CHAPTER ONE

Sets, Relations and Functions

By a *set* we mean any collection of objects. For example, we may speak of the set of all living Indians, the set of all letters of the English alphabet or the set of real numbers less than 5. The objects constituting a set are *elements* or *members* of the set.

If X is a set and an element x is a member of, or belongs to the set X, then this is expressed, as $x \in X$.

There are two methods of representing a set.

(a) Roster method (or tabular method) In this method, a set is represented by listing all its elements in curly brackets and separating them by commas.

For example, set having elements 1, 2, 3, 5 only is written as $\{1, 2, 3, 5\}$.

(b) Property method In this method a set is represented by stating all the properties which are satisfied by the elements of the set and not by other elements outside the set.

If X contains all values of 'x' for which the condition $P(x)$ is true, then we write $X = \{x : P(x)\}$.

Illustration 1

The set of all real roots of the equation $x^4 - 2x^2 - 3 = 0$ is denoted by

$\{x : x \in \mathbf{R}, x^4 - 2x^2 - 3 = 0\}$ or equivalently by $\left\{\sqrt{3}, -\sqrt{3}\right\}$.

Illustration 2

The set of integers strictly between 2 and 8 is represented by

$\{x : x \in \mathbf{I}, 2 < x < 8\}$ or equivalently by $\{3, 4, 5, 6, 7\}$.

Illustration 3

The set of Presidents of India can be represented by

$\{x : x$ is/was President of India$\}$ or equivalently

$A = \{$Rajendra Prasad, Sarvepalli Radhakrishnan, Zakir Hussain, V.V. Giri, …….$\}$

The number of elements of A or the cardinality of A is denoted by $n(A)$. In Illustration 3, $n(A) = 13$.

FINITE SET AND INFINITE SET

A set is called a *finite set* if it contains only finite number of elements. A set which does not contain finite number of elements is called *infinite set*.

Null set A set containing no element is called a *null set* or *empty set* or *void set* and is denoted by 'ϕ'. Equivalently $P(x)$ is a property satisfied by no object at all.

Singleton set A set containing exactly one element is called a *singleton set*.

If a set X has 'r' elements then we write $n(X)$ (number of elements in X) $= r$.

Subset A set 'A' is said to be subset of the set X if every element of A is an element of X and we write $A \subseteq X$. A is said to be a proper subset of X if $A \subseteq X$. and $A \neq X$, this will be written as $A \underset{\neq}{\subset} X$. The denial of A $\subset$ B is written as $A \not\subset B$.

For two sets A and B, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

The set of all subsets of X is called *Power set* of X denoted by $P(X)$ i.e. $P(X) = \{A : A \subseteq X\}$.

Some Basic Properties:

(i) $A \subseteq A$

(i) $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$

(iii) The only subset of ϕ is ϕ itself.

(iv) The subsets of $\{x\}$ are ϕ and $\{x\}$.

(v) The subsets of $\{x, y\}$ are ϕ, $\{x\}$, $\{y\}$, $\{x, y\}$

(vi) If $n(X) = r$ then number of all subsets of X will be 2^r i.e. $n(P(X)) = 2^r$.

ALGEBRA OF SETS

Union of two Sets

Union of two sets A and B is denoted by $A \cup B$

and, $A \cup B = \{x : x \in A$ or $x \in B\}$

Clearly $A \subseteq A \cup B$, $B \subseteq A \cup B$

and $A \cup B = B \cup A$, $A \cup A = A$,

$A \cup B = B$ if and only if $A \subseteq B$,

$(A \cup B) \cup C = A \cup (B \cup C)$ (Associativity).

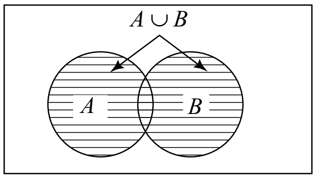


Fig. 1.1

Intersection of Two Sets

Intersection of two sets A and B is denoted by $A \cap B$ and, $A \cap B = \{x : x \in A$ and $x \in B\}$
Trivially, $A \cap B \subseteq A$, $A \cap B \subseteq B$ and $A \cap B = B \cap A$, $(A \cap B) \cap C = A \cap (B \cap C)$ (Associativity).

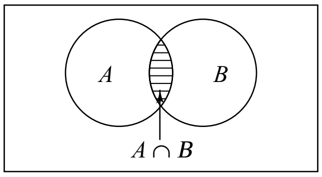


Fig. 1.2

Complement of a Set

Complement of a set B in a set A is written as

$$A \sim B = \{x : x \in A, x \notin B\}$$

e.g., $\mathbf{I} \sim \mathbf{N} = \ldots -3, -2, -1, 0\}$. The set of irrational numbers $= \mathbf{R} \sim \mathbf{Q}$

Where **I** is the set of integers
N is the set of natural numbers
R is the set of real number and
Q is the set of rational numbers

In any discussion involving sets and their operations, we presume that all these sets are subsets of a parent set called *Universal set*, U. The complement of A in U is denoted by A^c or $A' = \{x : x \notin A\}$.

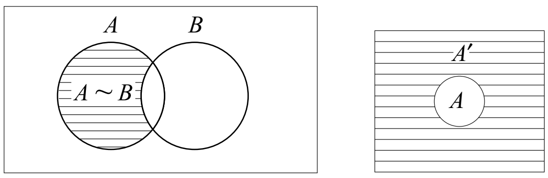


Fig. 1.3

SOME PROPERTIES OF OPERATIONS ON SETS

1. $A \cap (B_1 \cup B_2 \cup \ldots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)$ (Intersection is distributive over union)
2. $A \cup (B_1 \cap B_2 \cap \ldots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \ldots \cap (A \cup B_n)$ (Union is distributive over intersection)
3. $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$. (De Morgan's Laws)
4. $A \sim \left(\bigcap_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n}(A \sim A_i)$.
5. $A \sim \left(\bigcup_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n}(A \sim A_i)$.
6. $A \cap A^c = \phi$
7. $A \cup A^c = X$
8. $\phi^c = X$
9. $X^c = \phi$
10. $(A^c)^c = A$
11. $A \sim B = A \cap B^c$
12. $A \subseteq B$ if and only if $B^c \subseteq A^c$.
13. $A \sim A = \phi$.
14. $(A \sim B) \sim C = (A \sim C) \sim B$
15. $A \sim (B \cup C) = (A \sim B) \cap (A \sim C)$
16. If A and B are finite sets, then
    $n(A \cup B) = n(A) + n(B) - n(A \cap \mathrm{B})$
17. If A, B and C are finite sets then
    $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Symmetric Difference of Two Sets

$$A \Delta B = (A \sim B) \cup (B \sim A)$$

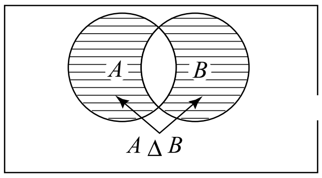


Fig. 1.4

$A \Delta A = \phi$, $A \Delta B = B \Delta A$

Cartesian Product

The cartesian product of sets A and B is denoted by $A \times B$ and, $A \times B = \{(a, b): a \in A, b \in B\}$

In general $A \times B \neq B \times A$, moreover,

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

and $$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Illustration 4

If $A = \{0, 1, 2, 3, 5\}$, $B = \{0, 1, 2, 3)$, $C = \{0, 1, 4, 5\}$ then $B \subseteq A$, $C \nsubseteq A$, $A \cap C = \{0, 1, 5\}$, $B \cap C = \{0, 1\}$, $A \sim B = \{5\}$, $A \sim C = \{2, 3\}$, $B \sim C = \{2, 3\}$, $C \sim B = \{4, 5\}$, $A \Delta B = \{5\}$, $A \Delta C = \{2, 3, 4\}$ and $B \Delta C = \{2, 3, 4, 5\}$

Illustration 5

Simplify $(A \cup B \cup C) \cap (A \cap B^c \cap C^c)^c \cap C^c$

$$(A \cup B \cup C) \cap (A \cap B^c \cap C^c)^c \cap C^c$$
$$= [(A \cup B \cup C) \cap (A^c \cup B \cup C)] \cap C^c$$
$$= [(A \cap A^c) \cup (B \cup C] \cap C^c$$
$$= [\phi \cup (B \cup C)] \cap C^c$$
$$= (B \cup C) \cap C^c = (B \cap C^c) \cup (C \cap C^c).$$
$$= (B \cap C^c) \cup \phi = B \cap C^c = B \sim C.$$

Relations

A relation from a set of X to a set Y is a subset R of $X \times Y$. If $(a, b) \in R$, we say a is related to b and often write it as $a\ R\ b$. If $X = Y$, we say that R is relation on X.

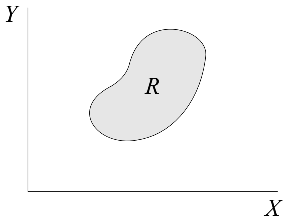


Fig. 1.5

Let R be a relation from X to Y. For each subset A of X, we write

$$R(A) = \{y \in Y: x\ R\ y \text{ for some } x \in A\}$$

and call it the (direct) image of A under R, and for each subset B of Y, we write $R^{-1}(B) = \{x \in X : x\ R\ y \text{ for some } y \in B\}$ and call it the *inverse image* of B under R.

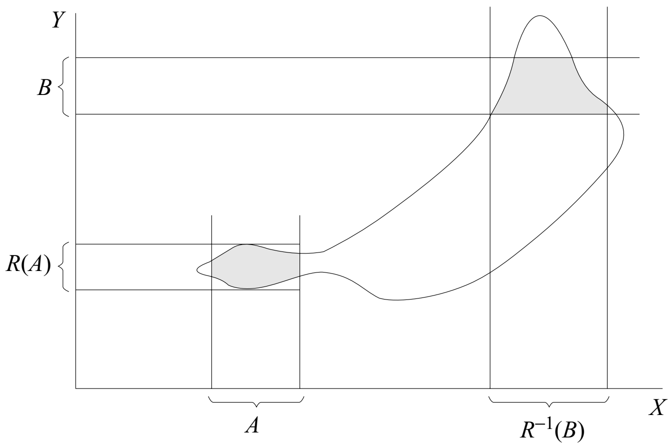


Fig. 1.6

One can think of R^{-1} as a relation from Y to X, where

$$yR^{-1}\, x \Leftrightarrow xRy$$

that is, $\quad R^{-1} = \{(y, x) : (x, y) \in R\}$

R^{-1} is called the *reverse* of the relation R.

Domain of a relation R from a set X to a set Y is the set of all first components of the elements of R, i.e., dom $R = \{a \in X: (a, b) \in R \text{ for some } b \in R\}$

Let R be a relation from X to Y and S from Y to Z, then $SoR = \{(x, z)| \ x \in X, z \in Z$ and $\exists\ y \in Y$ such that and $(x, y) \in R$ and $(y, z) \in S$.

If $n(A), n(B) < \infty$, then the number of relation from A to B will be $2^{n(A)\, n(B)}$

Equivalence Relation

A relation R on a set X (i.e. $R \subset X \times X$) is said to be

(i) *reflexive* if $x\ R\ x$ for all $x \in X$.
(ii) *symmetric* if $x\ R\ y \Rightarrow y\ R\ x$, where $x, y \in X$
(iii) *transitive* if $x\ R\ y$, $y\ R\ z \Rightarrow x\ R\ z$, where $x, y, z \in X$

Further, a relation R in a set X is said to be an equivalence relation if it is reflexive, symmetric and transitive. **R** is called anti-symmetric if $x\ R\ y$ and $y\ R\ x \Rightarrow x = y$.

Illustration 6

For $x, y \in \mathbf{I}$, write $x\ R\ y$ if $x - y$ divisible by 6; this is an equivalence relation and is usually written as $x \equiv y \pmod{6}$.

Illustration 7

The relation R on $\mathbf{R}$ defined as $R = \{(a, b): a \le b\}$ is not symmetric as $(1, 3) \in R$ but $(3, 1) \notin R$ but is transitive and reflexive.

Illustration 8

If $A = \{1, 2, 3\}$; let $R = \{1, 1), (2, 2)\ (3, 3), (1, 2)\ (2, 1)\}$ then R is reflexive, symmetric and transitive.

The number of equivalence relations that can be defined on a set containing k elements is given

$$B_k = \sum_{n=0}^{k-1} \binom{k-1}{n} B_n$$

E.g. $\quad B_0 = 1, B_1 = 1, B_2 = 2, B_3 = \binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2$
$$= 1 + 2 + 2 = 5$$

Functions

A relation F from A to B is said to be a function if for each $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in F$. b is called image of 'a' under F and is denoted by $F(a)$. Thus every function is a relation but the converse may not be true.

If $n(A) = n$ and $n(B) = m$ then total number of functions from A to B is n^m.

SOME DEFINITIONS

Let A and B be two non-empty sets. A function f from A to B can also be defined as a rule that assigns to each element in the set A, one and only one element of the set B. In general, the sets A and B need not be sets of real numbers. However, we consider only those functions for which A and B are both subsets of the real numbers. We shall denote by $\mathbf{R}$, the set of all real numbers.

The set A in the above definition is called the **domain** of the function f. We usually denote it by dom f. If x is an element in the domain of a function f, then the element that

f associates with x is denoted by the symbol $f(x)$, and is called the *image* of x under f, or the *value* of f at x. The set of all possible values of $f(x)$ as x varies over the domain is called the **range of f**. If $f: A \to B$, then the range of f is a subset of B and the set B is called **co-domain of f**.

Remark

- If x is an element in the domain of a function f, the definition of a function requires that f assigns one and only one value to x. This means that a function cannot be *multiple-valued*. For example, the expression $\pm\sqrt{x}$ does not define a function of x, since it assigns two values to each positive x.

ALGEBRAIC OPERATIONS ON FUNCTIONS

1. If f and g are two functions, then *sum of the functions* $f + g$, is defined for all $x \in \text{dom } f \cap \text{dom } g$ by
$$(f + g)(x) = f(x) + g(x).$$
2. If k is any real number and f is a function, then kf is defined for all $x \in \text{dom } f$ by $(kf)(x) = kf(x)$.
3. If f and g are two functions, then the *pointwise product* fg is defined for all $x \in \text{dom } f \cap \text{dom } g$ by
$$(fg)(x) = f(x)\, g(x).$$
4. If f and g are functions, then f/g is defined for all $x \in \text{dom}(f) \cap \text{dom}(g) \cap \{x: g(x) \neq 0\}$ by $(f/g)(x) = f(x)/g(x)$.
5. *Composition of functions* Let $f: A \to B$ and $g: B \to C$ be functions, then $gof: A \to C$ defined by $(gof)(x) = g(f(x))$.

We have the following formulae for domains of functions.

1. $\text{dom}(f + g) = \text{dom } f \cap \text{dom } g$
2. $\text{dom}(fg) = \text{dom } f \cap \text{dom } g$
3. $\text{dom}(f/g) = \text{dom } f \cap \text{dom } g \cap \{x : g(x) \neq 0\}$
4. $\text{dom}\sqrt{f} = \text{dom } f \cap \{x : f(x) \geq 0\}$

Note

Note that if gof is defined, then fog may not be defined. For example if $f: \mathbf{A} \to \mathbf{B}$ and $g: \mathbf{B} \to \mathbf{C}$ then gof is defined but fog is not. Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \cos x$ and $g(x) = x^3$. Then $gof(x) = g(f(x)) = g(\cos x) = \cos^3 x$ and $fog(x) = f(x^3) = \cos x^3$. Thus even if fog and gof are both defined it is not necessary that $fog = gof$.

DOMAINS AND RANGES OF SOME FUNCTIONS

1. *Constant functions* A function that assigns the same value to every member of its domain is called a constant function. The domain of the constant function $f(x) = c$ is $\mathbf{R}$ and its range is $\{c\}$.

