac{\sin x \cos^3 x}{1 + \cos^2 x}$ (2) $\dfrac{\sin 2x}{\sin^4 x + \cos^4 x}$

(3) $\dfrac{\cos 3x - \cos 4x}{\sin 3x + \sin 4x}$ (4) $\dfrac{x^2+1}{x^4+1}$

(5) $(3x+2)\sqrt{x-4}$ (6) $\dfrac{e^x+1}{e^x-1}$.

2. (1) $\dfrac{\sqrt{x^2-a^2}}{x}$ (2) $\dfrac{1}{x^2\sqrt{a^2+x^2}}$

(3) $\dfrac{1}{\sqrt{(a^2+x^2)^3}}$ (4) $\sqrt{\dfrac{a+x}{a-x}}$.

3. (1) $\dfrac{2\sin x + 3\cos x}{3\sin x + 4\cos x}$ (2) $\dfrac{1}{2+3\tan x}$

(3) $\dfrac{3e^x-4}{4e^x+5}$.

4. (1) $\dfrac{1}{1-\cot x}$ (2) $\dfrac{1}{1+\tan 2x}$.

5. (1) $\dfrac{\cos x}{\sin(x-a)}$ (2) $\dfrac{\cos(x+2a)}{\cos(x-2a)}$

(3) $\dfrac{\sin(x-a)}{\cos(x+b)}$ (4) $\dfrac{1}{\sin(x-a)\cdot\cos(x-b)}$

(5) $\dfrac{1}{\sin(x-a)\cdot\sin(x-b)}$.

ANSWERS

1. (1) $\dfrac{1}{2}\log|\cos^2 x+1| - \dfrac{1}{2}\cos^2 x + c$ (2) $\tan^{-1}(\tan^2 x) + c$

(3) $2\log\left|\sec\left(\dfrac{x}{2}\right)\right| + c$ (4) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x^2-1}{\sqrt{2}\,x}\right) + c$

(5) $\dfrac{28}{3}(x-4)^{\frac{3}{2}} + \dfrac{6}{5}(x-4)^{\frac{5}{2}} + c$ (6) $x + 2\log|1+e^{-x}| + c$.

2. (1) $\sqrt{x^2-a^2} - a\sin^{-1}\left(\dfrac{x}{a}\right) + c$ (2) $\dfrac{\sqrt{a^2+x^2}}{a^2x} + c$

(3) $\dfrac{x}{a^2\sqrt{a^2+x^2}} + c$ (4) $-a\cos^{-1}\left(\dfrac{x}{a}\right) - \sqrt{a^2-x^2} + c$.

3. (1) $\dfrac{18}{25}x + \dfrac{1}{25}\log|3\sin x + 4\cos x| + c$

(2) $\dfrac{2}{13}x + \dfrac{3}{13}\log|2\cos x + 3\sin x| + c$

(3) $-\frac{4}{5}x+\frac{31}{20}\log|4e^x+5|+c.$

4. (1) $\frac{x}{2}+\frac{1}{2}\log|\sin x-\cos x|+c$ (2) $\frac{x}{2}+\frac{1}{4}\log|\sin 2x+\cos 2x|+c.$

5. (1) $(\cos a)\cdot\log|\sin(x-a)|-x\sin a+c$

(2) $x\cos 4a+(\sin 4a)\cdot\log|\cos(x-2a)|+c$

(3) $\cos(a+b)\cdot\log|\sec(x+b)|-x\sin(a+b)+c$

(4) $\frac{1}{\cos(a-b)}\cdot\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+c$

(5) $\frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+c.$

Solved Examples	2 marks each

Ex. 12. Integrate the following w.r.t. x :

(1) $\frac{1}{4x^2-20x+17}$ **(2)** $\frac{1}{1+x-x^2}$ **(3)** $\frac{1}{ae^x+be^{-x}}$.

Solution :

(1) $\int\frac{1}{4x^2-20x+17}\,dx$

$$=\frac{1}{4}\int\frac{1}{x^2-5x+\frac{17}{4}}\,dx$$

$$=\frac{1}{4}\int\frac{1}{\left(x^2-5x+\frac{25}{4}\right)-\frac{25}{4}+\frac{17}{4}}\,dx$$

$$=\frac{1}{4}\int\frac{1}{\left(x-\frac{5}{2}\right)^2-(\sqrt{2})^2}\,dx$$

$$=\frac{1}{4}\times\frac{1}{2\sqrt{2}}\log\left|\frac{x-\frac{5}{2}-\sqrt{2}}{x-\frac{5}{2}+\sqrt{2}}\right|+c$$

$$=\frac{1}{8\sqrt{2}}\log\left|\frac{2x-5-2\sqrt{2}}{2x-5+2\sqrt{2}}\right|+c.$$

(2) Let $I = \int \frac{1}{1+x-x^2}\,dx$

$$1+x-x^2 = 1-(x^2-x)$$

$$= 1-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}$$

$$= \frac{5}{4}-\left(x^2-x+\frac{1}{4}\right)$$

$$= \left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2$$

$$\therefore\ I = \int \frac{1}{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$= \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\left|\frac{\frac{\sqrt{5}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}-\left(x-\frac{1}{2}\right)}\right|+c$$

$$= \frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right|+c.$$

---

(3) Let $I = \int \frac{1}{ae^x+be^{-x}}\,dx$

$$= \int \frac{1}{ae^x+\left(\frac{b}{e^x}\right)}\,dx$$

$$= \int \frac{e^x}{ae^{2x}+b}\,dx$$

Put $e^x = t$

$\therefore\ e^x\,dx = dt$

$$\therefore\ I = \int \frac{1}{at^2+b}\,dt$$

$$= \frac{1}{a}\int \frac{1}{t^2+\left(\frac{b}{a}\right)}\,dt$$

$$= \frac{1}{a}\int \frac{1}{t^2+\left(\sqrt{\frac{b}{a}}\right)^2}\,dt$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{\frac{b}{a}}} \tan^{-1}\left[\frac{t}{\left(\sqrt{\frac{b}{a}}\right)}\right] + c$$

$$= \frac{1}{a} \cdot \frac{\sqrt{a}}{\sqrt{b}} \tan^{-1}\left(\sqrt{\frac{a}{b}} \cdot t\right) + c$$

$$= \frac{1}{\sqrt{ab}} \tan^{-1}\left(\sqrt{\frac{a}{b}} \cdot e^x\right) + c.$$

Ex. 13. Integrate the following w.r.t. x :

(1) $\frac{1}{\sqrt{3x^2+5x+7}}$ **(2)** $\frac{1}{\sqrt{15+4x-4x^2}}$ **(3)** $\frac{1}{x^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}}-4}}$.

Solution :

(1) Let $I = \int \frac{1}{\sqrt{3x^2+5x+7}}\, dx$

$$3x^2+5x+7 = 3\left[x^2+\frac{5}{3}x+\frac{7}{3}\right]$$

$$= 3\left[\left(x^2+\frac{5x}{3}+\frac{25}{36}\right)+\left(\frac{7}{3}-\frac{25}{36}\right)\right]$$

$$= 3\left[\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2\right]$$

$$\therefore \sqrt{3x^2+5x+7} = \sqrt{3}\sqrt{\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2}$$

$$\therefore I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2}}\, dx$$

$$= \frac{1}{\sqrt{3}} \log\left| x+\frac{5}{6}+\sqrt{\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2}\right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left| x+\frac{5}{6}+\sqrt{x^2+\frac{5x}{3}+\frac{7}{3}}\right| + c.$$

---

(2) $\int \frac{1}{\sqrt{15+4x-4x^2}}\,dx$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{15}{4}+x-x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{15}{4}-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{(2)^2-\left(x-\frac{1}{2}\right)^2}}\,dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{2}\right)+c$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{4}\right)+c.$$

---

(3) Let $I = \int \frac{1}{x^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}}-4}}\,dx$

$$= \int \frac{x^{-\frac{2}{3}}}{\sqrt{\left(x^{\frac{1}{3}}\right)^2-4}}\,dx$$

Put $x^{\frac{1}{3}} = t$ $\quad \therefore \; \frac{1}{3}x^{-\frac{2}{3}}dx = dt \quad \therefore \; x^{-\frac{2}{3}}dx = 3\cdot dt$

$$\therefore \; I = \int \frac{1}{\sqrt{t^2-4}}\cdot 3dt = 3\int \frac{1}{\sqrt{t^2-4}}\,dt$$

$$= 3\log|t+\sqrt{t^2-4}|+c$$

$$= 3\log\left|x^{\frac{1}{3}}+\sqrt{x^{\frac{2}{3}}-4}\right|+c.$$

Examples for Practice	2 marks each

Integrate the following w.r.t. *x* :

1. $\frac{1}{9x^2+6x+10}$ **2.** $\frac{1}{5-4x-3x^2}$ **3.** $\frac{\sin x}{1+\cos^2 x}$

4. $\dfrac{1}{e^{3x}+e^{-3x}}$

5. $\dfrac{1}{\sqrt{2ax-x^2}}$

6. $\dfrac{1}{\sqrt{(x-3)(x+2)}}$

7. $\dfrac{1}{\sqrt{1+x-x^2}}$

8. $\dfrac{1}{\sqrt{8-3x+2x^2}}$

9. $\dfrac{a^x}{\sqrt{1-a^{2x}}}$

10. $\dfrac{\sec^2 x}{\sqrt{16+\tan^2 x}}$.

ANSWERS

1. $\dfrac{1}{9}\tan^{-1}\left(\dfrac{3x+1}{3}\right)+c$

2. $\dfrac{1}{2\sqrt{19}}\log\left|\dfrac{3x+2+\sqrt{19}}{3x+2-\sqrt{19}}\right|+c$

3. $\tan^{-1}(\cos x)+c$

4. $\dfrac{1}{3}\tan^{-1}(e^{3x})+c$

5. $\sin^{-1}\left(\dfrac{x-a}{a}\right)+c$

6. $\log\left|\left(x-\dfrac{1}{2}\right)+\sqrt{x^2-x-6}\right|+c$

7. $\sin^{-1}\left(\dfrac{2x-1}{\sqrt{5}}\right)+c$

8. $\dfrac{1}{\sqrt{2}}\log\left|\left(x-\dfrac{3}{4}\right)+\sqrt{x^2-\dfrac{3x}{2}+4}\right|+c$

9. $\dfrac{\sin^{-1}(a^x)}{\log a}+c$

10. $\log|\tan x+\sqrt{16+\tan^2 x}|+c.$

Solved Examples	3 or 4 marks each

Ex. 14. Integrate the following w.r.t. x :

(1) $\dfrac{\sin 2x}{3\sin^4 x-4\sin^2 x+1}$ **(2)** $\dfrac{\cos x-\sin x}{\sqrt{8-\sin 2x}}$ **(3)** $\dfrac{\sin x}{\sin 3x}$

(4) $\sqrt{\tan x}+\sqrt{\cot x}$.

