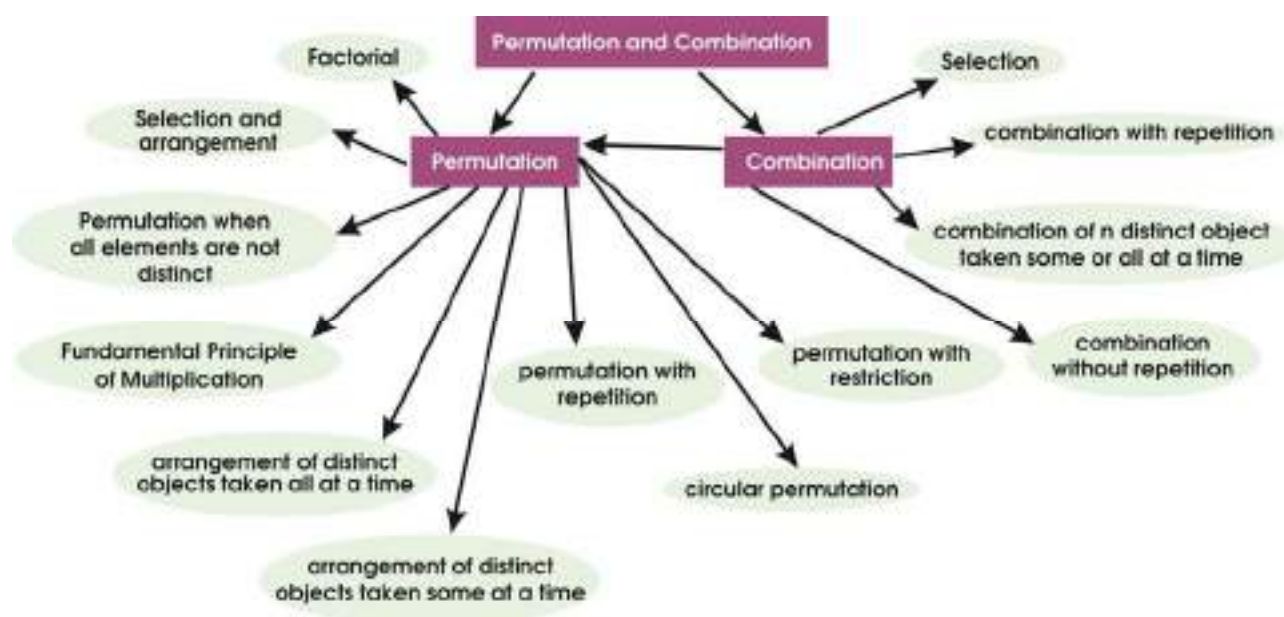# Chapter VI

Permutations and Combinations

Learning Objectives:

After completion of this unit you will be able to:

- Explain the difference between permutation and combination

- Solve problems based on permutation and combination

- Apply the formulas and concepts learnt to find solution of application problems

- Appreciate how to count without counting.

Concept MapP

Permutation and Combination

Factorial — Permutation — Combination — Selection

- Selection and arrangement
- Permutation when all elements are not distinct
- Fundamental Principle of Multiplication
- arrangement of distinct objects taken all at a time
- arrangement of distinct objects taken some at a time
- permutation with repetition
- permutation with restriction
- circular permutation
- combination with repetition
- combination of n distinct object taken some or all at a time
- combination without repetition

6.1   Introduction

Counting or enumeration is usually a common process when a student is studying arithmetic. But in later years very little attention is paid to counting when a student turns to difficult areas in mathematics, such as algebra, geometry and calculus.

A password on a computer system consists of eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain atleast one digit. How many such password are possible? The technique required to answer this and a wide variety of other counting problems will be introduced in this chapter.

Counting does not end with arithmetic. It has wide applications in probability, statistics, coding theory and analysis of algorithms.

Before we introduce the concept further we learn the meaning and notation of factorial.

6.2   Factorial

The product of first n natural numbers is called 'n factorial' and is denoted by n!

i.e., the product

$1 \times 2 \times 3 \times .... \times (n-1) \times n$ is denoted by n! and we read this symbol as 'n factorial'.

In particular

$$1 = 1!$$
$$1 \times 2 = 2!$$
$$1 \times 2 \times 3 = 3!$$
$$1 \times 2 \times 3 \times 4 = 4!$$
$$1 \times 2 \times 3 \times 4 \times 5 = 5! \qquad \text{and so on}$$

Thus $1 \times 2 \times 3 \times ... \times (n-1) \times n = n!$

Further we define 0! = 1

Note: The factorial of a negative integer or a fraction is not defined.

We can write $7! = 7 \times 6! = 7 \times 6 \times 5! = 7 \times 6 \times 5 \times 4!$ and so on

Thus clearly, for a natural number n

$n! = n\,(n-1)!$

$= n\,(n-1)\,(n-2)!$ (provided n > 2)

$= n\,(n-1)\,(n-2)\,(n-3)!$ (provided n > 3)

and so on

Example 1: Evaluate (i) 5!   (ii) 7!   (iii) 7! – 5!

Solution:

(i) $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

(iii) $7! - 5! = 7 \times 6 \times 5! - 5! = (42-1)\,5!$

$= 41 \times 5! = 4920$

Example 2: Compute (i) $\dfrac{11!}{9!}$ (ii) $\dfrac{13!}{(2!)(11!)}$

Solution:

(i) $\dfrac{11!}{9!} = \dfrac{11 \times 10 \times 9!}{9!} = 110$ (ii) $\dfrac{13!}{(2!)(11!)} = \dfrac{13 \times 12 \times (11!)}{2 \times (11!)} = 78$

Example 3: Find the LCM of 5!, 6! and 7!

Solution: LCM of 5!, 6! and 7!

$\qquad$ = LCM of 5!, $6 \times 5!$ and $7 \times 6 \times 5!$

$\qquad$ = $7 \times 6 \times 5! = 7!$

Example 4: Express the following in factorial notation.

(i) $\qquad 6 \times 7 \times 8 \times 9$

(ii) $\qquad 4 \times 5 \times 6^2 \times 7 \times 8$

Solution:

(i) $\qquad 6 \times 7 \times 8 \times 9 = \dfrac{1 \times 2 \times 3 \times 4 \times 5 \times (6 \times 7 \times 8 \times 9)}{1 \times 2 \times 3 \times 4 \times 5} = \dfrac{9!}{5!}$

(ii) $\qquad 4 \times 5 \times 6^2 \times 7 \times 8 = 6 \ X \ (4 \times 5 \times 6 \times 7 \times 8)$

$\qquad = 1 \times 2 \times 3 \times \ (4 \times 5 \times 6 \times 7 \times 8)$

$\qquad = 8!$

Example 5: If $\dfrac{1}{9!} + \dfrac{1}{10!} = \dfrac{x}{11!}$, find x

Solution: We have, $\dfrac{1}{9!} + \dfrac{1}{10 + 9!} = \dfrac{x}{11 \times 10 \times 9!}$

$$\Rightarrow \quad 1+\frac{1}{10}=\frac{x}{110}$$

$$\Rightarrow \quad \frac{11}{10}=\frac{x}{110} \text{ i.e., } x = 121$$

Example 6: Convert the following products into factorial notation

(i)   2.4.6.8... (2n)

(ii)   1.3.5... (2n–1)

Solution:

(i)   2.4.6.8 (2n)

$$= (2.1) \times (2.2) \times (2.3) \times (2.4)x... \times (2.n)$$

$$= 2^n (1.2.3.4...n)$$

$$= 2^n (n!)$$

(ii)   1.3.5... (2n –1)

$$= \frac{1.2.3.4.5...(2n-1)(2n)}{2.4.6...(2n)}$$

$$= \frac{(2n)!}{2^n [1.2.3...n]} = \frac{(2n)!}{2^n (n!)}$$

Example 7: Show that, 41! + 1 is not divisible by any number from 2 to 41.

Solution: Clearly 41! is divisible by every integer from 2 to 41. Thus 41! + 1 leaves 1 as remainder when divided by any integer from 2 to 41. Hence 41! + 1 is not divisible by any integer from 2 to 41.

Exercise 1.1

1. Evaluate

    (i)   6!

    (ii) $\dfrac{20!}{18!}$

    (iii) $\dfrac{9!-8!}{7!}$

2. Is 4! + 5! = 9!

3. Compute $\dfrac{n!}{(n-r)!}$ when

    (i)   n = 8, r = 2

    (ii)   n = 12, r = 3

4. If $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!}$, find x

5. Evaluate $\dfrac{m!}{r!(n-r)!}$, when

    (i)   n = 13, r = 2

    (ii)   n = 8, r = 5

6. Show that (n+2) n! = n! + (n+1)!

7. Find n if

    (i)   (n+1)! = 20 (n–1)!

    (ii)   (n+2)! = 12 (n!)

8. Show that, n (n–1) (n–2) ... (n–r+1) = $\dfrac{n!}{(n-r)!}$

9. If $\dfrac{n!}{2!(n-2)!}\div\dfrac{n!}{4!(n-4)!}=2$, find the value of n.

Answers:

1. (i) 720   (ii) 380   (iii) 64

2. No

3.   (i) 56   (ii) 1320

4.   64

5.   (i) 78   (ii) 56

7.   (i) 4   (ii) 2

9.   5

6.3   Fundamental Principles of Counting

The study of permutation and combination begins with two basic principles of counting the rules of sum and product. We will begin with simple applications of these rules. While solving difficult problems, we decompose them into simple parts solve them and finally obtain the final answer.

Our first principle of counting can be stated as follows:

The Rule of Sum (**Fundamental Principle of Addition**) A job occurs if either event E occurs or event F occurs exclusively.

If event E can occur in m ways and event F can occur in n ways, then the job can occur in (m + n) ways.

This rule can be further extended

Example 8: A school library has 13 and 7 books available on financial mathematics written by Indian and foreign authors respectively. A student wants to learn more about the subject and has only 1 ticket to borrow the book. In how many ways the book can be borrowed.

Solution: By rule of sum (i.e., the fundamental principle of addition) a student can select the book in = 13+7 = 20 ways.