2. *Polynomial functions* A function of the form cx^n, where c is a constant and n is a non-negative integer, is called a *monomial* in x. Examples are $2x^3$, $5x^4$, $-6x$ and x^8. The function $4x^{1/2}$ and x^{-3} and not monomials because the powers of x are not non-negative integers. A function that is expressible as the sum of finitely many monomials in x is called a *polynomial* in x. Thus
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$
is a polynomial. The domain of a polynomial is $\mathbf{R}$ and its range is a subset of $\mathbf{R}$.

3. The domain of $f(x) = \log_e x$ (= ln x) is $(0, \infty) = \{x \in \mathbf{R}, x > 0\} = \mathbf{R}^+$, and its range is $(-\infty, \infty)$ (i.e., the whole of $\mathbf{R}$).

4. The domain of $f(x) = e^x$ is $\mathbf{R}$ and its range is $\mathbf{R}^+$.

TYPES OF FUNCTIONS

1. *Rational function.* This function is defined as the ratio of two polynomials
$$y = \frac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{b_0 x^m + b_1 x^{m-1} + \cdots + b_m}$$
For example, $y = (x^2 + 3)/(x^3 + 4)$ is rational function.

2. *Irrational function.* If in the function $y = f(x)$, the operations of addition, subtraction, multiplication, division and raising to a power with rational non-integral exponents are performed on the right-hand side, the function $y = f(x)$ is said to be irrational. Examples are $y = (3x^2 + \sqrt{x})/\sqrt{2 + 4x}$ and $y = \sqrt{x}$.

3. *Even function.* A function $y = f(x)$ is said to be an even function if $f(-x) = f(x)\ \forall\ x \in \text{dom}(f)$. Examples are $y = |x|$, $y = \cos x$ and $y = x^{2n}$. The graph of an even function will be symmetric about y-axis

4. *Odd function.* A function $y = f(x)$ is said to be an odd function if $f(-x) = -f(x)\ \forall\ x \in \text{dom}(f)$. Examples are $y = \sin x$ and $y = x^{2n+1}$.

Clearly, $y = f(x) + f(-x)$ is always an even function, and $y = f(x) - f(-x)$ is always odd. Any function $y = f(x)$ can be expressed uniquely as the sum of an even and an odd function as follows:
$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$$

5. *Periodic function.* A function $y = f(x)$ is said to be periodic if there exists a number $T > 0$ such that $f(x + T) = f(x)$ for all x in the domain of f. The least such T is called the period of f. For example, the period of $\sin x$ and $\cos x$ is 2π, and that of $\tan x$ is π.

If $f(x)$ is a periodic function with period T, then the function $f(ax + b)$, $a > 0$, is periodic with period T/a. For example, $\sin 2x$ has a period π, $\cos 3x$ has a period $2\pi/3$ and $\tan(2x + 4)$ has a period $\pi/2$. The function

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbf{Q} \\ -1 \text{ if } x \in \mathbf{R} \sim \mathbf{Q} \end{cases}$$

is a periodic function without any period. The sum of two periodic functions may not be periodic e.g., $f(x) = \{x\}$, the fractional part of x and $g(x) = \sin x$.

6. *Onto function (or Surjective Function).* If a function $f: A \to B$ is such that each element in B is the f-image of at least one element in A, then we say that f is a function of A 'onto' B. Equivalently a function f is an onto function if co-domain of f = Range of f.

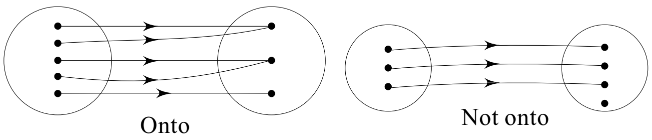


Fig. 1.7

For example, $f: \mathbf{R} \to [-1, 1]$ defined by $f(x) = \sin x$ is an onto function but $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin x$ is not onto since Range of $f = [-1, 1]$ and co-domain of $f = \mathbf{R}$.

In order to show that a function $f: A \to B$ is onto we start with any $y \in B$ and try to find $x \in A$ such that $f(x) = y$.

7. *One-to-one function (or injective function).* A function f is said to be *one-to-one* if it does not take the same value at two distinct points in its domain. For example, $f(x) = x^3$ is one-to-one, whereas $f(x) = x^2$ is not, as $f(1) = 1$ and $f(-1) = 1$. Note that a periodic function $f: R \to R$ cannot be one-to-one as $f(x + T) = f(x)$ for some $T > 0$.

In order to show that a function $f: A \to B$ is one-to-one, we may take any $x, y \in A$ such that $f(x) = f(y)$ and try to show that $x = y$.

If $n(A) = m$ and $n(B) = n (m \le n)$, then the number of injections (or one-one functions) from A to B is

$${}^{n}P_{m} = \frac{n!}{(n-m)!}.$$

8. *Bijective function (One-to-One and Onto).* If a function f is both one-to-one and onto, then f is said to be a *bijective function*. For example an identity function $i_A : A \to A$ defined by $i_A(x) = x$ is trivially a bijective function. The function $f: [-\pi/2, \pi/2] \to [-1, 1]$ defined by $f(x) = \sin x$ is a bijective function. The function $f: (-\pi/2, \pi/2) \to \mathbf{R}$ defined by $f(x) = \tan x$ is also a bijective function.

If A and B are finite sets and $f: A \to B$ is a bijection then $n(A) = n(B)$. If $n(A) = n$, then the number of bijections from A to B is the total number of arrangements of n items taken all at a time which is $n!$

9. *Bounded and unbounded functions* A function f defined on an interval I is said bounded on I if there is $k > 0$ such that $|f(x)| \le k$ for all $x \in I$. Equivalently there is m and M such that $m \le f(x) \le M$ for all $x \in I$. E.g. $f(x) = \dfrac{1}{x}$ is not bounded on $(0, 1)$ and $f(x) = x^2$ is bounded on $[0, 1]$. $f(x) = \sin x$ or $\cos x$ are bounded functions on $\mathbf{R}$ whereas $f(x) = \tan x$ is unbounded on $(-\pi/2, \pi/2)$.

GRAPHS OF SOME FUNCTIONS

1. *Constant function* $f(x) = c$ represents a constant function (Fig. 1.8).

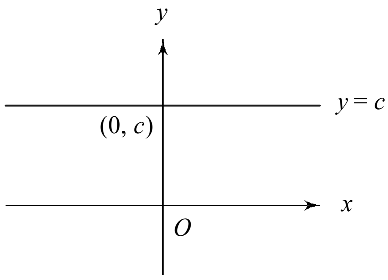


Fig. 1.8

2. *Proportional values*. If variables y and x are direct proportional, then the functional dependence between them is represented by the equation:

$$y = kx,$$

where k is a constant a *factor of proportionality*.

A graph of a direct proportionality is a straight line, going through an origin of coordinates and forming with an x-axis an angle α, a tangent of which is equal to k: $\tan \alpha = k$. Therefore, a factor of proportionality is called also a *slope*. These are shown in three graphs with $k = 1/3$, $k = 1$ and $k = -3$ on Fig.1.9

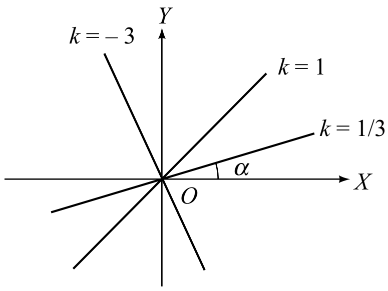


Fig. 1.9

3. *Linear function*. If variables y and x are related by the 1-st degree equation:

$$Ax + By = C,$$

(at least one of numbers A or B is non-zero), then a graph of the functional dependence is a straight line. If $C = 0$, then it goes through an origin of coordinates, otherwise – not. Graphs of linear functions for different combinations of A, B, C are represented on Fig.1.10

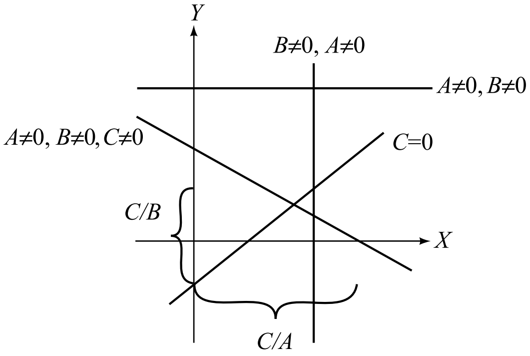


Fig. 1.10

4. *Inverse proportionality*. If variables y and x are *inverse proportional*, then the functional dependence between them is represented by the equation:

$$y = k/x,$$

where k is a constant.

A graph of an inverse proportionality is a curve, having two branches (Fig. 1.11). This curve is called a *hyperbola*. These curves are received at crossing a circular cone by a plane. As shown on Fig. 1.11, a product of coordinates of a hyperbola points is a constant value, equal in this case to 1. In general case this value is k, as it follows from a hyperbola equation:

$$xy = k$$

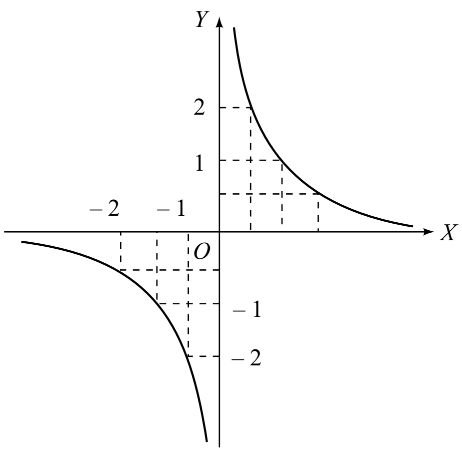


Fig. 1.11

The main characteristics and properties of hyperbola:

- the function domain: $x \neq 0$, and codomain: $y \neq 0$;
- the function is monotone (decreasing) at $x < 0$ and at $x > 0$, but it is not monotone on the whole, because of a point of discontinuity $x = 0$
- the function is unbounded, discontinuous at a point $x = 0$, odd, non-periodic;
- there are no zeros of the function.

5. *Quadratic function*. This is the function: $y = ax^2 + bx + c$, where a, b, c – constants, $a \neq 0$. In the simplest case we have $b = c = 0$ and $y = ax^2$. A graph of this function is a *quadratic parabola* – a curve going through an origin of coordinates. Every parabola has an axis of symmetry OY, which is called an *axis of parabola*. The point O of intersection of a parabola with its axis is a *vertex of parabola*.

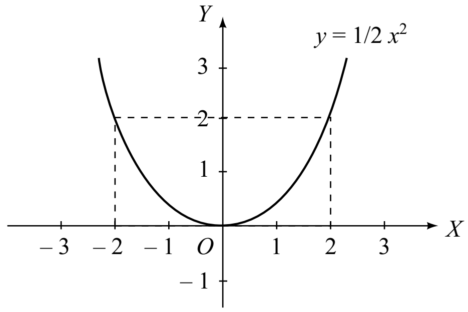


Fig. 1.12

A graph of the function $y = ax^2 + bx + c$ is also a quadratic parabola of the same shape, that $y = ax^2$, but its vertex is not an origin of coordinates, this is a point whose coordinates:

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$$

The form and location of a quadratic parabola in a coordinate system depends completely on two parameters: the coefficient a of x^2 and *discriminant* D : $D = b^2 - 4ac$. These properties follow from analysis of the quadratic equation roots. All possible different cases for a quadratic parabola are shown on Fig. 1.13

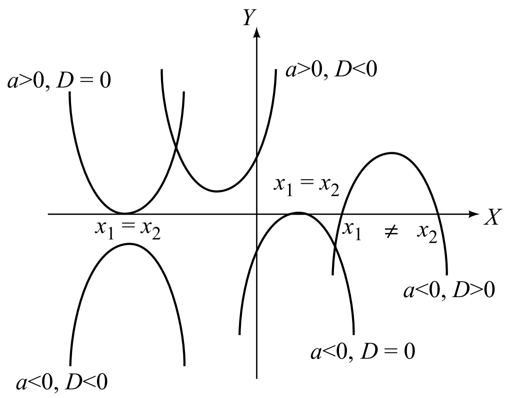


Fig. 1.13

The main characteristics and properties of a quadratic parabola:

- the function domain: $-\infty < x < +\infty$ (i.e. x is any real number)
- the function is not monotone on the whole, but to the right or to the left of the vertex it behaves as a monotone function;
- the function is unbounded, continuous in everywhere, even at $b = c = 0$, and non-periodic;
- the function has no zeros at $D < 0$.

6. *Power function*. This is the function: $y = ax^n$ where a, n are constants. At $n = 1$ we obtain the function, called a *direct proportionality*: $y = ax$; at $n = 2$ – a *quadratic parabola*; at $n = -1$ – an *inverse proportionality* or *hyperbola*. So, these functions are particular cases of a power function. We know, that a zero power of every non-zero number is 1, thus at $n = 0$ the power function becomes a constant: $y = a$, i.e. its graph is a straight line, parallel to an x-axis, except an origin of coordinates. All these cases (at $a = 1$) are shown on Fig. 1.14 ($n \geq 0$) and Fig. 1.15 ($n < 0$).

Negative values of x are not considered here, because then some of functions:

$$y = x^{1/2},\ \ y = x^{1/4}$$

lose a meaning.

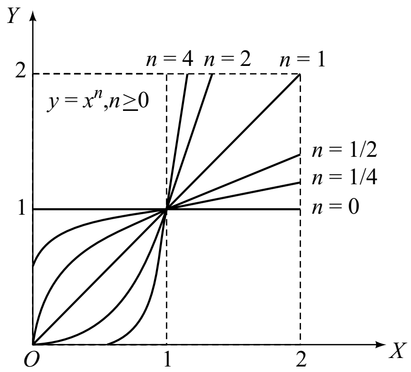


Fig. 1.14

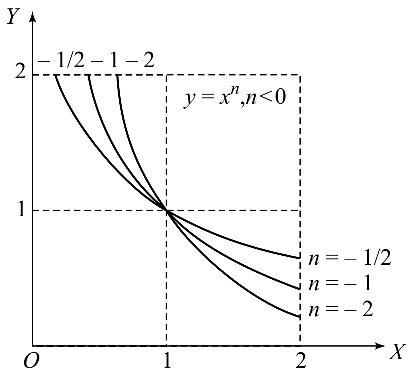


Fig. 1.15

If n – integer, power functions have a meaning also at $x < 0$, but their graphs have different forms depending on that is n an even or an odd number. In Fig. 1.16 two such power functions are shown: for $n = 2$ and $n = 3$.

At $n = 2$ the function is even and its graph is symmetric relatively an axis Y; at $n = 3$ the function is odd and its graph is symmetric relatively an origin of coordinates. The function $y = x^3$ is called a *cubic parabola*.

On Fig. 1.17 the function $y = \pm\sqrt{x}$ is represented. This function is inverse to the quadratic parabola $y = x^2$, its graph is received by rotating the quadratic parabola graph around a bisector of the 1-st coordinate angle. We see by the graph, that this is the two-valued function (the sign ± before the square root symbol says about this). Such functions are not studied in an elementary mathematics, therefore we consider usually as a function one of its branches: either an upper or a lower branch.

