dy/dx as Rate Measurer and Tangents, Normals Exercise 1 : Single Option Correct Type Questions

1. Consider the cubic equation $f(x) = x^3 - nx + 1 = 0$ where $n \geq 3, n \in N$, then $f(x) = 0$ has

(a) at least one root in $(0, 1)$ (b) at least one root in $(1, 2)$
(c) at least one root in $(-1, 0)$ (d) data insufficient

2. If the normal to $y = f(x)$ at $(0, 0)$ is given by $y - x = 0$, then $\lim_{x \to 0} \dfrac{x^2}{f(x^2) - 20f(9x^2) + 2f(99x^2)}$ is

(a) $1/19$ (b) $-1/19$ (c) $1/2$ (d) does not exist

3. Tangent to a curve intersects the Y-axis at a point. A line perpendicular to this tangent through P passes through another point $(1, 0)$. The differential equation of the curve is

(a) $y\dfrac{dy}{dx} - x\left(\dfrac{dy}{dx}\right)^2 = 1$ (b) $x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1$
(c) $y\dfrac{dy}{dx} + x = 1$ (d) None of these

4. The number of points in the rectangle $\{(x, y) : |x| \leq 9, |y| \leq 3\}$ which lie on the curve $y^2 = x + \sin x$ and at which the tangent to the curve is parallel to X-axis, is

(a) 3 (b) 2 (c) 4 (d) None of these

5. If $a, b, c, d \in R$ such that $\dfrac{a + 2c}{b + 3d} = -\dfrac{4}{3}$, then the equation $ax^3 + bx^2 + cx + d = 0$ has

(a) at least one root in $(-1, 0)$ (b) at least one root in $(0, 1)$
(c) no root in $(-1, 1)$ (d) no root in $(0, 2)$

6. If $3(a + 2c) = 4(b + 3d) \neq 0$, then the equation $ax^3 + bx^2 + cx + d = 0$ will have

(a) no real solution
(b) at least one real root in $(-1, 0)$
(c) at least one real root in $(0, 1)$
(d) None of the above

7. Let $f(x)$ be a differentiable function in the interval $(0, 2)$, then the value of $\int_0^2 f(x)\, dx$

(a) $f(c)$ where $c \in (0, 2)$ (b) $2f(c)$, where $c \in (0, 2)$
(c) $f'(c)$, where $c \in (0, 2)$ (d) None of these

8. Let $f(x)$ be a fourth differentiable function such that $f(2x^2 - 1) = 2xf(x), \forall x \in R$, then $f^{\text{iv}}(0)$ is equal to (where $f^{\text{iv}}(0)$ represents fourth derivative of $f(x)$ at $x = 0$)

(a) 0 (b) 1
(c) -1 (d) data insufficient

9. The curve $x + y - \ln(x + y) = 2x + 5$ has a vertical tangent at the point (α, β). Then, $\alpha + \beta$ is equal to

(a) -1 (b) 1 (c) 2 (d) -2

10. Let $y = f(x)$, $f : R \to R$ be an odd differentiable function such that $f'''(x) > 0$ and $g(\alpha, \beta) = \sin^8 \alpha + \cos^8 \beta + 2 - 4 \sin^2 \alpha \cos^2 \beta$. If $f''(g(\alpha, \beta)) = 0$, then $\sin^2 \alpha + \sin^2 \beta$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

11. A polynomial of 6th degree $f(x)$ satisfies $f(x) = f(2 - x), \forall x \in R$, if $f(x) = 0$ has 4 distinct and 2 equal roots, then sum of the roots of $f(x) = 0$ is

(a) 4 (b) 5 (c) 6 (d) 7

12. Let a curve $y = f(x)$, $f(x) \geq 0, \forall x \in R$ has property that for every point P on the curve length of subnormal is equal to abscissa of P. If $f(1) = 3$, then $f(4)$ is equal to

(a) $-2\sqrt{6}$ (b) $2\sqrt{6}$ (c) $3\sqrt{5}$ (d) None of these

13. If a variable tangent to the curve $x^2y = c^3$ makes intercepts, a, b on X and Y-axes respectively, then the value of a^2b is

(a) $27c^3$ (b) $\dfrac{4}{27}c^3$
(c) $\dfrac{27}{4}c^3$ (d) $\dfrac{4}{9}c^3$

14. Let $f(x) = \begin{vmatrix} 1 & 1 & 1 \\ 3 - x & 5 - 3x^2 & 3x^3 - 1 \\ 2x^2 - 1 & 3x^5 - 1 & 7x^8 - 1 \end{vmatrix}$. Then, the equation $f(x) = 0$ has

(a) no real root
(b) atmost one real root
(c) at least two real roots
(d) exactly one real root in $(0, 1)$ and no other real root

15. The graphs $y = 2x^3 - 4x + 2$ and $y = x^3 + 2x - 1$ intersect at exactly 3 distinct points. The slope of the line passing through two of these points is

(a) equal to 4 (b) equal to 6
(c) equal to 8 (d) not unique

16. In which of the following functions Rolle's theorem is applicable?

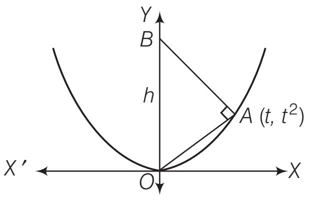
(a) $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$ on $[0, 1]$

(b) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \leq x < 0 \\ 0, & x = 0 \end{cases}$ on $[-\pi, 0]$

(c) $f(x)=\dfrac{x^2-x-6}{x-1}$ on $[-2, 3]$

(d) $f(x)=\begin{cases}\dfrac{x^3-2x^2-5x+6}{x-1}, & \text{if } x\neq 1\\ -6, & \text{if } x=1\end{cases}$ on $[-2, 3]$

17. The figure shows a right triangle with its hypotenuse OB along the Y-axis and its vertex A on the parabola $y=x^2$.



Let h represents the length of the hypotenuse which depends on the x-coordinate of the point A. The value of $\lim\limits_{t\to 0}(h)$ is equal to

(a) 0 (b) 1/2 (c) 1 (d) 2

18. Number of positive integral value(s) of 'a' for which the curve $y=a^x$ intersects the line $y=x$ is

(a) 0 (b) 1 (c) 2 (d) more than 2

19. If $f(x)=4-\left(\dfrac{1}{2}-x\right)^{2/3}$, $g(x)=\begin{cases}\dfrac{\tan[x]}{x}, & x\neq 0\\ 1, & x=0\end{cases}$

$h(x)=\{x\}$, $k(x)=5^{\log_2(x+3)}$, where $[x]$ and $\{x\}$ denote the greatest integer and fraction part function, then in $[0, 1]$, Lagrange's mean value theorem is not applicable to

(a) f, g, h (b) h, k (c) f, g (d) g, h, k

20. If the function $f(x)=x^4+bx^2+8x+1$ has a horizontal tangent and a point of inflection for the same value of x, then the value of b is equal to

(a) -1 (b) 1 (c) 6 (d) -6

21. Coffee is coming out from a conical filter, with height and diameter both are 15 cm into a cylindrical coffee pot with a diameter 15 cm. The rate at which coffee comes out from the filter into the pot is 100 cu cm/min.

The rate (in cm/min) at which the level in the pot is rising at the instance when the coffee in the pot is 10 cm, is

(a) $\dfrac{9}{16\pi}$ (b) $\dfrac{25}{9\pi}$ (c) $\dfrac{5}{3\pi}$ (d) $\dfrac{16}{9\pi}$

22. A horse runs along a circle with a speed of 20 km/h. A lantern is at the centre of the circle. A fence is there along the tangent to the circle at the point at which the horse starts. The speed with which the shadow of the horse moves along the fence at the moment when it covers 1/8 of the circle (in km/h) is

(a) 20 (b) 40 (c) 30 (d) 60

23. Water runs into an inverted conical tent at the rate of 20 cu ft/min and leaks out at the rate of 5 cu ft/min. The height of the water is three times the radius of the water's surface. The radius of the water surface is increasing when the radius is 5 ft, is

(a) $\dfrac{1}{5\pi}$ ft/min (b) $\dfrac{1}{10\pi}$ ft/min (c) $\dfrac{1}{15\pi}$ ft/min (d) None of these

24. Let $f(x)=x^3-3x^2+2x$. If the equation $f(x)=k$ has exactly one positive and one negative solution, then the value of k equals

(a) $-\dfrac{2\sqrt{3}}{9}$ (b) $-\dfrac{2}{9}$

(c) $\dfrac{2}{3\sqrt{3}}$ (d) $\dfrac{1}{3\sqrt{3}}$

25. The x-intercept of the tangent at any arbitrary point of the curve $\dfrac{a}{x^2}+\dfrac{b}{y^2}=1$ is proportional to

(a) square of the abscissa of the point of tangency

(b) square root of the abscissa of the point of tangency

(c) cube of the abscissa of the point of tangency

(d) cube root of the abscissa of the point of tangency

26. If $f(x)$ is continuous and differentiable over $[-2, 5]$ and $-4\le f'(x)\le 3$ for all x in $(-2, 5)$, then the greatest possible value of $f(5)-f(-2)$ is

(a) 7 (b) 9

(c) 15 (d) 21

27. A curve is represented parametrically by the equations $x=t+e^{at}$ and $y=-t+e^{at}$, where $t\in R$ and $a>0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are

(a) $(1, 0)$ (b) $(1/e, 0)$

(c) $(e, 0)$ (d) $(2e, 0)$

28. At any two points of the curve represented parametrically by $x=a(2\cos t-\cos 2t)$, $y=a(2\sin t-\sin 2t)$, the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by

(a) $2\pi/3$ (b) $3\pi/4$ (c) $\pi/2$ (d) $\pi/3$

29. Let $F(x)=\displaystyle\int_{\sin x}^{\cos x} e^{(1+\sin^{-1}(t))^2}\,dt$ on $\left[0, \dfrac{\pi}{2}\right]$, then

(a) $F''(c)=0$ for all $c\in\left(0, \dfrac{\pi}{2}\right)$

(b) $F''(c)=0$ for some $c\in\left(0, \dfrac{\pi}{2}\right)$

(c) $F'(c)=0$ for some $c\in\left(0, \dfrac{\pi}{2}\right)$

(d) $F(c)\neq 0$ for all $c\in\left(0, \dfrac{\pi}{2}\right)$

30. If $f'(1) = 1$ and $\frac{d}{dx}(f(2x)) = f'(x), \forall\ x > 0$. If $f'(x)$ is differentiable, then there exists a number $c \in (2, 4)$ such that $f''(c)$ is equal to

(a) $-1/4$
(b) $-1/8$
(c) $1/4$
(d) $1/8$

31. Let $f(x)$ and $g(x)$ be two functions which are defined and differentiable for all $x \geq x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$, then

(a) $f(x) < g(x)$ for some $x > x_0$
(b) $f(x) = g(x)$ for some $x > x_0$
(c) $f(x) > g(x)$ only for some $x > x_0$
(d) $f(x) > g(x)$ for all $x > x_0$

32. The range of values of m for which the line $y = mx$ and the curve $y = \frac{x}{x^2 + 1}$ enclose a region, is

(a) $(-1, 1)$ (b) $(0, 1)$
(c) $[0, 1]$ (d) $(1, \infty)$

33. Let S be a square with sides of length x. If we approximate the change in size of the area of S by $h \cdot \left.\frac{dA}{dx}\right|_{x = x_0}$, when the sides are changed from x_0 to $x_0 + h$, then the absolute value of the error in our approximation, is

(a) h^2 (b) $2hx_0$ (c) x_0^2 (d) h

34. Consider $f(x) = \int_1^x \left(t + \frac{1}{t}\right) dt$ and $g(x) = f'(x)$ for $x \in \left[\frac{1}{2}, 3\right]$. If P is a point on the curve $y = g(x)$ such that the tangent to this curve at P is parallel to a chord joining the points $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$ and $(3, g(3))$ of the curve, then the coordinates of the point P are

(a) can't be found out (b) $\left(\frac{7}{4}, \frac{65}{28}\right)$
(c) $(1, 2)$ (d) $\left(\sqrt{\frac{3}{2}}, \frac{5}{\sqrt{6}}\right)$

dy/dx as Rate Measurer & Tangents, Normals Exercise 2 : More than One Option Correct Type Questions

35. For the curve represented parametrically by the equation, $x = 2\log(\cot t) + 1$ and $y = \tan t + \cot t$, then

(a) tangent at $t = \frac{\pi}{4}$ is parallel to X-axis
(b) normal at $t = \frac{\pi}{4}$ is parallel to Y-axis
(c) tangent at $t = \frac{\pi}{4}$ is parallel to $y = x$
(d) normal at $t = \frac{\pi}{4}$ is parallel to $y = x$

36. Consider the curve $f(x) = x^{1/3}$, then

(a) the equation of tangent at $(0, 0)$ is $x = 0$
(b) the equation of normal at $(0, 0)$ is $y = 0$
(c) normal to the curve does not exist at $(0, 0)$
(d) $f(x)$ and its inverse meet at exactly 3 points

37. The angle at which the curve $y = ke^{kx}$ intersects Y-axis is

(a) $\tan^{-1}(k^2)$ (b) $\cot^{-1}(k^2)$
(c) $\sin^{-1}\left(\frac{1}{\sqrt{1 + k^4}}\right)$ (d) $\sec^{-1}(\sqrt{1 + k^4})$

38. Let $f(x) = 8x^3 - 6x^2 - 2x + 1$, then

(a) $f(x) = 0$ has no root in $(0, 1)$
(b) $f(x) = 0$ has at least one root in $(0, 1)$
(c) $f'(c)$ vanishes for some $c \in (0, 1)$
(d) None of the above

39. If $f(0) = f(1) = f(2) = 0$ and function $f(x)$ is twice differentiable in $(0, 2)$ and continuous in $[0, 2]$, then which of the following is/are definitely true?