Example 9: A man is advised to go in for a knee replacement surgery. This facility is available in 5 private or 3 government hospitals. Alternatively he can go abroad where his son who is an orthopedic surgeon can operate. Find the number of options available to him.

Solution: By rule of sum (i.e., the fundamental principle of addition) there are = 5 + 3 + 1, i.e., 9 options available to the person.

The solution of the following problem help us introduce second principle of counting.

Anirudh goes to a stadium to watch a cricket match. If there are 3 independent gates to enter the stadium and 2 independent gates to exit. List the different ways Anirudh can enter and exit the stadium if there is no restriction on the gate, one can use.

Let us name the three entrance gates as E_1, E_2, E_3 and the two exit gates as X_1, and X_2.

How many different pairs of entering and exciting the stadium are possible? There are 3 ways in which Anirudh can enter the stadium. For every choice of entering gate there are two choices of exiting. Therefore there are $3 \times 2 = 6$ different ways in which one can enter and exit the stadium.

Then the various possibilities of entering and exiting the stadium are illustrated in the following figure.
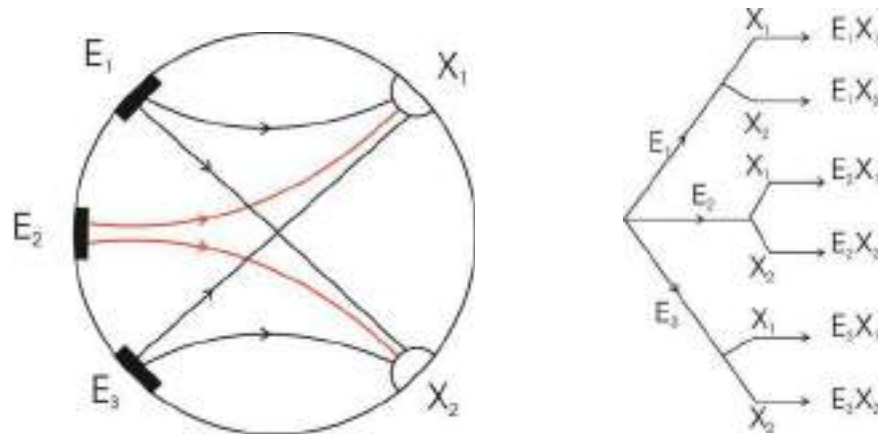

There are six possibilities as illustrated in the above figure.

Let us consider the following example to show that the rule can be extended to more than two tasks or events.

Reema goes to the market to buy school bag, lunch box and water bottle for her younger sister for the new academic session. The school bag can be chosen in 3 different ways. After the school bag is chosen, a lunch box can be chosen in 2 different ways. Hence, there are $3 \times 2 = 6$ pairs of school bag and lunch box to choose from. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are $6 \times 2 = 12$ different ways in which, Reema can buy these items for her sister.

If we name the 3 school bags as B_1, B_2 and B_3, the two lunch boxes as L_1, L_2 and the two water bottles as W_1, W_2, then the various possibilities can be shown with the arrow diagram shown below.
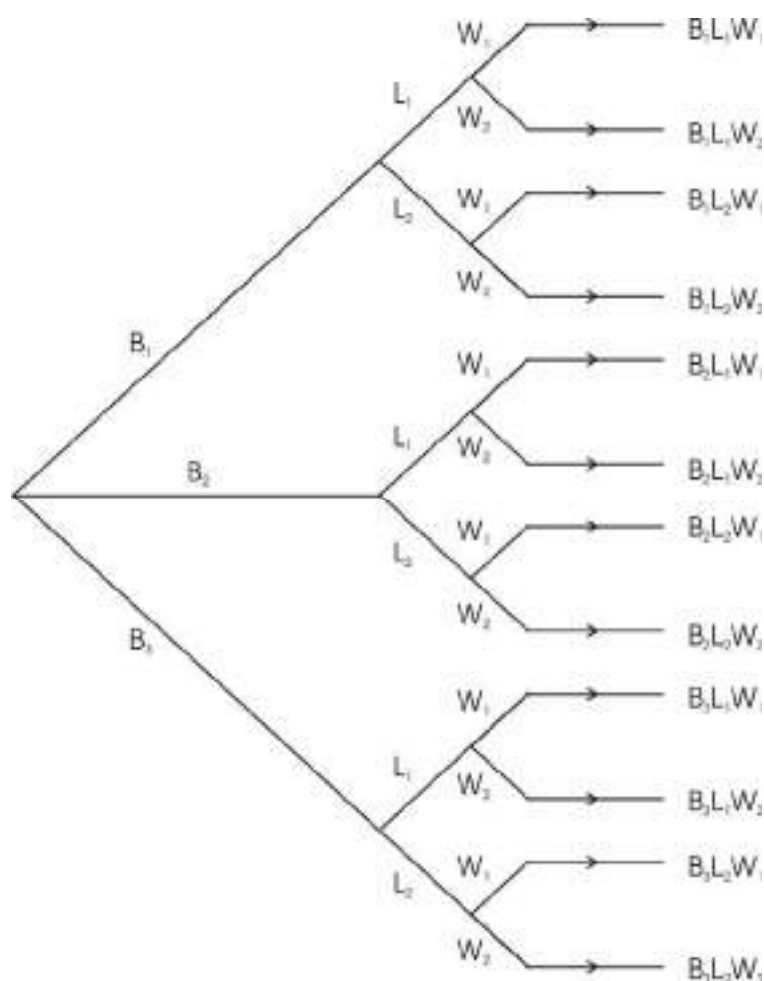
Total = 12 possibilities

In fact the problem of the above type can be solved by applying the following principle known as fundamental principle of multiplication or the Rule of Product.

The Rule of Product (Fundamental Principle of Multiplication)

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of ways in which the two events can occur in the given order is m × n."

The above principle can be generalized for any finite number of events. For example, for 3 events, the principle is as follows:

"If an event can occur m in the different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, them the total number of ways of occurrence of the events in the given order is m x n x p.

In the first problem the number of ways of visiting the stadium consists of the occurrence of the following events in succession.

(i)   the event of choosing the entrance gate i.e., 3

(ii)   the event of choosing the exit gate i.e., 2

There are various possible ways in which one can enter and exit the stadium. We have to choose any one of the possible ways which can be done by counting the different ways of occurrence of the events in succession.

We observe the number of possible ways the product of number of ways of occurrence of the successive events.

So the number of possible ways = 3 × 2 i.e. = 6 ways

By similar argument, we can conclude that the number of possible ways of choosing a school bag, lunch box and water bottle is 3 × 2 × 2 = 12 ways.

Example 10: In a class there are 20 boys and 15 girls. The teacher wants to select either a boy or a girl to represent the class in a competition. In how many ways can this be done.

Solution: By fundamental principle of addition the teacher can select a boy or a girl in = 20 + 15 i.e. = 35 ways.

Example 11: The dramatics club of a college has selected six boys and five girls for a play. From this group the director can cast his leading couple (a boy and a girl) in how many ways?

Solution: The director can cast a boy for leading role in = 6 ways

The director can cast a girl for leading role in = 5 ways

By fundamental principle of multiplication the director can cast the couple in $6 \times 5 = 30$ ways.

Example 12: The chairs in an auditorium are to be labelled with a letter and a positive integer not exceeding 10. What is the largest number of chairs that can be put in the auditorium?

Solution: The letter can be assigned in 26 ways.

The number can be assigned in 10 ways.

The different ways the chair can be labelled is = 26×10 i.e., 260 ways

Hence the capacity of the auditorium if all possible labelled chairs are there is 260.

Example 13: How many different license plates are possible, if each plate contains a sequence of three letters starting with D followed by four digit - non zero number.

Solution: Since the number plate starts with D, so there is only one choice for the first letter. There are 26 choices for each of the remaining 2 letters and 10 choices for each of the four digits (i.e., 0 to 9)

By fundamental principle of multiplication the total possibilities are

$$= 1 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6760000$$

The sequence of letters followed by four zeros

$$= 1 \times 26 \times 26 \times 1 \times 1 \times 1 \times 1$$

$$= 676$$

The license plates with 4 digit non zero numbers

= 6760000 – 676

= 6759324

Example 14: An investment banker finalizes the list of specific entities worthy of investment in each of the three instruments mentioned below:

–   3 Public Ltd. companies for direct equity investment.

–   5 Mutual fund schemes

–   2 Banks for Fixed Deposit (F.D)

In how many ways the investment can be made if the board decides to.

(i)   invest the entire fund in one entity (i.e., company, scheme or bank)

(ii)   invest in one entity of each of the three instruments.

Solution:

(i)   If the entire fund is invested in one entity.

By fundamental principle of addition (the rule of sum) the investment can be made in 3 + 5 + 2 = 10 ways

(ii)   Investment in one entity of each instruments

By fundamental principle of multiplication or (the rule of product)

Public Ltd. company can be chosen in = 3 ways

Mutual Fund scheme can be selected in = 5 ways

Bank can be selected in = 2 ways

Total number of ways = $3 \times 5 \times 2 = 30$ ways

Example 15: How many three digit numbers can be formed using the digits 0, 1, 3, 5 and 8.

(i)   If repetition of digits is not allowed

(ii)   If repetition of digits is allowed.

Solution:

There be as many numbers as there are ways of filling 3 vacant places $\boxed{H}\boxed{T}\boxed{U}$ in succession by the digits. Since 0 cannot come in hundred's (run on) place so it can be filled in = 4 ways

The ten's place can be filled in (as 0 can come) = 4 ways

The one's place can be filled in = 3 ways

Therefore, by fundamental principle of multiplication the required number of 3 digit numbers = $4 \times 4 \times 3 = 48$

(i)   If repetition of digits is allowed $\boxed{H}\boxed{T}\boxed{U}$

The hundred's place can be filled in = 4 ways

The ten's place can be filled in = 5 ways

The unit's place can be filled in = 5 ways

Therefore by fundamental principle of multiplication the required number of 3 digit numbers = $4 \times 5 \times 5 = 100$

Example 16: Find the number of different signals that can be generated by arranging at least 4 flags in order (one below the other) on a vertical staff, if 5 different flags are available.