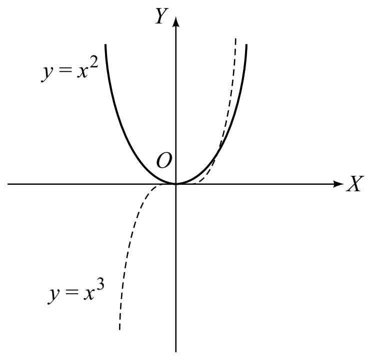


Fig. 1.16

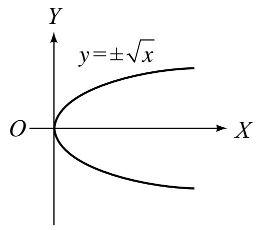


Fig. 1.17

7. *Exponential function*. The function $y = a^x$, where a is a positive constant number, is called an *exponential function*. The argument x adopts *any real* values; as the function values *only positive numbers* are considered, because otherwise we will have a multi-valued function. So, the function $y = 81^x$ has at $x = 1/4$ four different values: $y = 3$, $y = -3$, $y = 3i$ and $y = -3i$. But we consider as the function value only $y = 3$. Graphs of an exponential function for $a = 2$ and $a = 1/2$ are shown on Fig. 1.18. All they are going through the point (0, 1). At $a = 1$ we have as a graph a straight line, parallel to x-axis, i.e. the function becomes a constant value, equal to 1. At $a > 1$ an exponential function increases, and at $0 < a < 1$ – decreases.

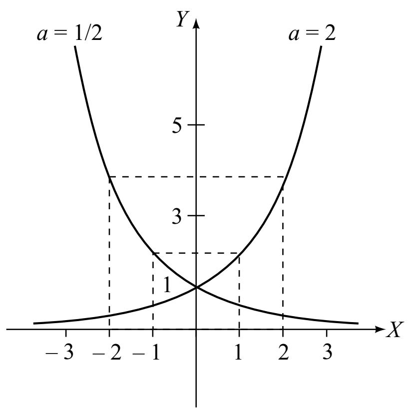


Fig. 1.18

The main characteristics and properties of an exponential function:

- the function domain: — $\infty < x < +\infty$ (i.e. x is any real number) and its codomain: $y > 0$;
- this is a monotone function: it increases at $a > 1$ and decreases at $0 < a < 1$;
- the function is unbounded, continuous in everywhere, non-periodic;
- the function has no zeros.

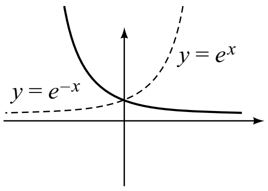


Fig. 1.19

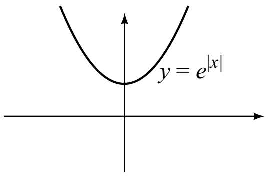


Fig. 1.20

8. *Logarithmic function*. The function $y = \log_a x$, where a is a positive constant number, not equal to 1 is called a *logarithmic function*. This is an inverse function relatively to an exponential function; its graph (Fig.1.19) can be obtained by rotating a graph of an exponential function around of a bisector of the 1-st coordinate angle.

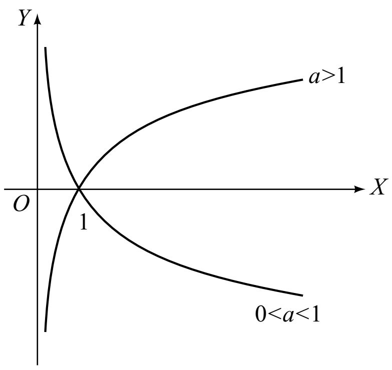


Fig. 1.21

The main characteristics and properties of a logarithmic function:

- the function domain: $x > 0$ and its codomain: $-\infty < y < +\infty$ (i.e. y is any real number);
- this is a monotone function: it increases for $a > 1$ and decreases for $0 < a < 1$;
- the function is unbounded, continuous in everywhere, non-periodic;
- the function has one zero: $x = 1$.

9. *Absolute-value function* is given by

$$y = |x| = \max\{x, -x\} = \begin{cases} x \ , & x \geq 0 \\ -x \ , & x < 0 \end{cases}$$

It is depicted in Fig. 1.22.

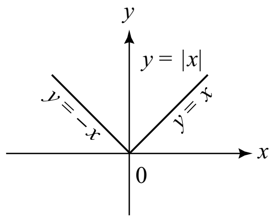


Fig. 1.22

Greatest Integer Function and **Fractional part function** $f(x) = [x]$ will denote the integral part of x or greatest integer less than or equal to x. E.g. $[5] = 5$, $[5.2], = 5$, $[-5.1] = -6$, $[\pi] = 3$, $[e] = 2$. The domain of this function is **R** and the range is **I**. For positive x, this function might represent, for example, the legal age of a person is a function of his chronological age x.

The *fractional part* of x is the fractional part $\{x\}$ of x so $\{x\} = x - [x]$ e.g. $\{5.04\} = 0.04$, $\{-4\} = 0$, $\{-1.7\} = 0.3$, $\{\pi\} = \pi - 3$. The domain of this function is **R** whereas the range is [0, 1).

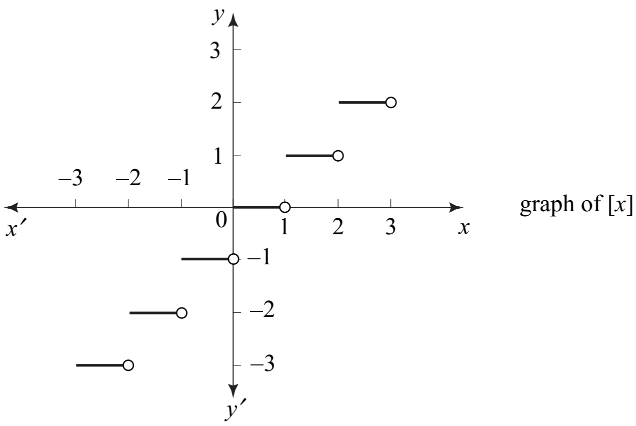


Fig. 1.23

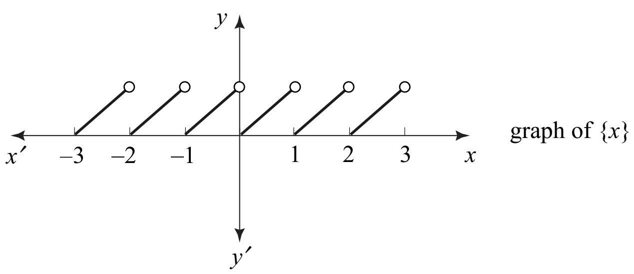


Fig. 1.24

Some Properties

1. $x - 1 < [x] \leq x$
2. $[\{x\}] = 0 = \{[x]\}$
3. If $n \leq x < n + 1$, $n \in$ **I**, then $[x] = n$ and conversely.
4. If $x \in$ I, then $[x] = x$ otherwise $[x] < x$.
5. $[-x] = \begin{cases} -[x] & , \text{ if } x \in \mathbf{I} \\ -[x] - 1 \ , & \text{if } x \notin \mathbf{I} \end{cases}$

$$\text{in other word, } [x] + [-x] = \begin{cases} 0, \text{ if } & x \in \mathbf{I} \\ -1, \text{ if } & x \notin \mathbf{I} \end{cases}$$

6. $[x + y] = [x] + [y]$, if x or $y \in \mathbf{I}$.

7. $[x + y] = \begin{cases} [x]+[y], & \text{if } \{x\}+\{y\}<1 \\ [x]+[y]+1, & \text{if } \{x\}+\{y\} \ge 1 \end{cases}$

   Moreover, $[x + y] \ge [x] + [y]$.

8. $\{x\} = \begin{cases} x, & \text{if } 0 < x < 1 \\ 0, & x \in \mathbf{I} \end{cases}$

9. $\{x\} + \{-x\} = \begin{cases} 0, & x \in \mathbf{I} \\ 1, & \text{if } x \notin \mathbf{I} \end{cases}$

10. $[x]+\left[x+\frac{1}{n}\right]+\cdots+\left[x+\frac{n-1}{n}\right]=[n\,x]$.

11. *Signum function* is given by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and it is depicted in Fig. 1.25

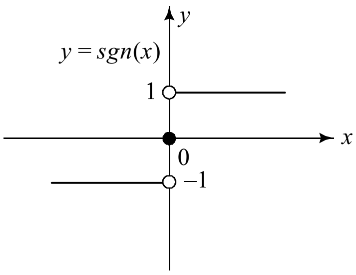


Fig. 1.25

12. The *least integer function*: The function whose value at any number x is the smallest integer greater than or equal to x is called *the least integer function*, e.g. if
$f(x) = [x]'$ = smallest integer, greater than or equal to x.
then $[1.5]' = 2$ $[1.99]' = 1$ $[1.4]' = 2$.

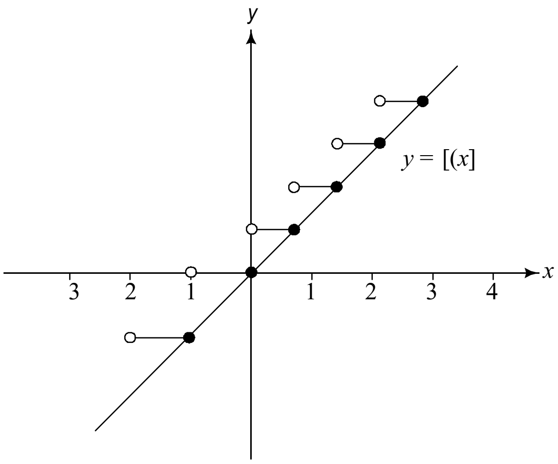


Fig. 1.26

Figure 1.26 shows the graph. For positive value of x, this function might represent for example, the cost of parking x hours in a parking lot which charges Rs 10 for each hour or part of an hour in this case $f(x) = 10\,[x]'$.

INVERSE OF A FUNCTION

The function $f(x) = x^{1/3}$ and $g(x) = x^3$ have the following property:

$$f(g(x)) = f(x^3) = (x^3)^{1/3} = x$$
$$g(f(x)) = g(x^{1/3}) = (x^{1/3})^3 = x$$

Similarly, the functions $f(x) = \log_e x$ and $g(x) = e^x$ cancel the effect of each other. If two functions f and g satisfy $f(g(x)) = x$ for every x in the domain of g, and $g(f(x)) = x$ for every x in the domain of f, we say that *f is the inverse of g and g is the inverse of f*, and we write $f = g^{-1}$, or $g = f^{-1}$. (**The symbol f^{-1} does not mean $1/f$**.) To find the inverse of f, write down the equation $y = f(x)$ and then solve x as a function of y. The resulting equation is $x = f^{-1}(y)$.

If f is one-to-one, then f has an inverse defined on its range and, conversely, if f has an inverse, then f is one-to-one. f^{-1} is defined on the range of f. If f is one-to-one from A to B, and g is one-to-one from B to C, then $f\,o\,g$ is one-to-one from A to C, and $(f\,o\,g)^{-1} = g^{-1}\,o\,f^{-1}$.

If $f : X \to Y$ is bijective then $f^{-1} : Y \to X$ is bijective and $\left(f^{-1}\right)^{-1} = f$. Moreover, $f^{-1}(\{y\}) = \{f^{-1}(y)\}$ for all $y \in Y$.

Illustration 9

Let $f(x) = \dfrac{ax+b}{cx+d}$. The domain of f is $\mathbf{R} \sim \left\{-\dfrac{d}{c}\right\}$.

Let $y = \dfrac{ax+b}{cx+d}$ so $cxy + dy = ax + b \Rightarrow x = \dfrac{b-dy}{cy-a}$.

Hence $f^{-1}(x) = \dfrac{b-dx}{cx-a}$.

DIRECT AND INVERSE IMAGES

Let f be a function with domain X and co-domain Y. If $A \subseteq X$, then the direct image of A under f is the subset of Y (denoted by $f(A)$) is defined to be $\{f(x) : x \in A\}$.

For example, let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = x^2$. If $A = \{-3, -1, 0, 1, 3\}$, then $f(A) = \{0, 1, 9\}$.

Note that if $f(X) = Y$, then the function $f : X \to Y$ is onto. The following relations can be verified easily.

(i) $A \subset B \Rightarrow f(A) \subset f(B)$
(ii) $f(A \cup B) = f(A) \cup f(B)$
(iii) $f(A \cap B) \subset f(A) \cap f(B)$. The inclusion may be strict.

Let f be a function with domain X and range Y and let B be a subset of Y. The inverse image of B under f is a subset of X (denoted by $f^{-1}(B)$) is defined to be $\{x : f(x) \in B\}$.

Illustration 10

If f: $\mathbf{R} \to \mathbf{R}$ is a function defined by $f(x) = |x|$ and $A = (-\infty, 0)$, then $f^{-1}(A) = \{x: f(x) \in A\} = \{x: |x| \in (-\infty, 0)\} = \phi$ (the empty set). If $B = (-3, 3)$, then $f^{-1}(B) = (-3, 3)$.

Note that if f is one-to-one, then $f^{-1}(\{x\})$ is a singleton set. The following relations can be verified easily for $A, B \subset Y$.

(i) $A \subset B = f^{-1}(A) \subset f^{-1}(B)$

(ii) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(iii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: Let **I** be set of integers, **N** = the set of non-negative integers, N_p = the set of non-positive integers. Then the sets A and B satisfying $A \cap B = \phi$ are

(a) $A = I \sim N_p, B = N \sim N_p$

(b) $A = I \sim N, B = I \sim N_p$

(c) $A = N \Delta N_p, B = I \sim N_p$

(d) $A = N \Delta N_p, B = (I \sim N) \cup \{0\}$

Ans. (b)

Solution: $I \sim \mathbf{N} = \{\ldots -3, -2, -1\}$, $I \sim N_p = \{1, 2, 3\ldots\}$, $\mathbf{N} \sim N_p = \{1, 2, \ldots\}$, $N \Delta N_p = (\mathbf{N} \sim N_p) \cup (N_p \sim \mathbf{N})$

$$= \{0, 1, 2, \ldots\}$$

$$(I \sim \mathbf{N}) \cup \{0\} = \{\ldots, -3, -2, -1, 0\}$$

The disjoint sets are $\mathbf{I} \sim \mathbf{N}$ and $\mathbf{I} \sim N_p$.

Example 2: Which of the following equality is not true.

(a) $A \cap (B \sim C) = A \cap B \sim (A \cap C)$

(b) $A \sim (A \cap B) = A \sim B$

(c) $A \sim (B \sim C) = (A \sim B) \cup (A \cap C)$

(d) $A \sim (B \Delta C) = (A \sim B) \Delta (A \sim C)$

Ans. (d)

Solution: For equality (a),

$$A \cap (B \sim C) = A \cap (B \cap C')$$
$$= \phi \cup (A \cap B \cap C')$$
$$= (A \cap B \cap A') \cup (A \cap B \cap C')$$
$$= A \cap B \cap (A' \cup C')$$
$$= A \cap B \sim (A \cap C)$$

For equality (b),

$$A \sim (A \cap B) = A \cap (A' \cup B')$$
$$= (A \cap A') \cup (A \cap B')$$
$$= \phi \cup (A \cap B') = A \cap B' = A \sim B$$

For equality (c)

$$A \sim (B \sim C) = A \sim (B \cap C') = A \cap (B' \cup C)$$
$$= (A \cap B') \cup (A \cap C)$$
$$= (A \sim B) \cup (A \cap C)$$

For (d) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5\}$, $C = \{1, 2, 3\}$

So, $B \Delta C = \{4, 5\} \cup \{1, 2\} = \{1, 2, 4, 5\}$

Thus $A \sim (B \Delta C) = \{3\}$

$$A \sim B = \{1, 2\}; A \sim C = \{4, 5\}$$

Therefore $(A \sim B) \Delta (A \sim C) = (\{1, 2\} \sim \{4, 5\}) \cup (\{4, 5\} \sim \{1, 2\})$

$$= \{1, 2\} \cup \{4, 5\} = \{1, 2, 4, 5\}.$$

Hence $A \sim (B \Delta C) \neq (A \sim B) \Delta (A \sim C)$

Example 3: A boating club consists of 82 members, each member is either a sailboat owner or a powerboat owner. If 53 members owned sailboats and 38 members owned powerboats, the number of members owned both sailboat and powerboat is

(a) 6 (b) 7

(c) 9 (d) 4

Ans. (c)

Solution: Let S = the set of all members owning sailboats and P = the set of all members owning powerboats

$$n(S \cup P) = n(S) + n(P) - n(S \cap P)$$
$$82 = 53 + 38 - n(S \cap P)$$
$$\Rightarrow \quad n(S \cap P) = 91 - 82 = 9.$$

Example 4: If, $B \subsetneq A'$, then which of the following sets is equal to A'.