Solution :

(1) Let $I=\displaystyle\int\frac{\sin 2x}{3\sin^4 x-4\sin^2 x+1}\,dx$

Put $\sin^2 x=t$ $\quad\therefore$ $2\sin x\cos x\,dx=dt$

$\therefore$ $\sin 2x\,dx=dt$

$\therefore$ $I=\displaystyle\int\frac{1}{3t^2-4t+1}\,dt$

$=\dfrac{1}{3}\displaystyle\int\frac{1}{t^2-\frac{4}{3}t+\frac{1}{3}}\,dt$

$$=\frac{1}{3}\int \frac{1}{\left(t^2-\frac{4}{3}t+\frac{4}{9}\right)-\frac{4}{9}+\frac{1}{3}}\,dt$$

$$=\frac{1}{3}\int \frac{1}{\left(t-\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}\,dt$$

$$=\frac{1}{3}\times\frac{1}{2\times\frac{1}{3}}\log\left|\frac{t-\frac{2}{3}-\frac{1}{3}}{t-\frac{2}{3}+\frac{1}{3}}\right|+c$$

$$=\frac{1}{2}\log\left|\frac{3t-3}{3t-1}\right|+c$$

$$=\frac{1}{2}\log\left|\frac{3\sin^2 x-3}{3\sin^2 x-1}\right|+c.$$

---

(2) Let $I=\int \frac{\cos x-\sin x}{\sqrt{8-\sin 2x}}\,dx$

Put $\sin x+\cos x=t$

$\therefore\ (\cos x-\sin x)\,dx=dt$

Also, $(\sin x+\cos x)^2=t^2$

$\therefore\ \sin^2 x+\cos^2 x+2\sin x\cos x=t^2$

$\therefore\ 1+\sin 2x=t^2 \qquad \therefore\ \sin 2x=t^2-1$

$$\therefore\ I=\int \frac{1}{\sqrt{8-(t^2-1)}}\,dt=\int \frac{1}{\sqrt{8-t^2+1}}\,dt$$

$$=\int \frac{1}{\sqrt{3^2-t^2}}\,dt$$

$$=\sin^{-1}\left(\frac{t}{3}\right)+c$$

$$=\sin^{-1}\left(\frac{\sin x+\cos x}{3}\right)+c.$$

---

(3) Let $I=\int \frac{\sin x}{\sin 3x}\,dx=\int \frac{\sin x}{3\sin x-4\sin^3 x}\,dx=\int \frac{1}{3-4\sin^2 x}\,dx$

Dividing both numerator and denominator by $\cos^2 x$, we get

$$I=\int \frac{\sec^2 x}{3\sec^2 x-4\tan^2 x}\,dx$$

$$= \int \frac{\sec^2 x}{3(1+\tan^2 x) - 4\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{3 - \tan^2 x} dx$$

Put $\tan x = t \quad \therefore \sec^2 x \, dx = dt$

$$\therefore I = \int \frac{1}{3 - t^2} dt = \int \frac{1}{(\sqrt{3})^2 - t^2} dt$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + t}{\sqrt{3} - t} \right| + c$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x} \right| + c.$$

---

(4) Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$$= \int \left(\sqrt{\tan x} + \frac{1}{\sqrt{\tan x}} \right) dx$$

$$= \int \frac{\tan x + 1}{\sqrt{\tan x}} dx$$

Put $\tan x = t^2 \qquad \therefore \sec^2 x \, dx = 2t\, dt$

$$\therefore dx = \frac{2t\, dt}{\sec^2 x} = \frac{2t\, dt}{1 + \tan^2 x} = \frac{2t\, dt}{1 + t^4}$$

$$\therefore I = \int \frac{t^2 + 1}{t} \cdot \frac{2t}{1 + t^4} dt = 2 \int \frac{t^2 + 1}{t^4 + 1} dt$$

$$= 2 \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt \qquad \text{... [Dividing N and D by } t^2\text{]}$$

$$= 2 \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t} \right)^2 + 2} dt$$

Put $t - \frac{1}{t} = u \qquad \therefore \left(1 + \frac{1}{t^2} \right) dt = du$

$$\therefore I = 2 \int \frac{1}{u^2 + 2} du = 2 \int \frac{1}{u^2 + (\sqrt{2})^2} du$$

$$= 2 \times \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c$$

$$= \sqrt{2} \tan^{-1}\left[\frac{t - \frac{1}{t}}{\sqrt{2}}\right] + c$$

$$= \sqrt{2} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) + c$$

$$= \sqrt{2} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2 \tan x}}\right) + c.$$

Ex. 15. Integrate the following w.r.t. x :

(1) $\dfrac{3x + 4}{x^2 + 6x + 5}$ **(2)** $\sqrt{\dfrac{9 - x}{x}}$.

Solution :

(1) Let $I = \displaystyle\int \frac{3x + 4}{x^2 + 6x + 5} dx$

Let $3x + 4 = A\left[\dfrac{d}{dx}(x^2 + 6x + 5)\right] + B$

$$= A(2x + 6) + B$$

$\therefore\ 3x + 4 = 2Ax + (6A + B)$

Comparing the coefficient of x and constant on both sides, we get

$2A = 3$ and $6A + B = 4$

$\therefore\ A = \dfrac{3}{2}$ and $6\left(\dfrac{3}{2}\right) + B = 4$

$\therefore\ B = -5$

$\therefore\ 3x + 4 = \dfrac{3}{2}(2x + 6) - 5$

$$\therefore\ I = \int \frac{\frac{3}{2}(2x + 6) - 5}{x^2 + 6x + 5} dx$$

$$= \frac{3}{2}\int \frac{2x + 6}{x^2 + 6x + 5} dx - 5\int \frac{1}{x^2 + 6x + 5} dx$$

$$= \frac{3}{2} I_1 - 5I_2$$

I_1 is of the type $\displaystyle\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

$\therefore\ I_1 = \log|x^2 + 6x + 5| + c_1$

$$I_2 = \int \frac{1}{x^2+6x+5}\,dx = \int \frac{1}{(x^2+6x+9)-4}\,dx$$

$$= \int \frac{1}{(x+3)^2-2^2}\,dx$$

$$= \frac{1}{2\times 2}\log\left|\frac{x+3-2}{x+3+2}\right| + c_2 = \frac{1}{4}\log\left|\frac{x+1}{x+5}\right| + c_2$$

$$\therefore\ I = \frac{3}{2}\log|x^2+6x+5| - \frac{5}{4}\log\left|\frac{x+1}{x+5}\right| + c, \text{ where } c = c_1 + c_2.$$

---

(2) Let $I = \int \sqrt{\frac{9-x}{x}}\,dx = \int \sqrt{\frac{9-x}{x}\cdot\frac{9-x}{9-x}}\,dx$

$$= \int \frac{9-x}{\sqrt{9x-x^2}}\,dx$$

Let $9-x = A\left[\frac{d}{dx}(9x-x^2)\right] + B$

$= A(9-2x) + B$

$\therefore\ 9-x = (9A+B) - 2Ax$

Comparing the coefficient of x and constant on both the sides, we get

$-2A = -1$ and $9A+B = 9$

$\therefore\ A = \frac{1}{2}$ and $9\left(\frac{1}{2}\right) + B = 9 \quad \therefore\ B = \frac{9}{2}$

$\therefore\ 9-x = \frac{1}{2}(9-2x) + \frac{9}{2}$

$$\therefore\ I = \int \frac{\frac{1}{2}(9-2x)+\frac{9}{2}}{\sqrt{9x-x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{9-2x}{\sqrt{9x-x^2}}\,dx + \frac{9}{2}\int \frac{1}{\sqrt{9x-x^2}}\,dx$$

$$= \frac{1}{2}I_1 + \frac{9}{2}I_2$$

In I_1, put $9x - x^2 = t \qquad \therefore\ (9-2x)\,dx = dt$

$$\therefore \; I_1 = \int \frac{1}{\sqrt{t}}\, dt = \int t^{-\frac{1}{2}}\, dt$$

$$= \frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c_1 = 2\sqrt{9x - x^2} + c_1$$

$$I_2 = \int \frac{1}{\sqrt{\frac{81}{4} - \left(x^2 - 9x + \frac{81}{4}\right)}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(\frac{9}{2}\right)^2 - \left(x - \frac{9}{2}\right)^2}}\, dx$$

$$= \sin^{-1}\left(\frac{x - \frac{9}{2}}{\left(\frac{9}{2}\right)}\right) + c_2 = \sin^{-1}\left(\frac{2x-9}{9}\right) + c_2$$

$$\therefore \; I = \sqrt{9x - x^2} + \frac{9}{2}\sin^{-1}\left(\frac{2x-9}{9}\right) + c, \text{ where } c = c_1 + c_2.$$

Ex. 16. Integrate the following w.r.t. x :

(1) $\dfrac{1}{2 + \cos x - \sin x}$ ***(March '22)*** **(2)** $\dfrac{1}{5 - 4\cos x}$ ***(Sept. '21)***

(3) $\dfrac{1}{2\sin 2x - 3}$ **(4)** $\dfrac{1}{1 + \cos\alpha \cdot \cos x}$.

Solution :

(1) Let $I = \displaystyle\int \frac{1}{2 + \cos x - \sin x}\, dx$

Put $\tan\left(\frac{x}{2}\right) = t \qquad \therefore \; x = 2\tan^{-1} t$

$$\therefore \; dx = \frac{2dt}{1+t^2} \text{ and } \sin x = \frac{2t}{1+t^2},\; \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore \; I = \int \frac{1}{2 + \left(\frac{1-t^2}{1+t^2}\right) - \left(\frac{2t}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1+t^2}{2 + 2t^2 + 1 - t^2 - 2t} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{t^2 - 2t + 3}\,dt = 2\int \frac{1}{(t^2 - 2t + 1) + 2}\,dt$$

$$= 2\int \frac{1}{(t-1)^2 + (\sqrt{2})^2}\,dt$$

$$= 2 \times \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t-1}{\sqrt{2}}\right) + c$$

$$= \sqrt{2}\tan^{-1}\left[\frac{\tan\left(\frac{x}{2}\right) - 1}{\sqrt{2}}\right] + c.$$

