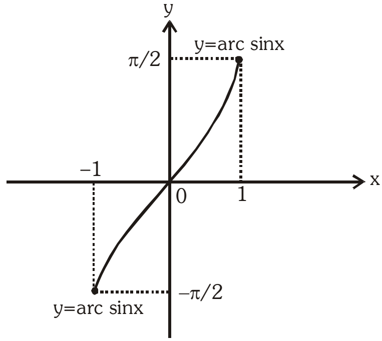
# *INVERSE TRIGONOMETRIC FUNCTION*

## 1. DOMAIN, RANGE & GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS :

**(a)** $f^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$,

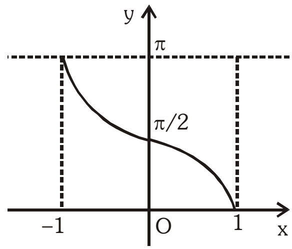
$f^{-1}(x) = \sin^{-1}(x)$




(y = $\sin^{-1}$x)

**(b)** $f^{-1} : [-1, 1] \to [0, \pi]$,

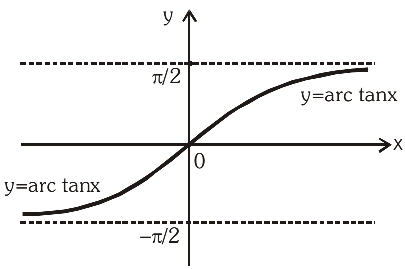
$f^{-1}(x) = \cos^{-1} x$




(y = $\cos^{-1}$x)

**(c)** $f^{-1} : R \to (-\pi/2, \pi/2)$,

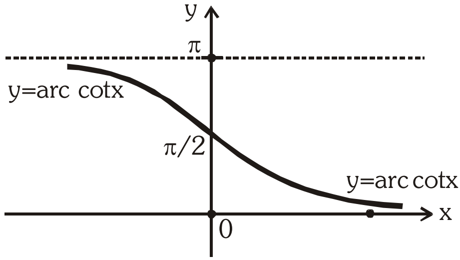
$f^{-1}(x) = \tan^{-1} x$




(y = $\tan^{-1}$x)

**(d)** $f^{-1} : R \to (0, \pi)$,

$f^{-1}(x) = \cot^{-1} x$

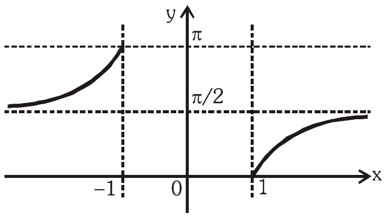



($y = \cot^{-1}x$)

**(e)** $f^{-1} : (-\infty, -1] \cup [1, \infty)$

$\to [0, \pi/2) \cup (\pi/2, \pi]$,

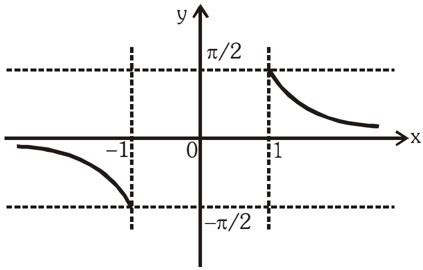
$f^{-1}(x) = \sec^{-1} x$




**(f)** $f^{-1} : (-\infty, -1] \cup [1, \infty)$

$\to [-\pi/2, 0) \cup (0, \pi/2]$,

$f^{-1}(x) = \text{cosec}^{-1} x$




## 2. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

**P-1 :**

**(i)** $y = \sin(\sin^{-1}x) = x$, $x \in [-1,1]$, $y \in [-1,1]$, y is aperiodic

**(ii)** $y = \cos(\cos^{-1} x) = x$ , $x \in [-1,1]$, $y \in [-1,1]$, y is aperiodic

**(iii)** $y = \tan(\tan^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic

**(iv)** $y = \cot(\cot^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic

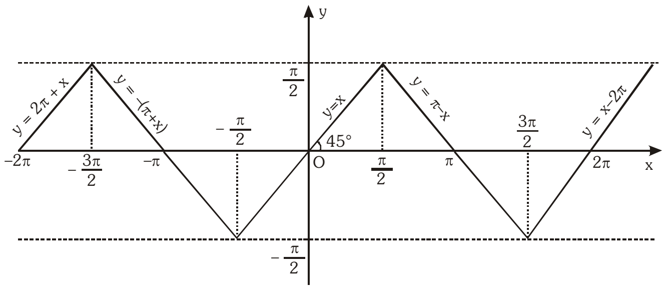
**(v)** $y = \text{cosec}(\text{cosec}^{-1} x) = x$, $|x| \geq 1$, $|y| \geq 1$, y is aperiodic

**(vi)** $y = \sec(\sec^{-1} x) = x$, $|x| \geq 1$ ; $|y| \geq 1$, y is aperiodic

**P-2 :**

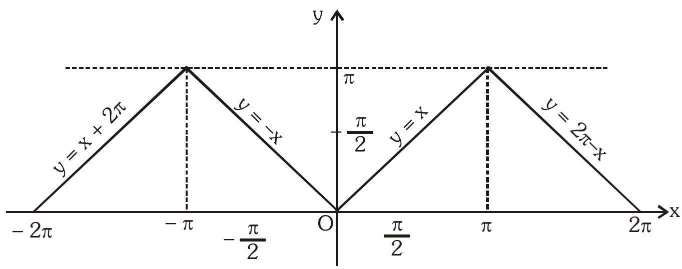
**(i)** $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Periodic with period $2\pi$.