(a) $(A \cap B) \cup B$ (b) $(A \cap B) \cup A'$

(c) $(A \cup B) \cap A'$ (d) $(A \cup B) \cap B$

Ans. (b)

Solution: For (a), $(A \cap B) \cup B = (A \cup B) \cap (B \cup B)$

$$= (A \cup B) \cap B = B.$$

For (b), $(A \cap B) \cup A' = (A \cup A') \cap (B \cup A')$

$$= X \cap A' = A'$$

For (c), $(A \cup B) \cap A' = (A \cap A') \cup (B \cap A')$

$$= \phi \cup (B \cap A')$$
$$= B \cap A' = B.$$

For (d) $(A \cup B) \cap B = B$.

Example 5: If $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] = \frac{5x}{6}$, then x is any term of the following

(a) 3, 6, 9, 12,… (b) 9, 18, 27, 36, …

(c) 6, 12, 18, 24,… (d) $\frac{6}{5}, \frac{12}{5}, \frac{18}{5}, \ldots$

Ans. (c)

◎ **Solution:** Since $\left[\frac{x}{2}\right], \left[\frac{x}{3}\right] \in \boldsymbol{I}$, so $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] \in \boldsymbol{I}$

Thus $\frac{5x}{6} \in \boldsymbol{I} \Rightarrow x = \frac{6}{5}n, n \in \boldsymbol{I}$.

Substituting this value in $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] = \frac{5x}{6}$, we have

$$\left[\frac{3}{5}n\right] + \left[\frac{2}{5}n\right] = n$$

$$\Rightarrow \quad \frac{3}{5}n - \left\{\frac{3}{5}n\right\} + \frac{2}{5}n - \left\{\frac{2}{5}n\right\} = n$$

$$\Rightarrow \quad \left\{\frac{3}{5}n\right\} + \left\{\frac{2}{5}n\right\} = 0$$

$$\Rightarrow \quad \left\{\frac{3}{5}n\right\} = 0 = \left\{\frac{2}{5}n\right\} \qquad (\text{Since } 0 \le \{x\} < 1)$$

Thus $3n = 5m_1$, $2n = 5m_2$, Therefore $xn = \frac{2n \cdot 3n}{5}$

$$= \frac{5m_1 \times 5m_2}{5} = 5m_1 m_2 \Rightarrow \frac{x \cdot 5m_1}{3} = 5m_1 m_2$$

$$\Rightarrow \quad x = 3m_2$$

Similarly $\frac{x \cdot 5m_2}{2} = 5m_1 m_2 \Rightarrow x = 2m_1$

Hence x is multiple of 2 and 3 so of 6 and $x \in \mathbf{I}$

◉ **Example 6:** The relation R defined by '>' on the set $\mathbf{N}$ is

(a) reflexive (b) symmetric
(c) transitive (d) equivalence relation

Ans. (c)

◎ **Solution:** $2 \not> 2$ so > is not reflexive, $3 > 2$ but $2 \not< 3$ so $(3, 2) \in R$ but $(2, 3) \notin R$. Thus R is not symmetric. If $(a, b) \in R$ and $(b, c) \in R$ then $a > b$, $b > c \Rightarrow a > c$ so $(a, c) \in R$. R is not an equivalence relation.

◉ **Example 7:** The relation $a\, R\, b$ defined by a is factor of b on $\boldsymbol{N}$ is not

(a) reflexive (b) transitive
(c) anti symmetric (d) symmetric

Ans. (d)

◎ **Solution:** For $a \in \boldsymbol{N}$, a is a factor of a so R is reflexive. If a is factor b and b is factor of c then a is factor of c so R is transitive. If a is factor of b and b is factor of a then $a = b$ so R is anti symmetric, 2 is factor of 4 but 4 is not a factor of 2.

◉ **Example 8:** The domain of the function $f(x) = \log_2 \sin x$ is

(a) $\mathbf{R}$ (b) $\mathbf{R} \sim \{n\pi : n \in \mathbf{I}\}$
(c) $R \sim \{n\pi : n \in \mathbf{N}\}$ (d) $\bigcup_{n\in\mathbf{I}} (2n\pi, (2n+1)\pi)$

Ans. (d)

◎ **Solution:** $f(x) = \log_2 \sin x$ is defined for all x for which $\sin x > 0$. But $\sin x > 0$ if $x \in (0, \pi) \cup (2\pi, 3\pi) \cup \ldots = \bigcup_{n\in\mathbf{I}} (2n\pi, (2n+1)\pi)$.

◉ **Example 9:** The domain of $y = \cos^{-1}\frac{1-2x}{4}$ is

(a) $\left[-\frac{3}{2}, \frac{5}{2}\right]$ (b) $[-1, 1]$
(c) $[0, 2]$ (d) $\left[-\frac{1}{2}, \frac{3}{2}\right]$

Ans. (a)

◎ **Solution:** The given function is defined if

$$-1 \le \frac{1-2x}{4} \le 1 \quad \text{i.e. if } -4 \le 1-2x \le 4,$$

$$\Rightarrow -5 \le -2x \le 3 \Rightarrow \frac{-3}{2} \le x \ge \frac{5}{2}$$

◉ **Example 10:** Which of the following functions is bounded

(a) $y = 1 - \log_{10}x$ (b) $y = e^{-2x}$
(c) $y = \sin^{-1}(2x + 1)$ (d) $y = \tan(4x + 1)$

Ans. (c)

◎ **Solution:** The range $\log_{10}x$ is $(-\infty, \infty)$ so $y = 1 - \log_{10}x$ is unbounded. $y = e^{-2x}$ is unbounded from below as $x \to -\infty$, $y \to \infty$. The range of $\sin^{-1}x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ so $y = \sin^{-1}(2x + 1)$ is a bounded function. The range of $\tan x$ is $\mathbf{R}$ so $y = \tan(4x + 1)$ is unbounded

◉ **Example 11:** A function out of the following whose period is not π is

(a) $\sin^2 x$ (b) $\cos^2 x$
(c) $\tan(2x + 3)$ (d) $y = |\sin x|$

Ans. (c)

◎ **Solution:** $y = \sin^2 x = \frac{1}{2}[1 - \cos 2x]$. Since period of $\cos x$ is 2π so period of $\cos 2x$ is π.

$y = 1 + \cos^2 x = 1 + \frac{1}{2}(1 + \cos 2x)$. The period of this is again π. The period of $\tan x$ is π so period of $\tan(2\pi + 3)$ is $\pi/2$. If $f(x) = |\sin x|$ then $f(x + \pi) = |\sin(x + \pi)| = |-\sin x| = |\sin x| = f(x)$. Thus the period of f is π.

◉ **Example 12:** Which of the following functions is an odd function

(a) $y = x^4 - 2x^2$ (b) $y = x - x^2$
(c) $y = \cos x$ (d) $y = x - \frac{x^3}{6} + \frac{x^5}{40}$

Ans. (d)

◎ **Solution:** If $f(x) = x^4 - 2x^2$ then $f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$. Hence f is an even function. If $u(x) = x - x^2$ then $u(-x) = x - x^2$ so u is neither even nor an odd function. $p(x) = \cos x$, $p(-x) = \cos(-x) = \cos x = p(x)$ so p is even function. If $s(x) = x - \frac{x^3}{6} + \frac{x^5}{40}$ then $s(-x) = -x - \frac{x^3}{6} - \frac{x^5}{40} = -\left(x - \frac{x^3}{6} + \frac{x^5}{40}\right) = -s(x)$, so s is an odd function.

LEVEL 1

Straight Objective Type Questions

Example 13: Let $X = \{x : x = n^3 + 2n + 1, n \in \mathbf{N}\}$ and $Y = \{x : x = 3n^2 + 7, n \in \mathbf{N}\}$ then

(a) $X \cap Y$ is a subset of $\{x : x = 3n + 5, n \in \mathbf{N}\}$
(b) $X \cap Y \subseteq \{x : x = n^2 + n + 1, n \in \mathbf{N}\}$
(c) $34 \in X \cap Y$
(d) none of these

Ans. (c)

Solution: If $n^3 + 2n + 1 = 3n^2 + 7$

$\Rightarrow \quad n^3 - 3n^2 + 2n - 6 = 0$

$\Rightarrow \quad (n - 3)(n^2 + 2) = 0$

$\Rightarrow \quad n = 3$ as $n \in \mathbf{N}$

So, $\quad x = 3 \times 3^2 + 7 = 34 \in X \cap Y$.

In (a) and (b) $x \neq 34$, for any $n \in \mathbf{N}$.

Example 14: A, B, C are the sets of letters needed to spell the words STUDENT, PROGRESS and CONGRUENT, respectively. Then $n(A \cup (B \cap C))$ is equal to

(a) 8 (b) 9
(c) 10 (d) 11

Ans. (b)

Solution:
$$A = \{D, E, N, S, T, U]$$
$$B = \{E, G, O, P, R, S\}$$
$$C = \{C, E, G, N, O, R, T, U\}$$

So $\quad A \cup (B \cap C)$
$$= \{D, E, N, S, T, U\} \cup \{E, G, O, R\}$$
$$= \{D, E, G, O, N, R, S, T, U\}$$

and $n[(A \cup (B \cap C)] = 9$.

Example 15: Let $A = \{x : x$ is a prime factor of $240\}$
$B = \{x : x$ is the sum of any two prime factors of $240\}$.

Then

(a) $5 \notin A \cap B$ (b) $7 \in A \cap B$
(c) $8 \in A \cap B$ (d) $8 \in A \cup B$

Ans. (d)

Solution: $240 = 2 \times 3 \times 5 \times 8$

So $\quad A = \{2, 3, 5\}, \quad B = \{5, 7, 8\}$.

Clearly $8 \in A \cup B$.

Example 16: A, B, C are three sets such that $n(A) = 25$, $n(B) = 20$, $n(c) = 27$, $n(A \cap B) = 5$, $n(B \cap C) = 7$ and $A \cap C = \phi$ then $n(A \cup B \cup C)$ is equal to

(a) 60 (b) 65
(c) 67 (d) 72.

Ans. (a)

Solution: $A \cap C = \phi \quad \Rightarrow \quad A \cap B \cap C = \phi$.

$$\therefore\ n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 25 + 20 + 27 - 5 - 7 - 0 + 0 = 60.$$

Example 17: Let $X = \{(x,y,z) \mid x,y,z \in \mathbf{N}.\ x + y + z = 10,\ x < y < z\}$ and $Y = \{(x,y,z) \mid x,y,z \in \mathbf{N},\ y = |x - z|\}$ then $X \cap Y$ is equal to

(a) $\{(2,3,5)\}$ (b) $\{1,4,5\}$
(c) $\{5,1,4\}$ (d) $\{(2,3,5), (1,4,5)\}$

Ans. (d)

Solution: $X = \{(1,2,7), (1,3,6), (1,4,5), (2,3,5)\}$.
Elements of X which belong to Y are (1,4,5) and (2,3,5) both so they belong to $X \cap Y$.

Example 18: If A,B,C are three non-empty sets such that $A \cap B = \phi$, $B \cap C = \phi$, then

(a) $A = C$ (b) $A \subset C$
(c) $C \subset A$ (d) none of these

Ans. (d)

Solution: Let $A = \{1,2,3,4,5\}$, $B = \{6,7,8,9\}$ and $C = \{11,12,13\}$ which satisfy the given conditions but none of (a), (b) or (c).

Example 19: Two finite sets have m and n elements respectively. The total number of subsets of first set is 56 more than the total number of subsets of the second set. The values of m and n respectively are

(a) 7, 6 (b) 6, 3
(c) 5, 1 (d) 8, 7

Ans. (b)

Solution: According to the given condition, we have $2^m = 2^n + 56$
$\Rightarrow 2^{m-3} - 2^{n-3} = 7 \Rightarrow 2^{n-3}(2^{m-n} - 1) = 7$. Since 7 is a prime number so we must have $n - 3 = 0$ (clearly $m \neq n$). Thus $n = 3$. Therefore, $2^m = 2^3 + 56 = 64 = 2^6 \Rightarrow m = 6$.

Example 20: Among employee of a company taking vacations last years, 90% took vacations in the summer, 65% in the winter, 10% in the spring, 7% in the autumn, 55% in winter and summer, 8% in the spring and summer, 6% in the autumn and summer, 4% in the winter and spring, 4% in winter and autumn, 3% in the spring and autumn, 3% in the summer, winter and spring, 3% in the summer, winter and autumn, 2% in the summer, autumn and spring, and 2% in the winter, spring and autumn. Percentage of employee that took vacations during every season:

(a) 4 (b) 3
(c) 2 (d) 8

Ans. (c)

◎ **Solution:** Suppose that number of employee taking vacations is 100.

Su – set of employee taking leave in Summer
W – set of employee taking leave in Winter
Sp – set of employee taking leave in Spring
A – set of employee taking leave in Autumn

$$n(Su) = 90, n(W) = 65, n(Sp) = 10, n(A) = 7$$
$$n(W \cap Su) = 55, n(Sp \cap Su) = 8, n(A \cap Su) = 6$$
$$n(W \cap Sp) = 4, n(W \cap Au) = 4, n(Sp \cap A) = 3$$
$$n(Su \cap A) = 3, n(Su \cap W \cap A) = 3$$
$$n(Su \cap W \cap Sp) = 3, n(Su \cap A \cap Sp) = 2$$
$$n(W \cap Sp \cap A) = 2$$

$$\begin{aligned} n(Su \cap Sp \cap W \cap A) &= n(Su) + n(Sp) + n(W) + n(A) - n(Su \cap Sp) \\ &\quad - n(Sp \cap W) - n(W \cap A) - n(Su \cap A) - n(Su \cap W) \\ &\quad - n(Sp \cap A) + n(Su \cap Sp \cap W) + n(Su \cap W \cap A) \\ &\quad + n(W \cap A \cap Su) + n(Su \cap Sp \cap A) \\ &\quad - n(Sp \cup Su \cup A \cup W) \\ &= 90 + 65 + 10 + 7 - 55 - 8 - 6 - 4 - 4 - 3 \\ &\quad + 3 + 3 + 2 + 2 - 100 = 2 \end{aligned}$$

◉ **Example 21:** If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$ then the number of elements in $(A \cup B) \times (A \cap B) \times (A \,\Delta\, B)$ is

(a) 5 (b) 30
(c) 10 (d) 4

Ans. (b)

◎ **Solution:** $A \cup B = \{1, 2, 3, 4, 5\}$. $n(A \cup B) = 5$

$$A \cap B = \{3, 4\}, n(A \cap B) = 2$$
$$A \,\Delta\, B = (A \sim B) \cup (B \sim A) = \{1, 2\} \cup \{5\} = \{1, 2, 5\}$$
$$n(A \,\Delta\, B) = 3. \text{ Hence } n((A \cup B) \times (A \cap B) \times (A \,\Delta\, B)\} = 5 \times 2 \times 3 = 30.$$

◉ **Example 22:** Let **I** be the set of integers. For $a, b \in \mathbf{I}$, $a\,R\,b$ if and only if $|a - b| < 1$, then

(a) R is not reflexive
(b) R is not symmetric
(c) $R = \{(a,a); a \in \mathbf{I}\}$
(d) R is not an equivalence relation.

Ans. (c)

◎ **Solution:** For any integers a, b, $|a - b| < 1$ if and only if $|a - b| = 0$ so $a = b$. Hence $R = \{(a,a); a \in \mathbf{I}\}$.Thus R is reflexive, symmetric and transitive.