---

(2) Let $\text{I} = \int \frac{1}{5 - 4\cos x}\,dx$

Put $\tan\left(\frac{x}{2}\right) = t \qquad \therefore\ x = 2\tan^{-1} t$

$\therefore\ dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore\ I = \int \frac{1}{5 - 4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1+t^2}{5 + 5t^2 - 4 + 4t^2} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{1 + 9t^2}\,dt = \frac{2}{9}\int \frac{1}{\frac{1}{9} + t^2}\,dt$$

$$= \frac{2}{9}\int \frac{1}{\left(\frac{1}{3}\right)^2 + t^2}\,dt$$

$$= \frac{2}{9} \times \frac{1}{\left(\frac{1}{3}\right)} \tan^{-1}\left[\frac{1}{\left(\frac{1}{3}\right)}\right] + c$$

$$= \frac{2}{3}\tan^{-1}(3t) + c$$

$$= \frac{2}{3}\tan^{-1}\left[3\tan\left(\frac{x}{2}\right)\right] + c.$$

---

(3) Let $I=\int \frac{1}{2\sin 2x-3}\,dx$

Put $\tan x=t \qquad \therefore\ x=\tan^{-1} t$

$\therefore\ dx = \frac{dt}{1+t^2}$ and $\sin 2x=\frac{2t}{1+t^2}$

$$\therefore\ I=\int \frac{1}{2\left(\frac{2t}{1+t^2}\right)-3}\cdot\frac{dt}{1+t^2}$$

$$=\int \frac{1+t^2}{4t-3-3t^2}\cdot\frac{dt}{1+t^2}$$

$$=\int \frac{1}{-3t^2+4t-3}\,dt$$

$$=-\frac{1}{3}\int \frac{1}{t^2-\frac{4}{3}t+1}\,dt$$

$$=-\frac{1}{3}\int \frac{1}{\left(t^2-\frac{4}{3}t+\frac{4}{9}\right)-\frac{4}{9}+1}\,dt$$

$$=-\frac{1}{3}\int \frac{1}{\left(t-\frac{2}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2}\,dt$$

$$=-\frac{1}{3}\times\frac{1}{\left(\frac{\sqrt{5}}{3}\right)}\tan^{-1}\left(\frac{t-\frac{2}{3}}{\left(\frac{\sqrt{5}}{3}\right)}\right)+c$$

$$=-\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{3t-2}{\sqrt{5}}\right)+c$$

$$=-\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{3\tan x-2}{\sqrt{5}}\right)+c.$$

---

(4) Let $I=\int \frac{1}{1+\cos\alpha\cdot\cos x}dx$

Put $\tan\left(\frac{x}{2}\right)=t \quad \therefore\ x=2\tan^{-1}t$

$$\therefore \; dx = \frac{2dt}{1+t^2} \text{ and } \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore \; I = \int \frac{1}{1+(\cos\alpha)\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1+t^2}{1+t^2+\cos\alpha - \cos\alpha \cdot t^2} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{(1+\cos\alpha)+(1-\cos\alpha)\,t^2}\,dt$$

$$= 2\int \frac{1}{2\cos^2\left(\frac{\alpha}{2}\right)+2\sin^2\left(\frac{\alpha}{2}\right)\cdot t^2}\,dt$$

$$= \frac{2}{2\sin^2\left(\frac{\alpha}{2}\right)} \int \frac{1}{\cot^2\left(\frac{\alpha}{2}\right)+t^2}\,dt$$

$$= \frac{1}{\sin^2\left(\frac{\alpha}{2}\right)} \times \frac{1}{\cot\left(\frac{\alpha}{2}\right)} \cdot \tan^{-1}\left[\frac{t}{\cot\left(\frac{\alpha}{2}\right)}\right] + c$$

$$= \frac{2}{2\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)} \cdot \tan^{-1}\left[\tan\left(\frac{\alpha}{2}\right)\cdot\tan\left(\frac{x}{2}\right)\right] + c$$

$$= 2\operatorname{cosec}\alpha \cdot \tan^{-1}\left[\tan\left(\frac{\alpha}{2}\right)\cdot\tan\left(\frac{x}{2}\right)\right] + c.$$

Examples for Practice	3 or 4 marks each

Integrate the following w.r.t. x :

1. (1) $\dfrac{3\cos x}{4\sin^2 x + 4\sin x - 1}$ (2) $\dfrac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}}$.

2. (1) $\dfrac{1}{4+5\sin^2 x}$ (2) $\dfrac{1}{3\cos^2 x + 5}$

(3) $\dfrac{1}{\cos 2x + 3\sin^2 x}$ (4) $\dfrac{1}{3+2\sin^2 x + 5\cos^2 x}$

(5) $\dfrac{\cos x}{\cos 3x}$.

3. (1) $\dfrac{2x+1}{x^2+4x-5}$ (2) $\dfrac{7x+3}{\sqrt{3+2x-x^2}}$

(3) $\dfrac{2x+1}{\sqrt{x^2+2x+3}}$ (4) $\sqrt{\dfrac{x-5}{x-7}}$.

4. (1) $\dfrac{1}{3+2\sin x}$ (2) $\dfrac{1}{1-2\sin x}$

(3) $\dfrac{1}{4-5\cos x}$ (4) $\dfrac{1}{3-2\sin x+5\cos x}$

(5) $\dfrac{1}{3-2\cos 2x}$ (6) $\dfrac{1}{3-5\sin 2x}$

(7) $\dfrac{1}{3+2\sin 2x+4\cos 2x}$ (8) $\dfrac{1}{\cos\alpha+\cos x}$.

ANSWERS

1. (1) $\dfrac{3}{4\sqrt{2}}\log\left|\dfrac{2\sin x+1-\sqrt{2}}{2\sin x+1+\sqrt{2}}\right|+c$

(2) $\log|(\sin x-1)+\sqrt{\sin^2 x-2\sin x-3}|+c$

2. (1) $\dfrac{1}{6}\tan^{-1}\left(\dfrac{3\tan x}{2}\right)+c$ (2) $\dfrac{1}{2\sqrt{10}}\tan^{-1}\left(\dfrac{\sqrt{5}\tan x}{2\sqrt{2}}\right)+c$

(3) $\dfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+c$ (4) $\dfrac{1}{2\sqrt{10}}\tan^{-1}\left(\dfrac{\sqrt{5}\tan x}{2\sqrt{2}}\right)+c$

(5) $\dfrac{1}{2\sqrt{3}}\log\left|\dfrac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+c.$

3. (1) $\log|x^2+4x-5|-\dfrac{1}{2}\log\left|\dfrac{x-1}{x+5}\right|+c$

(2) $-7\sqrt{3+2x-x^2}+10\sin^{-1}\left(\dfrac{x-1}{2}\right)+c$

(3) $2\sqrt{x^2+2x+3}-\log|(x+1)+\sqrt{x^2+2x+3}|+c$

(4) $\sqrt{x^2-12x+35}+\log|(x-6)+\sqrt{x^2-12x+35}|+c.$

4. (1) $\dfrac{2}{\sqrt{5}}\tan^{-1}\left[\dfrac{3\tan\left(\dfrac{x}{2}\right)+2}{\sqrt{5}}\right]+c$

(2) $\dfrac{1}{\sqrt{3}}\log\left|\dfrac{\tan\left(\dfrac{x}{2}\right)-2-\sqrt{3}}{\tan\left(\dfrac{x}{2}\right)-2+\sqrt{3}}\right|+c$

(3) $\frac{1}{3}\log\left|\dfrac{3\tan\left(\frac{x}{2}\right)-1}{3\tan\left(\frac{x}{2}\right)+1}\right|+c$

(4) $\frac{1}{2\sqrt{5}}\log\left|\dfrac{\sqrt{5}+1+\tan\left(\frac{x}{2}\right)}{\sqrt{5}-1-\tan\left(\frac{x}{2}\right)}\right|+c$

(5) $\frac{1}{\sqrt{5}}\tan^{-1}(\sqrt{5}\tan x)+c$

(6) $\frac{1}{8}\log\left|\dfrac{3\tan x-9}{3\tan x-1}\right|+c$

(7) $\frac{1}{2\sqrt{11}}\log\left|\dfrac{\sqrt{11}+\tan x-2}{\sqrt{11}-\tan x+2}\right|+c$

(8) $\operatorname{cosec}\alpha\cdot\log\left|\dfrac{\cos\left(\frac{\alpha}{2}\right)+\sin\left(\frac{\alpha}{2}\right)\tan\left(\frac{x}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)-\sin\left(\frac{\alpha}{2}\right)\tan\left(\frac{x}{2}\right)}\right|+c.$

10.3 INTEGRATION BY PARTS

Theory Questions | 3 marks each

Q. 2. If u and v are functions of x, then $\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx.$

(Sept. '21, March '22)

Proof : Let $\int v\,dx = w$. Then $\frac{dw}{dx}=v$

By the rule for the derivative of the product of two functions,

$$\frac{d}{dx}(uw)=u\frac{dw}{dx}+w\frac{du}{dx}=uv+w\frac{du}{dx}$$

$\therefore$ by the definition of indefinite integral,

$$\int\left(uv+w\frac{du}{dx}\right)dx=uw$$

$$\therefore\ \int uv\,dx+\int\left(w\frac{du}{dx}\right)dx=uw$$

$$\therefore\ \int uv\,dx=uw-\int\left(w\frac{du}{dx}\right)dx$$

$$\therefore\ \int uv\,dx=u\int v\,dx-\int\left(\frac{du}{dx}\cdot\int v dx\right)dx.$$

Q. 3. Prove that :

(1) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

(2) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c$

(3) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2 - a^2}| + c.$

Proof :

(1) Let $I = \int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - x^2} \cdot 1\, dx$

$$= \sqrt{a^2 - x^2} \cdot \int 1\, dx - \int \left[\frac{d}{dx}(\sqrt{a^2 - x^2}) \cdot \int 1\, dx\right] dx$$

$$= \sqrt{a^2 - x^2} \cdot x - \int \frac{-x}{\sqrt{a^2 - x^2}} \cdot x\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= x\sqrt{a^2 - x^2} - I + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore\ 2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore\ I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c.$$

---

(2) Let $I = \int \sqrt{x^2 + a^2}\, dx = \int \sqrt{x^2 + a^2} \cdot 1\, dx$

$$= \sqrt{x^2 + a^2} \cdot \int 1\, dx - \int \left[\frac{d}{dx}(\sqrt{x^2 + a^2}) \cdot \int 1\, dx\right] dx$$

$$= \sqrt{x^2 + a^2} \cdot x - \int \frac{x}{\sqrt{x^2 + a^2}} \cdot x\, dx$$

$$= x \cdot \sqrt{x^2 + a^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{x^2 + a^2}}\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= x\sqrt{x^2 + a^2} - I + a^2 \log|x + \sqrt{x^2 + a^2}|$$

$$\therefore\ 2I = x\sqrt{x^2 + a^2} + a^2 \log|x + \sqrt{x^2 + a^2}|$$

$$\therefore\ I = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c.$$

---

(3) Let $I = \int \sqrt{x^2 - a^2}\, dx = \int \sqrt{x^2 - a^2} \cdot 1\, dx$

$$= \sqrt{x^2 - a^2} \cdot \int 1\, dx - \int \left[\frac{d}{dx}(\sqrt{x^2 - a^2}) \cdot \int 1\, dx \right] dx$$

$$= \sqrt{x^2 - a^2} \cdot x - \int \frac{x}{\sqrt{x^2 - a^2}} \cdot x\, dx$$

$$= x \cdot \sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= x\sqrt{x^2 - a^2} - I - a^2 \log |x + \sqrt{x^2 - a^2}|$$

$$\therefore\ 2I = x\sqrt{x^2 - a^2} - a^2 \log |x + \sqrt{x^2 - a^2}|$$

$$\therefore\ I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c.$$

Solved Examples	2 marks each

Ex. 17. Integrate the following w.r.t. x :

(1) $x^2 \log x$ **(2) $x \sin x$ *(Sept. '21)*** **(3) $\dfrac{x}{1 + \cos 2x}$.**

Solution :

(1) Let $I = \int x^2 \log x\, dx = \int \log x \cdot x^2\, dx$

$$= (\log x) \int x^2\, dx - \int \left[\left\{ \frac{d}{dx}(\log x) \int x^2\, dx \right\} \right] dx$$

$$= (\log x) \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3}\, dx$$

$$= \frac{x^3 \log x}{3} - \frac{1}{3} \int x^2\, dx$$

$$= \frac{x^3 \log x}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$$

$$= \frac{x^3}{9}(3 \log x - 1) + c.$$

---

(2) $\int x \sin x\, dx = x \int \sin x\, dx - \int \left[\frac{d}{dx}(x) \int \sin x\, dx \right] dx$

$$= x\,(-\cos x) - \int 1\,(-\cos x)\, dx$$

$$= -x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + c.$$

---

(3) $\int \frac{x}{1+\cos 2x}\,dx = \int \frac{x}{2\cos^2 x}\,dx$

$$= \frac{1}{2}\int x \sec^2 x\,dx$$

$$= \frac{1}{2}\left[x\int \sec^2 x\,dx - \int\left\{\frac{d}{dx}(x)\int \sec^2 x\,dx\right\}dx\right]$$

$$= \frac{1}{2}\left[x\tan x - \int 1\cdot\tan x\,dx\right]$$

$$= \frac{1}{2}\left[x\tan x - \log|\sec x|\right] + c.$$

Ex. 18. Integrate the following w.r.t. x :

(1) $\log x$ *(March '22)*   (2) $\sin^{-1}x$.