(a) $f''(c) = 0; \forall\ c \in (0, 2)$
(b) $f'(c) = 0$; for at least two $c \in (0, 2)$
(c) $f'(c) = 0$; for exactly one $c \in (0, 2)$
(d) $f''(c) = 0$; for at least one $c \in (0, 2)$

40. Equation $\frac{1}{(x + 1)^3} - 3x + \sin x = 0$ has

(a) no real root
(b) two real and distinct roots
(c) exactly one negative root
(d) exactly one root between -1 and ∞

41. If f is an odd continuous function in $[-1, 1]$ and differentiable in $(-1, 1)$, then

(a) $f'(A) = f(1)$ for some $A \in (-1, 0)$
(b) $f'(B) = f(1)$ for some $B \in (0, 1)$
(c) $n(f(A))^{n-1} f'(A) = (f(1))^n$ for some $A \in (-1, 0), n \in N$
(d) $n(f(B))^{n-1} f'(B) = (f(1))^n$ for some $B \in (0, 1), n \in N$

42. The parabola $y = x^2 + px + q$ intersects the straight line $y = 2x - 3$ at a point with abscissa 1. If the distance between the vertex of the parabola and the X-axis is least, then

(a) $p = 0$ and $q = -2$
(b) $p = -2$ and $q = 0$

(c) least distance between the vertex of the parabola and X-axis is 2
(d) least distance between the vertex of the parabola and X-axis is 1

43. The abscissa of the point on the curve $\sqrt{xy} = a + x$, the tangent at which cuts off equal intersects from the coordinate axes, is $(a > 0)$

(a) $\frac{a}{\sqrt{2}}$ (b) $-\frac{a}{\sqrt{2}}$ (c) $a\sqrt{2}$ (d) $-a\sqrt{2}$

44. If the side of a triangle vary slightly in such a way that its circumradius remains constant, then $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C}$ is equal to

(a) $6R$ (b) $2R$
(c) 0 (d) $2R\,(dA + dB + dC)$

45. Let $f(x)$ satisfy the requirements of Lagrange's mean value theorem in $[0, 1]$, $f(0) = 0$ and $f'(x) \le 1 - x, \forall\ x \in (0, 1)$, then

(a) $f(x) \ge x$ (b) $|f(x)| \ge 1$
(c) $f(x) \le x(1 - x)$ (d) $f(x) \le 1/4$

46. For function $f(x) = \frac{\ln x}{x}$, which of the following statements are true?

(a) $f(x)$ has horizontal tangent at $x = e$
(b) $f(x)$ cuts the X-axis only at one point
(c) $f(x)$ is many-one function
(d) $f(x)$ has one vertical tangent

47. Equation of a line which is tangent to both the curves $y = x^2 + 1$ and $y = -x^2$ is

(a) $y = \sqrt{2}\,x + \frac{1}{2}$ (b) $y = \sqrt{2}\,x - \frac{1}{2}$
(c) $y = -\sqrt{2}\,x + \frac{1}{2}$ (d) $y = -\sqrt{2}\,x - \frac{1}{2}$

48. Let $F(x) = (f(x))^2 + (f'(x))^2$, $F(0) = 6$ where $f(x)$ is thrice differentiable function such that $|f(x)| \le 1$ for all $x \in [-1, 1]$, then choose the correct statement(s).

(a) There is at least one point in each of the intervals $(-1, 0)$ and $(0, 1)$ where $|f'(x)| \le 2$
(b) There is at least one point in each of the intervals $(-1, 0)$ and $(0, 1)$ where $F(x) \le 5$
(c) There is no point of local maxima of $F(x)$ in $(-1, 1)$
(d) For some $c \in (-1, 1)$, $F(c) \ge 6$, $F'(c) = 0$ and $F''(c) \le 0$

49. If the Rolle's theorem is applicable to the function f defined by

$$f(x) = \begin{cases} ax^2 + b, & |x| \le 1 \\ 1, & |x| = 1 \\ \frac{c}{|x|}, & |x| > 1 \end{cases}$$

in the interval $[-3, 3]$, then which of the following alternative(s) is/are correct?

(a) $a + b + c = 2$ (b) $|a| + |b| + |c| = 3$
(c) $2a + 4b + 3c = 8$ (d) $4a^2 + 4b^2 + 5c^2 = 15$

dy/dx as Rate Measurer & Tangents, Normals Exercise 3 : Statements I and II Type Questions

- **Directions** (Q. Nos. 50 to 56) *For the following questions, choose the correct answer from the options (a), (b), (c) and (d) defined as follows*

(a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

50. **Statement I** If $g(x)$ is a differentiable function $g(1) \ne 0$, $g(-1) \ne 0$ and Rolle's theorem is not applicable to $f(x) = \frac{x^2 - 1}{g(x)}$ in $[-1, 1]$, then $g(x)$ has atleast one root in $(-1, 1)$.

Statement II If $f(a) = f(b)$, then Rolle's theorem is applicable for $x \in (a, b)$.

51. **Statement I** Shortest distance between $|x| + |y| = 2$ and $x^2 + y^2 = 16$ is $4 - \sqrt{2}$.

Statement II Shortest distance between the two smooth curves lies along the common normal.

52. **Statement I** If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the n real roots of a polynomial equation of nth degree with real coefficients such that sum of the roots taken $r\,(1 \le r \le n)$ at a time is positive, then all the roots are positive.

Statement II The number of times sign of coefficients change while going left to right of a polynomial equation is the number of maximum positive roots.

53. **Statement I** Tangents at two distinct points of a cubic polynomial cannot coincide.

Statement II If $P(x)$ is a polynomial of degree $n\,(n \ge 2)$, then $P'(x) + k$ cannot hold for n or more distinct values of x.

54. **Statement I** For $f(x)=\begin{cases}\frac{1}{2}-x, & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2, & x\geq\frac{1}{2}\end{cases}$, mean value theorem is applicable in the interval [0, 1].

Statement II For application of mean value theorem, $f(x)$ must be continuous in [0, 1] and differentiable in (0, 1).

55. Let $f(x)=\ln(2+x)-\dfrac{2x+2}{x+3}$.

Statement I The function $f(x)=0$ has a unique solution in the domain of $f(x)$.

Statement II If $f(x)$ is continuous in $[a, b]$ and is strictly monotonic in (a, b), then f has a unique root in (a, b).

56. Consider the polynomial function

$$f(x)=\frac{x^7}{7}-\frac{x^6}{6}+\frac{x^5}{5}-\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}+x$$

Statement I The equation $f(x)=0$ cannot have two or more roots.

Statement II Rolle's theorem is not applicable for $y=f(x)$ on any interval $[a, b]$, where $a, b\in R$.

dy/dx as Rate Measurer & Tangents, Normals Exercise 4 : Passage Based Questions

Passage I (Q. Nos. 57 to 59)

We say an equation $f(x)=g(x)$ is consistent, if the curves $y=f(x)$ and $y=g(x)$ touch or intersect at at least one point. If the curves $y=f(x)$ and $y=g(x)$ do not intersect or touch, then the equation $f(x)=g(x)$ is said to be inconsistent, i.e. has no solution.

57. The equation $\cos x+\cos^{-1}x=\sin x+\sin^{-1}x$ is

(a) consistent and has infinite number of solutions
(b) consistent and has finite number of solutions
(c) inconsistent
(d) None of the above

58. The equation $\sin x=x^2+x+1$ is

(a) consistent and has infinite number of solutions
(b) consistent and has finite number of solutions
(c) inconsistent
(d) consistent and has unique solution

59. Among the following equations, which is consistent in $(0, \pi/2)$?

(a) $\sin x+x^2=0$
(b) $\cos x=x$
(c) $\tan x=x$
(d) All of these

Passage II (Q. Nos. 60 to 62)

To find the point of contact $P\equiv(x_1, y_1)$ of a tangent to the graph of $y=f(x)$ passing through origin O, we equate the slope of tangent to $y=f(x)$ at P to the slope of OP. Hence, we solve the equation $f'(x_1)=\dfrac{f(x_1)}{x_1}$ to get x_1 and y_1.

60. The equation $|\ln mx|=px$, where m is a positive constant, has a single root for

(a) $0<p<\dfrac{m}{e}$
(b) $p<\dfrac{e}{m}$
(c) $0<p<\dfrac{e}{m}$
(d) $p>\dfrac{m}{e}$

61. The equation $|\ln mx|=px$, where m is a positive constant, has exactly two roots for

(a) $p=\dfrac{m}{e}$
(b) $p=\dfrac{e}{m}$
(c) $0<p\leq\dfrac{e}{m}$
(d) $0<p\leq\dfrac{m}{e}$

62. The equation $|\ln mx|=px$, where m is a positive constant, has exactly three roots for

(a) $p<\dfrac{m}{e}$
(b) $0<p<\dfrac{m}{e}$
(c) $0<p<\dfrac{e}{m}$
(d) $p<\dfrac{e}{m}$

Passage III (Q. Nos. 63 to 64)

Consider the family of circles: $x^2+y^2-3x-4y-c_i=0, c_i\in N$ $(i=1, 2, 3, \ldots, n)$.
Also, let all circles intersects X-axis at integral points only and $c_1<c_2<c_3<c_4\ldots<c_n$. A point (x, y) is said to be integral point, if both coordinates x and y are integers.

63. If circle $x^2+y^2-3x-4y-(c_2-c_1)=0$ and circle $x^2+y^2=r^2$ have only one common tangent, then

(a) $r=1/2$
(b) tangent passes through (10, 0)
(c) (3, 4) lies outside the circle $x^2+y^2=r^2$
(d) $c_2=2r+c_1$

64. The ellipse $4x^2+9y^2=36$ and hyperbola $a^2x^2-y^2=4$ intersect orthogonally, then the equation of circle through the points of intersection of two conics is

(a) $x^2+y^2=(c_5)^2$
(b) $x^2+y^2=\dfrac{c_4}{7}$
(c) $x^2+y^2=c_3-2c_1$
(d) $x^2+y^2=c_7/14$

dy/dx as Rate Measurer & Tangents, Normals Exercise 5 :
Matching Type Questions

65. Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	The equation $x \log x = 3 - x$ has at least one root in	(p)	$(0, 1)$
(B)	If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has at least one root in	(q)	$(1, 3)$
(C)	If $c = \sqrt{3}$ and $f(x) = x + \frac{1}{x}$, then interval of x in which LMVT is applicable for $f(x)$ is	(r)	$(0, 3)$
(D)	If $c = \frac{1}{2}$ and $f(x) = 2x - x^2$, then the interval of x in which LMVT is applicable for $f(x)$, is	(s)	$(-1, 1)$

66. Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	A circular plate is expanded by heat from radius 5 cm to 5.06 cm. Approximate increase in area is	(p)	4
(B)	If an edge of a cube increases by 1%, then percentage increase in volume is	(q)	0.6π
(C)	If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is twice the rate of decrease of x, then x is equal to (rate of decrease is non-zero)	(r)	3
(D)	Rate of increase in area of equilateral triangle of side 15 cm, when each side is increasing at the rate of 0.1 cm/s, is	(s)	$\frac{3\sqrt{3}}{4}$

dy/dx as Rate Measurer & Tangents, Normals Exercise 6 :
Single Integer Answer Type Questions

67. A point is moving along the curve $y^3 = 27x$. The interval in which the abscissa changes at slower rate than ordinate, is (a, b). Then, $(a + b)$ is ....

68. The slope of the curve $2y^2 = ax^2 + b$ at $(1, -1)$ is -1. Then, $(a - b)$ is ....

69. Let $f(1) = -2$, $f'(x) \geq 4.2$ for $1 \leq x \leq 6$. The smallest possible value of $f(6) - 16$ is ....

70. Let $f(x) = \begin{cases} -x^2, & \text{for } x < 0 \\ x^2 + 8, & \text{for } x \geq 0 \end{cases}$. Then, the absolute value of x-intercept of the line that is tangent to the graph of $f(x)$, is ....

71. The tangent to the graph of the function $y = f(x)$ at that point with abscissa, $x = a$ forms with the X- axis at an angle of $\frac{\pi}{3}$ and the point with abscissa at $x = b$ at an angle $\frac{\pi}{4}$, then the value of $\left| \int_a^b f'(x) \cdot f''(x)\, dx \right|$ is ....

72. Two curves $C_1 : y = x^2 - 3$ and $C_2 : y = kx^2,\ k \in R$ intersect each other at two different points. The tangent drawn to C_2 at one of the points of intersection $A(a, y_1), (a > 0)$ meets C_1 again at $B(1, y_2)$.
The value of a is ....

73. Consider the function $f(x) = 8x^2 - 7x + 5$ on the interval $[-6, 6]$. The value of c that satisfies the conclusion of the mean value theorem, is ....

74. Suppose that f is differentiable for all x and that $f'(x) \leq 2$ for all x. If $f(1) = 2$ and $f(4) = 8$, then $f(2)$ has the value equal to ....

75. Suppose a, b and c are positive integers with $a < b < c$ such that $1/a + 1/b + 1/c = 1$. The value of $(a + b + c - 5)$ is ....