◉ **Example 23:** Let W denote the words in the English Dictionary. Define the relation **R** by $\mathbf{R} = \{(x, y) \in W \times W :$ the words x and y have at least one letter common$\}$, then **R** is

(a) reflexive, not symmetric and transitive
(b) not reflexive, symmetric and transitive
(c) reflexive, symmetric and not transitive
(d) reflexive, symmetric and transitive

Ans. (c)

◎ **Solution:** $(x, x) \in \mathbf{R} \quad \forall\; x \in W$ as all letters in both are common. If $(x, y) \in \mathbf{R}$ then x and y have a letter in common $\Rightarrow (y, x) \in \mathbf{R}$.

Next, let x = fix, y = six and z = son then$(x, y) \in \mathbf{R}$, $(y, z) \in \mathbf{R}$ but $(x, z) \notin \mathbf{R}$

So **R** is reflexive, symmetric but not transitive

◉ **Example 24:** If the relation $R : A \to B$, where $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$ is defined by $R = \{(x, y); x < y, x \in A, y \in B\}$ then RoR^{-1} is

(a) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
(b) $\{(3, 1), (5, 1), (5, 2), (5, 3), (5, 4)\}$
(c) $\{(3, 3), (3, 5), (5, 3), (5, 5)$
(d) none of these

Ans. (c)

◎ **Solution:** $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$

and $\quad R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$.

Thus $\quad RoR^{-1} = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$.

◉ **Example 25:** Let $A = \{x \in \mathbf{R} : [x + 3] + [x + 4] \leq 3\}$ and $B = \left\{x \in \mathbf{R} : 3^x \left(\sum_{r=1}^{\infty} \frac{3}{10^r}\right)^{x-3} < 3^{-3x}\right\}$ then

(a) $A = B$ (b) $A \subsetneq B$
(c) $B \subsetneq A$ (d) $A \cap B = \phi$

Ans. (a)

◎ **Solution:** Let $x \in A$, $[x + 3] + [x + 4] \leq 3$

$$\Rightarrow \quad [x] + 3 + [x] + 4 \leq 3$$
$$\Rightarrow \quad 2[x] \leq -4 \Rightarrow [x] \leq -2$$
$$\Rightarrow \quad x \in (-\infty, -1)$$
$$A = (-\infty, -1)$$

If $x \in B$ then $3^x\, 3^{x-3} \left(\sum_{r=1}^{\infty} \frac{1}{10^r}\right)^{x-3} < 3^{-3x}$

$$\Rightarrow \quad 3^{2x-3}\left(\frac{1/10}{1-1/10}\right)^{x-3} < 3^{-3x}$$
$$\Rightarrow \quad 3^{2x-3}\left(3^{-2}\right)^{x-3} < 3^{-3x}$$
$$\Rightarrow \quad 3^3 < 3^{-3x} \Rightarrow 3 < -3x$$

so $x \in (-\infty, -1)$

Hence $B = (-\infty, -1)$. Thus $A = B$.

◉ **Example 26:** The range of the function $f(x) = {}^{7-x}P_{x-3}$ is

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3\}$ (d) $\{1, 2, 3, 4, 5\}$

Ans. (c)

◎ **Solution:** $7 - x \geq 1, x - 3 \geq 0$

and $7 - x \geq x - 3$

$\Rightarrow \quad x \leq 6, x \geq 3, x \leq 5.$

Thus $3 \leq x \leq 5$

$\therefore \quad \text{Range} = \{4P_0, 3P_1, 2P_2\} = \{1, 3, 2\}$

● **Example 27:** The domain of the function

$f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(a) [1, 2] (b) [2, 3)
(c) [1, 3] (d) [1, 2]

Ans. (b)

◎ **Solution:** $-1 \leq x - 3 \leq 1$ and $9 - x^2 > 0$

$\Rightarrow \quad 2 \leq x \leq 4$ and $-3 < x < 3$.

So domain of f is [2, 3).

● **Example 28:** The solution of $8x \equiv 6 \pmod{14}$ is

(a) [8], [6] (b) [6], [14]
(c) [6], [13] (d) [8], [14], [16]

where $[a] = \{a + 14k : k \in \mathbf{I}\}$

Ans. (c)

◎ **Solution:** We need to solve $14|(8x - 6)$ i.e., $8x - 6 = 14k$, for $k \in I$. The values 6 and 13 satisfy this equation, while 8, 14 and 16 do not.

● **Example 29:** Let $R = \{(x, y): x, y \in A, x + y = 5\}$, where $A = \{1, 2, 3, 4, 5\}$ then

(a) R is not reflexive, symmetric and not transitive
(b) R is an equivalence relation
(c) R is reflexive, symmetric but not transitive
(d) R is not reflexive, not symmetric but transitive

Ans. (a)

◎ **Solution:** $R = \{(1, 4), (4, 1), (2, 3), (3, 2)\}$, so R is not reflexive as $(1, 1) \notin R$. R is symmetric by definition and R is not transitive as $(1, 4) \in R$, $(4, 1) \in R$ but $(1, 1) \notin R$.

● **Example 30:** Let R be a relation on a set A such that $R = R^{-1}$ then R is

(a) reflexive (b) symmetric
(c) transitive (d) an equivalence relation

Ans. (b)

◎ **Solution:** If $(a, b) \in R$ then $(b, a) \in R^{-1} = R$ so R is symmetric. The relation in Example 29 satisfy $R = R^{-1}$ but is neither reflexive nor transitive.

● **Example 31:** For $x, y \in \mathbf{R}$, define a relation R by $x R y$ if and only if $x - y + \sqrt{2}$ is an irrational number. Then R is

(a) reflexive (b) symmetric
(c) transitive (d) none of these

Ans. (a)

◎ **Solution:** Since $x - x + \sqrt{2} = \sqrt{2}$ which is an irrational number so $x R x$ for all $x \in \mathbf{R}$. Hence R is reflexive. R is not symmetric as $(\sqrt{2}, 1) \in R$ but $(1, \sqrt{2}) \notin R$. Again R is not transitive since $(\sqrt{2}, 1) \in R$ and $(1, 2\sqrt{2}) \in R$ but $(\sqrt{2}, 2\sqrt{2}) \notin R$.

● **Example 32:** If $n|q$ and $A = \{z \in C : z^n = 1\}$, $B = \{z : z^q = 1\}$ then

(a) $A = B$ (b) $A \cap B = \{1\}$
(c) $B \subseteq A$ (d) $A \subsetneq B$

Ans. (d)

◎ **Solution:** $q = p\,n$ for some $p \in \mathbf{N}$

$$z^q - 1 = (z^n)^p - 1 = (z^n - 1)\,(z^{(p-1)n} + \cdots + z^n + 1)$$

Every root of $z^n - 1$ is a root of $z^q - 1$ and every root of $z^{(p-1)n} + \cdots + z^n + 1 = 0$ is also a root of $z^q - 1$. Hence $A \subsetneq B$ and $A \cap B = A$.

● **Example 33:** If $A = \{z : (1 + 2i)\,\bar{z} + (1 - 2i)\,z + 2 = 0\}$ and $B = \{z : (3 + 2i)\,\bar{z} + (3 - 2i)\,z + 3 = 0\}$, then

(a) $A \cap B$ is a singleton set
(b) $A \subseteq B$
(c) $B \subseteq A$
(d) $A \cap B = \phi$.

Ans. (a)

◎ **Solution:** Equations in sets A and B represent straight line with $\bar{\alpha}_1 = 1 + 2i$ and $\bar{\alpha}_2 = 3 + 2i$. Since $\dfrac{\bar{\alpha}_1}{\alpha_1} \neq \dfrac{\bar{\alpha}_2}{\alpha_2}$ so the lines are intersecting, hence $A \cap B$ is a singleton set.

● **Example 34:** Let $x, y \in \mathbf{I}$ and suppose that a relation R on $\mathbf{I}$ is defined by $x R y$ if and only if $x \leq y$ then

(a) R is reflexive but not symmetric
(b) R is an equivalence relation
(c) R is neither reflexive nor symmetric
(d) R is symmetric but not transitive

Ans. (a)

◎ **Solution:** Since $x \leq x$ for all $x \in \mathbf{I}$ so R is reflexive but is not symmetric as $(1, 2) \in R$ and $(2, 1) \notin R$. Also R is transitive as $x \leq y, y \leq z \Rightarrow x \leq z$.

● **Example 35:** If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 + 1$, then value of $f^{-1}(17)$ and $f^{-1}(-3)$ are, respectively,

(a) ϕ, $\{4, -4\}$ (b) ϕ, $\{3, -3\}$
(c) $\{3, -3\}$, ϕ (d) $\{4, -4\}$, ϕ

Ans. (d)

◎ **Solution:** For any $A \subseteq \mathbf{R}$, we have $f^{-1}(A) = \{x \in \mathbf{R}: f(x) \in A\}$. Thus,

$f^{-1}(17) = \{x : f(x) \in \{17\}\} = \{x : f(x) = 17\}$
$= \{x : x^2 + 1 = 17\} = \{4, -4\}$,
and similarly, $f^{-1}(-3) = \{x \in \mathbf{R} : x^2 + 1 = -3\} = \phi$.

Example 36: The functions f and g are given by $f(x) = \{x\}$, the fractional part of x and $g(x) = \frac{1}{2}$ sin $[x]\pi$, where $[x]$ denotes the integral part of x. Then range of gof is

(a) $[-1, 1]$ (b) $\{0\}$
(c) $\{-1, 1\}$ (d) $[0, 1]$

Ans. (b)

Solution: $(gof)(x) = g(f(x)) = 1/2 \sin [\{x\}]\pi = 0$, for all $x \in \mathbf{R}$. Hence the range of gof is $\{0\}$.

Example 37: The period of the function $f(x) = \cos^2 3x + \tan 4x$ is

(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/6$ (d) π

Ans. (d)

Solution: $f(x) = (1/2)(1 + \cos 6x) + \tan 4x$. The period of $\cos 6x$ is $2\pi/6 = \pi/3$ and the period of $\tan 4x$ is $\pi/4$. Hence the period of f is l.c.m. of $\pi/3$ and $\pi/4 = \pi$.

Example 38: The domain of the function

$$f(x) = \sin^{-1}\left(\log_3 \frac{x}{3}\right) \text{ is}$$

(a) $[-1, 9]$ (b) $[1, 9]$
(c) $[-9, 1]$ (d) $[3, 9]$

Ans. (b)

Solution: The function f is defined only if $-1 \le \log_3 (x/3) \le 1$. This inequality is possible only if $1/3 \le x/3 \le 3$ i.e., $1 \le x \le 9$.

Example 39: The domain of the function

$$f(x) = \frac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{-x^2+2x+8}} \text{ is}$$

(a) $(1, 4)$ (b) $(-2, 4)$
(c) $(2, 4)$ (d) none of these

Ans. (c)

Solution: Since for, $0 < a < 1$, $\log_a x < 0$ for $x > 1$ so $\log_{0.3}(x - 1) < 0$ for $x > 2$. Also $-x^2 + 2x + 8 > 0$ if and only if $x \in (-2, 4)$. Hence the domain of the given function is $(2, 4)$.

Example 40: The function f: $[-1/2, 1/2] \to [-\pi/2, \pi/2]$ defined by $f(x) = \sin^{-1}(3x - 4x^3)$ is

(a) both one-one and onto
(b) neither one-one nor onto
(c) onto but not one-one
(d) one-one but not onto

Ans. (a)

Solution: Since $\sin^{-1}(3x - 4x^3) = 3\sin^{-1} x \in [-\pi/2, \pi/2]$ i.e., $\sin^{-1} x \in [-\pi/6, \pi/6]$ or $x \in [-1/2, 1/2]$ so f is onto. Also $f'(x) = \dfrac{3}{\sqrt{1-x^2}} > 0$ for $-1/2 < x < 1/2$. Therefore, f increases on $[-1/2, 1/2]$ and hence f is one-one.

Example 41: Given $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ and $g(x) = \dfrac{1}{\sqrt{x - |x|}}$ then

(a) dom $f \ne \phi$ and dom $g = \phi$
(b) dom $f = \phi$ and dom $g \ne \phi$
(c) f and g have the same domain
(d) dom $f = \phi$ and dom $g = \phi$

Ans. (a)

Solution: dom $f = \{x : |x| > x\}$ and dom $g = \{x : x > |x|\} = \phi$. Thus dom $f = \mathbf{R}^-$ (the set of all negative real numbers) and dom $g = \phi$.

Example 42: Which of the following functions is not onto

(a) f: $\mathbf{R} \to \mathbf{R}$, $f(x) = 3x + 4$
(b) f: $\mathbf{R} \to \mathbf{R}^+$, $f(x) = x^2 + 2$
(c) f: $\mathrm{R}^+ \to \mathrm{R}^+$, $f(x) = \sqrt{x}$
(d) none of these

Ans. (b)

Solution: The function $f(x) = 3x + 4$ is onto as for $y \in \mathbf{R}$, $f\left(\dfrac{y-4}{3}\right) = y$. The function $f : \mathbf{R}^+ \to \mathbf{R}^+$, $f(x) = \sqrt{x}$ is onto as for $y \in \mathbf{R}^+$, $f(y^2) = y$. $f : \mathbf{R} \to \mathbf{R}^+$, $f(x) = x^2 + 2$ is not onto as $1 \in \mathbf{R}^+$ has no pre-image.

Example 43: Which of the following functions is not one-one

(a) f: $\mathbf{R} \to \mathbf{R}$, $f(x) = 2x + 5$
(b) f: $[0, \pi] \to [-1, 1]$, $f(x) = \cos x$
(c) f: $[-\pi/2, \pi/2] \to [1, 7]$ $f(x) = 3 \sin x + 4$
(d) $f : \boldsymbol{R} \to [-1, 1]$, $f(x) = \sin x$

Ans. (d)

Solution: The function in (a) is one-one as $2x_1 + 5 = 2x_2 + 5$ is possible only if $x_1 = x_2$. The function in (b) is one-one as $\cos x_1 = \cos x_2$ if and only if $\sin \dfrac{x_1 - x_2}{2} = 0$ i.e., $x_1 = x_2$. Similarly the function in (c) is also one-one. The function in (d) is not one-one as $f(\pi) = f(2\pi) = 0$.

Example 44: Which of the following functions is non-periodic

(a) $f(x) = x - [x]$

(b) $f(x) = \begin{cases} 1 \text{ if } x \text{ is a rational number} \\ 0 \text{ if } x \text{ is an irrational number} \end{cases}$

(c) $f(x) = \sqrt{\dfrac{8}{1+\cos x} + \dfrac{8}{1-\cos x}}$

(d) none of these

Ans. (d)

◎ **Solution:** The function in (a) is periodic with period 1 and the function in (b) is also periodic since $f(x + r) = f(x)$ for every rational r. The function in (c) is equal to $\dfrac{4}{|\sin x|}$ and thus has period π.

◉ **Example 45:** Let $f(x) = \dfrac{\alpha x}{x+1}$, $x \neq -1$. Then, for what value of α is $f(f(x)) = x$?

(a) $\sqrt{2}$ (b) $-\sqrt{2}$
(c) 1 (d) -1

Ans. (d)

◎ **Solution:** $f(f(x)) = \dfrac{\alpha\, f(x)}{f(x)+1} = \dfrac{\alpha^2 x}{(\alpha+1)x+1}$

Thus $f(f(x)) = x \Leftrightarrow \alpha^2 x = (\alpha + 1)x^2 + x$
$\Leftrightarrow (\alpha^2 - 1)x = (\alpha + 1)\, x^2$
$\Leftrightarrow (\alpha + 1)\,((\alpha - 1) - (\alpha + 1)x) = 0$

Since $(\alpha - 1) - (\alpha + 1)\, x \neq 0$ for all x so $\alpha = -1$.