Solution :

(1) $\int \log x\,dx = \int (\log x)\cdot(1)\,dx$

$$= (\log x)\int 1dx - \int\left[\frac{d}{dx}(\log x)\int 1\,dx\right]dx$$

$$= (\log x)x - \int \frac{1}{x}\cdot x\,dx$$

$$= x\log x - \int 1\,dx = x\log x - x + c.$$

---

(2) Let $I = \int \sin^{-1}x\,dx = \int(\sin^{-1}x)\cdot(1)\,dx$

$$= (\sin^{-1}x)\int 1\,dx - \int\left[\frac{d}{dx}(\sin^{-1}x)\int 1\,dx\right]dx$$

$$= (\sin^{-1}x)x - \int \frac{1}{\sqrt{1-x^2}}\cdot x\,dx$$

$$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Put $1-x^2 = t$ $\therefore\ -2x\,dx = dt$ $\therefore\ x\,dx = \frac{-dt}{2}$

$$\therefore\ I = x\sin^{-1}x - \int \frac{1}{\sqrt{t}}\left(-\frac{dt}{2}\right)$$

$$= x\sin^{-1}x + \frac{1}{2}\int t^{-\frac{1}{2}}\,dt$$

$$= x\sin^{-1}x + \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c = x\sin^{-1}x + \sqrt{1-x^2} + c.$$

Examples for Practice	2 marks each

Integrate the following w.r.t. x :

1. (1) xe^x (2) $x\cos x$ (3) $x\sin 2x$

(4) $\dfrac{x}{1-\cos x}$ (5) $x\tan^2 x$ (6) $x^3 \log x$.

2. (1) $\tan^{-1}x$ (2) $\cos^{-1}x$ (3) $\sec^{-1}x$.

ANSWERS

1. (1) $xe^x - e^x + c$ (2) $x\sin x + \cos x + c$

(3) $-\dfrac{1}{2}x\cos 2x + \dfrac{1}{4}\sin 2x + c$ (4) $-x\cot\left(\dfrac{x}{2}\right) + 2\log\left|\sin\left(\dfrac{x}{2}\right)\right| + c$

(5) $x\tan x - \log|\sec x| - \dfrac{x^2}{2} + c$ (6) $\dfrac{1}{4}x^4\log x - \dfrac{1}{16}x^4 + c$.

2. (1) $x\tan^{-1}x - \dfrac{1}{2}\log|1+x^2| + c$ (2) $x\cos^{-1}x - \sqrt{1-x^2} + c$

(3) $x\sec^{-1}x - \log|x + \sqrt{x^2-1}| + c$.

Solved Examples	3 marks each

Ex. 19. Integrate the following w.r.t. x :

(1) $x\sin^2 x$ **(2)** $\dfrac{x}{1+\sin x}$ **(3)** $x^3\tan^{-1}x$.

Solution :

(1) $\int x\cdot\sin^2 x\,dx = \int x\left(\dfrac{1-\cos 2x}{2}\right)dx$

$$=\frac{1}{2}\int(x - x\cos 2x)\,dx$$

$$=\frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$$

$$=\frac{1}{2}\cdot\frac{x^2}{2} - \frac{1}{2}\left[x\int\cos 2x\,dx - \int\left\{\frac{d}{dx}(x)\int\cos 2x\,dx\right\}dx\right]$$

$$=\frac{x^2}{4} - \frac{1}{2}\left[x\cdot\frac{\sin 2x}{2} - \int 1\cdot\frac{\sin 2x}{2}\,dx\right]$$

$$=\frac{x^2}{4} - \frac{1}{4}x\cdot\sin 2x + \frac{1}{4}\int\sin 2x\,dx$$

$$=\frac{x^2}{4}-\frac{1}{4}x\cdot\sin 2x+\frac{1}{4}\cdot\frac{(-\cos 2x)}{2}+c$$

$$=\frac{x^2}{4}-\frac{1}{4}x\cdot\sin 2x-\frac{1}{8}\cos 2x+c$$

$$=\frac{1}{4}[x^2-x\sin 2x-\frac{1}{2}\cos 2x]+c.$$

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(2) $$\int\frac{x}{1+\sin x}\,dx=\int\frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$$

$$=\int\frac{x(1-\sin x)}{(1-\sin^2 x)}\,dx$$

$$=\int x\left(\frac{1-\sin x}{\cos^2 x}\right)dx$$

$$=\int x\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos x}\times\frac{1}{\cos x}\right)dx$$

$$=\int x(\sec^2 x-\sec x\tan x)\,dx$$

$$=\int x\sec^2 x\,dx-\int x\sec x\tan x\,dx$$

$$=x\int\sec^2 x\,dx-\int\left[\frac{d}{dx}(x)\int\sec^2 x\,dx\right]dx$$

$$-\left\{x\int\sec x\tan x\,dx-\int\left[\frac{d}{dx}(x)\int\sec x\cdot\tan x\,dx\right]dx\right\}$$

$$=x\tan x-\int 1\cdot\tan x\,dx-x\sec x+\int 1\cdot\sec x\,dx$$

$$=x\tan x-\log|\sec x|-x\sec x+\log|\sec x+\tan x|+c$$

$$=x(\tan x-\sec x)-\log\sec x+\log|\sec x+\tan x|+c.$$

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(3) Let $I=\int x^3\tan^{-1}x\,dx=\int(\tan^{-1}x)\cdot x^3\,dx$

$$=(\tan^{-1}x)\int x^3\,dx-\int\left[\left\{\frac{d}{dx}(\tan^{-1}x)\int x^3\,dx\right\}\right]dx$$

$$=(\tan^{-1}x)\left(\frac{x^4}{4}\right)-\int\left(\frac{1}{1+x^2}\right)\frac{x^4}{4}\,dx$$

$$=\frac{x^4\tan^{-1}x}{4}-\frac{1}{4}\int\frac{(x^4-1)+1}{x^2+1}\,dx$$

$$=\frac{x^4\tan^{-1}x}{4}-\frac{1}{4}\int\frac{(x^2-1)(x^2+1)+1}{x^2+1}\,dx$$

$$=\frac{x^4\tan^{-1}x}{4}-\frac{1}{4}\int\left[x^2-1+\frac{1}{x^2+1}\right]dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\left[\int x^2\, dx - \int 1\, dx + \int \frac{1}{x^2+1} dx \right]$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1} x \right] + c$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{\tan^{-1} x}{4} - \frac{x^3}{12} + \frac{x}{4} + c$$

$$= \frac{1}{4}(\tan^{-1} x)(x^4 - 1) - \frac{x}{12}(x^2 - 3) + c.$$

Ex. 20. Integrate the following w.r.t. x :

(1) $(\log x)^2$ (2) $\log(\log x) + \dfrac{1}{(\log x)^2}$.

Solution :

(1) $\int (\log x)^2\, dx = \int (\log x)^2 \cdot (1)\, dx = (\log x)^2 \int 1\, dx - \int \left[\frac{d}{dx}(\log x)^2 \cdot \int 1\, dx \right] dx$

$$= (\log x)^2 \cdot x - \int \frac{2 \log x}{x} \cdot (x)\, dx = x\,(\log x)^2 - 2 \int (\log x)\,(1)\, dx$$

$$= x\,(\log x)^2 - 2 \left\{ (\log x) \int 1\, dx - \int \left[\frac{d}{dx}(\log x) \int 1\, dx \right] dx \right\}$$

$$= x\,(\log x)^2 - 2(\log x)\,(x) + 2 \int \frac{1}{x} \cdot x\, dx$$

$$= x\,(\log x)^2 - 2x \log x + 2 \int 1\, dx$$

$$= x\,(\log x)^2 - 2x \log x + 2x + c = x\,[(\log x)^2 - 2\log x + 2] + c.$$

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(2) $\int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] dx$

$$= \int \log(\log x) \cdot 1\, dx + \int \frac{1}{(\log x)^2} dx$$

$$= \log(\log x) \cdot \int 1\, dx - \int \left\{ \frac{d}{dx} [\log(\log x)] \cdot \int 1\, dx \right\} dx + \int \frac{1}{(\log x)^2} dx$$

$$= \log(\log x) \cdot x - \int \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \cdot x\, dx + \int \frac{1}{(\log x)^2} dx$$

$$= x \log(\log x) - \int \frac{1}{\log x} \times \frac{1}{x} \times x\, dx + \int \frac{1}{(\log x)^2} dx$$

$$= x \log(\log x) - \int (\log x)^{-1} \cdot 1\, dx + \int \frac{1}{(\log x)^2} dx$$

$$= x\log(\log x) - \left[(\log x)^{-1}\cdot\int 1\,dx - \int\left\{\frac{d}{dx}(\log x)^{-1}\cdot\int 1\,dx\right\}dx\right] + \int\frac{1}{(\log x)^2}\,dx$$

$$= x\log(\log x) - \left[(\log x)^{-1}\cdot x - \int -1\,(\log x)^{-2}\cdot\frac{d}{dx}(\log x)\cdot x\,dx\right] + \int\frac{1}{(\log x)^2}\,dx$$

$$= x\log(\log x) - \frac{x}{\log x} - \int\frac{1}{(\log x)^2}\times\frac{1}{x}\times x\,dx + \int\frac{1}{(\log x)^2}\,dx$$

$$= x\log(\log x) - \frac{x}{\log x} - \int\frac{1}{(\log x)^2}\,dx + \int\frac{1}{(\log x)^2}\,dx$$

$$= x\log(\log x) - \frac{x}{\log x} + c.$$

Ex. 21. Integrate the following w.r.t. x :

(1) $\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$ **(2)** $\cos(\sqrt[3]{x})$.