Solution: A signal can consist of either 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 4 flag signals as there are ways of filling in 4 vacant place in succession by the 5 flags available.

By fundamental principle of multiplication, the number of ways is $5 \times 4 \times 3 \times 2 = 120$

Counting the same way the number of 5 flag signals is $5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required number of signals = 120 + 120 = 240

Exercise 1.2

1. Find the number of 4 letter words, with or without meaning, which can be formed using the letters of the word HONEST, when the repetition of the letters is not allowed.

2. How many 3 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if

    (i)   no letter is repeated

    (ii)   repetition of letters is allowed

4. A tennis club consists of 8 boys and 11 girls. In how many ways can a mixed doubles team be chosen?

5. There are 5 vacant seats in a row. In how many ways can 3 men sit.

6. Find the total number of ways of answering 6 multiple choice questions, if each question has 4 choices.

7. Find the number of three digit even positive integers.

8. Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if 5 different flags are available.

9. A coin is tossed 4 times and the outcomes are recorded. How many different outcomes are possible?

10. There are 5 true-false questions in a test. If no two students have answered the same sequence of answers and no student has given all correct answers. How many students are there is the class for this to happen?

11. If each user on a computer system has a password which is eight characters long where each character is an upper case letter or a digit.

Each password must contain at least one digit. How many password are possible.

12. In a class test a teacher decides to give 5 questions one each from first five exercises of the textbook. If the first five exercises have 7, 12, 6, 10 and 3 questions respectively. Find the number of ways in which the question paper can be set.

13. How many numbers are there between 100 and 1000 such that 7 is in the units place.

14. How many numbers having 5 digits can be formed with the digits 0, 2, 3, 4 and 5 if repetition of digits is not allowed. How many of these are divisible by 5?

15. There are 21 towns in district connected by railways. Find the number of tickets required by the railways so that a passenger can travel from one town to another.

Answers:

1. 720

2. 50

3. (i) 5040

   (ii) 10000

4. 88

5. 60

6. 4096

7. 450

8. 320

9. 16

10. 31

11. $(36)^8 - (26)^8$

12. 15120

13.   90

14.   96, 42

15.   420

6.4   Permutations

Definition 1: A permutation is an arrangement of objects in a definite order taken some or all at a time.

The word permutation means arrangements. Each of the different arrangements that can be made by taking some or all of a given set of objects at a time is known as their permutation.

Let us understand this with the help of an example.

The postal department has to inscribe different codes on the consignment being shipped using digits 5, 7 and alphabet A. How many different codes can be generated using one or more character out of 5, 7 and A.

(i)   Codes using one character   :   5, 7, A

(ii)   Codes using two character   :   57, 75, 5A, A5, 7A, A7

(iii)   Codes using three character   :   57A, 5A7, 75A, 7A5, A57, A75

Now each of the above three gives us number of permutations of 3 characters (2 digits and 1 alphabet) taken one, two or all three at a time.

If we have to select two characters from 5, 7 and A. It can be done by writing 57, 5A, 7A. Thus the number of ways of selecting two characters is 3. Now each of these selections made above give rise to two arrangements i.e., 57 can be arranged in two ways as 57 and 75; similarly 5A can be arranged as 5A and A5; 7A as 7A and A7. Thus the total number of arrangement of 3 characters taken two at a time is 6, there are:

        57, 75, 5A, A5, 7A, A7

Thus we have two types of classification (i) selection (ii) arrangement. Selections of objects are called combinations and arrangements of objects are called permutations.

Note: Permutation of objects taken some or all at a time, not only includes every combination of those objects but all other arrangements which each of these combination give rise to.

The number of permutations of n different things taken r at a time where repetition is not allowed is denoted by ^nP_r or P(n r) where n and r are positive integers and r ≤ n.

Permutations when all the objects are distinct.

Theorem 1: The number of permutations of n different objects taken r at a time, where o < r ≤ n and objects do not repeat is

$$^nP_r = n\ (n-1)\ (n-2).....\ (n-r+1) = \frac{n!}{(n-r)!}$$

Proof: There will be as many permutations as there are ways of filling r vacant places by n distinct objects.
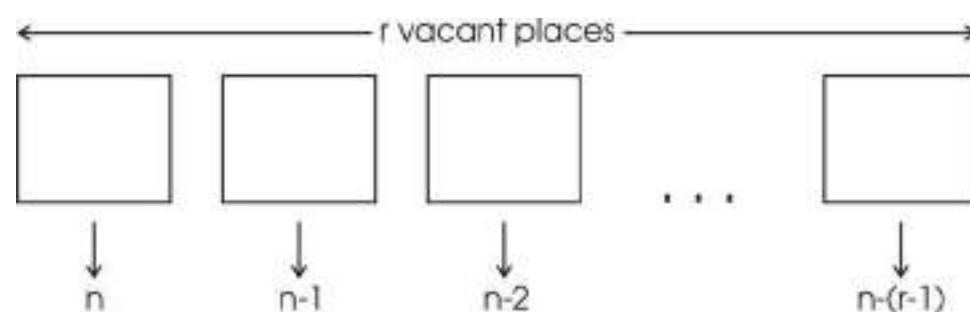
The first place can be filled in n ways, following which the second place can be filled in (n–1) ways and then the third place can be filled in (n–2) ways.

Continuing so on the rth place can be filled in (n–(r–1)) ways.

Therefore, the number of ways of filling r vacant places in succession is

n (n–1) (n–2) ..... (n–r+1)

= $n!$, if r = n

$$\frac{n\,(n-1)\,(n-2)\,...\,(n-r+1)\,(n-r)\,(n-r-1)...\,3.\,2.\,1}{(n-r)\,(n-r-1)\,...\,3.2.1}, \text{if } o < r < n$$

This expression for ^nP_r on simplifying gives

$$\Rightarrow\ ^nP_r = \frac{n!}{(n-r)!}, \text{ where } 0 < r < n$$

Note : As 0! = 1

$$^nP_r = \frac{n!}{(n-r)!} \text{ stands true even for r=n}$$

Counting permutations is merely counting the number of ways in which some or all objects at a time are arranged. Arranging no objects is same as leaving all objects behind and we knows this can be done in only one way. Hence, $^nP_o = 1$ makes sense. So $^nP_r = \frac{n!}{(n-r)!}$ stands true even for r = 0 Hence, $^nP_r = \frac{n!}{(n-r)!}, o \le r \le n.$

Theorem 2: The number of permutations of n different objects taken r at a time, when repetition is allowed, is n^r.

Proof: Since each of the r vacant places can be filled in n ways. Therefore the number of ways are

$$n \times n \times n \times .... \times n = n^r$$

Example 17: Find the value of n such that

(i) $^nP_4 = 20\ ^nP_2, n > 3$

(ii) $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3}, n > 4$

Solution:

(i) Given that $^nP_4 = 20 \, ^nP_2$

$\Rightarrow \quad \dfrac{n!}{(n-4)!} = 20 \dfrac{n!}{(n-2)!}$

$\Rightarrow \quad \dfrac{n!}{(n-4)!} = 20 \dfrac{n!}{(n-2)(n-3)(n-4)!}$

$\Rightarrow \quad (n-2)(n-3) = 5 \times 4 \qquad (\therefore n > 3)$

$\Rightarrow \quad n-2 = 5 \text{ or } n-3 = 4$

$\Rightarrow \quad n = 7$

(ii) Given that $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3}$

$\Rightarrow \quad 3 \, ^nP_4 = 5 \, ^{n-1}P_4$

$\Rightarrow \quad 3 \dfrac{n!}{(n-4)!} = 5 \dfrac{(n-1)!}{(n-5)!}$

$\Rightarrow \quad \dfrac{3n(n-1)!}{(n-4)(n-5)!} = \dfrac{5(n-1)!}{(n-5)!}$

$\Rightarrow \quad 3n = 5(n-4) \qquad \text{(Cancelling common factors on both sides)}$

$\Rightarrow \quad 3n = 5n - 20$

$\Rightarrow \quad 2n = 20 \qquad \Rightarrow n = 10$

Example 18: Find r if

(i) $^5P_r = 2 \, ^6P_{r-1}$

(ii) $5 \, ^4P_r = 6 \, ^5P_{r-1}$

Solution:

(i)   We have $^5P_r = 2\,{}^6P_{r-1}$

$\Rightarrow \qquad \dfrac{5!}{(5-r)!} = 2 \times \dfrac{6!}{(6-r+1)!}$

$\Rightarrow \qquad \dfrac{5!}{(5-r)!} = 2 \times \dfrac{6.5!}{(7-r)(6-r)(5-r)!}$

$\Rightarrow \qquad (7-r)\,(6-r) = 12$

$\Rightarrow \qquad r^2 - 10r - 3r + 30 = 0$

$\Rightarrow \qquad (r-10)\,(r-3) = 0$

$\Rightarrow \qquad r = 10 \text{ or } 3$

$\Rightarrow \qquad r = 3 \quad (\because r \le 5)$

(ii) $5\,{}^4P_r = 6\,{}^5P_{r-1}$

$\Rightarrow \qquad 5 \times \dfrac{4!}{(4-r)!} = 6 \times \dfrac{5!}{(5-r+1)!}$

$\Rightarrow \qquad \dfrac{5!}{(4-r)!} = \dfrac{6 \times 5!}{(6-r)(5-r)(4-r)!}$

$\Rightarrow \qquad (6-r)\,(5-r) = 6$

$\Rightarrow \qquad r^2 - 11r - 24 = 0$

$\Rightarrow \qquad (r-8)\,(r-3) = 0$

$\Rightarrow \qquad r = 8 \text{ or } 3$

$\Rightarrow \qquad r = 3 \qquad (\because r \le 4)$

Example 19: How many 4-digit numbers can be formed using the digits 1 to 9 if no digit is repeated.