$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x & , & -\frac{3\pi}{2} \le x \le -\frac{\pi}{2} \\ x & , & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x & , & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi & , & \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \\ 3\pi - x & , & \frac{5\pi}{2} \le x \le \frac{7\pi}{2} \\ x - 4\pi & , & \frac{7\pi}{2} \le x \le \frac{9\pi}{2} \end{cases}$$




**(ii)** $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0,\pi]$, periodic with period $2\pi$

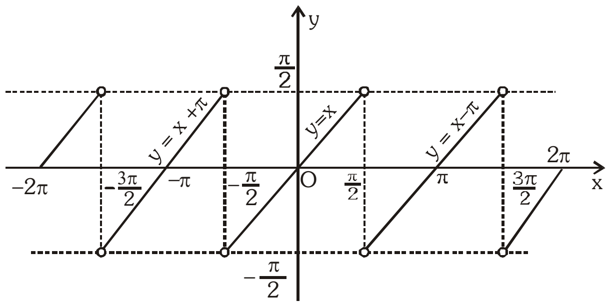
$$\cos^{-1}(\cos x) = \begin{cases} -x & , & -\pi \le x \le 0 \\ x & , & 0 \le x \le \pi \\ 2\pi - x & , & \pi \le x \le 2\pi \\ x - 2\pi & , & 2\pi \le x \le 3\pi \\ 4\pi - x & , & 3\pi \le x \le 4\pi \end{cases}$$



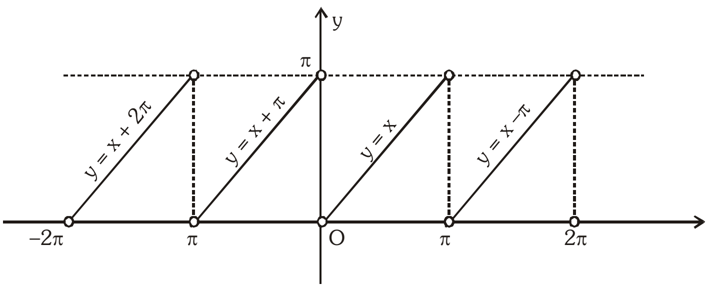
**(iii)** $y = \tan^{-1}(\tan x)$

$$x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}; \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ periodic with period } \pi$$

$$\tan^{-1}(\tan x) = \begin{cases} x + \pi & , \ -\dfrac{3\pi}{2} < x < -\dfrac{\pi}{2} \\ x & , \ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ x - \pi & , \ \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x - 2\pi & , \ \dfrac{3\pi}{2} < x < \dfrac{5\pi}{2} \\ x - 3\pi & , \ \dfrac{5\pi}{2} < x < \dfrac{7\pi}{2} \end{cases}$$

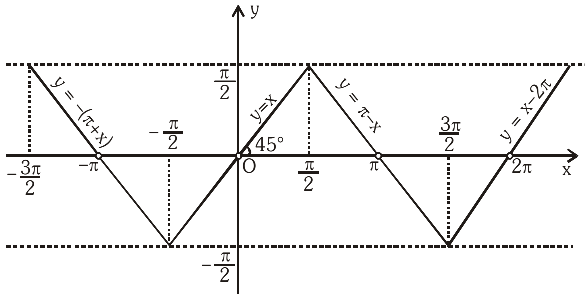



**(iv)** $y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi\}$, $y \in (0, \pi)$, periodic with period $\pi$



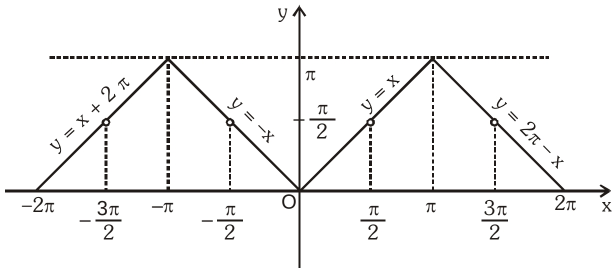

**(v)** $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in R - \{n\pi, n \in I\}$ $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$,

y is periodic with period $2\pi$.