◉ **Example 46:** Let $R = \{(x, y): x, y \in \mathbf{R}, x^2 + y^2 \leq 25\}$

$R' = \left\{(x, y): x, y \in \mathbf{R}, y \geq \dfrac{4}{9}x^2\right\}$ then

(a) dom $R \cap R' = [-4, 4]$
(b) range $R \cap R' = [0, 4]$
(c) range $R \cap R' = [0, 5]$
(d) $R \cap R'$ defines a function.

Ans. (c)

◎ **Solution:** The equation $x^2 + y^2 = 25$ represents a circle with centre (0, 0) and radius 5 and the equation $y = \dfrac{4}{9}x^2$ represents a parabola with vertex (0, 0) and focus (0, 1/9). Hence $R \cap R'$ is the set of points indicated in the Fig.1.27

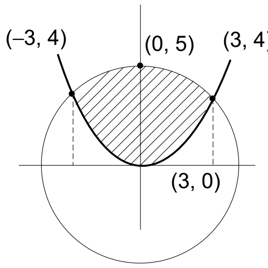


Fig. 1.27

$= \{(x, y): -3 \leq x \leq 3, 0 \leq y \leq 3]\}$. Thus
dom $R \cap R' = [-3, 3]$ and range $R \cap R' = [0, 5] \supset [0, 4]$
Since $(0, 0) \in R \cap R'$ and $(0, 5) \in R \cap R'$
$\therefore$ 0 is related to 0 as well as 5.
Hence $R \cap R'$ doesn't define a function.

◉ **Example 47:** In a factory 70% of the workers like oranges and 64% like apples. If x% like both oranges and apples, then

(a) $x \leq 34$ (b) $x \geq 64$
(c) $34 \leq x \leq 64$ (d) none of these.

Ans. (c)

◎ **Solution:** Let the total number of workers be 100. A, the set of workers who like oranges and B, the set of workers who like apples.

So $n(A) = 70, n(B) = 64, n(A \cap B) = x$.
Also $n(A \cup B) \leq 100$.
$\Rightarrow$ $n(A) + n(B) - n(A \cap B) \leq 100$
$\Rightarrow$ $70 + 64 - x \leq 100 \Rightarrow x \geq 34$
Since $n(A \cap B) \leq n(B) \Rightarrow x \leq 64$
Hence $34 \leq x \leq 64$.

◉ **Example 48:** The Cartesian product of $A \times A$ has 16 elements. $S = \{(a, b) \in A \times A | a < b\}$. (– 1, 2) and (0, 1) are two elements belonging to S. The remaining elements of S are given by.

(a) $\{(-1, 0), (-1, 1), (0, 2), (1, 2)\}$
(b) $\{(-1, 0), (1, 1), (2, -1), (1, 2)\}$
(c) $\{(0, -1), (1, -1), (0, 2), (1, 2)\}$
(d) none of these

Ans. (a)

◎ **Solution:** $(-1, 2) \in A \times A$
$\Rightarrow$ $-1 \in A$, $2 \in A$ and $(0, 1) \in A \times A \Rightarrow 0 \in A, 1 \in A$
So, $A = \{-1, 0, 1, 2\}$ as A has four elements
and $S = \{(-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}$.
Hence the required element of S are given by (a)

◉ **Example 49:** If R and R' are two symmetric relations (not disjoint) on a set A, then the relation $R \cap R'$ is

(a) reflexive (b) symmetric
(c) transitive (d) none of these.

Ans. (b)

◎ **Solution:**
Let $(a, b) \in R \cap R'$ for some $a, b \in A$.
$\Rightarrow$ $(a, b) \in R$ and $(a, b) \in R'$
$\Rightarrow$ $(b, a) \in R$ and $(b, a) \in R'$ as R and R' are symmetric
$\Rightarrow$ $(b, a) \in R \cap R'$ showing that $R \cap R'$ is symmetric.

◉ **Example 50:** α, β, γ denote respectively the sets containing the letters in the names Apoorv, Mannan and Manvi of three children playing together. Which of the following is not correct.

(a) $n(\alpha \cap \gamma) | n(\alpha \cup \beta \cup \gamma)$
(b) $n(\beta \cap \gamma) | n(\alpha \cup \beta \cup \gamma)$
(c) $n(\alpha \cup \beta \cup \gamma) = 8$
(d) $n(\alpha \cup \beta \cup \gamma) = n(\alpha \cup \gamma)$

Ans. (b)

◎ **Solution:** $\alpha = \{a, o, p, r, v\}$, $\beta = \{a, m, n\}$
$\gamma = \{a, i, m, n, v\}$
$\alpha \cap \beta = \{a\}$, $\alpha \cap \gamma = \{a, v\}$, $\beta \cap \gamma = \{a, m, n\}$
$\alpha \cup \beta \cup \gamma = \{a, i, m, n, o, p, r, v\} = \alpha \cup \gamma$.

Example 51: If $f: \mathbf{R} \to \mathbf{R}$, defined by $f(x) = x^3 + 7$, then the value of $f^{-1}(71)$ and $f^{-1}(-1)$ respectively are

(a) $\{4\}, \phi$ (b) $\phi, \{-2\}$
(c) $\{4\}, \{-2\}$ (d) $\{2\}, \{-4\}$

Ans (c)

Solution: $f(x) = x^3 + 7 = 71 \Rightarrow x^3 = 64 \Rightarrow x = 4$

$\Rightarrow \quad f^{-1}(71) = \{4\}$

and $f(x) = x^3 + 7 = -1 \Rightarrow x^3 = -8 \Rightarrow x = -2 \Rightarrow f^{-1}(-1) = \{-2\}$

Example 52: If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$ and $\mathbf{N}$, the set of natural numbers is the universal set, then $A' \cup [(A \cup B) \cap B']$ is

(a) A (b) A'
(c) B (d) $\mathbf{N}$.

Ans. (d)

Solution: $(A \cup B) \cap B' = A$ as $A \cap B = \phi$

$\Rightarrow \quad A' \cup [(A \cup B) \cap B'] = A' \cup A = \mathbf{N}$.

Example 53: If $X = \{1, 2, 3, 4,\}$, then a one-one onto mapping $f: X \to X$ such that $f(1) = 1, f(2) \neq 2$ and $f(4) \neq 4$ is given by

(a) $\{(1, 1), (2, 3), (3, 4), (4, 2)\}$
(b) $\{(1, 1), (2, 4), (3, 1), (4, 2)\}$
(c) $\{(1, 2), (2, 4), (3, 2), (4, 3)\}$
(d) none of these

Ans. (a)

Solution: f in (a) is clearly one-one and onto also satisfies $f(1) = 1, f(2) = 3 \neq 2, f(4) = 2 \neq 4$.

Example 54: Let $f(x) = (x + 1)^2 - 1, x \geq -1$ then the set $\{x : f(x) = f^{-1}(x)\}$ is equal to

(a) $\{0, -1\}$ (b) $\{0, 1\}$
(c) $\{-1, 1\}$ (d) $\{0\}$

Ans. (a)

Solution: Let $y = (x + 1)^2 - 1, x \geq -1$

$\Rightarrow \quad (x + 1)^2 = y + 1$

$\Rightarrow \quad x + 1 = \sqrt{y+1}$ as $x \geq -1$

$\Rightarrow \quad x = -1 + \sqrt{y+1}\ , y \geq -1$

Thus $\quad f^{-1}(x) = -1 + \sqrt{x+1}$

So $\quad f(x) = f^{-1}(x)$

$\Rightarrow \quad (x + 1)^2 - 1 = -1 + \sqrt{x+1}$

$\Rightarrow \quad \sqrt{x+1} = 0$ or $(x + 1)^{3/2} = 1$

$\Rightarrow \quad x = -1$ or $x = 0$

Example 55: Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets, then

(a) $P \subset Q$ and $Q \sim P \neq \phi$ (b) $Q \not\subset P$
(c) $P \not\subset Q$ (d) $P = Q$.

Ans. (d)

Solution: $\sin\theta - \cos\theta = \sqrt{2}\cos\theta$

$\Leftrightarrow \quad \sin\theta = (\sqrt{2}+1)\cos\theta$

$\Leftrightarrow \quad \cos\theta = (\sqrt{2}-1)\sin\theta$

$\Leftrightarrow \quad \sin\theta + \cos\theta = \sqrt{2}\sin\theta$

$\Rightarrow \quad P = Q$.

Example 56: If A, B, C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(a) $B = C$ (b) $A \cap B = \phi$
(c) $A = B$ (d) $A = C$

Ans. (a)

Solution: Let $x \in B \Rightarrow x \in A \cup B$

$\Rightarrow \quad x \in A \cup C \Rightarrow x \in A$ or $x \in C$

If $x \in A$, then $x \in A \cap B = A \cap C \Rightarrow x \in C$

So $\quad x \in B \Rightarrow x \in C$.

Similarly $\quad x \in C \Rightarrow x \in B$, Hence $B = C$.

Example 57: The domain of the function

$$f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}} \text{ is}$$

(a) $[1, 2]$ (b) $[2, 3]$
(c) $[1, 3]$ (d) $[1, 2]$

Ans. (b)

Solution: $x^2 < 9 \Rightarrow -3 < x < 3$

and $-1 \leq x - 3 \leq 1 \Rightarrow 2 \leq x < 3$

Example 58: Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X, Z \subseteq X$, and $Y \cap Z$ is empty is

(a) 2^5 (b) 5^3
(c) 5^2 (d) 3^5

Ans. (d)

Solution: For each $x \in X$, we have three choices

$x \in Y,\ x \notin Z; \quad x \notin Y,\ x \in Z; \quad x \notin Y,\ x \notin Z$

So the required number of ordered pairs is 3^5.

Example 59: Let $X = \{1, 2, 3, 4\}$. The number of equivalence relations that can be defined on X is

(a) 10 (b) 15
(c) 16 (d) 8

Ans. (b)

Solution: The number of equivalence relations

$$B_k = \sum_{n=0}^{k-1} \binom{k-1}{n} B_n; \quad B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5$$

$$B_4 = \binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 + \binom{3}{3}B_3$$
$$= 1 + 3 + 3 \times 2 + 1 \times 5 = 15$$

Example 60: The function $f(x) = \sin\dfrac{\pi x}{n!} - \cos\dfrac{\pi x}{(n+1)!}$ is

(a) non periodic
(b) periodic with period $2(n!)$
(c) periodic with period $2\,(n+1)!$
(d) periodic with period $n!$

Ans. (c)

Solution: Since the period of $\sin x$ is 2π so the period of $\sin\frac{\pi x}{n!}$ is $\frac{2\pi n!}{\pi} = 2(n!)$. The period of $\cos\frac{\pi x}{(n+1)!}$ is $2(n+1)!$ The period of $f(x) = \text{l.c.m.}\ (2(n!),\ 2(n+1)!) = 2(n+1)!$

Example 61: If $f(x)^2\ f\left(\frac{1-x}{1+x}\right) = x^3,\ x \neq -1,\ 1$ and $f(x) \neq 0$, then $\{f(-2)\}$ (the fractional part of $f(-2)$) is equal to

(a) $2/3$ (b) $1/3$

(c) $1/2$ (d) 0

Ans. (a)

Solution: Replacing x by $\frac{1-x}{1+x}$ in the equation

$$(f(x))^2\ f\left(\frac{1-x}{1+x}\right) = x^3 \qquad \text{...(i)}$$

We have

$$\left(f\left(\frac{1-x}{1+x}\right)\right)^2 f(x) = \left(\frac{1-x}{1+x}\right)^3 \qquad \text{... (ii)}$$

Solving (i) and (ii), we have

$$\left(\frac{x^3}{(f(x))^2}\right)^2 f(x) = \left(\frac{1-x}{1+x}\right)^3$$

$$\Rightarrow \quad \frac{(f(x))^3}{x^6} = \left(\frac{1+x}{1-x}\right)^3$$

$$\Rightarrow \quad f(x) = x^2\left(\frac{1+x}{1-x}\right)$$

$$f(-2) = 4\left(\frac{-1}{3}\right) = -\frac{4}{3}$$

$$\{f(-2)\} = -\frac{4}{3} - \left[-\frac{4}{3}\right] = -\frac{4}{3} + 2 = \frac{2}{3}.$$

Example 62: Let $f(x)$ be a function such that $f(x-1) + f(x+1) = \sqrt{2}f(x)$ for all $x \in \mathbf{R}$. If $f(3) = 5$ then $\sum_{r=0}^{10} f(3+8r)$ is equal to

(a) 50 (b) 55
(c) 0 (d) 10

*Ans***.** (b)

Solution: The given equation is

$$f(x-1) + f(x+1) = \sqrt{2}f(x) \qquad \text{... (i)}$$

Replace x by $x-1$ in (i)

$$f(x-2) + f(x) = \sqrt{2}f(x-1) \qquad \text{... (ii)}$$

Replace x by $x+1$ in (ii)

$$f(x) + f(x+2) = \sqrt{2}f(x+1) \qquad \text{... (iii)}$$

Adding (ii) and (iii), we have

$$f(x-2) + f(x+2) + 2f(x) = \sqrt{2}\,(f(x-1) + f(x+1))$$
$$= \sqrt{2}\ \sqrt{2}f(x) = 2f(x)$$

$\Rightarrow \quad f(x-2) = -f(x+2)$

Replacing x by $x+2$, we have

$$f(x) = -f(x+4) = -(f(x+8)) = f(x+8)$$

$$\sum_{r=0}^{10} f(3+8r) = 11 \times f(3) = 11 \times 5 = 55.$$

Example 63:

$$\text{Let } f(\theta) = \begin{vmatrix} \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \end{vmatrix}$$

then f is

(a) a non periodic function
(b) periodic with period π
(c) periodic with period $\pi/2$
(d) odd function

Ans. (c)

Solution: $f(\theta) = \begin{vmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1+\sin^2\theta & \cos^2\theta & 4\sin 2\theta \end{vmatrix}$

$$(R_1 \to R_1 - R_3$$
$$R_2 \to R_2 - R_3)$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1+\sin^2\theta + 4\sin 4\theta & \cos^2\theta & 4\sin 4\theta \end{vmatrix} \qquad (C_1 \to C_1 + C_3)$$

$$= -(\cos^2\theta + 1 + \sin^2\theta + 4\sin 4\theta)$$
$$= -2\,(1 + 2\sin 4\theta)$$

which is periodic function with period $\frac{2\pi}{4} = \frac{\pi}{2}$.

Assertion-Reason Type Questions

Example 64: Let **R** be the real line. Consider the following subsets of the plane $\mathbf{R} \times \mathbf{R}$:

$$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$$
$$T = \{(x, y) : x - y \text{ is an integer}\}$$

Statement-1 : T is an equivalence relation on R but S is not an equivalence relation on R.
Statement-2 : S is neither reflexive nor symmetric but T is reflexive, symmetric and transitive.
Ans. (a)

Solution: Since $x \neq x + 1$, $(x, x) \notin S$, so S is not reflexive. Next $x, y \in S$, $\Rightarrow y = x + 1 \Rightarrow x = y - 1 \Rightarrow (y, x) \notin S$, so is not symmetric.
Since $x - x = 0$ is an integer $(x, x) \in T\ \forall\ x \in T$
$\Rightarrow$ T is reflexive.
Again $(x, y) \in T \Rightarrow x - y$ is an integer
$\Rightarrow$ $y - x$ is also an integer $\Rightarrow (y, x) \in T$
So T is symmetric.
Also $(x, y) \in T$, $(y, z) \in T$.
$\Rightarrow$ $x - y$ and $y - z$ are integers
$\Rightarrow$ $x - z = (x - y) - (y - z)$ is also an integer
$\Rightarrow$ $(x, z) \in T$
So T is Transitive.
Which shows that statement-2 is true and hence statement-1 is also true.