Solution: All possible 4-digit numbers are arrangements of 9 digits taken 4 at a time i.e., $^9P_4$

$$^9P_4 = \dfrac{9!}{(9-4)!} = \dfrac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024$$

Example 20: How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

Solution: Number of arrangement of 8 distinct letters taken all at a time is

$$= {}^{8}P_8 = 8! = 40{,}320$$

Example 21: How many 4-digit numbers with each digit even can be formed if:

(i)   repetition of digits is not allowed

(ii)   repetition of digits is allowed

Solution:

(i)   Four digit even digit numbers if repetition is not allowed: $\boxed{Th}\boxed{H}\boxed{T}\boxed{U}$

Even digits are 0, 2, 4, 6 and 8.

The thousand's place can be filled in 4 ways ($\because$ 0 can not come)

The remaining 3 places (i.e. hundred's, ten's & unit)

Can be filled in ${}^{4}P_3$ ways   ($\because$ 0 can come)

All possible 4 digit numbers = $4 \times {}^{4}P_3$

$$= 4 \left[\frac{4!}{(4-3)!} \right] = 96$$

(ii)   Four digit numbers with each digit even (with repetition):

The thousand place can be filled in 4 ways (except 0)

Then each of the remaining 3 digits can be filed in 5 ways

So all possible 4 digit numbers with all digits even and with repetition

= $4 \times 5 \times 5 \times 5 = 500$

Example 22: In how many different ways can the letters of the word SUNDAY be arranged? How many of these begin with S? How many of these arrangements begin with S but does not end with Y.

Solution: The word SUNDAY contains 6 distinct letters. So the number of different ways are arrangement of the 6 letters taken all at a time = $^6P_6$ = 6! = 720

The arrangements that begin with S can be obtained by keeping S fixed in the first position and then arranging the remaining 5 letters. This can be done in $^5P_5$ = $\lfloor 5$ = 120 ways.

Also the arrangements that begin with S and end with Y can be obtained by keeping S and Y fixed in the first and last position respectively and arranging the remaining 4 letters in $^4P_4$ = 4! = 24 ways.

Number of arrangements that begin with S but does not end with Y

= (Number of arrangements that begin with S) − (Number of arrangements that begin with S and end with Y)

$\qquad = {}^5P_5 - {}^4P_4 = \lfloor 5 - \lfloor 4 = 96$

Example 23: In how many ways can a group of 8 friends having different heights stand in a row for a group photograph if

(i)   they stand in ascending order of their heights (from left to right)

(ii)   the tallest and the shortest should not stand together.

Solution:

(i)   If the students stand in ascending order of their heights from left to right then each of them can stand in only one position according to their height.

        So number of ways = $1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$ = 1 way

(ii)   Number of ways in which the tallest and the shortest friend would stand together can be obtained by considering those two persons as a single unit. Then we need to arrange 7 in all which can be done in $^7P_7$ = 7! ways.

        Now the tallest and the shortest which come together can be arranged among themselves in 2! ways.

        So the total number of arrangements when the tallest and the shortest are together is 7! × 2!

Number of arrangements when the tallest and the shortest are not together

= (All possible arrangements of 8 friends) – (Arrangements when tallest and shortest are together)

= 8! – 7! × 2!

= 8.7! – 7! × 2

= 7! (8–2)

= 6 × 7!

Example 24: Find the number of arrangements of the letters of the word FRAGILE. In how many of these arrangements

(i)   do all the vowels occur together.

(ii)   do all the vowels occur together and all the consonants occur together.

(iii) do all the vowels never occur together.

(iv)   do the vowels occupy only even places.

(v)   do the vowels occupy only odd places.

Solution: Number of arrangements of the letters of the word FRAGILE = $^7P_7$ = 7!

(i)   There are 7 letters in the word FRAGILE, in which there are 3 vowels, namely A, I and E. Since the vowels have to occur together, we can assume them as single object (AIE). This single object along with 4 remaining letters (objects) will be counted as 5 objects.

Permutation of these 5 objects taken all at a time = $^5P_5$ = 5!

But A, I, E can be arranged among themselves in = $^3P_3$ = 3!

Hence by multiplication principle number of permutation

= 5! × 3! = 120 × 6 = 720

(ii)   One of the arrangement of all vowels coming together and all consonants coming together is (AIE) (FRGL)

Considering (AIE) to be one group and (FRGL) to be another group the two objects can be arranged in $^2P_2 = 2!$ ways

The vowels A, I, E can be arranged among themselves in $= {}^3P_3 = 3!$ ways

The consonants F, R, G, L can be arranged among themselves in $= {}^4P_4 = 4!$ ways

So by multiplication rule the total number of arrangements are $= 2! \times 3! \times 4!$

$= 2 \times 6 \times 24 = 288$ ways

(iii)   All vowels never occur together = (All the arrangements of the word FRAGILE) – (the arrangements with all vowels together)

$= 7! - 5! \times 3!$

$= 7 \times 6 \times 5! - 5! \times 6$

$= 5! (42 - 6)$

$= 120 \times 36 = 4320$

(iv)   The word FRAGILE consists of 7 letters of which 3 are vowels and 4 are consonants.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |

The vowels can occupy even places if they occur at second, fourth and sixth positions.

The three vowels can occupy the three positions in $^3P_3 = 3!$ ways

The remaining four positions can be filled up by 4 consonants in $^4P_4 = 4!$ ways

So the required number of ways $= 3! \times 4! = 144$

(v)   The three vowels can occupy the 4 odd places in $^4P_3$ ways. The remaining 4 places can be filled up by 4 consonants in $^4P_4$ ways.

So the required arrangements = $^4P_3 \times {}^4P_4 = \dfrac{4!}{(4-3)!} \times 4!$

$= 4! \times 4! = 24 \times 24 = 576$

Exercise 1.3

1.   Find n if $^{n-1}P_3 : {}^nP_4 = 1 : 9$

2.   Find r if

(i) $^9P_r = 3024$

(ii) $^5P_r = 2\,{}^6P_{r-1}$

3.   Prove the following:

(i) $^nP_n = 2 \cdot {}^nP_{n-2}$

(ii) $^{n-1}P_r + r \cdot {}^{n-1}P_{r-1} = {}^nP_r$

4.   How many 3-digit numbers are there with no digit repeated?

5.   How many 4-digit even numbers can be formed using the digits 1, 2, 3, 5, 7 and 8 if repetition of digits is not allowed?

6.   How many numbers between 6000 and 7000 formed with the digits 0, 1, 5, 6, 7 and 9 are divisible by 5 if

(i)   repetition of digits is allowed

(ii)   repetition of digits is not allowed

7.   A family of 6 brothers and 4 sisters is to be arranged for a photograph in one row. In how many ways can they be seated so that

(i)   all the sisters sit together

(ii)   no two sisters sit together

8. How many words, with or without meaning can be made from the letters of the word TUESDAY, assuming that no letter is repeated, if

    (i) all letters are used at a time

    (ii) 5 letters are used at a time

    (iii) all letters are used but first and last letter is a vowel

9. In how many ways can the letters of the word PERMUTATIONS be arranged if the

    (i) words start with P and end with S

    (ii) there are 5 letters between P and S

    (iii) vowels are all together

10. How many words with or without meaning can be formed using all the letters of the word LAUGHTER if

    (i) the words start with L but does not end with R

    (ii) no two vowels come together

    (iii) the relative positions of vowels and consonants remains unchanged

11. In how many ways can 5 Mathematics, 4 English and 3 Accountancy books can be arranged in a shelf if

    (i) all books on the same subject are together

    (ii) No two books on the same subject are together

12. Find the rank of the word LATE, if the letters of the word LATE are permuted and words so formed are arranged as in dictionary.

13. Find the numbers of words with or without meaning which can be made using all the letters of the word AGAIN. If all these words are arranged as in dictionary what will be the 49th word, 50th word?

14. Determine the number of paths in the xy–plane from (1, 2) to (7, 5), where each such path is made up of individual steps going one unit to the right (R) or one unit upwards (U)
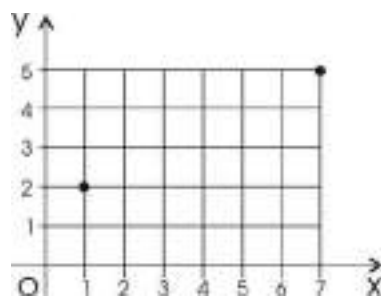
15. How many positive integers greater than 5,000,000 can be formed using the digits 2, 3, 3, 5, 5, 6, 8?

16. The board of directors of a pharmaceutical company has 10 members. An upcoming stockholder's meeting is scheduled to approve a new president, vice president, secretary and a treasurer. How many different ways the four can be appointed.

17. In how many ways can 9 people be arranged around a circular table if two people insist on sitting next to each other.

18. If the letters of the word ADINI are arranged as in dictionary, then what is the 46th word?

19. If the letters of the word SCHOOL are arranged as in dictionary, then find the rank of the word SCHOOL.

Answers:

1. n = 9

2. (i) r = 4   (ii) r = 3

4. 648

5. 120

6.   (i) 71   (ii) 24

7.   (i) $\underline{|7} \times \underline{|4} = 120960$

    (ii) $\underline{|6} \times {}^7P_4 = 604800$

8.   (i) 5040   (ii) 2520   (iii) 720

9.   (i) 1814400   (ii) 21,772,800   (iii) 2419200

10.   (i) 4320   (ii) $5! \times {}^6P_3 = 14400$ (iii) $3! \times 5! = 6 \times 120 = 720$

11.   (i) 103680   (ii) ${}^5P_5 \times {}^6P_4 \times {}^{10}P_3 = 31104000$

12.   14

13.   NAAGI & NAAIG

14.   84

15.   720

16.   5040

17. $7! \times 2 = 10,080$

18.   INDIA

19.   303

We can state without proof the following theorems:

Theorem 1: The number of permutations of n objects, where p objects are of the same kind and rest, if any, are all different = $\dfrac{n!}{p!}$

We have a more general theorem.

Theorem 2: The number of permutations of n objects, where p_1 objects are of one kind, p_2 objects are of second kind, ..., p_k objects are of kth kind and rest if any, are all distinct is $\dfrac{n!}{p_1! p_2! ... p_k!}$.

Example 25: Find the number of permutations of the letters of the following words.

(i)   COMMERCE

(ii)   MATHEMATICS

Solution:

(i)   The word COMMERCE has 8 letters, in which C appears twice, M appears twice, E appears twice and the remaining letters only once.