Subjective Type Questions

76. Show that a tangent to an ellipse whose segment intercepted by the axes is the shortest, is divided at the point of tangency into two parts respectively, is equal to the semi-axes of the ellipse.

77. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that points of contact lie on $y^2 = \dfrac{x^2}{1+x^2}$.

78. If f is a continuous function with $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2$.

79. Find the equation of the straight line which is a tangent at one point and normal at another point to the curve $y = 8t^3 - 1$, $x = 4t^2 + 3$.

80. Let a curve $y = f(x)$ passes through (1, 1), at any point P on the curve tangent and normal are drawn to intersect the X-axis at Q and R, respectively. If $QR = 2$, find the equation of all such possible curves.

81. Show that the angle between the tangent at any point P and the line joining P to the origin 'O' is the same at all points of the curve $\log(x^2 + y^2) = c \tan^{-1}(y/x)$, where c is constant.

82. If the equation of two curves are $y^2 = 4ax$ and $x^2 = 4ay$

(i) Find the angle of intersection of two curves.

(ii) Find the equation of common tangents to these curves.

83. A straight line intersects the three concentric circles at A, B, C. If the distance of the line from the centre of the circles is 'P', prove that the area of the triangle formed by the tangents to the circle at A, B, C is $\left(\dfrac{1}{2P} \cdot AB \cdot BC \cdot CA\right)$.

84. Find the equation of all possible curves such that length of intercept made by any tangent on X-axis is equal to the square of x-coordinate of the point of tangency. Given that the curve passes through $(2, 1)$.

85. The tangent to the curve $y = x - x^3$ at a point P meets the curve again at Q. Prove that one point of trisection of PQ lies on the Y-axis. Find the locus of the other points of trisection.

86. Determine all the curves for which the ratio of the length of the segment intercepted by any tangent on the Y-axis to the length of the radius vector is constant.

87. If t be a real number satisfying $2t^3 - 9t^2 + 30 - a = 0$, then the values of the parameter a for which the equation $x + \dfrac{1}{x} = t$ gives six real and distinct values of x.

dy/dx as Rate Measurer & Tangents, Normals Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

88. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is **[Integer Type Question 2014]**

89. Let $f(x) = 2 + \cos x$, for all real x.

Statement I For each real t, there exists a point c in $[t, t + \pi]$, such that $f'(c) = 0$. Because

Statement II $f(t) = f(t + 2\pi)$ for each real t. **[Assertion and Reason 2007]**

(a) Statement I is correct, Statement II is also correct; Statement II is the correct explanation of Statement I

(b) Statement I is correct, Statement II is also correct; Statement II is not the correct explanation of Statement I

(c) Statement I is correct; Statement II is incorrect

(d) Statement I is incorrect; Statement II is correct

90. If $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2, \forall x_1, x_2 \in R$. Find the equation of tangent to the curve $y = f(x)$ at the point (1, 2). **[Analytical Descriptive 2005]**

91. The point(s) on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical, is (are) **[One Correct Option 2002]**

(a) $\left(\pm\dfrac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\sqrt{\dfrac{11}{3}}, 0\right)$ (c) $(0, 0)$ (d) $\left(\pm\dfrac{4}{\sqrt{3}}, 2\right)$

92. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive X-axis, then $f'(3)$ is equal to **[One Correct Option 2000]**

(a) −1 (b) −3/4 (c) 4/3 (d) 1

(ii) JEE Main & AIEEE

93. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the Y-axis passes through the point **[2017 JEE Main]**

(a) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (b) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, -\frac{1}{3}\right)$

94. The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1,1) **[2015 JEE Main]**

(a) does not meet the curve again
(b) meets in the curve again the second quadrant
(c) meets the curve again in the third quadrant.
(d) meets the curve again in the fourth quadrant

Answers

Chapter Exercises

1. (a) **2.** (b) **3.** (a) **4.** (b) **5.** (b)
6. (b) **7.** (b) **8.** (a) **9.** (b) **10.** (b)
11. (c) **12.** (b) **13.** (c) **14.** (c) **15.** (c)
16. (d) **17.** (c) **18.** (b) **19.** (a) **20.** (d)
21. (d) **22.** (b) **23.** (a) **24.** (a) **25.** (c)
26. (d) **27.** (d) **28.** (a) **29.** (b) **30.** (b)
31. (d) **32.** (b) **33.** (a) **34.** (d)
35. (a, b) **36.** (a, b, d) **37.** (b, c) **38.** (b, c)
39. (b, d) **40.** (b, c, d)
41. (a, b, d) **42.** (b, d) 43. (a, b) **44.** (c, d)
45. (c, d) **46.** (a, b, c) **47.** (a, c)
48. (a, b, d) **49.** (a, b, c, d) **50.** (c)
51. (d) **52.** (a) **53.** (d) **54.** (d) **55.** (c)
56. (a) **57.** (b) **58.** (c) **59.** (b) **60.** (d) **61.** (a)
62. (b) **63.** (b) **64.** (d)
65. (A) $\rightarrow$ (q); (B) $\rightarrow$ (r); (C) $\rightarrow$ (q); (D) $\rightarrow$ (p)
66. (A) $\rightarrow$ (q); (B) $\rightarrow$ (r); (C) $\rightarrow$ (p); (D) $\rightarrow$ (s)
67. (0) **68.** (2) **69.** (3) **70.** (1) **71.** (1)
72. (3) **73.** (0) **74.** (4) **75.** (6)
79. $\pm \sqrt{2}(27x - 105)$
80. $\log y - x = \pm \left(\log - \left| \frac{1 - \sqrt{1 - y^2}}{y} \right| + \sqrt{1 - y^2} \right) - 1$
82. (i) $Q = \tan^{-1}\left(\frac{3}{4}\right)$ (ii) $x + y = a = 0$
84. Possible curve are $y = \frac{x}{2(x-1)}$ or $y = \frac{3x}{2(1+x)}$.
85. $y = x - 5x^3$ **86.** $(y + \sqrt{x^2 + y^2})\, x^{k-1} = c_1$
87. No real value **88.** 8 **89.** (b) **90.** $y - 2 = 0$
91. (d) **92.** (d) **93.** (c) **94.** (d)

Solutions

1. Here, $f(x) = x^3 - nx + 1$

$$f(0) = 1 \text{ and } f(1) = 1 - n + 1 = 2 - n$$

$\because \quad n \ge 3$

$\therefore \quad f(1) < 0$ and also we have, $f(0) > 0$

$\therefore \quad f(x)$ must have at least one real root in $(0, 1)$.

2. Given, slope of normal to $y = f(x)$ is 1.

$$\Rightarrow \quad \left(-\frac{1}{f'(x)}\right)_{(0,\,0)} = 1$$

$$\Rightarrow \quad f'(0) = -1 \qquad \text{...(i)}$$

Now, $\displaystyle\lim_{x\to 0} \frac{x^2}{f(x^2) - 20f(9x^2) + 2f(99x^2)}$

$$= \lim_{x\to 0} \frac{2x}{f'(x^2)\cdot 2x - 20f'(9x^2)\cdot 18x + 2f'(99x^2)\cdot(198x)}$$

[using L' Hospital's rule]

$$= \lim_{x\to 0} \frac{1}{f'(x^2) - 180f'(9x^2) + f'(99x^2)\cdot(198)}$$

$$= \frac{1}{f'(0) - 180\cdot f'(0) + 198\cdot f'(0)}$$

$$= \frac{1}{-1 + 180 - 198}$$

$$= -\frac{1}{19}$$

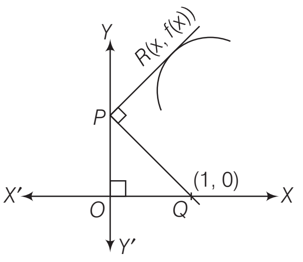
3. The equation of the tangent at the point $R(x, f(x))$ is

$$Y - f(x) = f'(x)(X - x) \qquad \text{...(i)}$$

The point of intersection on Y-axis, is $P(0, f(x) - x f'(x))$.

The slope of line perpendicular to tangent at R, is

$$m_{PQ} = \frac{(f(x) - xf'(x)) - 0}{0 - 1} \qquad \text{...(ii)}$$



$$\therefore \quad f'(x)\cdot m_{PQ} = -1$$

$$\Rightarrow \quad f'(x)\cdot \frac{(f(x) - xf'(x))}{-1} = -1$$

$$\Rightarrow \quad f(x)f'(x) - x(f'(x))^2 = 1$$

4. Here, $y^2 = x + \sin x$

$$\Rightarrow \quad 2y\,dy/dx = 1 + \cos x$$

For horizontal tangent, $dy/dx = 0$

$$\therefore \quad \cos x = -1 \Rightarrow x = (2n+1)\pi$$

Since, $\quad y^2 = x + \sin x$ and $|y| \le 3$

$$\Rightarrow \quad 0 \le x \le 9$$

$$\Rightarrow \quad 0 \le (2n+1)\pi \le 9$$

$$\therefore \quad n = 0 \Rightarrow x = \pi$$

$$\Rightarrow \quad y^2 = \pi \text{ or } y = \pm\sqrt{\pi}$$

Thus, points are $(\pi, \sqrt{\pi})$ and $(\pi, -\sqrt{\pi})$.

$\therefore$ Number of points is 2.

5. Here, $\dfrac{a+2c}{b+3d} = -\dfrac{4}{3}$

$$\Rightarrow \quad 3a + 6c = -4b - 12d$$

$$\Rightarrow \quad 3a + 4b + 6c + 12d = 0$$

$$\text{or} \quad \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0 \qquad \text{...(i)}$$

Consider, $\quad f(x) = \dfrac{ax^4}{4} + \dfrac{bx^3}{3} + \dfrac{cx^2}{2} + dx$

Then, $\quad f(0) = 0 = f(1) \qquad$ [using Eq. (i)] ...(ii)

$\therefore$ $f(x)$ satisfy the condition of Rolle's theorem in $[0, 1]$.

Hence, $f'(x) = 0$ has at least one solution in $(0, 1)$.

6. Let $f(x) = \dfrac{ax^4}{4} + \dfrac{bx^3}{3} + \dfrac{cx^2}{2} + dx$,

which is continuous and differentiable.

Also, $\quad f(0) = 0,\ f(-1) = \dfrac{a}{4} - \dfrac{b}{3} + \dfrac{c}{2} - d$

$$= \frac{1}{4}(a + 2c) - \frac{1}{3}(b + 3d) = 0$$

So, according to Rolle's theorem, there exists at least one root of $f'(x) = 0$ in $(-1, 0)$.

7. Let us consider $g(t) = \displaystyle\int_0^t f(x)\,dx$

Applying LMVT in (0, 2), we get

$$\frac{g(2) - g(0)}{2 - 0} = g'(c), \text{ where } c \in (0, 2)$$

$$\Rightarrow \quad \int_0^2 f(x)\,dx = 2f(c), \text{ where } c \in (0, 2)$$

8. Replace x by $-x$

$\Rightarrow \quad x[f(x) + f(-x)] = 0 \Rightarrow f(x)$ is an odd function.

$\Rightarrow \quad f^{iv}(x)$ is also odd $\Rightarrow f^{iv}(0) = 0$

9. Given, $x + y - \ln(x + y) = 2x + 5$

$$\Rightarrow \quad 1 + \frac{dy}{dx} - \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = 2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = \frac{\alpha + \beta + 1}{\alpha + \beta - 1} = \infty, \text{ when } \alpha + \beta = 1$$

10. Since, $f''(x)$ is odd function.

$\therefore \quad g(\alpha, \beta) = 0$

$\Rightarrow \quad (\sin^4 \alpha - 1)^2 + (\cos^4 \beta - 1)^2 + 2(\sin^2 \alpha - \cos^2 \beta)^2 = 0$

$\Rightarrow \quad \sin^2 \alpha + \sin^2 \beta = 1$

11. Let α be the root of $f(x) = 0$.

$\Rightarrow \quad f(\alpha) = f(2 - \alpha) = 0$

$\therefore$ $f(x)$ has 4 distinct and 2 equal roots.

$\therefore$ Sum of the roots = 6

12. Given, $y \dfrac{dy}{dx} = x$

$\Rightarrow \quad y\,dy = x\,dx \Rightarrow y^2 = x^2 + c$

$\because \quad f(1) = 3$

$\therefore \quad 9 = 1 + c \Rightarrow c = 8 \Rightarrow y^2 = x^2 + 8$

$\Rightarrow \quad f(x) = \sqrt{x^2 + 8}$

$\Rightarrow \quad f(4) = \sqrt{16 + 8} = 2\sqrt{6}$

13. Given, $x^2 y = c^3$

$\Rightarrow \quad x^2 \dfrac{dy}{dx} + 2xy = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{2y}{x}$

Equation of tangent at (x, y), is $Y - y = -\dfrac{2y}{x}(X - x)$

$Y = 0$, gives $X = \dfrac{3x}{2} = a$

and $\quad X = 0$, gives $Y = 3y = b$

Now, $\quad a^2 b = \dfrac{9x^2}{4} \cdot 3y = \dfrac{27}{4} x^2 y = \dfrac{27}{4} c^3$

14. Clearly, $f(0) = f(1) = 0$ and $f(x)$ is a polynomial of degree 10.

Therefore, by LMVT, we must have at least one root in (0, 1).

Since, the degree of $f(x)$ is even.