Example 65: Consider the following relations.
$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$
$S = \left\{\left(\dfrac{m}{n}, \dfrac{p}{q}\right)\right\}$ m, n, p, q, are integer such that $n \cdot q \neq 0$ and $qm = pn\}$
Statement-1: S is an equivalence relation but R is not an equivalence relation.
Statement-2: R and S both are symmetric.
Ans. (c)

Solution: Since $(0, 1) \in R$ but $(1, 0) \notin R$, R is not symmetric and hence is not an equivalence relation so statement-2 is false.
Next, For the relation S, $qm = pn \Rightarrow \dfrac{m}{n} = \dfrac{p}{q}$

Thus $\left(\dfrac{m}{n}, \dfrac{p}{q}\right) \in S \Rightarrow \dfrac{m}{n} = \dfrac{p}{q}$ which shows that S is reflexive and symmetric

Again, $\left(\dfrac{m}{n}, \dfrac{p}{q}\right) \in S$ and $\left(\dfrac{p}{q}, \dfrac{r}{s}\right) \in S$

$$\Rightarrow \quad \frac{m}{n} = \frac{p}{q} = \frac{r}{s} \quad \Rightarrow \left(\frac{m}{n}, \frac{r}{s} \in S\right)$$

Thus S is transitive and hence S is an equivalence relation. So, statement 1 is true.

Example 66: Let R be a relation on the set N of natural numbers defined by $n\ Rm \Leftrightarrow n$ is a factor of m (*i.e.* $n \mid m$).
Statement-1: R is not an equivalence relation
Statement-2: R is not symmetric
Ans. (a)

Solution: Statement-2 is true as $2 \mid 6 \Rightarrow 2\mathbf{R}6$ but 6 does not divide 2 so R is not symmetric $\Rightarrow R$ is not an equivalence relation and the statement-1 is also true.

Example 67: Let $A = \{1, 2, 3\}$ and $B = \{3, 8\}$
Statement-1: $(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (8, 3)\}$
Statement-2: $(A \times B) \cap (B \times A) = \{(3, 3)\}$
Ans. (b)

Solution: $A \cup B = \{1, 2, 3, 8\}$, $A \cap B = \{3\}$
$\Rightarrow$ $(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (8, 3)\}$
$\Rightarrow$ Statement-1 is True.
$(x, y) \in (A \times B) \cap (B \times A)$
$\Rightarrow$ $(x, y) \in A \times B$ and $(x, y) \in B \times A$
$\Rightarrow$ $x \in A \cap B$, $y \in A \cap B$
$\Rightarrow$ $\{(3, 3)\} = (A \times B) \cap (B \times A) \Rightarrow$ Statement-2 is also true but is not a correct explanation for statement-1.

Example 68: Statement-1: The number of bijective functions from the set A containing 100 elements to itself is 2^{100}.
Statement-2: The total number of bijections from a set containing n elements to itself is $n!$
Ans. (d)

Solution: Statement-2 is true and so, statement-1 is False.

Example 69: Statement-1: $f : R \to R$ is a function defined by $f(x) = 5x + 3$. If $g = f^{-1}$, then $g(x) = \dfrac{x-3}{5}$.
Statement-2: If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then fog is the identity function on A.
Ans. (c)

Solution: Let $y = 5x + 3 \Rightarrow x = \dfrac{y-3}{5}$.

$\Rightarrow$ $g(x) = \dfrac{x-3}{5}$ is the inverse of f, so statement-1 is True.

Statement-2 is false because $g : B \to A$ and $f : A \to B$

$\Rightarrow$ $fog : B \to B$ and $g = f^{-1} \Rightarrow fog$ is an identity function on B.

Example 70: Let X and Y be two sets.

Statement-1: $X \cap (Y \cup X)' = \phi$

Statement-2: If $X \cup Y$ has m elements and $X \cap Y$ has n elements then symmetric difference $X \Delta Y$ has $m - n$ elements

Ans. (b)

Solution: $X \cap (Y \cup X)' = X \cap (Y' \cap X') = X \cap X' \cap Y' = \phi$.

$\Rightarrow$ Statement-1 is True.

$X \Delta Y = (X \sim Y) \cup (Y \sim X) = (X \cup Y) \sim (X \cap Y)$

$\Rightarrow$ number of elements in $X \Delta Y = m - n$.

$\Rightarrow$ Statement-2 is True but does explain statement-1.

Example 71: Let f be a function defined by

$f(x) = (x - 1)^2 + 1, (x \geq 1)$

Statement-1: The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$

Statement-2: f is a bijection and $f^{-1}(x) = 1 + \sqrt{x - 1}$, $x \geq 1$

Ans. (a)

Solution: Let $y = f(x) = (x - 1)^2 + 1$

$\Rightarrow$ $y - 1 = (x - 1)^2 \Rightarrow x = 1 + \sqrt{y - 1}$, $y \geq 1$

Thus $f^{-1}(x) = 1 + \sqrt{x - 1}$, $x \geq 1$. So statement-2 is true.

Now $f(x) = f^{-1}(x)$

$\Rightarrow$ $(x - 1)^2 = \sqrt{x - 1}$

$\Rightarrow$ $\sqrt{x - 1}\,[(x - 1)^{3/2} - 1] = 0$

$\Rightarrow$ $x = 1, 2$.

So statement-1 is true and statement-2 is a correct explanation for statement-1.

Example 72: Let R be the set of real numbers

Statement-1: $A = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y + x \text{ is an integer}\}$ is an equivalence relation on $\mathbf{R}$.

Statment-2: $B = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = \alpha x$ for some rational number $\alpha\}$ is not equivalence relation on $\mathbf{R}$.

Ans. (d)

Solution: A is neither reflexive nor transitive as $x + x$ may not be integer $\forall\, x \in \mathbf{R}$ and if $x + y$ and $y + z$ are integers, $x + z$ may not be an integer for $x, y, z \in \mathbf{R}$.

So statement-1 is false.

Statememt-2 is true as B is not symmetric, because $(\sqrt{3}, 0) \in B$ as $0 = \sqrt{3} \times 0$ for $\alpha = 0$, but $(0, \sqrt{3}) \notin B$.

Example 73: Consider the following relation R on the set of real square matrices of order 3.

$R = \{ (A, B): A = P^{-1}BP$ for some invertible matrix $P\}$

Statement-1: R is an equivalence relation.

Statement-2: For any two invertible 3×3 matrices M and N, $(MN)^{-1} = N^{-1}M^{-1}$.

Ans. (b)

Solution: Statement-2 in true (See Text.)

In statement-1, $A = I^{-1} A I$

for all real square matrices A of order 3.

$\Rightarrow$ $(A, A) \in R \Rightarrow R$ is reflexive,

Next, let $(A, B) \in R$

$\Rightarrow$ $\exists$ a invertible matrix P of order 3.

such that $A = P^{-1} B P$

$\Rightarrow$ $B = P A P^{-1} = (P^{-1})^{-1} A\, (P^{-1})$

$\Rightarrow$ R in symmetric

If Now $(A, B) \in R$ and $(B, C) \in R$

Then $\exists$ invertible matrices P and Q

of order 3 such that

$A = P^{-1} B P$ and $B = Q^{-1} C Q$.

$\Rightarrow$ $A = P^{-1} Q^{-1} C Q P = (QP)^{-1} C QP$ (From statement-2)

$\Rightarrow$ $(A, C) \in R$ and thus R in transitive. Hence R is an equivalence relation and the statement-1 in also true but statement-2 is not a correct explanation for it.

LEVEL 2

Straight Objective Type Questions

Example 74: From 50 students taking examinations in Mathematics, Physics and Chemistry, 37 passed Mathematics, 24 Physics and 43 Chemistry. At most 19 passed Mathematics and Physics, at most 29 Mathematics and Chemistry and at most 20 Physics and Chemistry. The largest possible number that could have passed all three exams is

(a) 10 (b) 12

(c) 9 (d) none of these.

Ans. (d)

Solution: The given conditions can be expressed as

$n(M \cup P \cup C) = 50$, $n(M) = 37$, $n(P) = 24$, $n(C) = 43$, $n(M \cap P) \leq 19$,

$n(M \cap C) \leq 29$ and $n(P \cap C) \leq 20$,

$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$

$\Rightarrow$ $50 = 37 + 24 + 43 - n(M \cap P) - n(P \cap C)$

$$- n(M \cap C) + n(M \cap P \cap C)$$

$$\Rightarrow \quad n(M \cap P \cap C) \le n(M \cap P) + n(M \cap C) + n(P \cap C) - 54.$$

Therefore, the number of students that could have passed all three exams is at most
$19 + 29 + 20 - 54 = 14$.

Example 75: Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2 \ldots, B_n$ are n sets each with 3 elements, let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of the A_i's and exactly 9 of the B_j's. Then n is equal to

(a) 15 (b) 3
(c) 45 (d) none of these

Ans. (c)

Solution: $S = \bigcup_{i=1}^{30} A_i$, so $n(S) = \frac{1}{10}(5 \times 30) = 15$ (since element in the union S belongs to exactly 10 of the A_i's).
Again $S = \bigcup_{i=1}^{n} B_i$ so

$$n(S) = 1/9(3 \times n) = n/3 = 15 \Rightarrow n = 45$$

Example 76: Let $R = \{(3, 3), (6, 6), (9, 9) (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set A = $\{3, 6, 9, 12\}$.
The relation is

(a) an equivalence relation.
(b) reflexive and symmetric only.
(c) reflexive and transitive only
(d) reflexive only.

Ans. (c)

Solution: R is reflexive as
$(3, 3), (6, 6), (9, 9), (12, 12) \in R$.
R is not symmetric as $(6, 12) \in R$ but $(12, 6) \notin R$.
R is transitive as the only pair which needs verification is $(3, 6)$ and $(6, 12) \in R \Rightarrow (3, 12) \in R$.

Example 77: Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set A = $\{1, 2, 3, 4\}$, the relation R is

(a) not symmetric (b) transitive
(c) a function (d) reflexive

Ans. (a)

Solution: R is not symmetric as $(2, 3) \in R$ but $(3, 2) \notin R$. It is not transitive as $(1, 3), (3, 1) \in R$. but $(1, 1) \notin$ R so is not reflexive also.
Again as $(2, 4)$ and $(2, 3) \in R$, it is not a function.

Example 78: Let **I** be the set of integers, **N** the set of non-negative integers; Np the set of non-positive integers ; E is the set of even integers and P is set of prime numbers. Then

(a) $\mathbf{N} \cap Np = \phi$ (b) $\mathbf{I} - \mathbf{N} = Np$
(c) $\mathbf{N} \Delta Np = I - \{0\}$ (d) $E \cap P = \phi$.

Ans. (c)

Solution: $N \cap Np = \{0\}$,

$$\mathbf{I} - N = \{\ldots, -2, -1\}$$
$$N \Delta Np = \{N - Np) \cup (Np - N)$$
$$= \{1, 2, 3 - -\} \cup \{\ldots, -3, -2, -1\}$$
$$= \mathbf{I} \sim \{0\}$$

and $E \cap P = \{2\}$.

Example 79: If $n(A) = n$ then $n\{(x, y, z); x, y, z \in A, x \ne y, y \ne z, z \ne x\} =$

(a) n^3 (b) $n(n-1)^2$
(c) $n^2(n-2)$ (d) $n^3 - 3n^2 + 2n$

Ans. (d)

Solution: There are n choices for the first coordinate, $n - 1$ choices for second coordinate and $n - 2$ choices for the third coordinate, hence $n(\{(x, y, z); x, y, z \in A, x \ne y \ne z\}) = n(n-1)(n-2) = n^3 - 3n^2 + 2n$.

Example 80: If $A = \{x : x \in \mathbf{I}, -2 \le x \le 2\}$, $B = \{x \in \mathbf{I}, 0 \le x \le 3\}$, $C = \{x: x \in \mathbf{N}, 1 \le x \le 2\}$ and $D = \{x, y) \in \mathbf{N} \times \mathbf{N}; x + y = 8\}$. Then

(a) $n(A \cup (B \cup C) = 5$ (b) $n(D) = 6$
(c) $n(B \cup C) = 5$ (d) none of these

Ans. (d)

Solution: $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 2, 3\}$, $C = \{1, 2\}$

so $B \cup C = \{0, 1, 2, 3\}$,
$A \cup (B \cup C) = \{-2, -1, 0, 1, 2, 3\}$
so $n(A \cup (B \cup C) = 6, n(B \cup C) = 4$
and $D = \{(1, 7), (2, 6), (3, 5), (4, 4), (5,3), (6, 2), (7, 1)\}$ so
$n(D) = 7$.

Example 81: If $A = \{4^n - 3n - 1 | n \in \mathbf{N}\}$ and $B = \{9n - 9: n \in \mathbf{N}\}$, then $A \cup B$ is equal to

(a) B (b) A
(c) $\mathbf{N}$ (d) none of these

Ans. (a)

Solution: It can be shown by induction that $9 \mid 4^n - 3n - 1$ for every $n \in \mathbf{N}$. Thus $A \subseteq B$. Clearly $A \ne B$ as $27 \in B$ but $27 \notin A$. Thus $A \cup B = B$.

Example 82: A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. The number of relations which can be defined from A to B is

(a) 2^5 (b) $2^{10} - 1$
(c) 2^{12} (d) none of these.

Ans. (c)

Solution: The number of elements in $A \times B$ is 12. Hence the number of subsets of $A \times B$ is 2^{12}, which includes the empty set

Example 83: If R and S are two symmetric relations then
(a) $R \, o \, S$ is a symmetric relation
(b) $S \, o \, R$ is a symmetric relation
(c) $R \, o \, S^{-1}$ is a symmetric relation
(d) $R \, o \, S$ is a symmetric relation if and only if $RoS = SoR$.

Ans. (d)

Solution: Since R and S are symmetric relations so $R^{-1} = R$ and $S^{-1} = S$. But $(R \, o \, S)^{-1} = S^{-1} \, o \, R^{-1} = S \, o \, R$. Thus RoS is symmetric if and only if $RoS = SoR$.

Example 84: Let A be the set of all determinants of order 3 with entries 0 or 1 only, B the subset of A consisting of all determinants with value 1, and C the subset consisting of all determinants with value –1. Then if $n(B)$ and $n(C)$ denote the number of elements in B and C, respectively, we have
(a) $C = \phi$ (b) $n(B) = n(C)$
(c) $A = B \cup C$ (d) $n(B) = 2n(C)$

Ans. (b)

Solution: C cannot be the empty set because, for instance,

$$-1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \in C. \text{ We also have } \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2,$$

so $A \neq B \cup C$. In general, the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{22}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}, \end{matrix}$$

with the a's being 0 or 1, equals 1 only if $a_{11}a_{22}a_{33} = 1$ and the remaining terms are zero; if $a_{12}\, a_{23}\, a_{31} = 1$ and the remaining terms are zero; or if $a_{13}\, a_{21}\, a_{32} = 1$ and the remaining terms are zero. Since there are three similar relations for determinants that equal – 1, we must have $n(B) = n(C)$.

Example 85: The domain of the function

$$f(x) = \log_2 \left(-\log_{1/2} \left(1 + \frac{1}{\sqrt[4]{x}} \right) - 1 \right) \text{ is}$$

(a) $0 < x < 1$ (b) $0 < x \leq 1$
(c) $x \geq 1$ (d) $x > 1$

Ans. (a)

Solution: For f to be defined we must have $\log_{1/2} \left(1 + \frac{1}{\sqrt[4]{x}} \right) < -1 \Leftrightarrow 1 + \frac{1}{\sqrt[4]{x}} > (2^{-1})^{-1} = 2$ which is possible if and only if $\frac{1}{\sqrt[4]{x}} > 1$ i.e., $0 < x < 1$.