So total number of arrangements = $\dfrac{8!}{2!\,2!\,2!}$ = 5040

(ii)   The word MATHEMATICS has 11 letters, in which A, M and T occurs twice and remaining letters are distinct.

So total number of arrangements = $\dfrac{11!}{2!\,2!\,2!}$ = 4989600

Example 26: Find the number of permutations of the letters of the word ENGINEERING. In how many of these arrangements:

(i)   do the words begin with E and end with G

(ii)   do all the vowels come together

(iii)   do all the vowels never come together

(iv)   no two vowels come together

Solution: The word ENGINEERING has 11 letters. Writing all identical letters together we get ÉEEIINNNGGR'.

It contains 3E's, 2I's, 3N's, 2G's and 1R

So the number of arrangements = $\dfrac{11!}{3!\,2!\,3!\,2!}$ = 277200

(i)   If we fix the first and last letter as E and G the remaining 9 letters contains 2E's, 3N's, 2I's, 1G and 1R.

So the number of arrangements = $\dfrac{9!}{3!\,2!\,2!}$ = 15,120

(ii) Let us consider all vowels as one letter (i.e. object). So in all we have 7 letters in which we have 3N's, 2G's, 1R and combination of vowels EEEII as 1 object.

The number of ways these 7 letters can be arranged among themselves in $= \dfrac{7!}{3!\,2!}$ ways and for each of these arrangements the 5 vowels can be arranged among themselves $= \dfrac{5!}{2!\,3!}$ ways.

So all vowels can come together in $= \dfrac{7!}{3!\,2!} \times \dfrac{5!}{2!\,3!} = 4200$

(iii) All vowels not coming together = All arrangements – Arrangements when all vowels come together

$= 277200 - 4200$

(iv) The word ENGINEERING consists of 5 vowels EEEII and 6 consonants NNNGGR.

Let us arrange the 6 consonants first. Since it contains 3N's, 2G's and IR, therefore the number of arrangements are $\dfrac{6!}{3!\,2!} = 60$ ways.

$* \ C \ * \ C \ * \ C \ * \ C \ * \ C \ * \ C \ *$

Now once the consonants 'C' are arranged the vowels EEEII can be placed in any of the 5 positions out of 7 shown by * symbol in $^{7}C_{5}$ ways. Further the vowels can be arranged among themselves in $\dfrac{5!}{3!\,2!}$ ways.

So the number of ways in which vowels can be arranged is

$^{7}C_{5} \times \dfrac{5!}{3!\,2!} = 21 \times 10 = 210$ ways

By multiplication rule the arrangements of the word ENGINEERING in which all vowels come together is

$60 \times 210 = 12600$

Example 27: In how many ways can the letters of the word ASSASSINATION be arranged? In how many of these arrangements the four S's do not come together.

Solution: The word ASSASSINATION has 13 letters. It contains 3A's, 4S's, 2I's, 2N's, 1T and 1O.

So the number of arrangements = $\dfrac{(13)!}{3!\,4!\,2!\,2!}$

$$= \dfrac{(13)!\,12\times11\times10\times9\times8\times7\times6\times5\times4!}{3!\,4!\,2!\,2!}$$

$$= 10810800$$

To find the number of arrangement if 4S's come together, we consider the four S's to be one letter. Then we shall have 10 letters in all with 3A's, 2I's, 2N's, 1T, 1 O and 1 combined S.

So number of arrangements when 4S's come together is $\dfrac{10!}{3!\,2!\,2!}$ = 151200

Hence number of arrangements when four S's do not come together is

$$= 10810800 - 151200 = 10659600$$

Example 28: A boy is to walk from P to Q. However, he can take a right step or an upward step, but not necessarily in the order shown in given figure. Find the number of possible paths he can take.

Solution: In all paths from P to Q the boy has to take a total of 11 steps five upward and six right steps. So number of paths is arrangement of 11 steps $\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \uparrow \uparrow \uparrow \uparrow \uparrow$
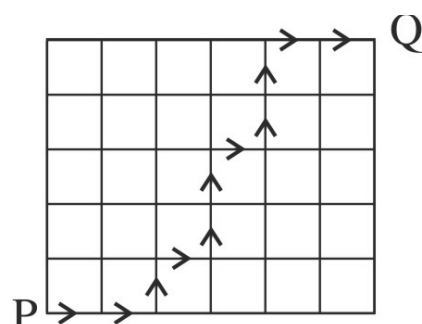
Which can be done in $\dfrac{11!}{5!\,6!} = \dfrac{11\times10\times9\times8\times7}{5\times4\times3\times2\times1} = 462$

Example 29: The MASSASAUGA is a brown and white venomous snake found in North America. How many arrangements can be made from this word. Find the number of arrangements if all vowels are together.

Solution: The word 'MASSASAUGA' has 10 letters. It contains 4A's, 3S's and one each of M, U and G.

So the number of arrangements are $\dfrac{10!}{4!\,3!} = 25200$

To find number of arrangements when all vowels are together we consider 4A's and IU i.e., AAAAU as one letter.

So now there are 6 letters having 3S's, 1M, 1G, and 1 combined vowels, which can be arranged in $\dfrac{6!}{3!}$

But the vowels AAAAU can be arranged among themselves in $\dfrac{5!}{4!}$ ways.

Hence number of arrangements when all vowels come together is $\dfrac{6!}{3!} \times \dfrac{5!}{4!} = 600$

Example 30: How many arrangements of the letters of word SOCIOLOGICAL are there if A and G are adjacent.

Solution: Since A and G are adjacent, therefore we consider them as one unit
(letter). The word SOCIOLOGICAL has 11 letters in which 3O's, 2C's, 2I's, 2L's, 1S and 1 combined GA.

210

Hence number of arrangements are $\dfrac{11!}{3!\,2!\,2!\,2!}$

But GA can also be written as AG if they are adjacent.

Therefore number of arrangements of SOCIOLOGICAL if A and G are adjacent is

$2! \times \dfrac{11!}{3!\,2!\,2!\,2!} = 1663200$

6.5   Circular Permutation

Till now we have done linear permutation i.e., arrangement in a line. In case the objects are arranged in the form of a circle it is known as circular permutation. Since a circle does not have any extremity therefore the number of arrangements in a circle depends on the relative position of the objects.

Theorem 1: The number of permutations of n objects in a circle taken all at a time is (n–1)!

Arrangements of beads to form a garland is also a circular permutation, but a garland can be turned upside down. So the two orders of arrangements i.e. clockwise and anticlockwise in not distinguishable, further reducing the number of arrangements to half. So in arrangements of beads in a necklace or arrangements of flowers in a garland we have the following therem.

Theorem 2: If clockwise and anticlockwise order of arrangements are not distinguishable,

Then number of permutations of n distinct objects is $\dfrac{(n-1)!}{2}$

Example 31: In how many ways can 7 students be made to sit in a (i) line (ii) circle.

Solution:

(i)   The number of ways in which 7 students can be made to sit in a line = $^7P_7$ = 7! = 5040

(ii)   The number of ways in which 7 students can be made to sit in a circle = (7–1)! = 6! = 720

Example 32: In how many ways can 4 men and 4 women be seated at a round table if

(i)   no restriction is imposed

(ii)   two particular women must sit together

(iii)   all women must sit together

Solution:

(i)   Total number of persons = 4 + 4 = 8

        Number of ways in which 8 persons can be seated at a round table

        = (8 − 1)! = 7! = 5040

(ii)   Since the two particular women are to sit together we consider them as one. Now total number of arrangements of 7 persons at a round table is (7−1)! = 6!

        The two women who are sitting together can be arranged among themselves in 2! ways.

        So the total number of arrangements = 2! × 6! = 1440

(iii)   The number of arrangements in which all the women would sit together can be obtained by considering all 4 women to be single entity. Then there will be total of (4 + 1) i.e. 5 persons which can be selated in 4! ways. Now the 4 women who are sitting together can be arranged among themselves in 4! ways.

        So the total number of arrangements at a round table if the 4 women sit together is 4! × 4!

                = 24 × 24 = 576 ways.

Example 33: 17 persons are invited to a party. How many seating arrangements are possible at a round table if

(i)   the host can sit any where

(ii)   the bigger chair is fixed for the host

(iii)   there are two pairs of indistinguishable twins among the guests.

Solution:

(i)   Number of ways in which 18 persons (17 invitees and 1 host) can be seated at a round table without any restriction is (18–1)! = 17!

(ii)   The bigger chair is fixed for the host, the remaining 17 persons can be seated at a round table in 17! ways.

(iii)   18 persons can sit at a round table in 17! ways. If there are 2 pairs of indistinguishable twins, than the number of arrangements are $\dfrac{17!}{2!\,2!}$.

Example 34: Find the number of ways in which 8 different beads can be arranged to form a necklace.

Solution: Eight different beads can be arranged in a circle in (8–1)! = 7! ways. Since there is no distinction between clockwise and anticlockwise arrangements, so the required number of arrangements are $\dfrac{7!}{2}$.

Example 35: How many necklaces can be made using 20 beads, 8 being blue, 5 green, 5 yellow and 2 red.

Solution: We known 20 distinct beads can be arranged in 19! ways. But since 8 of them are blue, 5 green, 5 yellow and 2 red therefore the number of arrangements are $\dfrac{19!}{8!\,5!\,5!\,2!}$.

Now, since clockwise and anticlockwise arrangements are identical, therefore to total number of arrangements are $\dfrac{1}{2}\left(\dfrac{19!}{8!\,5!\,5!\,2!}\right)$.

Example 36: Find the number of words with or without meaning which can be made using all the letters of the word MOTHER. If these words are written as a dictionary find the rank of the word mother.

Solution: There are 6 letters in the word MOTHER, all of which are distinct. So the required number of words is $^6P_6$ = 6! = 720. The letters of the word MOTHER in alphabetical order are E, H, M, O, R and T.

To obtain the number of words starting with E, we fix the letter E at the extreme left position, we then rearrange the remaining 5 letters taken all at a time in $^5P_5$ = = 5! ways. The same number of words we will obtain if we start with H. So before the first letter starting with M, we have 2 × 5! = 240 words.