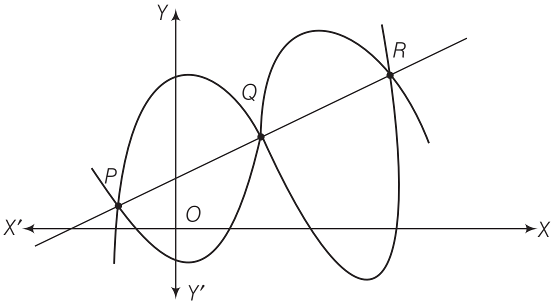
$\therefore$ It has at least two real roots.

15. Let (x_1, y_1) and (x_2, y_2) be two intersect points.

Given, $\quad y = x^3 + 2x - 1$ and $y = 2x^3 - 4x + 2$

$\therefore \quad y_1 = 2x_1^3 - 4x_1 + 2 \quad$ ...(i)

and $\quad 2y_1 = 2x_1^3 + 4x_1 - 2 \quad$ ...(ii)



Subtracting Eq. (i) from Eq. (ii), we get

$y_1 = 8x_1 - 4 \quad$ ...(iii)

Similarly, $\quad y_2 = 8x_2 - 4 \quad$ ...(iv)

Now, from Eqs. (iii) and (iv), we get $y_2 - y_1 = 8(x_2 - x_1)$

$\therefore \quad \dfrac{y_2 - y_1}{x_2 - x_1} = 8$

16. (a) Discontinuous at $x = 1$, therefore not applicable.

(b) Discontinuous at $x = 0$, therefore not applicable.

(c) Discontinuity at $x = 1 \Rightarrow$ not applicable.

(d) Note that $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$.

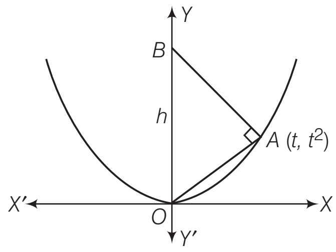
Hence, $f(x) = x^2 - x - 6$, if $x \neq 1$ and $f(1) = -6$

$\Rightarrow$ f is continuous at $x = 1$.

So, $f(x) = x^2 - x - 6$ is continuous throughout the interval $[-2, 3]$.

Also, note that $f(-2) = f(3) = 0$. Hence, Rolle's theorem is applicable.

17. Let $A = (t, t^2)$. Then, $m_{OA} = t, m_{AB} = -\dfrac{1}{t}$



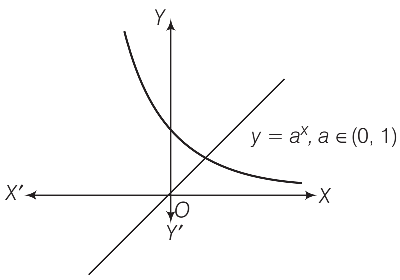
Equation of AB, $\quad y - t^2 = -\dfrac{1}{t}(x - t)$

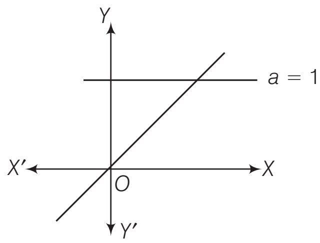
Putting $\quad x = 0$

$\Rightarrow \quad h = t^2 + 1$

Now, $\quad \lim\limits_{t \to 0} (h) = \lim\limits_{t \to 0} (1 + t^2) = 1$

18. For $0 < a \leq 1$, the line always intersects $y = a^x$



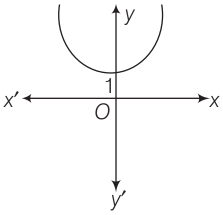


For $a > 1$, say $a = e$. Consider $f(x) = e^x - x$, $f'(x) = e^x - 1$

$f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$

$\therefore$ $f(x)$ is increasing $(\uparrow)$ for $x > 0$

and decreasing $(\downarrow)$ for $x < 0$



$y = e^x$ always lies above $y = x$, i.e. $e^x - x \geq 1$ for $a > 1$

Hence, the line $y = x$ intersect when $a \in (0, 1]$.

19. $\because$ f is not differentiable at $x = \frac{1}{2}$,

g is not continuous in $[0, 1]$ at $x = 0$ and 1,

and h is not continuous in $[0, 1]$ at $x = 1$

$$k(x) = (x+3)^{\ln_2 5} = (x+3)^p, \text{ where } 2 < p < 3$$

$\therefore$ None of these, f, g, h follows Lagrange's mean value theorem.

20. $f'(x) = 0$ and $f''(x) = 0$ for the same $x = x_1$ [say]

Now, $\quad f'(x) = 4x^3 + 2bx + 8$

$$f'(x_1) = 2[2x_1^3 + bx_1 + 4] = 0 \quad \text{...(i)}$$

$$f''(x_1) = 2[6x_1^2 + b] = 0 \quad \text{...(ii)}$$

From Eq. (ii), $\quad b = -6x_1^2$

Substituting this value of b in Eq. (i), we get

$$2x_1^3 + (-6x_1^3) + 4 = 0$$

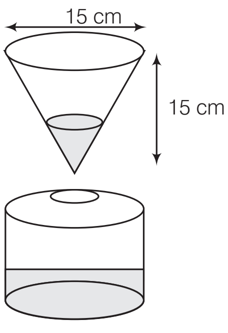
$\Rightarrow \quad 4x_1^3 = 4$

$\Rightarrow \quad x_1 = 1$

Hence, $\quad b = -6$

21. For a cylindrical pot, $V = \pi r^2 h$

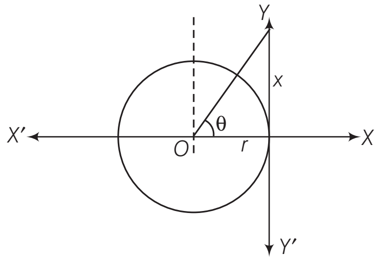
$$\Rightarrow \quad \frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h \cdot 2r\frac{dr}{dt}\right] \quad \left[r = \text{constant}, \frac{dr}{dt} = 0\right]$$



$$\Rightarrow \quad 100 = \pi r^2 \frac{dh}{dt} \Rightarrow 100 = \pi \cdot \frac{225}{4} \cdot \frac{dh}{dt} \quad \left[\because r = \frac{15}{2} \text{ cm}\right]$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{400}{225\pi} = \frac{16}{9\pi} \text{ cm/min}$$

22. $\tan\theta = x/r \Rightarrow x = r\tan\theta$

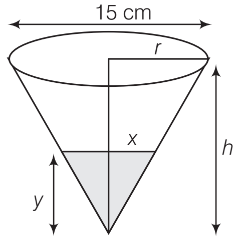


$\Rightarrow \quad dx/dt = r\sec^2\theta\,(d\theta/dt) = r\,\omega\sec^2\theta = v\sec^2\theta$

when, $\quad \theta = 2\pi/8,\ dx/dt = v\sec^2(\pi/4) = 2v = 40$ km/h.

23. $\frac{dV}{dt} = 15,\ h = 3r,\ V = \frac{1}{3}\pi x^2 y,\ \frac{dx}{dt} = ?$, when $x = 5$

$$\frac{x}{y} = \frac{r}{h} = \frac{1}{3} \Rightarrow V = \frac{2}{3}\pi r^2 3x = \pi x^3$$



$$\frac{dV}{dt} = 3\pi x^2\frac{dx}{dt}$$

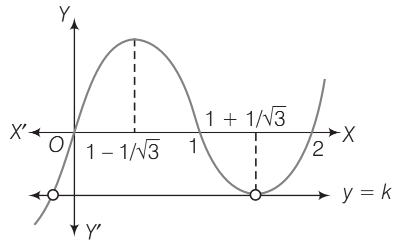
$$\Rightarrow \quad 15 = 3\pi \cdot 25\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{5\pi}$$

24. Given, $f(x) = x(x^2 - 3x + 2)$

$\Rightarrow f(x) = x(x-2)(x-1)$

Graph of $y = f(x)$ is shown as



Now, for exactly one positive and one negative solution of the equation $f(x) = K$

We should have, $\quad k = f\left(1 + \frac{1}{\sqrt{3}}\right)$

$$\left[\because -\frac{1}{\sqrt{3}} \text{ and } 1 + \frac{1}{\sqrt{3}} \text{ are the roots of } f'(x) = 0\right]$$

$$\therefore \quad k = \underbrace{\left(1 + \frac{1}{\sqrt{3}}\right)}_{x}\underbrace{\left(\frac{1}{\sqrt{3}} - 1\right)}_{x-2}\underbrace{\left(\frac{1}{\sqrt{3}}\right)}_{x-1}$$

$$= \left(\frac{1}{3} - 1\right)\left(\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$$

$$= -\frac{2\sqrt{3}}{3}$$

25. We have, $\dfrac{a}{x^2} + \dfrac{b}{y^2} = 1$

$$\Rightarrow \quad -\frac{2a}{x^3} - \frac{2b}{y^3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ay^3}{bx^3} \quad \text{...(i)}$$

Equation of tangent is $Y - y = -\dfrac{ay^3}{bx^3}(X - x)$

For x-intercept, put $Y = 0$

$$\therefore \quad X = \frac{bx^3}{ay^2} + x$$

$$\Rightarrow \quad X = x\left[\frac{bx^2 + ay^2}{ay^2}\right] = x\left[\frac{x^2y^2}{ay^2}\right] = \frac{x^3}{a}$$

[using Eq. (i)]

$\Rightarrow$ x-intercept is proportional to cube of abscissa.

26. Using LMVT in $[-2, 5]$, we get

$$-4 \le \frac{f(5) - f(-2)}{7} \le 3$$

$$\Rightarrow \quad -28 \le f(5) - f(-2) \le 21$$

27. Given, $x = t + e^{at}$, $y = -t + e^{at}$

$$\Rightarrow \quad \frac{dx}{dt} = 1 + ae^{at}, \frac{dy}{dt} = -1 + ae^{at}, \frac{dy}{dx} = \frac{-1 + ae^{at}}{1 + ae^{at}}$$

At the point A, $y = 0$ and $\dfrac{dy}{dx} = 0$ for some $t = t_1$

$$\therefore \quad ae^{at_1} = 1 \quad \text{...(i)}$$

$$\text{Also,} \quad 0 = -t_1 + e^{at_1}$$

$$\therefore \quad e^{at_1} = t_1 \quad \text{...(ii)}$$

Putting this value in Eq. (i), we get

$$at_1 = 1 \Rightarrow t_1 = \frac{1}{a}$$

Now from Eq. (i), $ae = 1$

$$\Rightarrow \quad a = \frac{1}{e}$$

Hence, $x_A = t_1 + e^{at_1} = e + e = 2e$

$$\Rightarrow \quad A \equiv (2e, 0)$$

28. Given, $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos t - \cos 2t}{\sin 2t - \sin t} = 0$$

$$\Rightarrow \quad \cos 2t = \cos t$$

$$\Rightarrow \quad \cos 2t = \cos(2\pi - t)$$

$$\Rightarrow \quad t = 2\pi / 3$$

29. $F'(x) = e^{(1 + \sin^{-1}(\cos x))^2} \cdot (-\sin x) - e^{(1 + x)^2} \cdot \cos x$

$$\therefore \quad F'(0) = 0 - e = -e$$

$$\text{and} \quad F'\left(\frac{\pi}{2}\right) = -e - 0 = -e$$

Hence, Rolle's theorem is applicable for the function $F'(x)$.

So, there exists c in $\left(0, \dfrac{\pi}{2}\right)$ for which $F''(c) = 0$ as

Rolle's theorem is applicable for $F'(x)$ in $\left[0, \dfrac{\pi}{2}\right]$.

Also, $\quad F(0) = \displaystyle\int_0^1 f(t)\,dt$ and $F\left(\dfrac{\pi}{2}\right) = \displaystyle\int_1^0 f(t)\,dt$

Hence, $F(0)$ and $F\left(\dfrac{\pi}{2}\right)$ have opposite signs.

$$\Rightarrow \quad F(c) = 0 \text{ for some } c \in \left(0, \frac{\pi}{2}\right)$$

30. Given, $f'(1) = 1, 2 \cdot f'(2x) = f'(x)$

Put $x = 1$, $f'(2) = \dfrac{f'(1)}{2} = \dfrac{1}{2}$

and $\quad f'(4) = \dfrac{1}{2} f'(2) = \dfrac{1}{4}$

Applying LMVT for $y = f'(x)$ in $[2, 4]$, we get

$$f''(c) = \frac{f'(4) - f'(2)}{2} = \frac{\frac{1}{4} - \frac{1}{2}}{2} = -\frac{1}{8}$$

31. Consider $\phi(x) = f(x) - g(x) \Rightarrow \phi'(x) = f'(x) - g'(x) > 0$.

Clearly, $\phi(x)$ is also continuous and derivable in $[x_0, x]$.