Solution :

(1) Let $I = \int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int x\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$

Put $\sin^{-1}x = t$ $\therefore\ \frac{1}{\sqrt{1-x^2}}\,dx = dt$ and $x = \sin t$

$\therefore\ I = \int(\sin t)\cdot t\,dt = \int t\sin t\,dt$

$$= t\int\sin t\,dt - \int\left[\frac{d}{dt}(t)\int\sin t\,dt\right]dt$$

$$= t(-\cos t) - \int 1\cdot(-\cos t)\,dt$$

$$= -t\cos t + \int\cos t\,dt = -t\cos t + \sin t + c$$

$$= -t\sqrt{1-\sin^2 t} + \sin t + c$$

$$= -\sin^{-1}x\cdot\sqrt{1-x^2} + x + c$$

$$= -\sqrt{1-x^2}\cdot\sin^{-1}x + x + c.$$

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(2) Let $I = \int\cos(\sqrt[3]{x})\,dx$

Put $\sqrt[3]{x} = t$ $\therefore\ x = t^3$ $\therefore\ dx = 3t^2\,dt$

$\therefore\ I = \int 3t^2\cos t\,dt$

$$= 3t^2\int\cos t\,dt - \int\left[\frac{d}{dt}(3t^2)\int\cos t\,dt\right]dt$$

$$= 3t^2\sin t - \int 6t\sin t\,dt$$

$$= 3t^2 \sin t - \left[6t \int \sin t \, dt - \int \left\{ \frac{d}{dt}(6t) \int \sin t \, dt \right\} dt \right]$$

$$= 3t^2 \sin t - [6t(-\cos t) - \int 6(-\cos t)\, dt]$$

$$= 3t^2 \sin t + 6t \cos t - 6 \sin t + c$$

$$= 3(t^2 - 2) \sin t + 6t \cos t + c$$

$$= 3\,(x^{\frac{2}{3}} - 2) \sin (\sqrt[3]{x}) + 6(\sqrt[3]{x}\,) \cos (\sqrt[3]{x}) + c.$$

Examples for Practice	**3 marks each**

Integrate the following w.r.t. *x* :

1. $x^2 e^{2x}$ **2.** $x^2 \cos^2 x$ **3.** $\dfrac{x + \sin x}{1 - \cos x}$ **4.** $x^2 \sin 3x$

5. $x \sec^{-1} x$ **6.** $x \sin^{-1} x$ **7.** $x^2 \tan^{-1} x$ **8.** $\sin (\sqrt[3]{x})$

9. $\tan^{-1}(\sqrt{x})$ **10.** $\dfrac{\log (\log x)}{x}$.

ANSWERS

1. $\dfrac{e^{2x}}{4}(2x^2 - 2x + 1) + c$

2. $\dfrac{x^3}{6} + \dfrac{x^2 \sin 2x}{4} + \dfrac{x \cos 2x}{4} - \dfrac{\sin 2x}{8} + c$

3. $-x \cot\left(\dfrac{x}{2}\right) + 4 \log \left|\sin\left(\dfrac{x}{2}\right)\right| + c$

4. $-\dfrac{1}{3} x^2 \cos 3x + \dfrac{2}{9} x \sin 3x + \dfrac{2}{27} \cos 3x + c$

5. $\dfrac{x^2 \sec^{-1} x}{2} - 9^{\frac{1}{2}} \sqrt{x^2 - 1} + c$

6. $\dfrac{x^2 \sin^{-1} x}{2} + \dfrac{1}{4} x\sqrt{1 - x^2} - \dfrac{1}{4} \sin^{-1} x + c$

7. $\dfrac{x^3 \tan^{-1} x}{3} - \dfrac{x^2}{6} + \dfrac{1}{6} \log |x^2 + 1| + c$

8. $-3x^{\frac{2}{3}} \cos (x^{\frac{1}{3}}) + 6x^{\frac{1}{3}} \cdot \sin (x^{\frac{1}{3}}) + 6 \cos(x^{\frac{1}{3}}) + c$

9. $(x + 1) \tan^{-1}(\sqrt{x}) - \sqrt{x} + c$

10. $(\log x)\cdot[\log (\log x) - 1] + c.$

Solved Examples	3 or 4 marks each

Ex. 22. Integrate the following w.r.t. x :

(1) $e^{2x}\cos 3x$ (2) $\sin(\log x)$ (3) $\sec^3 x$.

Solution :

(1) Let $I=\int e^{2x}\cos 3x\,dx$

$$=e^{2x}\int \cos 3x\,dx-\int\left[\frac{d}{dx}(e^{2x})\int \cos 3x\,dx\right]dx$$

$$=e^{2x}\cdot\frac{\sin 3x}{3}-\int e^{2x}\times 2\times\frac{\sin 3x}{3}\,dx$$

$$=\frac{1}{3}e^{2x}\sin 3x-\frac{2}{3}\int e^{2x}\sin 3x\,dx$$

$$=\frac{1}{3}e^{2x}\sin 3x-\frac{2}{3}\left[e^{2x}\int \sin 3x\,dx-\int\left\{\frac{d}{dx}(e^{2x})\int \sin 3x\,dx\right\}dx\right]$$

$$=\frac{1}{3}e^{2x}\sin 3x-\frac{2}{3}\left[e^{2x}\cdot\left(\frac{-\cos 3x}{3}\right)-\int e^{2x}\times 2\times\left(\frac{-\cos 3x}{3}\right)dx\right]$$

$$=\frac{1}{3}e^{2x}\sin 3x+\frac{2}{9}e^{2x}\cos 3x-\frac{4}{9}\int e^{2x}\cos 3x\,dx$$

$$\therefore\ I=\frac{1}{3}e^{2x}\sin 3x+\frac{2}{9}e^{2x}\cos 3x-\frac{4}{9}I$$

$$\therefore\ \left(1+\frac{4}{9}\right)I=\frac{1}{3}e^{2x}\sin 3x+\frac{2}{9}e^{2x}\cos 3x$$

$$\therefore\ \frac{13}{9}I=\frac{e^{2x}}{9}(3\sin 3x+2\cos 3x)$$

$$\therefore\ I=\frac{e^{2x}}{13}(2\cos 3x+3\sin 3x)+c.$$

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(2) Let $I=\int \sin(\log x)\,dx$

Put $\log x=t$. Then $x=e^t$ $\therefore\ dx=e^t\,dt$

$$\therefore\ I=\int(\sin t)\,e^t\,dt=\int e^t\sin t\,dt$$

$$=e^t\int \sin t\,dt-\int\left[\frac{d}{dt}(e^t)\int \sin t\,dt\right]dt$$

$$= e^t(-\cos t) - \int [e^t(-\cos t)]\,dt$$

$$= -e^t\cos t + \int e^t\cos t\,dt$$

$$= -e^t\cos t + e^t\int\cos t\,dt - \int\left[\frac{d}{dt}(e^t)\int\cos t\,dt\right]dt$$

$$= -e^t\cos t + e^t\sin t - \int e^t\sin t\,dt$$

$$\therefore\ I = -e^t\cos t + e^t\sin t - I$$

$$\therefore\ 2I = -e^t\cos t + e^t\sin t = e^t(\sin t - \cos t)$$

$$\therefore\ I = \frac{e^t}{2}(\sin t - \cos t) + c = \frac{x}{2}[\sin(\log x) - \cos(\log x)] + c.$$

---

(3) Let $I = \int \sec^3 x\,dx = \int \sec x\cdot\sec^2 x\,dx$

$$= \sec x\int\sec^2 x\,dx - \int\left[\frac{d}{dx}(\sec x)\int\sec^2 x\,dx\right]dx$$

$$= \sec x\tan x - \int \sec x\tan x\cdot\tan x\,dx$$

$$= \sec x\tan x - \int \sec x\tan^2 x\,dx$$

$$= \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x\tan x - \int (\sec^3 x - \sec x)\,dx$$

$$= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\therefore\ I = \sec x\tan x - I + \log|\sec x + \tan x| + c_1$$

$$\therefore\ 2I = \sec x\tan x + \log|\sec x + \tan x| + c_1$$

$$\therefore\ I = \frac{1}{2}\sec x\tan x + \frac{1}{2}\log|\sec x + \tan x| + c,\ \text{where } c = \frac{c_1}{2}$$

Examples for Practice	**3 or 4 marks each**

Integrate the following w.r.t. x :

1. $e^x\cdot\cos x$ **2.** $e^{2x}\cdot\sin 3x$ **3.** $\cos(2\log x)$

4. $\sin x\cdot\log(\cos x)$ **5.** $\mathrm{cosec}^3 x$.

ANSWERS

1. $\frac{e^x}{2}(\sin x + \cos x) + c$ **2.** $\frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + c$

3. $\frac{x}{5}[\cos(2\log x) + 2\sin(2\log x)] + c$ **4.** $-\cos x[\log(\cos x) - 1] + c$

5. $-\frac{1}{2}\mathrm{cosec}\,x\cot x + \frac{1}{2}\log\left|\tan\left(\frac{x}{2}\right)\right| + c.$

Solved Examples	3 marks each

Ex. 23. Show that $\int e^x\,[f(x)+f'(x)]\,dx = e^x\cdot f(x)+c$.

Hence, evaluate :

(1) $\displaystyle\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)dx$ **(2)** $\displaystyle\int \frac{(x^2+1)\,e^x}{(x+1)^2}\,dx$

(3) $\displaystyle\int e^{\sin^{-1}x}\left[\frac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]dx$ **(4)** $\displaystyle\int \frac{\log x}{(1+\log x)^2}\,dx$

(5) $\int [\sin(\log x)+\cos(\log x)]\,dx$.

Solution : By the rule of integration by parts, we have

$$\int e^x\cdot f'(x)\,dx = e^x\int f'(x)\,dx - \int\left[\frac{d}{dx}(e^x)\int f'(x)\,dx\right]dx$$

$$= e^x f(x) - \int e^x\cdot f(x)\,dx$$

$\therefore\ \int e^x\cdot f(x)\,dx + \int e^x\cdot f'(x)\,dx = e^x\cdot f(x)$

$\therefore\ \int e^x\,[f(x)+f'(x)]\,dx = e^x\cdot f(x)+c.$

Alternative Method :

$$\frac{d}{dx}[e^x\cdot f(x)] = e^x\cdot\frac{d}{dx}[f(x)] + f(x)\cdot\frac{d}{dx}(e^x)$$

$$= e^x\cdot f'(x) + f(x)\cdot e^x = e^x\,[f(x)+f'(x)]$$

$\therefore$ by the definition of indefinite integration,

$\int e^x\,[f(x)+f'(x)]\,dx = e^x\cdot f(x)+c.$

[This result can be used when the multiplier of e^x can be expressed as $f(x)+f'(x)$.]