Now considering the first two letters of the words starting with M, we get ME, MH before MO.

So to obtain the number of words starting with ME we fix the first two letters ME and arrange the remaining 4 letter taken all at a time in $^4P_4$ = 4! ways. The same number of words we will obtain with MH. So before MO, the number of words starting with ME and MH are 2 × 4! = 48 words.

Now the sequence of first three letters before MOT will be MOE, MOH and MOR, each one of these when fixed will give $^3P_3$ = 3! words. Thus the number of words starting with MOE, MOH and MOR are 3 × 3! = 18. Next, the first 3 words starting with MOT are MOTEHR, MOTERH & MOTHER.

Thus rank of word MOTHER = 240 + 48 + 18 + 3 = 309

6.5   Permutations with Restrictions

A restricted permutation is one in which certain objects are always included or excluded. It also includes those permutations in which some objects have to be placed in designated place or certain objects come together.

Let us state some of these theorems.

Theorem 1: The number of permutations of n different things taken r at a time in which m particular things never occur is obtained by arranging r objects out of the remaining n – m objects in $^{n-m}P_r$ ways, where n – m ≥ r.

Example 37: Find the number of ways of selecting a president, secretary and a treasurer from a board of 8 members, if 2 members who were holding one of these posts earlier cannot be nominated again.

Solution: Since 2 members are ruled out, so number of ways of selecting a president, secretary and a treasurer from among 6 members is number of arrangements of 6 members taken 3 at a time i.e., $^6P_3 = \dfrac{6!}{(6-3)!} = 120$.

Theorem 2: The number of permutations of n different things taken r at a time in which m particular things always occur is $^rP_m \times {}^{n-m}P_{r-m}$ ways.

Example 38: Find how many 4 digit numbers with distinct digits can be formed using the digits 1, 2, 3, 4, 5, 7 and 9 if each number contains two even digits.

Solution: Since 2 and 4 are the only two even digits given, so all number should contain them. So these two even digits can be placed in a 4 digit number in $^4P_2$ ways.

Now once these even digits are placed, the remaining 2 digits can be arranged in $^5P_2$ ways. So the number of 4 digit numbers are:

$$^4P_2 \times {}^5P_2$$

$$= \dfrac{4!}{(4-2)!} \times \dfrac{5!}{(5-2)!} = \dfrac{4!}{2!} \times \dfrac{5!}{3!}$$

$$= 3 \times 4 \times 4 \times 5 = 240$$

6.6   Combination

A school library offers two books for an academic year from a list of 12 books. If you have to put a right mark √ against a pair of books you desire in a list mentioning all possible pairs of books. Can you think of the number of choices you have to choose from.

What difference the order will make if you borrow the same two books. Give it a thought! Clearly, in this case the order is not important. Such an arrangement of objects in which order makes no difference is called a combination.

Let us consider some more illustrations.

On the first day of a class each of the 25 students shook hands with each other. How many handshakes took place. X shaking hands with Y and Y with X are not two different handshakes. Here again order is not important. There will be as many handshakes as there are combinations of 25 different students taken 2 at a time.

Now, we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by ^nC_r.

Let us assume there are 4 students A, B, C, D of a school who actively participates in various quiz competitions. A team of two students is to be selected to represent the school in an inter school quiz contest. In how many ways can we do so?

In fact the various possibilities of selecting the 2 students of the team are AB, AC, AD, BC, BD and CD. Here AB and BA are the same combination as order does not alter the combination. This is the reason we have not included BA and likewise some other arrangements in the list.

Here each selection is a combination of 4 different objects taken 2 at a time. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., $^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be arranged in 2! ways.

Hence the number of permutations = $^4C_2 \times 2!$

On the other hand, the number of permutations of 4 different things taken 2 at a time = $^4P_2$

Therefore $^4P_2 = {}^4C_2 \times 2!$ $\Rightarrow$ $^4C_2 = \dfrac{^4P_2}{2!}$

or $^4C_2 = \dfrac{4!}{(4-2)!\,2!}$

Similarly, if we have 5 different objects and we have to make combinations taking 3 at a time, we can show

$$^{5}C_3 = \frac{5!}{(5-3)!\,3!}$$

These examples suggest the following relationship between permutation and combination.

Remark: $^{n}P_r = {}^{n}C_r\ r!$, where $0 < r \le n$

Corresponding to each combination in $^{n}C_r$, we have $r!$ permutations, because r objects in every combination can be arranged in $r!$ ways.

Therefore the total number of permutations of n different things taken r at a time is $^{n}C_r \times r!$. On the other hand we know it is $^{n}P_r$.

Thus, $^{n}P_r = {}^{n}C_r \times r!$, $0 < r \le n$

This result gives us an important theorem of combination.

Theorem: The number of combinations of n different objects taken r at a time is given by

$$^{n}C_r = \frac{n!}{r!(n-r)!}$$

Note 1: It can be easily seen that

$$^{n}C_0 = 1 = {}^{n}C_n$$

$$^{n}C_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1 \qquad (\because 0! = 1)$$

$$^{n}C_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1 \qquad (\because 0! = 1)$$

Moreover, $^{n}C_n$ means the number of ways of selecting n objects from n distinct objects i.e. the whole group which can be done in 1 way.

Also $^{n}C_0$ means selecting nothing at all and it is equivalent to leaving behind all the objects and we know there is only one way of doing so.

Hence $^{n}C_0 = 1$

Note 2: $^nC_{n-r} = \dfrac{n!}{(n-r)!\left(n-(n-r)\right)!} = \dfrac{n!}{(n-r)!\,r!} = {}^nC_r$

Thus, $^nC_r = {}^nC_{n-r}$

This formula can be logically explained like this. The number of combination of n distinct objects taken r at a time is selecting a group of r objects out of n. Each time n objects are selected, we leave behind the remaining (n – r) objects. So number of combination of n objects taken r at a time is equal to number of combination of n objects taken (n – r) at a time.

Note 3: We know $^nC_r = \dfrac{n!}{r!(n-r)!}$

$\Rightarrow \quad {}^nC_r = \dfrac{n(n-1)(n-2).....(n-r+1)(n-r)!}{r!(n-r)!}$

$\Rightarrow \quad {}^nC_r = \dfrac{n(n-1)(n-2).....(n-r+1)}{r!}$

Thus, $^5C_2 = \dfrac{5\times 4}{2\times 1} = 10$, $^{51}C_3 = \dfrac{51\times 50\times 49}{3\times 2\times 1} = 20825$

Note 4: If $^nC_a = {}^nC_b \Rightarrow$ Either a = b or a = n – b i.e., n = a + b

Theorem: $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Proof: $^nC_r + {}^nC_{r-1}$

$= \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$= \dfrac{n!}{r(r-1)!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)(n-r)!}$

$= \dfrac{n!}{(r-1)!(n-r)!}\left[\dfrac{1}{r} + \dfrac{1}{n-r+1}\right]$

$= \dfrac{n!}{(r-1)!(n-r)!} \times \dfrac{n-r+1+r}{r(n-r+1)}$

$= \dfrac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!} = \dfrac{(n+1)!}{r!(n-r+1)!}$

$$= {}^{n+1}C_r$$

Example 39: If $n \times {}^{19}C_4 = {}^{19}P_4$, find n.

Solution: $n \times {}^{19}C_4 = {}^{19}C_4 \times 4!$

$$\Rightarrow n = 24$$

Example 40: Verify that ${}^{10}C_5 + {}^{10}C_6 = {}^{11}C_6$

Solution: LHS $= {}^{10}C_5 + {}^{10}C_6$

$$= \frac{10!}{5!\,5!} + \frac{10!}{6!\,4!}$$

$$= \frac{10!}{5! \times 5 \times 4!} + \frac{10!}{6 \times 5! \times 4!}$$

$$= \frac{10!}{5! \times 4!} \left[\frac{1}{5} + \frac{1}{6}\right]$$

$$= \frac{10!}{5! \times 4!} \times \frac{11}{5 \times 6}$$

$$= \frac{11 \times 10!}{6 \times 5! \times 5 \times 4!} = \frac{11!}{6! \times 5!} = {}^{11}C_6 = RHS$$

Hence verified

Example 41: Evaluate ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$

Solution: $({}^{15}C_8 + {}^{15}C_9) - ({}^{15}C_6 + {}^{15}C_7)$

$$= {}^{16}C_9 - {}^{16}C_7 \qquad (\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r)$$

$$= {}^{16}C_7 - {}^{16}C_7 \qquad (\because {}^{16}C_9 = {}^{16}C_{16-9})$$

$$= 0$$

Example 42: Prove that $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

Solution: LHS = $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n!}{r!(n-r4)!} \times \dfrac{(r-1)!(n-r+1)!}{n!}$

$= \dfrac{(r-1)!}{r(r-1)!} \dfrac{(n-r+1)(n-r)!}{(n-r)!} = \dfrac{n-r+1}{r} = RHS$

Example 43: If $^nC_4, ^nC_5$ and $^nC_6$ are in AP, find n

Solution: Since, $^nC_4, ^nC_5$ and $^nC_6$ are in AP

Therefore, $^nC_4 + ^nC_6 = 2(^nC_5)$

$= \dfrac{n!}{4!(n-4)!} + \dfrac{n!}{6!(n-6)!} = 2\left[\dfrac{n!}{5!(n-5)!}\right]$

$= \dfrac{1}{(n-4)(n-5)} + \dfrac{1}{30} = \dfrac{2}{5(n-5)}$

$= \dfrac{1}{30} = \dfrac{1}{n-5}\left[\dfrac{2}{5} - \dfrac{1}{n-4}\right]$

$= \dfrac{1}{6} = \dfrac{2n-13}{n^2 - 9n + 20}$

$= n^2 - 21n + 98 = 0$

$= (n-7)(n-14) = 0$

$= n = 7$ or 14

Example 44: If (i) $^nC_4 = ^nC_7$, find n

(ii) $^{11}C_r = ^{11}C_{r+3}$, find r

Solution:

(i) $^nC_4 = ^nC_7$

        $= ^nC_{n-4} = ^nC_7$ $(\because ^nC_r = ^nC_{n-r})$

$= n - 4 = 7$

$= n = 11$

(ii) $\quad {}^{11}C_r = {}^{11}C_{r+3}$

$\Rightarrow 11 = r + (r + 3) \quad (\because r \neq r+3, \text{ for non-negative integer r})$

$\Rightarrow 2r = 8$

$\Rightarrow r = 4$

Example 45: In how many ways can one select a cricket team of eleven from 17 players? If only 7 players can bowl and the team must include exactly 5 bowlers then how many different combinations are possible?