Using LMVT for $\phi(x)$ in $[x_0, x]$, we get

$$\phi'(c) = \frac{\phi(x) - \phi(x_0)}{x - x_0}$$

Since, $\quad \phi'(x) = f'(x) - g'(x) > 0$ for all $x > x_0$

$$\therefore \quad \phi'(c) > 0$$

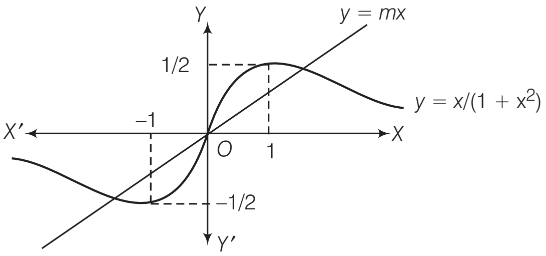
$$\text{Hence,} \quad \phi(x) - \phi(x_0) > 0$$

$$\Rightarrow \quad \phi(x) > \phi(x_0)$$

$$\Rightarrow \quad \phi(x) > 0 \quad [\because \phi(x_0) = f(x_0) - g(x_0) = 0]$$

$$\Rightarrow \quad f(x) - g(x) > 0$$

32. Solving, $\quad mx = \dfrac{x}{x^2 + 1} \Rightarrow x^2 + 1 = \dfrac{1}{m}$ or $x = 0$



$$\Rightarrow \quad x^2 = \frac{1}{m} - 1 > 0 \text{ for a region}$$

$$\Rightarrow \quad \frac{m - 1}{m} < 0 \Rightarrow m \in (0, 1)$$

Remark

For $m = 0$ or 1, the line does not enclose a region.

33. $\because$ Side $= x$

$\therefore$ Area $A = x^2 \Rightarrow \dfrac{dA}{dx} = 2x$. So, $\left\{\left(\dfrac{dA}{dx}\right)_{x = x_0} \times h\right\} = 2x_0h$

The exact change in the area of S when x is changed from x_0 to $x_0 + h$, is

$$(x_0 + h)^2 - x_0^2 = x_0^2 + 2x_0h + h^2 - x_0^2 = 2x_0h + h^2$$

The difference between the exact change and the approximate change is $2x_0h + h^2 - 2x_0h = h^2$

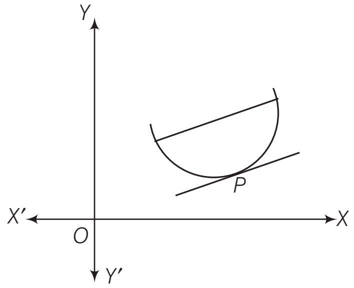
34. $f(x)=\int_1^x\left(t+\frac{1}{t}\right)dt \Rightarrow f'(x)=x+\frac{1}{x}$

$\therefore \quad g(x)=x+\frac{1}{x}$

For $\quad x\in\left[\frac{1}{2},3\right]$,

$$g\left(\frac{1}{2}\right)=2+\frac{1}{2}=\frac{5}{2},\ g(3)=3+\frac{1}{3}=\frac{10}{3}$$

Let $\quad P\equiv(c,g(c)),\ c\in\left[\frac{1}{2},3\right]$



By LMVT, $\quad g'(c)=\dfrac{g(3)-g\left(\frac{1}{2}\right)}{3-\frac{1}{2}}$

$\therefore \quad 1-\dfrac{1}{c^2}=\dfrac{\frac{10}{3}-\frac{5}{2}}{3-\frac{1}{2}}$

$\Rightarrow \quad c^2=\dfrac{3}{2}$

$\Rightarrow \quad c=\sqrt{\dfrac{3}{2}}$

$\therefore \quad g(c)=\sqrt{\dfrac{3}{2}}+\dfrac{1}{\sqrt{\frac{3}{2}}}=\dfrac{5}{\sqrt{6}}$

Thus, $\quad P\equiv\left(\sqrt{\dfrac{3}{2}},\dfrac{5}{\sqrt{6}}\right)$

35. Here, $\dfrac{dx}{dt}=\dfrac{2(-\operatorname{cosec}^2 t)}{\cot t}$

$\Rightarrow \quad \left.\dfrac{dx}{dt}\right|_{\text{at } t=\frac{\pi}{4}}=-4$

and $\quad \dfrac{dy}{dt}=\sec^2 t-\operatorname{cosec}^2 t$

$\Rightarrow \quad \left.\dfrac{dy}{dt}\right|_{\text{at } t=\frac{\pi}{4}}=0$

$\therefore \quad \dfrac{dy}{dx}=0$

Hence, tangent is parallel to X-axis and its normal is parallel to Y-axis.

36. Here, $f'(x)=\dfrac{1}{3x^{2/3}}$

$\Rightarrow f'(0)\to\infty$ and tangent is vertical at $x=0$.

Equation of tangent at $(0,0)$ is $x=0$.

Equation of normal is $y=0$.

Now, $\quad f(x)=f^{-1}(x)\Rightarrow x^{1/3}=x^3$

$\Rightarrow \quad x^9=x \Rightarrow x=0,1,-1$

37. Here, $\dfrac{dy}{dx}=k^2e^{kx}$

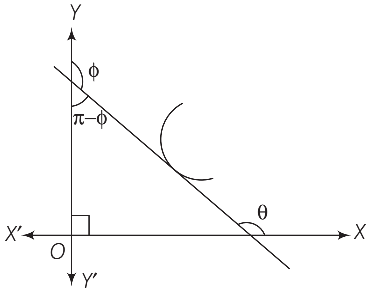
$\Rightarrow \left(\dfrac{dy}{dx}\right)_{x=0}=k^2=\tan\theta$, where θ is angle made by X- axis.

Let ϕ be the angle made by Y-axis.

$\therefore \quad \tan\theta=\tan\left(\dfrac{3\pi}{2}-\phi\right)=\cot\phi$

$\Rightarrow \quad \cot\phi=k^2 \Rightarrow \phi=\cot^{-1}k^2$

$\Rightarrow \quad \phi=\sin^{-1}\left(\dfrac{1}{\sqrt{1+k^4}}\right)$



38. As, $f(x)$ is continuous in $[0,1]$ and differentiable in $(0,1)$ and $f(0)=f(1)=1$.

$\therefore$ By Rolle's theorem, there must exist at least one $x=c\in(0,1)$ such that $f'(c)=0$

$\therefore f'(c)$ vanishes for some $c\in(0,1)$.

Now, $f(0)=1,\ f(1/2)=-\dfrac{1}{2}$ and $f(1)=1$

$\therefore$ By intermediate value theorem, $f(x)$ must have one root belongs to $\left(0,\dfrac{1}{2}\right)$ and other in the interval $\left(\dfrac{1}{2},1\right)$.

39. Here, $f(0)=f(1)$ and f is continuous in $[0,1]$ and derivable in $(0,1)$.

$\therefore f'(c_1)=0$ for at least one $c_1\in(0,1)$

Similarly, as $f(1)=f(2)$

$\therefore\ f'(c_2)=0$ for at least one $c_2\in(1,2)$. $\Rightarrow f'(c_1)=f'(c_2)$

$\Rightarrow\ f''(c)=0$ for at least one $c\in(c_1,c_2)$.

40. Let $f(x)=\dfrac{1}{(x+1)^3}-3x+\sin x$

Domain of f is $(-\infty,-1)\cup(-1,\infty)$.

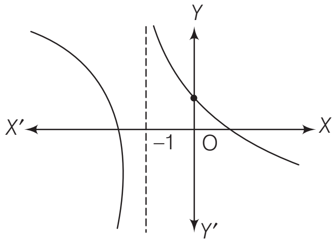
$$f'(x)=-3\left[\frac{1}{(x+1)^4}+1\right]+\cos x \Rightarrow f'(x)<0$$

$\Rightarrow f$ is decreasing.

Also, $\quad \lim\limits_{x\to-1^+}f(x)\to\infty,\quad \lim\limits_{x\to-1^-}f(x)\to-\infty$

and $\quad \lim\limits_{x\to\infty}f(x)\to-\infty,\quad \lim\limits_{x\to-\infty}f(x)\to\infty$

$\Rightarrow f(x) = 0$ has exactly two roots.



41. Clearly, $f(-1) = -f(1)$ and $f(0) = 0$. For (a) and (b) apply LMVT for the function $f(x)$ in $(-1, 0)$ and $(0, 1)$, respectively. For (d) apply LMVT for $(f(x))^n$ in $(0, 1)$.

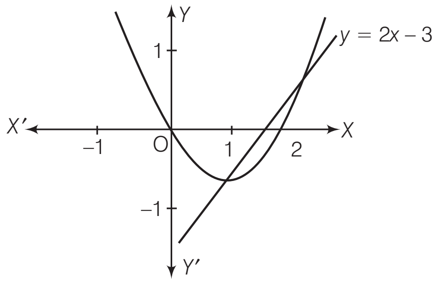
42. When $x = 1, y = -1$ [from the line]

Thus, it must lie on the parabola $y = x^2 + px + q$

$\Rightarrow \ -1 = 1 + p + q \ \Rightarrow \ p + q = -2$

$\therefore$ Now, distance of the vertex of the parabola from the X-axis is

$$d = f\left(-\frac{p}{2}\right) = \frac{p^2}{4} - \frac{p^2}{2} + q = q - \frac{p^2}{4}$$



Substituting $q = -2 - p$, we get

$$d = -2 - p - \frac{p^2}{4}$$

Now, take $g(p) = -2 - p - \dfrac{p^2}{4}$

So, $\quad g'(p) = -1 - \dfrac{p}{2} = 0 \ \Rightarrow \ p = -2$

Hence, $\quad q = 0$

Note that least distance of the vertex from X-axis is 1.

43. Given, $\sqrt{xy} = a + x \ \Rightarrow \ xy = a^2 + x^2 + 2ax$

$$\Rightarrow \quad y = \frac{a^2}{x} + x + 2a$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{a^2}{x^2} + 1 = -1$$

$$\Rightarrow \quad 2x^2 = a^2$$

$$\Rightarrow \quad x = \pm \frac{a}{\sqrt{2}}$$

44. Given, $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ [say]

$\therefore \quad da = 2R \cos A \, dA, \; db = 2R \cos B \, dB,$
$dc = 2R \cos C \, dC$

$$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC) \quad \text{...(i)}$$

Also, $\quad A + B + C = \pi$

So, $\quad dA + dB + dC = 0 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$$

45. $\dfrac{f(x) - f(0)}{x - 0} = f'(c) \le 1 - x$ for some $c \in (0, 1)$.

$\Rightarrow \ f(x) \le x(1 - x) \le 1/4$

46. Here, $f(x) = \dfrac{\ln x}{x}$ ...(i)

$\therefore \quad$ Domain is R^+.

$$\because \quad f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

(a) For horizontal tangent, $f'(x) = 0$

$\Rightarrow \quad \ln x = 1 \ \Rightarrow \ x = e$ [true]

(b) If $f(x)$ intersects the X-axis.

$\dfrac{\ln x}{x} = 0 \ \Rightarrow \ x = 1$ [true]

(c) $f'(x)$ is positive, if $x \in (0, e)$ and $f'(x)$ is negative, if $x \in (e, \infty)$

$\therefore \ f(x)$ is not monotonic.

$\therefore \ f(x)$ is many-one.

(d) For vertical tangent, $f'(x) = \infty$ [true]

$\Rightarrow \ \dfrac{1 - \ln x}{x^2} = \infty \ \Rightarrow \ \dfrac{x^2}{1 - \ln x} = 0$

$\Rightarrow \ x = 0$ which is not in the domain of $f(x)$. [false]

47. Let the tangent line be $y = ax + b$

The equation for its intersection with the upper parabola is

$$x^2 + 1 = ax + b$$

$$\Rightarrow \quad x^2 - ax + (1 - b) = 0$$

This has a double root when $a^2 - 4(1 - b) = 0$

or $\quad a^2 + 4b = 4$

For the lower parabola, $\ ax + b = -x^2$

$$\Rightarrow \quad x^2 + ax + b = 0$$

This has a double root when $a^2 - 4b = 0$

On subtracting these two equations, we get $8b = 4$ or $b = \dfrac{1}{2}$

On adding these equations, we get $2a^2 = 4$ or $a = \pm\sqrt{2}$

The tangent lines are $y = \sqrt{2}\,x + \dfrac{1}{2}$ and $y = -\sqrt{2}\,x + \dfrac{1}{2}$

48. For some $\alpha \in (0, 1), |f'(\alpha)| = \left|\dfrac{f(1) - f(0)}{1 - 0}\right| \le |f(1)| + |f(0)|$

$\Rightarrow \quad |f'(\alpha)| \le 1 + 1 = 2$

Similarly, for some $\beta \in (-1, 0), |f'(\beta)| \le 2$

Also, $\quad F(x) = (f(x))^2 + (f'(x))^2$

$\Rightarrow \quad F(\alpha) = (f(\alpha))^2 + (f'(\alpha))^2 \le 1 + 4 \le 5$

Similarly, $F(\beta) \le 5$ for some $\beta \in (-1, 0)$

As, $F(0) = 6$, so there must be a point of local maxima for $F(x)$ in $(-1, 1)$ and at the point of maxima, say $x = c$,

$$F(c) \geq 6 \Rightarrow F'(c) = 0 \text{ and } F''(c) \leq 0$$

49. As, Rolle's theorem is applicable, the function should be continuous and differentiable in $[-3, 3]$.

So, at $x = 1$ it is continuous

$$\Rightarrow \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad a + b = c = 1 \quad \text{...(i)}$$

Since, differentiable at $x = 1$, therefore

$$f'(1^+) = f'(1^-)$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{\frac{c}{1+h} - 1}{h} = 2a$$

$$\Rightarrow \quad 2a = \lim_{h\to 0} \frac{c - 1 - h}{h(h+1)} \text{ exists only when } c = 1$$

$$\Rightarrow \quad 2a = \lim_{h\to 0} \frac{-1}{(1+h)} = -1$$

$$\therefore \quad a = -1/2 \text{ and } c = 1 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$b = 3/2 \quad \text{...(iii)}$$

$$\therefore \quad a + b + c = 2 \quad \text{...(iv)}$$

$$|a| + |b| + |c| = 3 \quad \text{...(v)}$$

$$2a + 4b + 3c = 8 \quad \text{...(vi)}$$

$$4a^2 + 4b^2 + 5c^2 = 15 \quad \text{...(vii)}$$

50. **Statement I** As, $f(-1) = f(1)$ and Rolle's theorem is not applicable, then it implies that $f(x)$ is either discontinuous or $f'(x)$ does not exist at at least one point in $(-1, 1)$.