(1) Let $\displaystyle I=\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)dx$

$$=\int e^x\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)dx$$

$$=\int e^x\left(\frac{2}{2\cos^2 x}+\frac{2\sin x\cos x}{2\cos^2 x}\right)dx$$

$$=\int e^x(\sec^2 x+\tan x)\,dx$$

Put $f(x)=\tan x$

$\therefore\ f'(x)=\dfrac{d}{dx}(\tan x)=\sec^2 x$

$\therefore\ I=\int e^x\,[f(x)+f'(x)]\,dx$

$= e^x \cdot f(x) + c$

$= e^x \cdot \tan x + c$

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(2) Let $= \int \dfrac{(x^2+1)\, e^x}{(x+1)^2}\, dx$

$= \int e^x \left[\dfrac{(x^2-1)+2}{(x+1)^2} \right] dx$

$= \int e^x \left[\dfrac{x^2-1}{(x+1)^2} + \dfrac{2}{(x+1)^2} \right] dx$

$= \int e^x \left[\dfrac{x-1}{x+1} + \dfrac{2}{(x+1)^2} \right] dx$

Put $f(x) = \dfrac{x-1}{x+1}$

$\therefore\ f'(x) = \dfrac{d}{dx}\left(\dfrac{x-1}{x+1}\right) = \dfrac{(x+1)\cdot \dfrac{d}{dx}(x-1) - (x-1)\cdot \dfrac{d}{dx}(x+1)}{(x+1)^2}$

$= \dfrac{(x+1)\,(1-0) - (x-1)\,(1+0)}{(x+1)^2}$

$= \dfrac{x+1-x+1}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

$\therefore\ I = \int e^x [f(x) + f'(x)] + c$

$= e^x \cdot f(x) + c$

$= e^x \left(\dfrac{x-1}{x+1}\right) + c.$

---

(3) Let $I = \int e^{\sin^{-1} x} \left[\dfrac{x + \sqrt{1-x^2}}{\sqrt{1-x^2}} \right] dx$

$= \int e^{\sin^{-1} x} [x + \sqrt{1-x^2}\,] \cdot \dfrac{1}{\sqrt{1-x^2}}\, dx$

Put $\sin^{-1} x = t$ $\qquad \therefore\ \dfrac{1}{\sqrt{1-x^2}}\, dx = dt$

and $x = \sin t$

$\therefore\ I = \int e^t [\sin t + \sqrt{1-\sin^2 t}\,]\, dt$

$= \int e^t [\sin t + \sqrt{\cos^2 t}\,]\, dt$

$= \int e^t (\sin t + \cos t)\, dt$

Let $f(t) = \sin t \qquad \therefore \; f'(t) = \cos t$

$\therefore \; I = \int e^t [f(t) + f'(t)] \, dt$

$= e^t \cdot f(t) + c = e^t \cdot \sin t + c$

$= e^{\sin^{-1} x} \cdot x + c = x \cdot e^{\sin^{-1} x} + c.$

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(4) Let $I = \int \frac{\log x}{(1 + \log x)^2} \, dx$

Put $\log x = t$. Then $x = e^t \quad \therefore \; dx = e^t \, dt$

$\therefore \; I = \int \frac{t}{(1+t)^2} e^t \, dt = \int e^t \left[\frac{(1+t) - 1}{(1+t)^2} \right] dt$

$= \int e^t \left[\frac{1}{1+t} - \frac{1}{(1+t)^2} \right] dt$

If $f(t) = \frac{1}{1+t}$, then $f'(t) = \frac{-1}{(1+t)^2}$

$\therefore \; I = \int e^t \, [f(t) + f'(t)] \, dt = e^t \, f(t) + c = \frac{e^t}{1+t} + c = \frac{x}{1 + \log x} + c.$

---

(5) Let $I = \int [\sin(\log x) + \cos(\log x)] \, dx$

Put $\log x = t$. Then $x = e^t \quad \therefore \; dx = e^t \, dt$

$\therefore \; I = \int (\sin t + \cos t) \, e^t \, dt$

If $f(t) = \sin t$, then $f'(t) = \cos t$

$\therefore \; I = \int e^t \, [f(t) + f'(t)] \, dt$

$= e^t \, f(t) + c = e^t \, \sin t + c$

$= x \cdot \sin(\log x) + c.$

Examples for Practice	2 or 3 marks each

Integrate the following w.r.t. *x* :

1. (1) $e^x (2 + \cot x + \cot^2 x)$ (2) $e^x \left(\frac{\cos x - \sin x}{\sin^2 x} \right)$

(3) $e^x \left(\frac{1 + \sin x}{1 + \cos x} \right)$.

2. (1) $e^x \cdot \frac{x + 2}{(x+3)^2}$ (2) $e^x \cdot \frac{(x-1)}{(x+1)^3}$ (3) $e^x \cdot \left(\frac{1 - x}{1 + x^2} \right)^2$.

3. (1) $\left[\frac{\log x - 1}{1 + (\log x)^2} \right]^2$ (2) $e^{\tan^{-1} x} \left(\frac{1 + x + x^2}{1 + x^2} \right)$ (3) $e^{5x} \left[\frac{5x \log x + 1}{x} \right]$.

ANSWERS

1. (1) $e^x(2+\cot x)+c$ (2) $-e^x \cdot \operatorname{cosec} x + c$ (3) $e^x \tan\left(\frac{x}{2}\right)+c.$

2. (1) $\frac{e^x}{x+3}+c$ (2) $\frac{e^x}{(x+1)^2}+c$ (3) $\frac{e^x}{1+x^2}+c.$

3. (1) $\frac{x}{1+(\log x)^2}+c$ (2) $x \cdot e^{\tan^{-1} x}+c$ (3) $e^{5x} \cdot \log x + c.$

10.4 INTEGRATION BY PARTIAL FRACTIONS

Remember :

Integral of the type $\int \frac{P(x)}{Q(x)}\,dx$, where

(i) $P(x)$ and $Q(x)$ are polynomials in x

(ii) degree $P(x) <$ degree $Q(x)$

(iii) no common polynomial factors in $P(x)$ and $Q(x)$.

Case (i) : If the denominator $Q(x)$ consists of n distinct linear factors

i.e. $\frac{P(x)}{Q(x)} = \frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\ \dots\ (a_nx+b_n)}$,

then we need to find the constants $A_1, A_2, A_3, \dots, A_n$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\dots+\frac{A_n}{a_nx+b_n}$$

Case (ii) : If the denominator has repeated linear factor,

i.e. $Q(x) = (x-a)^k(x-a_1)(x-a_2)\ \dots\ (x-a_r)$, then we assume

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\dots+\frac{A_k}{(x-a)^k}+\frac{B_1}{x-a_1}+\frac{B_2}{x-a_2}+\dots+\frac{B_r}{x-a_r}$$

where $A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_r$ are constants.

Case (iii) : If the denominator $Q(x)$ has non-repeated quadratic factors, then corresponding to each quadratic factor ax^2+bx+c, we assume the partial fraction $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants.

Remark : If degree $P(x) >$ degree $Q(x)$, then divide $P(x)$ by $Q(x)$ till the degree of remainder $f(x)$ is less than $Q(x)$.

$\therefore \dfrac{P(x)}{Q(x)} = r(x) + \dfrac{f(x)}{Q(x)}$, where $r(x)$ is the quotient.

Solved Examples	3 or 4 marks each

Ex. 24. Integrate the following w.r.t. x :

(1) $\dfrac{x^2+2}{(x-1)(x+2)(x+3)}$ **(2)** $\dfrac{x^2+x-1}{x^2+x-6}$.

Solution :

(1) Let $I = \displaystyle\int \frac{x^2+2}{(x-1)(x+2)(x+3)}\, dx$

Let $\dfrac{x^2+2}{(x-1)(x+2)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

$\therefore x^2+2 = A\,(x+2)(x+3) + B\,(x-1)(x+3) + C\,(x-1)(x+2)$ ... (1)

Put $x-1=0$, i.e. $x=1$ in (1), we get

$1+2 = A(3)(4) + B(0)(4) + C(0)(3)$

$\therefore\ 3 = 12A$

$\therefore\ A = \dfrac{1}{4}$

Put $x+2=0$, i.e. $x=-2$ in (1), we get

$4+2 = A(0)(1) + B(-3)(1) + C(-3)(0)$

$\therefore\ 6 = -3B$

$\therefore\ B = -2$

Put $x+3=0$, i.e. $x=-3$ in (1), we get

$9+2 = A(-1)(0) + B(-4)(0) + C(-4)(-1)$

$\therefore\ 11 = 4C$

$\therefore\ C = \dfrac{11}{4}$

$$\therefore\ \frac{x^2+2}{(x-1)(x+2)(x+3)} = \frac{\left(\frac{1}{4}\right)}{x-1} + \frac{(-2)}{x+2} + \frac{\left(\frac{11}{4}\right)}{x+3}$$

$$\therefore\ I=\int\left[\frac{\left(\frac{1}{4}\right)}{x-1}+\frac{(-2)}{x+2}+\frac{\left(\frac{11}{4}\right)}{x+3}\right]dx$$

$$=\frac{1}{4}\int\frac{1}{x-1}\,dx-2\int\frac{1}{x+2}\,dx+\frac{11}{4}\int\frac{1}{x+3}\,dx$$

$$=\frac{1}{4}\log|x-1|-2\log|x+2|+\frac{11}{4}\log|x+3|+c.$$

---

(2) Let $I=\int\frac{x^2+x-1}{x^2+x-6}\,dx$

$$=\int\frac{(x^2+x-6)+5}{x^2+x-6}\,dx$$

$$=\int\left[1+\frac{5}{x^2+x-6}\right]dx$$

$$=\int 1\,dx+5\int\frac{1}{x^2+x-6}\,dx$$

Let $\frac{1}{x^2+x-6}=\frac{1}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$

$\therefore\ 1=A(x-2)+B(x+3)$ ... (1)

Put $x+3=0$, i.e. $x=-3$ in (1), we get

$1=A(-5)+B(0)\qquad\therefore\ A=-\frac{1}{5}$

Put $x-2=0$, i.e. $x=2$ in (1), we get

$1=A(0)+B(5)\qquad\therefore\ B=\frac{1}{5}$

$$\therefore\ \frac{1}{x^2+x-6}=\frac{\left(-\frac{1}{5}\right)}{x+3}+\frac{\left(\frac{1}{5}\right)}{x-2}$$

$$\therefore\ I=\int 1\,dx+5\int\left[\frac{\left(-\frac{1}{5}\right)}{x+3}+\frac{\left(\frac{1}{5}\right)}{x-2}\right]dx$$

$$=\int 1\,dx-\int\frac{1}{x+3}\,dx+\int\frac{1}{x-2}\,dx$$

$$=x-\log|x+3|+\log|x-2|+c.$$

Ex. 25. Integrate the following w.r.t. x :

(1) $\dfrac{x^2+2x+6}{(x+1)(x^2+4)}$ **(2)** $\dfrac{1}{x^3(1-x)}$.