Solution: Number of combinations of 17 players taken 11 at a time is ${}^{17}C_{11} = {}^{17}C_6$

$$= \frac{17.16.15.14.13.12}{6.5.4.3.2.1} = 12376$$

Further the team has 7 bowlers out of which we have to choose 5 and 10 other players out of which we have to choose 6, which can be done in

$\quad {}^{7}C_5 \times {}^{10}C_6 \text{ ways}$

$\quad = {}^{7}C_2 \times {}^{10}C_4$

$$= \frac{7 \times 6}{2 \times 1} \times \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$$

$\quad = 4410$

Example 46: If there are seven distinct points on the circumference of a circle. Find

(i) How many chords can be drawn joining the points in all possible ways.

(ii) How many triangles can be drawn using any 3 of these 7 points as vertices.

(iii) How many quadrilaterals can be drawn joining any 4 points on the circle.

Solution:

(i)   Number of chords = $^7C_2 = \dfrac{7 \times 6}{2 \times 1} = 21$

(ii) Number of triangles = $^7C_3 = \dfrac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$

(iii) Number of quadrilaterals = $^7C_4 = 35$

Example 47: To attend a joint summit, two countries send 5 delegates each to a negotiating table. A rectangular table is used with 5 chairs on each side. If each country is assigned a long side of the table, how much seating arrangements are possible.

Solution: No. of arrangements = $5! \times 5! \times 2$

$= 120 \times 120 \times 2 = 28800$

Example 48: There are 9 points in a plane of which only 5 are collinear. Find the number of

(i)   straight lines that can be formed joining two points.

(ii) triangles that can be formed joining any three points.

Solution:

(i)   A straight line is formed by combination of 2 points and combination of 9 points taken 2 at a time is $^9C_2$. But the 5 collinear points will contribute only one line among themselves.

Hence the number of straight line = $^9C_2 - {}^5C_2 + 1$

$= \dfrac{9 \times 8}{2 \times 1} - \dfrac{5 \times 4}{2 \times 1} + 1 = 27$

Hence 27 lines can be drawn

(ii) We know a triangle is formed by combination of 3 non collinear points and combination of 9 points taken 3 at a time is $^9C_3$. But 5 collinear points among themselves will not form any triangle as all 3 points in them will be collinear.

Hence number of triangle = $^9C_3 - {}^5C_3 = {}^9C_3 - {}^5C_2$

$$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = \frac{5 \times 4}{2 \times 1} = 84 - 10 = 74$$

Hence the number of triangles are 74

Example 49: What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these:

(i)   four cards are of the same suit

(ii)   four cards belong to different suits

(iii)   three cards are of same colour and one different.

Solution: There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different objects, taken 4 at a time.

Therefore the required number of ways = $^{52}C_4$

$$= \frac{52!}{4! \, 48!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4} = 270725$$

(i)   There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are $^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are $^{13}C_4$ ways of choosing 4 clubs and $^{13}C_4$ ways each of choosing 4 spades and similarly 4 hearts.

Therefore the required number of ways

$$= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4 \times \frac{13!}{4! \, 9!} = 2860$$

(ii)   There are 13 cards in each suit. So. There are $^{13}C_1$ ways at choosing 1 card from 13 cards of each suit. By multiplication rule, the no of ways at selecting 4 cards belong to four different suits is $^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = (13)^4$.

(iii)   Since, there are only two colours i.e., red and black so there are two possibilities i.e., 3 red and 1 black or 3 black and 1 red. We know there are 26 red and 26 black cards.

Hence the number of ways = $^{26}C_3 \times {}^{26}C_1 + {}^{26}C_1 \times {}^{26}C_3$

$$= 2 \left[\frac{26 \times 25 \times 24}{3 \times 2 \times 1} \times 26 \right] = 135200$$

Example 50: A committee of 5 is to be selected from amongst 6 gentlemen and 5 ladies. Determine the number of ways if it is to contain at least 1 gentleman and 1 lady.

Solution: If the committee is to contain at least one gentleman and 1 lady, the various possibilities are:

1G and 4L or 2G and 3L or 3G and 2L or 4G and 1L where G stands for gentleman and L stands for an lady.

The required number of ways are:

$$^{6}C_1 \times {}^{5}C_4 + {}^{6}C_2 \times {}^{5}C_3 + {}^{6}C_3 \times {}^{5}C_2 + {}^{6}C_4 \times {}^{5}C_1$$

$$= {}^{6}C_1 \times {}^{5}C_1 + {}^{6}C_2 \times {}^{5}C_2 + {}^{6}C_3 \times {}^{5}C_2 + {}^{6}C_2 \times {}^{5}C_1$$

$$= 6 \times 5 + \frac{6 \times 5}{2 \times 1} \times \frac{5 \times 4}{2 \times 1} + \frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1} + \frac{6 \times 5}{2 \times 1} \times 5$$

$$= 30 + 150 + 200 + 75$$

$$= 455$$

Exercise 1.4

1.   If $^{18}C_r = {}^{18}C_{r+2}$, find $^{r}C_5$

2.   If $^{n}C_r : {}^{n}C_{r+1} = 1:2$ and $^{n}C_{r+1} : {}^{n}C_{r+2} = 2:3$, find n and r

3.   Show that, $^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{n+r}C_r = {}^{n+r+1}C_r$ (Hint: Write $^{n}C_0 = {}^{n+1}C_0$)

4.   How many chords can be drawn through 17 points on a circle?

5. The number of diagonals of a polygon is twice the number of its sides. Find the number of sides of the polygon.

6. A box contains 6 red and 7 white balls. Determine the number of ways in which 4 red and 3 white balls can be selected.

7. In how many ways can a committee of 5 is to be formed from 4 teachers and 6 students so as to include at least 2 students.

8. A cricket team of 11 players is to be formed from 15 players. In how many different ways the team can be selected if

    (i) Two players who scored maximum runs and took maximum wickets respectively must be included.

    (ii) One who is not in form should be excluded.

9. In how many ways can a student choose a programme of 5 courses if 10 courses are available and 2 language courses are compulsory for every student.

10. In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two (–) signs are together.

11. Twenty points no four of which are coplanar are in space. How many triangles do they determine? How many planes? How many tetrahedrons?

Answers:

1. r = 56

2. r = 4, n =14

4. 136

5. 7

6. 525

7. 246

8.   (i) 715   (ii) 364

9.   56

10.   56

11.   1140, 1140, 4845

6.7   Combination with Repetition

We have seen number of arrangements of n distinct objects taken r at a time with repetition in n^r. Do we have situations where combinations of objects with repetition is a possibility.

Let us consider a practical example. During lunch break 5 friends go to school canteen where each one of them has one of the following: a burger, cold coffee or noodles. How many different purchases are possible? Let b, c and n represent a burger, cold coffee and noodles respectively.

Here we are concerned with how many of each items are purchased, not with the order in which they are purchased.

So the problem is one of selections or combinations with repetition. Some possible combinations are b b b c c b c c n n c c c c c etc.

Now we give an important theorem.

Theorem: Number of combination of r objects from a set with n objects when repetition of elements is allowed = $^{n+r-1}C_r$

In the problem discussed above, we have

   n = 3 (possible food or drink)

   r = 5 (friends)

Note: r > n is possible here

Number of combinations = $^{n+r-1}C_r$

   $= {}^{3+5-1}C_5$

$$= {}^7C_5 = \frac{7 \times 6}{2 \times 1} = 21$$

Example 51: How many ways one can select three gift vouchers from a box containing Rs. 100, Rs. 500, Rs. 1000 and Rs. 2000 gift vouchers? Assume that the order in which the vouchers are chosen does not matter, the vouchers of each denomination are identical and there are at least three gift vouchers of each type.

Solution: Since the order in which the vouchers are selected does not matter and vouchers of 4 different amounts are there and 3 gift vouchers have to be selected.

This problem involves counting of 3-combinations with repetition from a set of four elements.

Thus we have, n = 4, r = 3

Number of combinations = ${}^{n+r-1}C_r = {}^{4+3-1}C_3$

$$= {}^6C_3 = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

Let us Recollect Formulas of Permutation and Combination

S. No.	Type	Repetition	Formula
1.	Permutation of n distinct objects taken r at a time	Not allowed	${}^nP_r = \dfrac{n!}{(n-r)!}$
2.	Permutation of n distinct objects taken r at a time	Allowed	n^r
3.	Combination of n distinct objects taken r at a time	Not allowed	${}^nC_r = \dfrac{n!}{r!(n-r)!}$
4.	Combination of n distinct objects taken r at a time	Allowed	${}^{n+r-1}C_r = \dfrac{(n+r-1)!}{r!(n-1)!}$

Note: A word of Advice!

When dealing with counting problem, we should think is order important in the problem? When order is required we think in terms of permutations, arrangements and fundamental principle of multiplication. When order is not relevant, combination plays a key role in solving the problem.

Miscellaneous Examples

Example 52: How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER?

Solution: In the word DAUGHTER, there are 3 vowels and 5 consonants.

The number of ways of selecting 2 vowels out of 3 = $^3C_2$

The number of ways of selecting 3 consonants out of 5 = $^5C_3$

Therefore, the number of combinations of 2 vowels and 3 consonants is

$$^3C_2 \times {}^5C_3 = {}^3C_1 \times {}^5C_2 = 3 \times 10 = 30$$

Now each of these combinations has 5 letters which can be arranged among themselves in 5! ways.

Therefore the required number of words is $30 \times 5! = 3600$

Example 53: How many numbers greater than 2000000 can be formed using the digits 1, 3, 0, 3, 2, 3, 2?