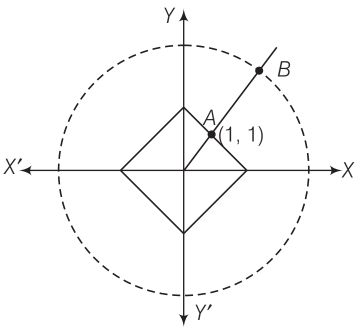
$\Rightarrow$ $g(x) = 0$ for at at least one value of x in $(-1, 1)$.

Statement II is false. Consider the example in Statement I.

51. Common normal is $y = x$

Solving, $y = x$ with $x + y = 2$, we get $A(1, 1)$

and with $x^2 + y^2 = 16$, we get $B(2\sqrt{2}, 2\sqrt{2})$



The shortest distance between the given curves is $AB = 4 - \sqrt{2}$.

But as the curves are not smooth, check at slope points. The coordinates in 1st quadrant are (2, 0) and (4, 0) and here distance = 2.

$\therefore$ $4 - \sqrt{2}$ is not the shortest.

52. If $P(x) = 0$ is a polynomial equation, then $P(-x) = 0$ has no positive root.

$\Rightarrow$ $P(x) = 0$ cannot have negative roots.

53. Let $A(a, P(a))$, $B(b, P(b))$, then slope of $AB = P'(a) = P'(b)$ from LMVT there exists $c \in (a, b)$, where $P'(c) =$ slope of AB.

54. $$f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -1, & x < \frac{1}{2} \\ 2\left(\frac{1}{2} - x\right)(-1), & x > \frac{1}{2} \end{cases}$$

Left hand derivative at $x = 1/2$ is (-1) and right hand derivative at $x = 1/2$ is 0, so the function is not differentiable at $x = 1/2$.

55. $f(x) = \ln(2 + x) - \dfrac{2x + 2}{x + 3}$ is continuous in $[-2, \infty)$.

$$f'(x) = \frac{1}{x+2} - \frac{4}{(x+3)^2} = \frac{(x+3)^2 - 4(x+2)}{(x+2)(x+3)^2}$$

$$= \frac{x^2 + 2x + 1}{(x+2)(x+3)^2} = \frac{(x+1)^2}{(x+2)(x+3)^2} > 0$$

$$[f'(x) = 0 \text{ at } x = -1]$$

$\Rightarrow$ f is increasing in $(-2, \infty)$.

Also, $\displaystyle\lim_{x\to -2^+} f(x) \to -\infty$

and $\displaystyle\lim_{x\to \infty} f(x) \to \infty \Rightarrow$ unique root.

56. Let $f(x) = 0$ has two roots say $x = r_1$ and $x = r_2$, where $r_1, r_2 \in [a, b]$.

$$\Rightarrow \quad f(r_1) = f(r_2)$$

Hence, there must exist some $c \in (r_1, r_2)$, where $f'(c) = 0$

But $f'(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$

for $x \geq 1$, $f'(x) = (x^6 - x^5) + (x^4 - x^3) + (x^2 - x) + 1 > 0$

for $x \leq 1$, $f'(x) = (1 - x) + (x^2 - x^3) + (x^4 - x^5) + x^6 > 0$

Hence, $f'(x) > 0$ for all x.

$\therefore$ Rolle's theorem fails.

$\Rightarrow$ $f(x) = 0$ cannot have two or more roots.

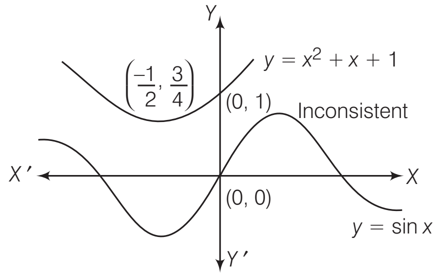
57. Let $f(x) = \cos x - \sin x + \cos^{-1} x - \sin^{-1} x$, $x \in [-1, 1]$

$\because$ $f(-1)\, f(1) < 0$

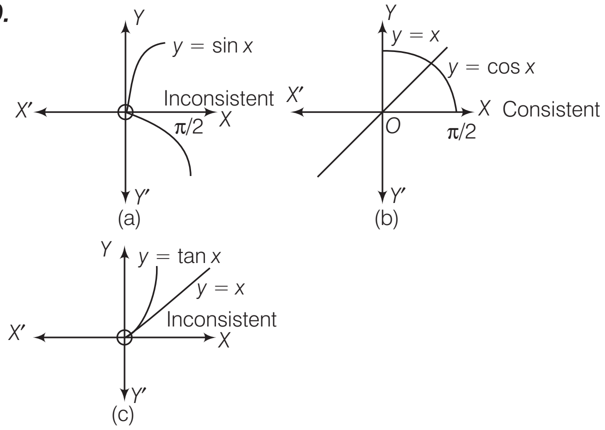
$\therefore$ There exists at least one $c \in (-1, 1)$ such that $f(c) = 0$.

Hence, the curves $y = \cos x + \cos^{-1} x$ and $y = \sin x + \sin^{-1} x$ intersect each other at at least one point.

58. Given, $\sin x = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$



59.

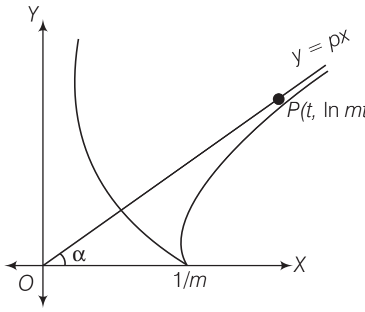


Sol. (Q. Nos. 60 to 62)

Slope of tangent at P = Slope of OP

$$\Rightarrow \quad \frac{1}{t} = \frac{\ln mt}{t}$$

$$\Rightarrow \quad t = \frac{e}{m}$$



$$\Rightarrow \quad P \equiv \left(\frac{e}{m}, 1\right)$$

$$\Rightarrow \quad \tan\alpha = p = \frac{m}{e}$$

60. $p > m/e$ **61.** $p = m/e$

62. $0 < p < m/e$

63. Putting $y = 0$, we get $x^2 - 3x - c_i = 0$

As roots are integers and $D = 9 + 4c_i$ must be perfect square, therefore $9 + 4c_i = (2\lambda + 1)^2, \lambda \in I$.

$$\Rightarrow \quad c_i = \lambda^2 + \lambda - 2$$

$$\Rightarrow \quad c_k = k(k+3), \quad k = 1, 2, 3, \ldots$$

$$\therefore \quad c_1 = 4, c_2 = 10, \ldots$$

$$\Rightarrow \quad x^2 + y^2 - 3x - 4y - (c_2 - c_1) = 0$$

or $\quad x^2 + y^2 - 3x - 4y - 6 = 0$

and $x^2 + y^2 = r^2$ will touch each other, if

$$\sqrt{9/4 + 4} = |r - \sqrt{9/4 + 10}|$$

$$\Rightarrow \quad |r - 7/2| = 5/2$$

$$\Rightarrow \quad r = 6$$

$\therefore$ Common tangent is $3x + 4y - 30 = 0$ and passes through (10, 0).

64. An ellipse and hyperbola intersect orthogonally.

They must be confocal $\Rightarrow a = 2$

Let point $P(\alpha, \beta)$ lies on both the curves, then

$$4\alpha^2 + 9\beta^2 = 36 \quad \text{...(i)}$$

and $$4\alpha^2 - \beta^2 = 4 \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$8\alpha^2 + 8\beta^2 = 40$$

$$\Rightarrow \quad \alpha^2 + \beta^2 = 5$$

or $$x^2 + y^2 = \frac{c_7}{14}, \text{ as } c_7 = 70.$$

65. (A) $f'(x) = \log x - \dfrac{3}{x} + 1$

$$\Rightarrow \quad f(x) = (x - 3)\log x + c$$

$$\Rightarrow \quad f(1) = f(3)$$

(B) $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$$\Rightarrow \quad f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$\Rightarrow \quad f(0) = f(3) \qquad [\because 27a + 9b + 3c + d = 0]$$

(C) $\dfrac{f(b) - f(a)}{b - a} = f'(\sqrt{3}) = \dfrac{2}{3}$

$$\Rightarrow \quad \frac{ab - 1}{ab} = \frac{2}{3}$$

(D) $\dfrac{f(b) - f(a)}{b - a} = f'\left(\dfrac{1}{2}\right) = 1$

$$\Rightarrow \quad a + b = 1$$

66. (A) Here, $r = 5$ cm, $\Delta r = 0.06$

$$\because \quad A = \pi r^2,$$

$$\therefore \quad dA = 2\pi r\, dr = 2\pi r \cdot \Delta r = 10\pi \times 0.06 = 0.6\,\pi$$

(B) $v = x^3, dv = 3x^2\, dx$

$$\frac{dv}{v} \times 100 = 3\frac{dx}{x} \times 100 = 3 \times 1 = 3$$

(C) $(x - 2)\dfrac{dx}{dt} = 2\dfrac{dx}{dt} \Rightarrow x = 4$

(D) $A = \dfrac{\sqrt{3}}{4}x^2$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \cdot x \cdot \frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{2} \cdot 15 \cdot \frac{1}{10} = \frac{3\sqrt{3}}{4} \text{ cm}^2/\text{s}$$

67. $\because \left|\dfrac{dx}{dt}\right| < \left|\dfrac{dy}{dt}\right| \Rightarrow \left|\dfrac{dy}{dx}\right| > 1$

and $$3y^2 \cdot \frac{dy}{dx} = 27 \quad \text{or} \quad \frac{dy}{dx} = \frac{9}{y^2}$$

$$\therefore \quad \frac{9}{y^2} > 1 \Rightarrow y^2 < 9$$

$$\Rightarrow \quad -3 < y < 3 \Rightarrow -27 < y^3 < 27$$

$$\Rightarrow \quad -27 < 27x < 27 \Rightarrow -1 < x < 1$$

$$\therefore \quad x \in (-1, 1) \Rightarrow a + b = 0$$

68. Here, $4y\dfrac{dy}{dx} = 2ax$

$\therefore \quad \left.\dfrac{dy}{dx}\right|_{(1,\,-1)} = \dfrac{-a}{2} = -1$

$\Rightarrow \quad a = 2$

Also, $2y^2 = ax^2 + b$ at $(1, -1)$ is

$2 = a + b$

$\Rightarrow \quad b = 0$

$\therefore \quad a - b = 2$

69. Using LMVT, for some $c \in (1, 6)$, we get

$$f'(c) = \frac{f(6) - f(1)}{5}$$

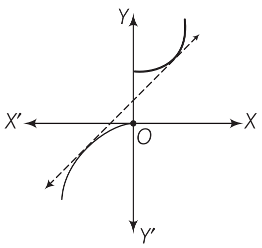
$$= \frac{f(6) + 2}{5} \geq 4.2$$

$\Rightarrow \quad f(6) \geq 19$

$\Rightarrow \quad f(6) - 16 \geq 3$

$\therefore$ Least value of $f(6) - 16 = 3$

70. Let $y = mx + c$ be a tangent to $f(x)$.



For, $x \geq 0$, intersection point is given by

$mx + c = x^2 + 8 \quad [\because\ y = x^2 + 8, \text{ for } x \geq 0]$

$\Rightarrow \quad x^2 - mx + (8 - c) = 0$

For line to be tangent, $D = 0$

$\therefore \quad m^2 = 4(8 - c) \quad$ ...(i)

Again, for $x < 0$

$mx + c = -x^2 \quad [\because\ y = -x^2, \text{ for } x < 0]$

$\Rightarrow \quad x^2 + mx + c = 0$

Now, $\quad D = 0$

$\Rightarrow \quad m^2 = 4c \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$c = 4, m = 4$

$\therefore$ Tangent is $y = 4x + 4$

Putting $\quad y = 0$, we get

$x = -1$

$\therefore$ Absolute value of x-intercept is 1.

71. Here, $f'(a) = \sqrt{3}$ and $f'(b) = 1$

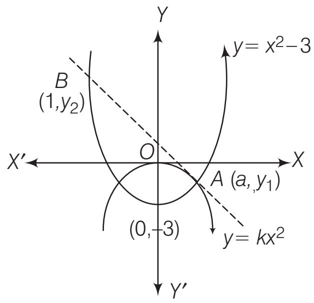
$$\therefore \quad \left|\int_a^b f'(x) \cdot f''(x)\, dx\right| = \left|\left(\frac{(f'(x))^2}{2}\right)_a^b\right| = \frac{1}{2}\left|(f'(b))^2 - (f'(a))^2\right|$$

$$= \frac{1}{2}|1 - 3| = 1$$

72. Point $A(a, y_1)$ lies on C_1 and C_2.

$\therefore \quad y_1 = a^2 - 3$ and $y_1 = ka^2$.