Solution :

(1) $\displaystyle\int \frac{x^2+2x+6}{(x+1)(x^2+4)}\,dx$

$$=\int \frac{(x^2+4)+2(x+1)}{(x+1)(x^2+4)}\,dx=\int\left(\frac{1}{x+1}+\frac{2}{x^2+4}\right)dx$$

$$=\int \frac{1}{x+1}\,dx+2\int \frac{1}{x^2+4}\,dx$$

$$=\log|x+1|+2\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$$

$$=\log|x+1|+\tan^{-1}\left(\frac{x}{2}\right)+c.$$

---

(2) $\displaystyle\int \frac{1}{x^3(1-x)}\,dx=\int \frac{(1-x^3)+x^3}{x^3(1-x)}\,dx$

$$=\int\left[\frac{1-x^3}{x^3(1-x)}+\frac{1}{1-x}\right]dx$$

$$=\int\left[\frac{(1-x)(1+x+x^2)}{x^3(1-x)}+\frac{1}{1-x}\right]dx$$

$$=\int\left[\frac{1+x+x^2}{x^3}+\frac{1}{1-x}\right]dx$$

$$=\int\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}+\frac{1}{1-x}\right)dx$$

$$=\int x^{-3}\,dx+\int x^{-2}\,dx+\int\frac{1}{x}\,dx+\int\frac{1}{1-x}\,dx$$

$$=\frac{x^{-2}}{-2}+\frac{x^{-1}}{-1}+\log|x|+\frac{\log|1-x|}{-1}+c$$

$$=\log\left|\frac{x}{1-x}\right|-\frac{1}{2x^2}-\frac{1}{x}+c.$$

Ex. 26. Integrate the following w.r.t. x :

(1) $\dfrac{2x^2-3}{(x^2-5)(x^2+4)}$ **(2)** $\dfrac{2x^2-1}{(x^2+4)(x^2-5)}$.

Solution :

(1) Let $I = \int \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)}\, dx$

Consider, $\frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)}$

For finding partial fractions, put $x^2 = t$.

$$\therefore \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} = \frac{2t - 3}{(t - 5)(t + 4)} = \frac{A}{t - 5} + \frac{B}{t + 4}$$

$$\therefore 2t - 3 = A\,(t + 4) + B\,(t - 5) \qquad \text{... (1)}$$

Put $t + 4 = 0$ i.e. $t = -4$ in (1), we get

$$2\,(-4) - 3 = A\,(0) + B\,(-9)$$

$$\therefore -11 = -9B \qquad \therefore B = \frac{11}{9}$$

Put $t - 5 = 0$ i.e. $t = 5$ in (1), we get

$$2\,(5) - 3 = A\,(9) + B\,(0)$$

$$\therefore 7 = 9A \qquad \therefore A = \frac{7}{9}$$

$$\therefore \frac{2t - 3}{(t - 5)(t + 4)} = \frac{\left(\frac{7}{9}\right)}{t - 5} + \frac{\left(\frac{11}{9}\right)}{t + 4}$$

$$\therefore \frac{2x^2 - 3}{(x^2 - 5)\,(x^2 + 4)} = \frac{\left(\frac{7}{9}\right)}{x^2 - 5} + \frac{\left(\frac{11}{9}\right)}{x^2 + 4}$$

$$I = \int \left[\frac{\left(\frac{7}{9}\right)}{x^2 - 5} + \frac{\left(\frac{11}{9}\right)}{x^2 + 4}\right] dx$$

$$= \frac{7}{9}\int \frac{1}{x^2 - (\sqrt{5})^2}\, dx + \frac{11}{9}\int \frac{1}{x^2 + 2^2}\, dx$$

$$= \frac{7}{9} \times \frac{1}{2\sqrt{5}} \log\left|\frac{x - \sqrt{5}}{x + \sqrt{5}}\right| + \frac{11}{9} \times \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= \frac{7}{18\sqrt{5}} \log\left|\frac{x - \sqrt{5}}{x + \sqrt{5}}\right| + \frac{11}{18} \tan^{-1}\left(\frac{x}{2}\right) + c.$$

---

(2) Let $I=\int \frac{2x^2-1}{(x^2+4)(x^2-5)}\,dx$

$$=\int \frac{(x^2-5)+(x^2+4)}{(x^2+4)(x^2-5)}\,dx$$

$$=\int\left(\frac{1}{x^2+4}+\frac{1}{x^2-5}\right)dx$$

$$=\int \frac{1}{x^2+2^2}\,dx+\int \frac{1}{x^2-(\sqrt{5})^2}\,dx$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+c.$$

Ex. 27. Integrate the following w.r.t. x :

(1) $\frac{3x-2}{(x+1)^2\,(x+3)}$ **(2)** $\frac{8}{(x+2)\,(x^2+4)}$.

Solution :

(1) Let $I=\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$

Let $\frac{3x-2}{(x+1)^2(x+3)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$

$\therefore\ 3x-2=A\,(x+1)(x+3)+B\,(x+3)+C\,(x+1)^2$ ... (1)

Put $x+1=0$, i.e. $x=-1$ in (1), we get

$-3-2=A\,(0)(2)+B\,(2)+C\,(0)$

$\therefore\ -5=2B\quad\therefore\ B=-\frac{5}{2}$

Put $x+3=0$, i.e. $x=-3$ in (1), we get

$-9-2=A\,(-2)(0)+B\,(0)+C\,(-2)^2$

$\therefore\ -11=4C\quad\therefore\ C=-\frac{11}{4}$

Put $x=0$ in (1), we get

$-2=A\,(1)(3)+B\,(3)+C\,(1)$

$\therefore\ -2=3A+3B+C$

$\therefore\ -2=3A-\frac{15}{2}-\frac{11}{4}$

$\therefore\ 3A=-2+\frac{15}{2}+\frac{11}{4}=\frac{-8+30+11}{4}=\frac{33}{4}\qquad\therefore\ A=\frac{11}{4}$

$$\therefore \quad \frac{3x-2}{(x+1)^2(x+3)} = \frac{\left(\frac{11}{4}\right)}{x+1} + \frac{\left(-\frac{5}{2}\right)}{(x+1)^2} + \frac{\left(-\frac{11}{4}\right)}{x+3}$$

$$\therefore \quad I = \int \left[\frac{\left(\frac{11}{4}\right)}{x+1} + \frac{\left(-\frac{5}{2}\right)}{(x+1)^2} + \frac{\left(-\frac{11}{4}\right)}{x+3}\right] dx$$

$$= \frac{11}{4}\int \frac{1}{x+1}\, dx - \frac{5}{2}\int (x+1)^{-2}\, dx - \frac{11}{4}\int \frac{1}{x+3}\, dx$$

$$= \frac{11}{4}\log|x+1| - \frac{5}{2}\cdot\frac{(x+1)^{-1}}{-1}\cdot\frac{1}{1} - \frac{11}{4}\log|x+3| + c$$

$$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + c.$$

---

(2) Let $I = \int \frac{8}{(x+2)(x^2+4)}\, dx$

Let $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

$\therefore \; 8 = A(x^2+4) + (Bx+C)(x+2)$ ... (1)

Put $x+2=0$, i.e. $x=-2$ in (1), we get

$8 = A(4+4) + (-2B+C)(0)$

$\therefore \; 8 = 8A \qquad \therefore \; A = 1$

Put $x = 0$ in (1), we get

$8 = A(4) + (0+C)(2)$

$\therefore \; 8 = 4 + 2C$ ... $[\because \; A = 1]$

$\therefore \; 2C = 2 \qquad \therefore \; C = 1$

Comparing the coefficient of x^2 on both the sides of (1), we get

$0 = A + B \quad \therefore \; B = -A = -1 \quad \therefore \; \frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{(-x+1)}{x^2+4}$

$$\therefore \; I = \int\left[\frac{1}{x+2} + \frac{(-x+1)}{x^2+4}\right] dx$$

$$= \int \frac{1}{x+1}\, dx - \frac{1}{2}\int \frac{2x}{x^2+4}\, dx + \int \frac{1}{x^2+4}\, dx$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

... $\left[\because \; \frac{d}{dx}(x^2+4) = 2x \text{ and } \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$

Ex. 28. Evaluate the following :

(1) $\displaystyle\int \frac{5e^x}{(e^x+1)(e^{2x}+9)}\,dx$ **(2)** $\displaystyle\int \frac{1}{\sin x + \sin 2x}\,dx$.

Solution :

(1) Let $I=\displaystyle\int \frac{5e^x}{(e^x+1)(e^{2x}+9)}\,dx$

Put $e^x=t$ $\therefore\ e^x\,dx=dt$

$$\therefore\ I=5\int \frac{1}{(t+1)(t^2+9)}\,dt$$

Let $\dfrac{1}{(t+1)(t^2+9)}=\dfrac{A}{t+1}+\dfrac{Bt+C}{t^2+9}$

$\therefore\ 1=A\,(t^2+9)+(Bt+C)(t+1)$ ... (1)

Put $t+1=0$, i.e. $t=-1$ in (1), we get

$1=A\,(1+9)+(-B+C)\,(0)$ $\therefore\ A=\dfrac{1}{10}$

Put $t=0$ in (1), we get

$1=A\,(9)+C\,(1)$ $\therefore\ C=1-9A=1-\dfrac{9}{10}=\dfrac{1}{10}$

Comparing coefficients of t^2 on both the sides in (1), we get

$0=A+B$ $\therefore\ B=-A=-\dfrac{1}{10}$

$$\therefore\ \frac{1}{(t+1)(t^2+9)}=\frac{\left(\frac{1}{10}\right)}{t+1}+\frac{\left(-\frac{1}{10}t+\frac{1}{10}\right)}{t^2+9}$$

$$\therefore\ I=5\int\left[\frac{\left(\frac{1}{10}\right)}{t+1}+\frac{\left(-\frac{1}{10}t+\frac{1}{10}\right)}{t^2+9}\right]dt$$

$$=\frac{1}{2}\int\frac{1}{t+1}\,dt-\frac{1}{2}\int\frac{t}{t^2+9}\,dt+\frac{1}{2}\int\frac{1}{t^2+9}\,dt$$

$$=\frac{1}{2}\log|t+1|-\frac{1}{4}\int\frac{2t}{t^2+9}\,dt+\frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)$$

$$= \frac{1}{2}\log|t+1| - \frac{1}{4}\int \frac{\frac{d}{dt}(t^2+9)}{t^2+9}\,dt + \frac{1}{6}\tan^{-1}\left(\frac{t}{3}\right)$$

$$= \frac{1}{2}\log|t+1| - \frac{1}{4}\log|t^2+9| + \frac{1}{6}\tan^{-1}\left(\frac{t}{3}\right) + c$$

$$\ldots \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= \frac{1}{2}\log|e^x+1| - \frac{1}{4}\log|e^{2x}+9| + \frac{1}{6}\tan^{-1}\left(\frac{e^x}{3}\right) + c.$$

---

(2) Let $I = \int \frac{1}{\sin x + \sin 2x}\,dx$

$$= \int \frac{1}{\sin x + 2\sin x\cos x}\,dx$$

$$= \int \frac{dx}{\sin x(1+2\cos x)} = \int \frac{\sin x\,dx}{\sin^2 x(1+2\cos x)}$$

$$= \int \frac{\sin x\,dx}{(1-\cos^2 x)(1+2\cos x)}$$

$$= \int \frac{\sin x\,dx}{(1-\cos x)(1+\cos x)(1+2\cos x)}$$

Put $\cos x = t$ $\quad\therefore\; -\sin x\,dx = dt$

$\therefore\; \sin x\,dx = -dt$

$$\therefore\; I = \int \frac{-dt}{(1-t)(1+t)(1+2t)}$$

$$= -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

Let $\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t}$

$\therefore\; 1 = A(1+t)(1+2t) + B(1-t)(1+2t) + C(1-t)(1+t)$ ... (1)