**(vi)** $y = \sec^{-1}(\sec x)$, y is periodic with period $2\pi$

$$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\},\ \ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$




**P-3 :**

**(i)** $\text{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$ ; $x \leq -1$ or $x \geq 1$

**(ii)** $\sec^{-1} x = \cos^{-1}\frac{1}{x}$ ; $x \leq -1$ or $x \geq 1$

**(iii)** $\cot^{-1}x = \tan^{-1}\frac{1}{x}$ ; $x > 0$

$= \pi + \tan^{-1}\frac{1}{x}$ ; $x < 0$

**P-4 :**

**(i)** $\sin^{-1}(-x) = -\sin^{-1} x$ , $-1 \leq x \leq 1$

**(ii)** $\tan^{-1}(-x) = -\tan^{-1} x$ , $x \in R$

**(iii)** $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $-1 \leq x \leq 1$

**(iv)** $\cot^{-1}(-x) = \pi - \cot^{-1} x$, $x \in R$

**(v)** $\sec^{-1}(-x) = \pi - \sec^{-1} x,\ x \le -1 \text{ or } x \ge 1$

**(vi)** $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x,\ x \le -1 \text{ or } x \ge 1$

**P-5 :**

**(i)** $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2},\ -1 \le x \le 1$

**(ii)** $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2},\ x \in R$

**(iii)** $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2},\ |x| \ge 1$

**P-6 :**

**(i)** $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy}, \text{ where } x > 0,\ y > 0 \ \&\ xy < 1 \\ \pi + \tan^{-1}\dfrac{x+y}{1-xy}, \text{ where } x > 0,\ y > 0 \ \&\ xy > 1 \\ \dfrac{\pi}{2}, \text{ where } x > 0,\ y > 0 \ \&\ xy = 1 \end{cases}$$

**(ii)** $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x-y}{1+xy}$, where $x > 0$, $y > 0$

**(iii)** $\sin^{-1} x + \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,

where $x > 0,\ y > 0 \ \&\ (x^2 + y^2) < 1$

Note that : $x^2 + y^2 < 1 \Rightarrow 0 < \sin^{-1} x + \sin^{-1} y < \dfrac{\pi}{2}$

**(iv)** $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,

where $x > 0,\ y > 0 \ \&\ x^2 + y^2 > 1$

**Note that :** $x^2 + y^2 > 1 \Rightarrow \dfrac{\pi}{2} < \sin^{-1} x + \sin^{-1} y < \pi$

**(v)** $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0,\ y > 0$

**(vi)** $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$, where $x > 0,\ y > 0$

**(vii)** $$\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ;\quad x < y,\quad x, y > 0 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ;\quad x > y,\quad x, y > 0 \end{cases}$$

**(viii)** $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$

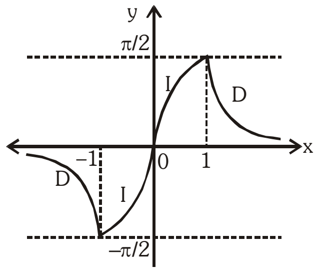
if $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$

**Note :** In the above results x & y are taken positive. In case if these are given as negative, we first apply P-4 and then use above results.

## 3. SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS :

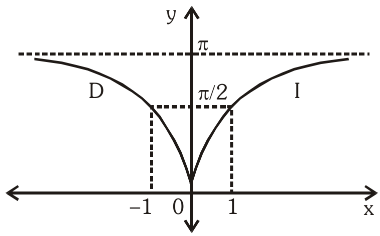
**(a)** $y = f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$= \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$$



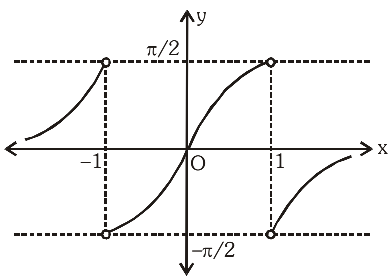

**(b)** $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$$= \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$



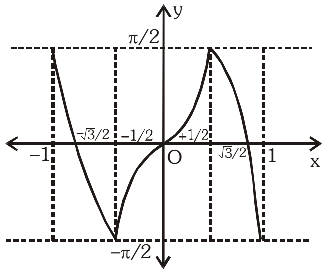

**(c)** $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2}$

$$= \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$$



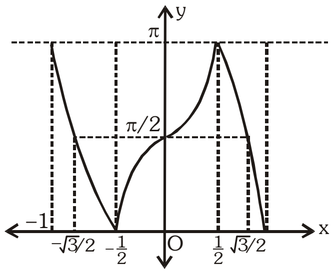
**(d)** $y = f(x) = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -(\pi + 3\sin^{-1} x) & \text{if} \quad -1 \le x \le -\frac{1}{2} \\ 3\sin^{-1} x & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1} x & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$



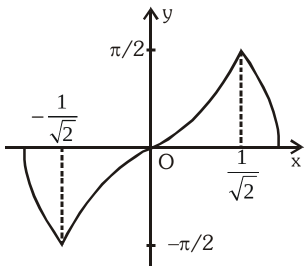

**(e)** $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1} x - 2\pi & \text{if} \quad -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1} x & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$




**(f)** $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$= \begin{cases} -\left(\pi + 2\sin^{-1} x\right) & -1 \le x \le -\frac{1}{\sqrt{2}} \\ 2\sin^{-1} x & -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1} x & \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$$




**(g)** $\cos^{-1}(2x^2 - 1)$

$$= \begin{cases} 2\cos^{-1} x & 0 \le x \le 1 \\ 2\pi - 2\cos^{-1} x & -1 \le x \le 0 \end{cases}$$