$\Rightarrow \quad a^2 - 3 = ka^2 \quad$ ...(i)



Now, $\quad y = kx^2$

$\Rightarrow \quad \dfrac{dy}{dx} = 2kx$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{(a,\, y_1)} = 2ka = \dfrac{y_2 - y_1}{1 - a}$

But, $\quad y_2 = 1 - 3 = -2$

$\therefore \quad 2ka = \dfrac{-2 - (a^2 - 3)}{1 - a}$

$\Rightarrow \quad 2ka = \dfrac{1 - a^2}{1 - a} = 1 + a$

$\Rightarrow \quad 2ak = 1 + a \quad$ ...(ii)

Substituting $k = \dfrac{a^2 - 3}{a^2}$ from Eq. (i) in Eq. (ii), we get

$$\frac{2a(a^2 - 3)}{a^2} = 1 + a$$

$\Rightarrow \quad 2a^2 - 6 = a + a^2$

$\Rightarrow \quad a^2 - a - 6 = 0$

$\Rightarrow \quad a = 3, -2$

$\therefore \quad a = 3 \quad [\because\ -2 \text{ is rejected as } a > 0]$

73. $f'(c) = 16c - 7 = \dfrac{f(6) - f(-6)}{12}$

$$= \frac{(8 \cdot 36 - 7 \cdot 6 + 5) - (8 \cdot 36 + 7 \cdot 6 + 5)}{12}$$

$$= -\frac{2 \cdot 7 \cdot 6}{12} = -7$$

$\Rightarrow \ 16c = 0 \ \Rightarrow \ c = 0$

74. Using LMVT for f in $[1, 2]$, we get, for some $c \in (1, 2)$

$$\frac{f(2) - f(1)}{2 - 1} = f'(c) \leq 2$$

$f(2) - f(1) \leq 2$

$\Rightarrow \quad f(2) \leq 4 \quad$ ...(i)

Again, using LMVT in $[2, 4]$, we get, for some $d \in (2, 4)$

$\Rightarrow \quad \dfrac{f(4) - f(2)}{4 - 2} = f'(d) \leq 2$

$\therefore \quad f(4) - f(2) \le 4, \quad 8 - f(2) \le 4, \quad 4 \le f(2)$

$\Rightarrow \qquad f(2) \ge 4$ ...(ii)

From Eqs. (i) and (ii), we get $f(2) = 4$

75. As, a, b and c are positive integers.

We must have $\frac{1}{a} < 1$, so $a > 1$. $\left[\text{similarly, } \frac{1}{b} < 1, \frac{1}{c} < 1\right]$

Since, $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$

$\Rightarrow \quad \frac{1}{a} > \frac{1}{3} \Rightarrow a < 3 \Rightarrow a = 2$

$\therefore \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$, where $2 < b < c$.

Similarly, $\frac{1}{b} > \frac{1}{4}$, so $b < 4 \Rightarrow b = 3$

Now, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$

$\Rightarrow \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{c} = 1$

$\Rightarrow \quad c = 6$

$\therefore \quad (a + b + c - 5) = 2 + 3 + 6 - 5 = 6$

76. Equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of tangent at $(a \cos \theta, b \sin \theta)$ is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

Intercept on the X-axis $= (a \sec \theta)$

Intercept on the Y-axis $= (b \operatorname{cosec} \theta)$

Length of intercept of the tangent by the axes

$$= \sqrt{a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta}$$

Let $\quad l = a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta$

$\Rightarrow \quad \frac{dl}{d\theta} = 2a^2 \sec^2\theta \tan \theta - 2b^2 \operatorname{cosec}^2\theta \cot \theta$

Now, $\quad \frac{dl}{d\theta} = 0$

$\Rightarrow \quad a^2 \sin^4 \theta = b^2 \cos^4 \theta \Rightarrow \frac{a}{b} = \cot^2\theta$

$\Rightarrow \quad \sin^2 \theta = \frac{b}{a + b}, \; \cos^2 \theta = \frac{a}{a + b}$

Distance between $(a \sec \theta, 0)$ and point of tangency $(a \cos \theta, b \sin \theta)$ is

$$= \sqrt{a^2 (\sec \theta - \cos \theta)^2 + b^2 \sin^2 \theta}$$

$$= \sqrt{a^2 \cos^2 \theta (\sec^2 \theta - 1)^2 + b^2 \sin^2 \theta}$$

$$= \sqrt{a^2 \cos^2 \theta \left(\frac{a + b}{a} - 1\right)^2 + b^2 \sin^2 \theta}$$

$$= \sqrt{a^2 \cos^2 \theta \frac{b^2}{a^2} + b^2 \sin^2 \theta} = b$$

Similarly, distance between $(0, b \operatorname{cosec} \theta) = a$

77. Let (x_1, y_1) be a point of contact of tangents from the origin (0, 0) to the curve $y = \sin x$.

Here, $\quad y = \sin x$

$\therefore \quad \frac{dy}{dx} = \cos x$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \cos x_1$

Now, equation of tangent at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$\Rightarrow \quad y - y_1 = (\cos x_1)(x - x_1)$

$\because$ It passes through (0, 0).

$\therefore \quad -y_1 = (\cos x_1)(-x_1)$ ...(i)

Also, (x_1, y_1) lies on the curve.

So, $\quad y_1 = \sin x_1$ ...(ii)

Squaring and adding Eqs. (i) and (ii), we get

$$\left(\frac{y_1}{x_1}\right)^2 + y_1^2 = \cos^2 x_1 + \sin^2 x_1 = 1$$

or $\quad y_1^2 + x_1^2 y_1^2 = x_1^2$

i.e. $\quad (x_1^2 + 1) y_1^2 = x_1^2$

$\therefore$ The point of contact (x_1, y_1) lies on the curve $y^2 = \frac{x^2}{x^2 + 1}$.

78. We must show that for a given $m \in R$ these exists $x \in R$ such that,

$$m^2x^2 + \int_0^x f(t)\, dt = 2$$

Let $\quad f(x) = \int_0^x [2m^2t + f(t)]\, dt - 2, \; x \in R$

Since, $f(x)$ is continuous and $2m^2t^2$ is continuous, therefore

$$\int_0^x [2m^2t + f(t)]\, dt \text{ continuous on } R,$$

$\therefore \quad f$ is continuous on R, also

$$f(0) = \int_0^0 [2m^2t + f(t)]\, dt - 2 = 0 - 2 = -2$$

and $\quad f(x) = \int_0^x [2m^2t + f(t)]\, dt - 2$

where, $\quad f(x) = m^2x^2 + \int_0^x f(t)\, dt - 2 \to \infty$

$\left[\because \int_0^x f(t)\, dt \to \infty\right]$

As, $\quad |x| \to \infty$

Thus, these exists some $a \in R$ such that;

$$f(x) > 1, \; \text{for } |x| > a$$

Note that f is a continuous on $[0, a + 1]$ and $f(0)\, f(a + 1) < 0$. By the intermediate value theorem of continuous functions, we have that there exists some $b \in (0, a + 1)$ such that $f(b) = 0$, i.e. there exists a real β which satisfies the equation

$$m^2x^2 + \int_0^x f(t)\, dt = 2$$

79. Tangent at any point $P(t_1)$, i.e. $(4t_1^2 + 3, 8t_1^3 - 1)$ be normal to the curve at $Q(t_2)$, i.e. $(4t_2^2 + 3, 8t_2^3 - 1)$.

The equation of the tangent at t_1 is

$$y-(8t_1^3-1)=\left(\frac{dy}{dx}\right)_{t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=\left(\frac{dy/dt}{dx/dt}\right)_{t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=\frac{24t_1^2}{8t_1}\cdot\{x-(4t_1^2+3)\}$$

or $$y-(8t_1^3-1)=3t_1\,\{x-(4t_1^2+3)\} \quad \text{...(i)}$$

Clearly, slope of tangent at t_1 = slope of tangent at t_2.

$$\therefore \quad \left(\frac{dy}{dx}\right)_{t_1}=\frac{-1}{\left(\frac{dy}{dx}\right)_{t_2}} \quad i.e.\, 3t_1=\frac{-1}{3t_2} \quad \text{...(ii)}$$

$\Rightarrow$ Equation of normal at t_2 is $y-(8t_2^{\,3}-1)=3t_1\{x-(4t_2^{\,3}+3)\}$ ...(iii)

On subtracting Eq. (iii) from Eq. (i), we get

$$(8t_2^3-1)-(8t_1^3-1)=3t_1\,\{(4t_2^2+3)-(4t_1^2+3)\}$$

$$\Rightarrow \quad 2t_2^2=t_1t_2+t_1^2$$

$$\Rightarrow \quad 2\cdot\left(\frac{-1}{9t_1}\right)^2=-\frac{1}{9}+t_1^2 \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad 2=-9t_1^2+81t_1^4$$

$$\therefore \quad 81t_1^4-9t_1^2-2=0$$

$$\Rightarrow \quad t_1=\pm\frac{\sqrt{2}}{3}$$

Putting in Eq. (i), the equation is

$$27(y+1)\mp 16\sqrt{2}=\pm\sqrt{2}\,(27x-105)$$

80. Equation of tangent at (x, y) is

$$Y-y=\frac{dx}{dy}(X-x)$$

$$\therefore \quad Q=\left(x-y\frac{dx}{dy},\,0\right)$$

Equation of normal at (x, y) is

$$Y-y=-\frac{dx}{dy}(X-x)$$

$$\therefore \quad R=\left(x+y\frac{dy}{dx},\,0\right)$$

Given, $\quad QR=2$

$$\Rightarrow \quad y\frac{dy}{dx}+y\frac{dx}{dy}=2$$

$$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)+y=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2\pm\sqrt{4-4y^2}}{2y}=\frac{1\pm\sqrt{1-y^2}}{y}$$

$$\Rightarrow \quad \frac{y\,dy}{1\pm\sqrt{1-y^2}}=dx \text{ or } \frac{1\mp\sqrt{1-y^2}}{y}dy=dx$$

On integrating both the sides, we get

$$\Rightarrow \quad \log y\mp\left(\log\left|\frac{1-\sqrt{1-y^2}}{y}\right|+\sqrt{1-y^2}\right)=x+c$$

The curve passes through (1,1), so $c=-1$

Hence, the possible curves

$$\log y-x=\pm\left(\log\left|\frac{1-\sqrt{1-y^2}}{y}\right|+\sqrt{1-y^2}\right)-1$$

81. Let the point $P(x, y)$ be on the curve,

$$\log(x^2+y^2)=c\tan^{-1}\left(\frac{y}{x}\right)$$

Differentiating both the sides w.r.t. 'x', we get

$$\frac{2x+2yy'}{(x^2+y^2)}=\frac{c\,(xy'-y)}{(x^2+y^2)}$$

$$\Rightarrow \quad y'=\frac{2x+cy}{cx-2y}=m_1 \text{ (say)}$$

Slope of $OP=\frac{y}{x}=m_2$ (say)

Let the angle between the tangent at P and OP be θ.

Then, $$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{2x+cy}{cx-2y}-\frac{y}{x}}{1+\frac{2xy+cy^2}{x(cx-2y)}}\right|=\frac{2}{c}$$

$$\therefore \quad \theta=\tan^{-1}\left(\frac{2}{c}\right)=\text{constant.}$$

Hence, the angle between the tangent at any point P and the line joining P to the origin O is the same.

82. (i) The given curves are

$$y^2=4ax \quad \text{...(i)}$$

and $$x^2=4ay \quad \text{...(ii)}$$

Point of intersection of Eqs. (i) and (ii) are $(0, 0)$ and $(4a, 4a)$.

From Eq. (i), $$\frac{dy}{dx}=\frac{2a}{y}=m_1 \text{ (say)}$$

From Eq. (ii), $$\frac{dy}{dx}=\frac{x}{2a}=m_2 \text{ (say)}$$

Let the angle of intersection of two curves is θ, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{2a}{y}-\frac{x}{2a}}{1+\frac{x}{y}}\right|=\left|\frac{4a^2-xy}{2a\,(x+y)}\right|$$

$$\therefore \quad (\tan\theta)_{(0,0)}=\infty \quad \text{or} \quad \theta=90^\circ$$

and $$(\tan\theta)_{(4a,\,4a)}=\left|\frac{\frac{1}{2}-2}{1+1}\right|=\left|-\frac{3}{4}\right|=\frac{3}{4}$$

Hence, $$\theta=\tan^{-1}\left(\frac{3}{4}\right)$$

(ii) The given curves are $y^2 = 4ax$ ...(i)

and $x^2 = 4ay$ ...(ii)

Tangents of Eq. (i) in terms of slope m is

$$y = mx + \frac{a}{m} \quad \text{...(iii)}$$

Now, Eq. (iii) is also tangent of Eq. (ii).

Eliminating y from Eqs. (ii) and (iii), we have

$$x^2 = 4a\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow \quad x^2 - 4amx - \frac{4a^2}{m} = 0 \quad \text{...(iv)}$$

$$\Rightarrow \quad B^2 - 4AC = 0$$

$$\Rightarrow \quad 16a^2m^2 - 4\left(-\frac{4a^2}{m}\right) = 0$$

$$\Rightarrow \quad m^3 = -1 \text{ or } m = -1$$

From Eq. (iii) common tangent is

$$y = -x - a \text{ or } y + x + a = 0$$

Hence, the common tangent is $x + y + a = 0$.

83. Let the coordinate system be chosen such that the given straight line is $x = p$ and the equations of the circles are $x^2 + y^2 = a^2, x^2 + y^2 = b^2, x^2 + y^2 = c^2$.

The line $x = p$ cuts these circles at A, B and C, respectively.

The coordinates of these points are $A(p, \sqrt{a^2 - p^2})$, $B(p, \sqrt{b^2 - p^2})$ and $C(p, \sqrt{c^2 - p^2})$.