Put $1-t=0$, i.e. $t=1$ in (1), we get

$1 = A(2)(3) + B(0)(3) + C(0)(2)$ $\quad\therefore\; A = \frac{1}{6}$

Put $1+t=0$, i.e. $t=-1$ in (1), we get

$1 = A(0)(-1) + B(2)(-1) + C(2)(0)$

$\therefore\; B = -\frac{1}{2}$

Put $1 + 2t = 0$, i.e. $t = -\frac{1}{2}$ in (1), we get

$$1 = A(0) + B(0) + C\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \quad \therefore \; C = \frac{4}{3}$$

$$\therefore \; \frac{1}{(1-t)(1+t)(1+2t)} = \frac{\left(\frac{1}{6}\right)}{1-t} + \frac{\left(-\frac{1}{2}\right)}{1+t} + \frac{\left(\frac{4}{3}\right)}{1+2t}$$

$$\therefore \; I = -\int \left[\frac{\left(\frac{1}{6}\right)}{1-t} + + \frac{\left(-\frac{1}{2}\right)}{1+t} + \frac{\left(\frac{4}{3}\right)}{1+2t}\right] dt$$

$$= -\frac{1}{6}\int \frac{1}{1-t}\,dt + \frac{1}{2}\int \frac{1}{1+t}\,dt - \frac{4}{3}\int \frac{1}{1+2t}\,dt$$

$$= -\frac{1}{6}\cdot\frac{\log|1-t|}{-1} + \frac{1}{2}\log|1+t| - \frac{4}{3}\cdot\frac{\log|1+2t|}{2} + c$$

$$= \frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + c$$

$$= \frac{1}{2}\log|\cos x + 1| + \frac{1}{6}\log|\cos x - 1| - \frac{2}{3}\log|2\cos x + 1| + c.$$

Examples for Practice	3 or 4 marks each

Integrate the following w.r.t. x :

1. (1) $\frac{12x+3}{6x^2+13x-63}$ (2) $\frac{3x^2+4x-5}{(x^2-1)(x+2)}$

(3) $\frac{5x+2}{x^2-3x+2}$.

2. (1) $\frac{x^2+3x+5}{(x+2)(x^2+2x+3)}$ (2) $\frac{1-3x^2}{x(1-x^2)}$

(3) $\frac{1}{x^2(1-2x)}$.

3. (1) $\frac{2x^2-1}{(x^2+5)(x^2+4)}$ (2) $\frac{x^2+3}{(x^2-1)(x^2-2)}$

(3) $\frac{x^2}{(x^2+1)(x^2-2)(x^2+3)}$.

4. (1) $\frac{3x-2}{(x+1)^2(x+2)}$ (2) $\frac{x^2+x+1}{(x-1)^3}$

(3) $\dfrac{2x+7}{(x-4)^2}$.

5. (1) $\dfrac{x}{(x-1)\,(x^2+1)}$ (2) $\dfrac{5x^2}{(x+1)\,(x^2+4)}$

(3) $\dfrac{1}{x^3-1}$.

6. (1) $\dfrac{2x}{(2+x^2)(3+x^2)}$ (2) $\dfrac{1}{x\cdot\log x\cdot(2+\log x)}$

(3) $\dfrac{2\,\log x+3}{x\,(3\,\log x+2)\,[(\log x)^2+1]}$ (4) $\dfrac{1}{\sin x\,(3+2\cos x)}$

(5) $\dfrac{1}{2\cos x+\sin 2x}$ (6) $\dfrac{1}{x\,(x^5+1)}$.

ANSWERS

1. (1) $\dfrac{51}{41}\log|2x+9|+\dfrac{31}{41}\log|3x-7|+c$

(2) $\dfrac{1}{3}\log|x-1|+3\log|x+1|-\dfrac{1}{3}\log|x+2|+c$

(3) $12\log|x-2|-7\log|x-1|+c$.

2. (1) $\log|x+2|+\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right)+c$

(2) $\log|x-x^3|+c$

(3) $2\log\left|\dfrac{x}{1-2x}\right|-\dfrac{1}{x}+c$.

3. (1) $-\dfrac{9}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+\dfrac{11}{\sqrt{5}}\tan^{-1}\left(\dfrac{x}{\sqrt{5}}\right)+c$

(2) $2\log\left|\dfrac{x+1}{x-1}\right|+\dfrac{5}{2\sqrt{2}}\log\left|\dfrac{x-\sqrt{2}}{x+\sqrt{2}}\right|+c$

(3) $\dfrac{1}{6}\tan^{-1}x+\dfrac{5}{15\sqrt{2}}\log\left|\dfrac{x-\sqrt{2}}{x+\sqrt{2}}\right|-\dfrac{\sqrt{3}}{10}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)+c$.

4. (1) $8\log\left|\dfrac{x+1}{x+2}\right|+\dfrac{5}{(x+1)}+c$

(2) $\log|x-1|-\dfrac{3}{x-1}-\dfrac{3}{2\,(x-1)^2}+c$

(3) $2 \log |x-4| - \frac{15}{x-4} + c.$

5. (1) $\frac{1}{2} \log |x-1| - \frac{1}{4} \log |x^2+1| + \frac{1}{2} \tan^{-1} x + c$

(2) $\log |(x+1)(x^2+4)^2| - 2 \tan^{-1}\left(\frac{x}{2}\right) + c$

(3) $\frac{1}{3} \log |x-1| - \frac{1}{6} \log |x^2+x+1| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c.$

6. (1) $\log\left|\frac{2+x^2}{3+x^2}\right| + c$

(2) $\frac{1}{2} \log\left|\frac{\log x}{2+\log x}\right| + c$

(3) $\frac{5}{13} \log|3 \log x + 2| - \frac{5}{26} \log|(\log x)^2 + 1| + \frac{12}{13} \tan^{-1} (\log x) + c$

(4) $\frac{1}{10} \log|1-\cos x| - \frac{1}{2} \log|1+\cos x| + \frac{2}{5} \log|3+2\cos x| + c$

(5) $\frac{1}{8} \log\left|\frac{1+\sin x}{1-\sin x}\right| - \frac{1}{4(1+\sin x)} + c$

(6) $\frac{1}{5} \log\left|\frac{x^5}{x^5+1}\right| + c.$

MULTIPLE CHOICE QUESTIONS	**2 marks each**

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. $\int \frac{x-\sin x}{1-\cos x} dx = \dots\dots\dots$

(a) $x \cot\left(\frac{x}{2}\right) + c$ (b) $-x \cot\left(\frac{x}{2}\right) + c$

(c) $\cot\left(\frac{x}{2}\right) + c$ (d) $x \tan\left(\frac{x}{2}\right) + c$

2. $\int \frac{\sin 3x}{\sin x} dx = \dots\dots\dots$

(a) $x - \sin 2x + c$ (b) $x + \sin 2x + c$

(c) $x + \cos 2x + c$ (d) $x - \cos 2x + c.$

3. $\int \frac{\cos(2x)-1}{\cos(2x)+1}\,dx =$ ....

(a) $\tan x - x + c$ (b) $x + \tan x + c$

(c) $x - \tan x + c$ (d) $-x - \cot x + c$ ***(Sept. '21)***

4. $\int \frac{dx}{4x^2-1} = A \log\left(\frac{2x-1}{2x+1}\right) + c$, then $A =$ ......

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $\frac{1}{4}$ ***(March '22)***

5. $\int \frac{\sqrt{\cot x}}{\sin x \cdot \cos x}\,dx =$ ....

(a) $2\sqrt{\cot x} + c$ (b) $-2\sqrt{\cot x} + c$

(c) $\frac{1}{2}\sqrt{\cot x} + c$ (d) $\sqrt{\cot x} + c$

6. $\int \frac{\sin x}{1 - \sin x}\,dx =$ ....

(a) $\sec x - \tan x + c$ (b) $\sec x + \tan x - x + c$

(c) $\sec x - \tan x - x + c$ (d) $\sec x - \tan x + x + c$

7. $\int \frac{dx}{x^2 (x^4+1)^{\frac{3}{4}}}$ is

(a) $-\left(1 + \frac{1}{x^4}\right)^{\frac{1}{4}} + c$ (b) $\frac{1}{4} \log\left|\frac{x^2}{x^4+1}\right| + c$

(c) $\frac{1}{2} \log|x^4+1| + c$ (d) $\left(1 + \frac{1}{x^4}\right)^{\frac{3}{4}} + c$

8. If $\int \frac{\log(\log x)}{7x}\,dx = k \log x[1 - \log(\log x)] + c$, then $k =$ ....

(a) $\frac{1}{7}$ (b) 1 (c) -1 (d) $-\frac{1}{7}$

9. $\int \frac{e^{2x} + e^{-2x}}{e^x}\,dx =$ ....

(a) $e^x - \frac{1}{3e^{3x}} + c$ (b) $e^x + \frac{1}{3e^{3x}} + c$

(c) $e^{-x}+\frac{1}{3e^{3x}}+c$ (d) $e^{-x}-\frac{1}{3e^{3x}}+c$

10. $\int \frac{dx}{\sqrt{\sin^3 x \cos x}} = \text{.........}$

(a) $2\sqrt{\tan x}+c$ (b) $2\sqrt{\cot x}+c$

(c) $-\frac{2}{\sqrt{\tan x}}+c$ (d) $\frac{2}{\sqrt{\cot x}}+c$

11. $\int \frac{1}{x+x^5}dx = f(x)+c$, then $\int \frac{x^4}{x+x^5}dx = \text{.........}$

(a) $\log x - f(x)+c$ (b) $f(x)+\log x+c$

(c) $f(x)-\log x+c$ (d) $\frac{1}{5}x^5 \cdot f(x)+c$

12. $\int \frac{\log x}{(\log ex)^2}dx = \text{.........}$

(a) $\frac{x}{1+\log x}+c$ (b) $x(1+\log x)+c$

(c) $\frac{1}{1+\log x}+c$ (d) $\frac{1}{1-\log x}+c$

ANSWERS

1. (b) $-x\cot\left(\frac{x}{2}\right)+c$ **2.** (b) $x+\sin 2x+c$

3. (c) $x-\tan x+c$ **4.** (d) $\frac{1}{4}$

5. (b) $-2\sqrt{\cot x}+c$ **6.** (b) $\sec x+\tan x-x+c$

7. (a) $-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+c$ **8.** (d) $-\frac{1}{7}$

9. (a) $e^x-\frac{1}{3e^{3x}}+c$ **10.** (c) $-\frac{2}{\sqrt{\tan x}}+c$

11. (a) $\log x-f(x)+c$ **12.** (a) $\frac{x}{1+\log x}+c.$