Solution: The numbers have to be greater than 2000000, so they can begin either with 2 or 3.

The numbers beginning with 2 = $\dfrac{6!}{3!}$ = $4 \times 5 \times 6 = 120$

Total numbers beginning with 3 = $\dfrac{6!}{2!\,2!}$ = $\dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (2 \times 1)}$ = 180

The required number of numbers = 120 + 180 = 300

Example 54: From a group of 20 students of an environmental club 9 are to be chosen for an educational tour. There are 3 friends among these students who

decide that either all of them will join or none of them will join. In how many ways can be students for the educational tour be chosen?

Solution: Let us consider the case when the three students doesn't join. In that case 9 students have to be chosen from the remaining 17 students, i.e.,

$$^3C_0 \times {}^{17}C_9 = {}^{17}C_9 \text{ ways}$$

In case the three students join the tour we chose the remaining 6 students from 17 students, which can be done in $^3C_3 \times {}^{17}C_6 = {}^{17}C_6$ ways.

Total number of combination satisfying the given condition = $^{17}C_9 + {}^{17}C_6$

$$= 24310 + 12376$$

$$= 36,686$$

Example 55: Find the number of (i) combinations and (ii) permutations of the letters of the word ACCOUNTANCY taken 4 at a time.

Solution: The given letters of the word are

(CCC), (AA), (NN), O, U, T and Y. There are 11 letters of which 7 are distinct.

The various choices of 4 letters at a time are

(i)   3 identical and 1 different letter

$$^3C_3 \times {}^6C_1 = 1 \times 6 = 6 \text{ ways}$$

(ii)   2 pairs of identical letters

$$^3C_2 = 3 \text{ ways (i.e., CCAA, CCNN, AANN)}$$

(iii)   One pair of identical letters and 2 different letters.

$$^3C_1 \times {}^6C_2 = 3 \times \frac{6 \times 5}{2 \times 1} = 45 \text{ ways}$$

(iv)   4 different letters.

$$7C4 = \frac{7 \times 6 \times 5 \times 4}{4 \times 3 \times 2 \times 1} = 35 \text{ ways}$$

So total number of combinations = 6 + 3 + 45 + 35 = 89

To find the permutations we have to arrange the letters in all the above cases:

(i) $\quad 6 \times \dfrac{4!}{3!} = 24$

(ii) $\quad 3 \times \dfrac{4!}{2! \times 2!} = 18$

(iii) $\quad 45 \times \dfrac{4!}{2!} = 540$

(iv) $\quad 35 \times 4! = 840$

The total number of permutations are = 24 + 18 + 540 + 840 = 1422

Example 56: There are 8 members in a committee. In how many ways we can choose:

(i) a subcommittee consisting of 3 members?

(ii) a chairperson, a secretary and a treasurer assuming that one person cannot hold more than one position?

Solution:

(i) To form a subcommittee we have to choose 3 members out of 8 which can be done in $^8C_3 = \dfrac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$ ways

(ii) To choose a chairperson, a secretary and a treasurer the order is also important, so it can be done in $^8P_3 = \dfrac{8!}{(8-3)!} = \dfrac{8!}{5!} = 8 \times 7 \times 6 = 336$ ways

Example 57: A company has 5 senior officers and 7 junior commissioned officers (JCO's). A team of 4 is to be sent on a special mission. In how many ways it can be formed so that it comprises of

(i) any 4 officers?

(ii) 4 senior officers?

(iii) 2 senior and 2 junior officers (JCO's)

230

(iv)   at least 2 senior officers

Solution:

(i)   Any 4 officers can be selected in $^{12}C_4$ = 495 ways

(ii)   4 senior officers can be chosen in $^5C_4$ = 5 ways.

(iii)   2 senior and 2 junior officers can be selected in check for sub-script

$$^5C_2 \times {}^7C_2 = \frac{5 \times 4}{2 \times 1} \times \frac{7 \times 6}{2 \times 1} = 10 \times 21 = 210 \text{ ways}$$

(iv)   The team of At least 2 senior officers can constitute of 2S and 2J or 3S, 1J or 4S

$$\Rightarrow {}^5C_2 \times {}^7C_2 + {}^5C_3 \times {}^7C_1 + {}^5C_4$$

$$= \frac{5 \times 4}{2 \times 1} \times \frac{7 \times 6}{2 \times 1} + \frac{5 \times 4}{2 \times 1} \times 7 + 5$$

$$= 10 \times 21 + 70 + 5$$

$$= 210 + 70 + 5 = 285$$

Example 58: From a total of 9 players a basketball team of playing 5 is to be selected. How many teams are possible if

(i)   the distinct positions of the playing 5 are to be taken into consideration.

(ii)   the distinct position of the playing 5 are not taken into consideration.

(iii)   the distinct positions are not taken into consideration, but two players either Krish or Rohit (but not both) should be in the playing 5.

Solution:

(i)   If the distinct position of the playing 5 are to be taken into consideration, then number of ways = $^9P_5 = \frac{9!}{4!}$ = 15120

(ii) If the distinct position of the playing 5 are not taken into consideration, then number of ways = $^9C_5 = 126$

(iii) Number of ways, when either Krish or Rohit (but not both) are in the playing 5

$= 2 \times {}^1C_1 \times {}^7C_4$

$= 2 \times 1 \times 35 = 70$

Example 59: Quality Control:

A medical store receives a shipment of 24 infrared temperature guns, including 5 that are defective. Three of these guns are to be sent to a private hospital.

(i) How many selections can be made

(ii) How many of these selections will contain no defective guns?

Solution:

(i) Number of selections = $^{24}C_3 = 2024$

(ii) Number of selections which contains no defective guns = $^{19}C_3 = 969$

Miscellaneous Exercise 1.5

1. An investment banker finalises the list of specific entities worthy of investment in each of the 3 instruments given below:

   3 private limited companies for direct equity investment

    5 mutual fund schemes

    2 banks for making fixed deposits

    In how many ways the investment can be made if:

    (i) the banker decides to invest the entire fund only in entity

    (ii) the banker chooses to invest in one entity of each of the three instruments.

2. A cookie shop has five different kind of cookies. How many different ways can six cookies be chosen assuming that only the type of cookie and not the individual cookies or the order in which they are chosen matters.

3. In an examination, a question paper consists of 12 questions divided into two sections i.e. A and B, containing 7 and 5 questions, respectively. A student is required to attempt 8 questions in all and first question of section A is compulsory. In how many ways can be student select the questions if at least 3 questions are to be attempted from each section.

4. How many 4 digit numbers can be formed from the digits 1, 1, 2, 2, 3, 3, 4 and 5?

5. How many 5 letter word can be formed using 3 letters of the word ALGORITHM and 2 letters from the world DUES.

6. Life Sciences (Medicine): There are 8 standard classification of blood type. An examination for prospective laboratory technicians consists of having each candidate determine the type of 3 blood samples.

    (i)   How many different examinations can be given if no. 2 samples are of the same type.

    (ii) How many different examination papers can be given if 2 or more samples can have the same type.

7. Find the number of parallelograms in the following figure.
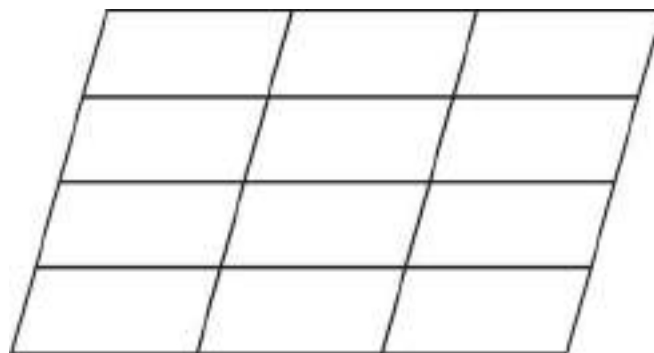
8. The number of incorrect predictions of 4 successive football matches is

    (i)   81   (ii)   64

(iii)   80   (iv)   63

9.   Number of ways in which 15 different children can sit in a merry-go-round relative to one another is

(i)   1/2 (14!)   (ii)   14!

(iii)   1/2 (15!)   (iv) $2 \times 14!$

10.   Number of diagonals of a convex hexagon are:

(i)   3   (ii)   6

(iii)   9   (iv)   15

11.   Number of divisors of 10,000,000 are:

(i)   7   (ii)   8

(iii)   49   (iv)   64

12.   A donuts shop offers 20 kinds of donuts. The shop has at least a dozen donuts of each kind if person enters the shop he can select dozen donuts in

(i) $^{31}C_{12}$ ways   (ii) $^{30}C_{12}$ ways

(iii) $^{32}C_{12}$ ways   (iv)   240 ways

13.   The number of permutations of n different things taken r at a time in which m particular things are placed in m given places in definite order is:

(i) $^{n-m}P_{r-m} \times m!$ (ii) $(n - m + 1)!$

(iii) $^{n-m}P_{r-m}$ (iv) $^{n}P_{r} - m!$

Answers:

1.   (i) 10   (ii) 30

2. $^{5+6-1}C_{6} = 210$

3.   265 ways

4.   354

5. $^9C_3 \times {}^4C_2 \times 5! = 60480$

6.   (i) 56   (ii) 512

7. $^4C_2 \times {}^5C_2 = 60$

8.   (iii)

9.   (ii)

10.   (iii)

11.   (iv)

12.   (i)

13.   (iii)

Summary

- There are two basic principles of counting

    1) The Rule of Sum (Fundamental Principle of Addition)

    If an event E can occur in m ways and another event F can occur in n ways, further the two events cannot occur simultaneously, then the either event E or F can occur in (m+n) ways.

    2) The Rule of Product (Fundamental Principle of Multiplication)

    If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is mxn.

- **Factorial**

    n! = 1 x 2 x 3 x … x (n-1) x n

    n! = n x (n-1) !

- The number of permutations of n different objects taken r at a time, where o < r ≤ n and objects do not repeat is given by $^nP_r = \dfrac{n!}{(n-r)!}$ where o ≤ r ≤ n.

- The number of permutations of n different objects taken r at a time, when repetition is allowed is n^r.