Equations of the tangents at these points are

$$px + \sqrt{a^2 - p^2}\,y = a^2, px + \sqrt{b^2 - p^2}\,y = b^2$$

and $px + \sqrt{c^2 - p^2}\,y = c^2$.

These tangents intersect at

$$\left[\frac{p^2 - \sqrt{a^2 - p^2}\sqrt{b^2 - p^2}}{p}, \sqrt{a^2 - p^2} + \sqrt{b^2 - p^2}\right],$$

$$\left[\frac{p^2 - \sqrt{b^2 - p^2}\sqrt{c^2 - p^2}}{p}, \sqrt{b^2 - p^2} + \sqrt{c^2 - p^2}\right],$$

$$\left[\frac{p^2 - \sqrt{c^2 - p^2}\sqrt{c^2 - p^2}}{p}, \sqrt{c^2 - p^2} + \sqrt{a^2 - p^2}\right].$$

Area of Δ formed by the tangents at A, B, C is

$$\Delta = \frac{1}{2}\begin{vmatrix} \frac{p^2 - \sqrt{a^2 - p^2}\sqrt{b^2 - p^2}}{p} & \sqrt{a^2 - p^2} + \sqrt{b^2 - p^2} & 1 \\ \frac{p^2 - \sqrt{b^2 - p^2}\sqrt{c^2 - p^2}}{p} & \sqrt{b^2 - p^2} + \sqrt{c^2 - p^2} & 1 \\ \frac{p^2 - \sqrt{c^2 - p^2}\sqrt{c^2 - p^2}}{p} & \sqrt{c^2 - p^2} + \sqrt{a^2 - p^2} & 1 \end{vmatrix}$$

$$= \frac{(\sqrt{a^2 - p^2} - \sqrt{c^2 - p^2})(\sqrt{c^2 - p^2} - \sqrt{b^2 - p^2})(\sqrt{b^2 - p^2} - \sqrt{a^2 - p^2})}{2p}$$

$$= \frac{CA \cdot BC \cdot AB}{2p}$$

84. Let the curve be $y = f(x)$ and tangent drawn at $P(x, y)$ meets the X-axis at T.

We have, $OT = x^2$

Equation of tangent at $P(x, y)$;

$$Y - y = f'(x)(X - x)$$

$$\Rightarrow \quad T \equiv \left(\left|x - \frac{y}{f'(x)}\right|, 0\right)$$

$$\Rightarrow \quad \left|x - \frac{y}{f'(x)}\right| = x^2$$

$$\Rightarrow \quad x - \frac{f(x)}{f'(x)} = \pm x^2$$

$$\Rightarrow \quad \frac{x f'(x) - f(x)}{x^2} = \pm f'(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \pm f'(x)$$

On integrating both the sides, we get $\frac{f(x)}{x} = \pm f(x) + c$

Since, the curve passes through (2, 1).

$$\therefore \quad \frac{1}{2} = \pm 1 + c$$

$$\Rightarrow \quad c = -\frac{1}{2}, \frac{3}{2}$$

$$\Rightarrow \quad f(x) = \frac{x}{2(x-1)} \text{ or } f(x) = \frac{3x}{2(1+x)}$$

Hence, possible curves are $y = \frac{x}{2(x-1)}$ and $y = \frac{3x}{2(1+x)}$.

85. For $y = x - x^3, \frac{dy}{dx} = 1 - 3x^2$

Therefore, the equation of the tangent at the point $P(x_1, y_1)$ is

$$y - y_1 = (1 - 3x_1^2)(x - x_1)$$

It meets the curve again at $Q(x_2, y_2)$.

Hence, $\quad x_2 - x_2^3 - (x_1 - x_1^3) = (1 - 3x_1^2)(x_2 - x_1)$

$$\Rightarrow (x_2 - x_1)[1 - (x_2^2 + x_1x_2 + x_1^2)] = (x_2 - x_1)(1 - 3x_1^2)$$

$$\Rightarrow \quad 1 - x_2^2 - x_1x_2 - x_1^2 = 1 - 3x_1^2$$

$$\Rightarrow \quad x_2^2 + x_1x_2 - 2x_1^2 = 0$$

$$\Rightarrow \quad \left(x_2 + \frac{x_1}{2}\right)^2 = \frac{9x_1^2}{4} \Rightarrow x_2 + \frac{x_1}{2} = \pm\frac{3x_1}{2}$$

Since, $x_1 \neq x_2$, we have $x_2 = -2x_1$

$\Rightarrow \quad Q$ is $(-2x_1, -2x_1 + 8x_1^3)$.

If $L_1(\alpha, \beta)$ is the point of trisection of PQ, then $\alpha = \frac{2x_1 - 2x_1}{3} = 0$. Hence, L_1 lies on the Y-axis. If $L_2(h, k)$ is the other point of trisection, then $h = \frac{x_1 - 4x_1}{3} = -x_1$ and

$$k = \frac{y_1 - 4x_1 + 16x_1^3}{3}$$

i.e. $\quad k = \frac{x_1 - x_1^3 - 4x_1 + 16x_1^3}{3} = -x_1 + 5x_1^3$

$$\Rightarrow \quad k = h - 5h^3$$

$\therefore$ Locus of (h, k) is $y = x - 5x^3$.

86. Let the curve be $y = f(x)$

The equation of the tangent at any point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

Its intercept on the y-axis is given by $(X = 0)$

$$Y = y - x\frac{dy}{dx} = k\sqrt{x^2 + y^2}$$

So, $\quad x\frac{dy}{dx} - y + k\sqrt{x^2 + y^2} = 0$ is the differential equation governing the curve. This can be written as

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}$$

Let $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

The differential equation becomes

$$v + x\frac{dv}{dx} = v - k\sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} + k\frac{dx}{x} = 0$$

$$\Rightarrow \quad \log|v + \sqrt{1 + v^2}| + k\log|x| = c \quad \text{(on integrating)}$$

$$\Rightarrow \quad \log\left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| + k\log|x| = c$$

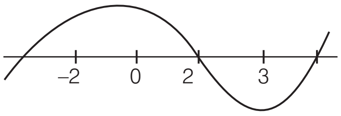
$$\Rightarrow \quad \log|y + \sqrt{x^2 + y^2}| + (k-1)\log|x| = c$$

$$\Rightarrow \quad (y + \sqrt{x^2 + y^2})\, x^{k-1} = c_1$$

87. We have, $2t^3 - 9t^2 + 30 - a = 0$

Any real root t_0 of this equation gives two real and distinct values of x if $|t_0| > 2$.

Thus, we need to find the condition for the equation in t to have three real and distinct roots none of which lies in $[-2, 2]$.



Let $\quad f(t) = 2t^3 - 9t^2 + 30 - a$

$\Rightarrow \quad f'(t) = 6t^2 - 8t = 0 \quad \Rightarrow \quad t = 0, 3$

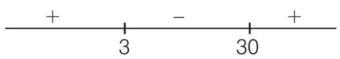
So, the equation $f(t) = 0$ has three real and distinct roots, if $f(0) \cdot f(3) < 0$

$\Rightarrow \quad (30 - a)(54 - 81 + 30 - a) < 0$

$\Rightarrow \quad (30 - a)(3 - a) < 0$

$\Rightarrow \quad (a - 3)(a - 30) < 0$, using number line rule i.e.,

$\Rightarrow \quad a \in (3, 30)$ ...(i)



Also, none of the roots lie in $[-2, 2]$

if $\quad f(-2) > 0 \quad$ and $\quad f(2) > 0$

$\Rightarrow \quad (-16 - 36 + 30 - a) > 0 \quad$ and $\quad (16 - 36 + 30 - a) > 0$

$\Rightarrow \quad a + 22 < 0 \quad$ and $\quad a - 10 < 0$

$\Rightarrow \quad a < -22$ and $\quad a < 10$

$\Rightarrow \quad a < -22$ ...(ii)

From Eqs. (i) and (ii) no real value of a exists.

88. Slope of tangent at the point (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$.

Given curve, $\quad (y - x^5)^2 = x(1 + x^2)^2$

$$\Rightarrow \quad 2(y - x^5)\left(\frac{dy}{dx} - 5x^4\right) = (1 + x^2)^2 + 2x(1 + x^2) \cdot 2x$$

Put $x = 1$ and $y = 3$, then

$$dy/dx = 8$$

89. Given, $f(x) = 2 + \cos x, \forall x \in R$

Statement I There exists a point $\in [t, t + \pi]$, where $f'(c) = 0$

Hence, Statement I is true.

Statement II $f(t) = f(t + 2\pi)$ is true. But Statement II is not correct explanation for Statement I.

90. As, $\quad |f(x_1) - f(x_2)| \le (x_1 - x_2)^2, \forall x_1, x_2 \in R$

$\Rightarrow \quad |f(x_1) - f(x_2)| \le |x_1 - x_2|^2 \quad$ [as $x^2 = |x|^2$]

$$\therefore \quad \left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| \le |x_1 - x_2|$$

$$\Rightarrow \quad \lim_{x_1 \to x_2}\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| \le \lim_{x_1 \to x_2}|x_1 - x_2|$$

$$\Rightarrow \quad |f'(x_1)| \le 0, \forall x_1 \in R$$

$\therefore \quad |f'(x)| \le 0$, which shows $|f'(x)| = 0$

[as modulus is non-negative or $|f'(x)| \ge 0$]

$\therefore \quad f'(x) = 0$ or $f(x)$ is constant function.

$\Rightarrow$ Equation of tangent at (1, 2) is

$$\frac{y - 2}{x - 1} = f'(x)$$

or $\quad y - 2 = 0 \quad$ [$\because f'(x) = 0$]

$\therefore \quad y - 2 = 0$ is required equation of tangent.

91. Given, $\quad y^3 + 3x^2 = 12y$ ...(i)

On differentiating w.r.t. x, we get

$$3y^2\frac{dy}{dx} + 6x = 12\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x}{12 - 3y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{12 - 3y^2}{6x}$$

For vertical tangent,

$$\frac{dx}{dy} = 0$$

$$\Rightarrow \quad 12 - 3y^2 = 0$$

$$\Rightarrow \quad y = \pm 2$$

On putting, $y = 2$ in Eq. (i), we get $x = \pm\frac{4}{\sqrt{3}}$ and again putting $y = -2$ in Eq. (i), we get $3x^2 = -16$, no real solution.

So, the required point is $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$.

92. Slope of tangent to the curve, $y = f(x)$ is

$$\left.\frac{dy}{dx}\right|_{(3, 4)} = f'(x)_{(3, 4)}$$

Therefore, slope of normal $= -\dfrac{1}{f'(x)_{(3,4)}} = -\dfrac{1}{f'(3)}$

But $\quad -\dfrac{1}{f'(3)} = \tan\left(\dfrac{3\pi}{4}\right)$ [given]

$\Rightarrow \quad \dfrac{-1}{f'(3)} = \tan\left(\dfrac{\pi}{2} + \dfrac{\pi}{4}\right) = -1$

$f'(3) = 1$

93. We have, $y = \dfrac{x+6}{(x-2)(x-3)}$

Point of intersection with Y-axis (0, 1)

$$y' = \frac{(x^2 - 5x + 6)(1) - (x+6)(2x-5)}{(x^2 - 5x + 6)^2}$$

$\Rightarrow$ $y' = 1$ at point (0, 1).

$\therefore$ Slope of normal is -1.

Hence, equation of normal is $x + y = 1$.

$\therefore$ $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ satisfy it.

94. Given equation of curve is

$$x^2 + 2xy - 3y^2 = 0 \quad \text{...(i)}$$

On differentiating w.r.t. x, we get

$$2x + 2xy' + 2y - 6yy' = 0 \Rightarrow y' = \frac{x+y}{3y-x}$$

At $\quad x = 1, y = 1, y' = 1$ i.e. $\left(\dfrac{dy}{dx}\right)_{(1,1)} = 1$

Equation of normal at (1, 1) is

$y - 1 = -\dfrac{1}{1}(x-1) \Rightarrow y - 1 = -(x-1)$

$\Rightarrow \quad x + y = 2 \quad$ ...(ii)

On solving Eqs. (i) and (ii) simultaneously, we get

$\Rightarrow \quad x^2 + 2x(2-x) - 3(2-x)^2 = 0$

$\Rightarrow \quad x^2 + 4x - 2x^2 - 3(4 + x^2 - 4x) = 0$

$\Rightarrow \quad -x^2 + 4x - 12 - 3x^2 + 12x = 0$

$\Rightarrow \quad -4x^2 + 16x - 12 = 0$

$\Rightarrow \quad 4x^2 - 16x + 12 = 0$

$\Rightarrow \quad x^2 - 4x + 3 = 0$

$\Rightarrow \quad (x-1)(x-3) = 0$

$\therefore \quad x = 1, 3$

Now, when $x = 1$, then $y = 1$

and when $x = 3$, then $y = -1$.

$\therefore \quad P = (1, 1)$ and $Q = (3, -1)$

Hence, normal meets the curve again at $(3, -1)$ in fourth quadrant.

Aliter

Given, $\quad x^2 + 2xy - 3y^2 = 0$

$\Rightarrow \quad (x-y)(x+3y) = 0$

$\Rightarrow \quad x - y = 0$ or $x + 3y = 0$

Equation of normal at (1, 1) is

$$y - 1 = -1(x-1)$$

$\Rightarrow \quad x + y - 2 = 0$

It intersects $x + 3y = 0$ at $(3, -1)$ and hence normal meets the curve again in fourth quadrant.