Hence the domain of the given function is $\{x : 0 < x < 1\}$.

Example 86: The domain of definition of

$$f(x) = \sqrt{\log_{0.4} \left(\frac{x-1}{x+5} \right)} \times \frac{1}{x^2 - 36} \text{ is}$$

(a) $\{x : x < 0, x \neq -6\}$
(b) $\{x : x > 0, x \neq 1, x \neq 6\}$
(c) $\{x : x > 1, x \neq 6\}$
(d) $\{x : x \geq 1, x \neq 6\}$

Ans. (c)

Solution: For $\sqrt{\log_{0\cdot4} \left(\frac{x-1}{x+5} \right)}$ to be defined, we must have $0 < \frac{x-1}{x+5} < 1$, which is true if $x > 1$. Morever, $\frac{1}{x^2 - 36}$ is defined for $x \neq \pm 6$. Hence the domain of f is $\{x : x > 1, x \neq 6\}$.

Example 87: The set of all x for which $f(x) = \log_{\frac{x-2}{x+3}} 2$ and $g(x) = \frac{1}{\sqrt{x^2 - 9}}$ are both not defined is
(a) $(-3, 2)$ (b) $[-3, 2)$
(c) $(-3, 2]$ (d) $[-3, 2]$

Ans. (d)

Solution: The function f is not defined for $-3 \leq x \leq 2$ and g is not defined for those x for which $x^2 - 9 \leq 0$ i.e., $x \in [-3, 3]$. Thus f and g are not defined on $[-3, 2]$.

Example 88: If $f(x)$ is a polynomial satisfying $f(x) f(1/x) = f(x) + f(1/x)$ and $f(3) = 28$, then $f(4)$ is given by
(a) 63 (b) 65
(c) 67 (d) 68

Ans. (b)

Solution: Any polynomial satisfying the functional equation $f(x) \cdot f(1/x) = f(x) + f(1/x)$ is of the form $x^n + 1$ or $-x^n + 1$. If $28 = f(3) = -3^n + 1$ then $3^n = -27$ which is not possible for any n. Hence $28 = f(3) = 3^n + 1 \Rightarrow 3^n = 27 \Rightarrow n = 3$. Thus $f(x) = x^3 + 1$, so $f(4) = 4^3 + 1 = 65$.

Example 89: Part of the domain of the function

$f(x) = \sqrt{\frac{\cos x - 1/2}{6 + 35x - 6x^2}}$ lying in the interval $[-1, 6]$ is
(a) $[-1/6, \pi/3] \cup [5\pi/3, 6]$
(b) $(-1/6, \pi/3] \cup [5\pi/3, 6)$
(c) $(-1/6, 6)$
(d) none of these

Ans. (a)

Solution: The function f is meaningful only if $\cos x - 1/2 \geq 0$, $6 + 35x - 6x^2 > 0$ or $\cos x - 1/2 \leq 0$, $6 + 35x - 6x^2 < 0$ i.e., $\cos x \geq 1/2$, $(6 - x)(1 + 6x) > 0$ or $\cos x \leq 1/2$, $(6 - x)(1 + 6x) < 0$. These inequalities are satisfied if $x \in (-1/6, \pi/3] \cup [5\pi/3, 6)$.

Example 90: Let f: $\mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$. Then
(a) f is both one-one and onto
(b) f is one-one but not onto

(c) f is onto but not one-one
(d) f is neither one-one nor onto

Ans. (d)

Solution: f is not one-one as $f(0) = 0$ and $f(-1) = 0$. f is also not onto as for $y = 1$ there is no $x \in \mathbf{R}$ such that $f(x) = 1$. If there is such a $x \in \mathbf{R}$ then $e^{|x|} - e^{-x} = e^{x} + e^{-x}$, clearly $x \neq 0$. For $x > 0$, this equation gives $-e^{-x} = e^{-x}$ which is not possible. For $x < 0$, the above equation gives $e^{x} = -e^{-x}$ which is also not possible.

Example 91: Let $f(x) = x^2$ and $g(x) = 2^x$ then the solution set of $fog\ (x) = g\ of(x)$ is

(a) $\mathbf{R}$ (b) $\{0\}$
(c) $\{0, 2\}$ (d) none of these

Ans. (c)

Solution: $fog\ (x) = f(g(x)) = f(2^x) = (2^x)^2 = 2^{2x}$ and $gof\ (x) = g\ (f(x)) = g\ (x^2) = 2^{x^2}$. Thus the solution of $2^{x^2} = 2^{2x}$ is given by $x^2 = 2x$ which is $x = 0, 2$.

Example 92: A function $f: \mathbf{R} \to \mathbf{R}$ satisfies the equation $f(x) f(y) - f(xy) = x + y$ for all $x, y \in \mathbf{R}$ and $f(1) > 0$, then

(a) $f(x) = x + 1/2$ (b) $f(x) = (1/2)x + 1$
(c) $f(x) = (1/2)x - 1$ (d) $f(x) = x + 1$

Ans. (d)

Solution: Taking $x = y = 1$, we get

$$f(1) f(1) - f(1) = 1 + 1 \Rightarrow f(1)^2 - f(1) - 2 = 0$$

$\Rightarrow\ (f(1) - 2)(f(1) + 1) = 0 \Rightarrow f(1) = 2\ (\because f(1) > 0)$

Taking $y = 1$, we get

$$f(x) f(1) - f(x) = x + 1 \Rightarrow 2f(x) - f(x) = x + 1$$

$$\Rightarrow \quad f(x) = x + 1.$$

Example 93: If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + 1/x$ then $f^{-1}(x)$ equals

(a) $\dfrac{x + \sqrt{x^2 - 4}}{2}$ (b) $\dfrac{x}{1 + x^2}$

(c) $\dfrac{x - \sqrt{x^2 - 4}}{2}$ (d) $1 + \sqrt{x^2 - 4}$

Ans. (a)

Solution: $y = x + 1/x \Rightarrow x^2 - xy + 1 = 0$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

Since $x \in [1, \infty)$ so $x = \dfrac{y + \sqrt{y^2 - 4}}{2}$.

Hence $f^{-1}(x) = \dfrac{x + \sqrt{x^2 - 4}}{2}$.

Example 94: Consider the function $f = \{(x, \sin x) | -\infty < x < \infty\}$. Let $A = \left[0, \dfrac{\pi}{6}\right]$ and $B = \left[0, \dfrac{\pi}{2}\right]$ then

(a) $f(A \cap B) = f(A) \cap f(B)$ (b) $f(A \cap B) = [0, 1]$

(c) $f(A) \cap f(B) = [0, 1]$ (d) $f(A) \cup f(B) = \left[0, \dfrac{1}{2}\right]$

Ans. (a)

Solution: $f(x) = \sin x$ is an increasing function on $[0, \pi/2]$ so $f(A) = \left[0, \sin\dfrac{\pi}{6}\right] = \left[0, \dfrac{1}{2}\right]$ and $f(B) = [0, 1]$.

Thus $f(A) \cap f(B) = \left[0, \dfrac{1}{2}\right]$. Also $A \cap B = \left[0, \dfrac{\pi}{6}\right]$, so $f(A \cap B) = \left[0, \sin\dfrac{\pi}{6}\right] = \left[0, \dfrac{1}{2}\right]$. Thus $f(A) \cap f(B) = \left[0, \dfrac{1}{2}\right] = f(A \cap B)$,

Also $f(A) \cup f(B) = [0, 1]$.

Example 95: Let $f(x)$ be a polynomial of even degree satisfying $f(2x)\left(1 - f\left(\dfrac{1}{2x}\right)\right) + f\left(16x^2 y\right) = f(-2) - f(4xy)$ for all $x, y \in \mathbf{R} \sim \{0\}$ and $f(4) = -255$, $f(0) = 1$. Then the value of $\left|\dfrac{f(2) + 1}{2}\right|$ is

(a) 4 (b) 5
(c) 7 (d) 6

Ans. (c)

Solution: Replacing y by $\dfrac{1}{8x^2}$ in the given functional equation, we obtain

$$f(2x)\left(1 - f\left(\frac{1}{2x}\right)\right) + f(2) = f(-2) - f\left(\frac{1}{2x}\right).$$

Since f is an even function so $f(2) = f(-2)$,

so $$f(2x) - f(2x) f\left(\frac{1}{2x}\right) = -f\left(\frac{1}{2x}\right)$$

$$\Rightarrow \quad f(2x) + f\left(\frac{1}{2x}\right) = f(2x)\, f\left(\frac{1}{2x}\right)$$

Replacing x by $\dfrac{x}{2}$, we have

$$f(x) + f\left(\frac{1}{x}\right) = f(x)\, f\left(\frac{1}{x}\right)$$

Since f is a polynomial, so $f(x) = \pm x^n + 1$ (see Example 88) But $-255 = f(4) = \pm 4^n + 1$. Only negative sign is possible, thus $4^n = 256 \Rightarrow n = 4$. i.e. $f(x) = -x^4 + 1$. $f(2) = -2^4 + 1 = -15$

$$\left|\frac{f(2) + 1}{2}\right| = \left|-\frac{14}{2}\right| = 7.$$

Example 96: The domain of the function $f(x) = \dfrac{1}{\sqrt{\sin x}} + \sqrt[3]{\sin x} + \log_{10} \dfrac{x - 5}{x^2 - 10x + 24}$ is

(a) $\{(2k\pi, (2k + 1)\pi) : k \in \mathbf{I}\}$
(b) $\{(2k\pi, (2k + 1)\pi) : k \in \mathbf{N}\}$
(c) $(6, \infty) \cup (4, 5)$
(d) $\{(2k\pi, (2k + 1)\pi) : k \in \mathbf{I}\} \cup (6, \infty)$

Ans. (b)

◎ **Solution:** The domain of $\sqrt[3]{\sin x}$ is **R**.

The domain of $\dfrac{1}{\sqrt{\sin x}} = \{x : \sin x > 0\}$

$= \{(2k\pi, (2k+1)\pi) : k \in \mathbf{I}\}$

The domain of $\log_{10} \dfrac{x-5}{x^2 - 10x + 24} = \log_{10} \dfrac{x-5}{(x-6)(x-4)}$

$= \{x : x \neq 5, x > 5, x > 6, x > 4$ or $x \neq 5, x < 5, 4 < x < 6\}$

$= (4, 5) \cap (6, \infty)$

Thus the domain of $f(x)$

$= \{(2k\pi, (2k+1)\pi) : k \in \mathrm{I}\} \cap \{(4, 5) \cup (6, \infty)\}$

$= \{(2k\pi, (2k+1)\pi) : k \in \mathbf{N}\}$

(since (4, 5) does not intersect $\{(2k\pi, 2k+1)\pi : k \in \mathbf{I}\}$)

EXERCISE

Concept-based Straight Objective Type Questions

1. Let U be a Universal set and $n(U) = 12$. If $A, B \subseteq U$ are such that $n(B) = 6$ and $n(A \cap B) = 2$ then $n(A \cup B')$ is equal to
   (a) 6 (b) 10
   (c) 7 (d) 8
2. Let R be a relation on **R** defined as $a\ R\ b$ if $|a| \leq b$. Then, relation R is
   (a) reflexive (b) symmetric
   (c) transitive (d) not antisymetric
3. Let $f(x) = ax + b$, $x \in \mathbf{R}$, and $g(x) = x + d$, $x \in \mathbf{R}$, then $fog = gof$ if and only if
   (a) $f(a) = g(c)$ (b) $f(d) = g(b)$
   (c) $f(b) = g(d)$ (d) $f(c) = g(a)$
4. The domain of the function $f(x) = \dfrac{\sin x}{\sqrt{|x| - x}}$ is
   (a) **R** (b) **R** ~ {0}
   (c) $\mathbf{R}^+$ (d) $\mathbf{R}^-$
   ($\mathbf{R}^+$ is the set of positive real numbers and $\mathbf{R}^-$ is the set of negative real numbers)
5. The domain of $y = \cos^{-1}(1 - 2x)$ is
   (a) [– 1, 1] (b) [0, 1]
   (c) [–1, 0] (d) $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
6. If $f(x) = \sin x - \cos x$ is written as $f_1(x) + f_2(x)$ where $f_1(x)$ is even and $f_2(x)$ is odd then
   (a) $f_1(x) = \cos x$ (b) $f_1(x) = -\cos x$
   (c) $f_2(x) = -\sin x + \cos x$ (d) $f_2(x) = \sin(2\pi - x)$
7. The range of $y = 1 - \sin x$ is
   (a) [–1, 1] (b) [0, 1]
   (c) [–1, 2] (d) [0, 2]
8. The function $f(x) = x^2 \log \dfrac{1-x}{1+x}$ is
   (a) a periodic function (b) a bounded function
   (c) an odd function (d) an even function

LEVEL 1

Straight Objective Type Questions

9. If $A = \{2, 3, 4, 5, 7\}$, $B = \{1, 2, 4, 7, 9\}$ then $((A \sim B) \cup (B \sim A)) \cap A$ is equal to
   (a) {3, 5} (b) {2, 4}
   (c) {3, 7} (d) {2, 7}
10. If $A = \{2x : x \in N\}$, $B = \{3x : x \in N\}$ and $C = \{5x : x \in N)$ then $A \cap (B \cap C)$ is equal to
    (a) {15, 30, 45, ... }
    (b) {10, 20, 30 ... }
    (c) {30, 60, 90, ... }
    (d) {7, 14, 21 ... }.
11. If X and Y are two sets such that $X \cap Y = X \cup Y$, then
    (a) $X \subset Y, X \neq Y$ (b) $Y \subset X, Y \neq X$
    (c) $X = Y$ (d) none of these
12. If X, Y and A are three sets such that $A \cap X = A \cap Y$ and $A \cup X = A \cup Y$ then
    (a) $X \subset Y$ (b) $Y \subset X$
    (c) $X = Y$ (d) none of these
13. If $A = \{x : x \in R$ and satisfy $x^2 - 15x + 56 = 0\}$
    $B = \{x : x \in N$ and $x + 5 \leq 14\}$
    and $C = \{x : x \in N$ and $x/112\}$.
    Then $A \cup (B \cap C)$ is equal to
    (a) {1, 2, 3,....8} (b) {8, 7}
    (c) {1, 2, 8, 7} (d) {1, 2, 4, 7, 8}

14. If A is the set of letters needed to spell "MATHEMATICS" and B is the set of letters needed to spell STATISTICS, then
    (a) $A \subset B$ (b) $A \sim B = \phi$
    (c) $A \Delta B = A \sim B$ (d) none of these
15. If A and B are two sets such that $n(A \cup B) = 36$, $n(A \cap B) = 16$ and $n(A \sim B) = 15$, then $n(B)$ is equal to
    (a) 21 (b) 31
    (c) 20 (d) 52
16. The maximum number of sets obtainable from A and B by applying union and difference operations is
    (a) 5 (b) 6
    (c) 7 (d) 8
17. If A and B both contain same number of elements and are finite sets then
    (a) $n(A \cup B) = n(A \cap B)$ (b) $n(A \sim B) = n(B \sim A)$
    (c) $n(A \Delta B) = n(B)$ (d) $n(A \sim B) = n(A)$
18. If $A \Delta B = A \cup B$ then
    (a) $A = B$ (b) $A \cap B = \phi$
    (c) $A \Delta B = \phi$ (d) $A \Delta B = A \sim B$
19. In a class 60% passed their Physics examination and 58% passed in Mathematics. Atleast what percentage of students passed both their Physics and Mathematics examination?
    (a) 18% (b) 17%
    (c) 16% (d) 2%
20. If the relation $R: A \to B$, where $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ is defined by $R = \{(x, y): x < y, x \in A, y \in B\}$, then
    (a) $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}$
    (b) $R = \{(1, 1), (1, 5), (2, 3), (3, 5)\}$
    (c) $R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 3)\}$
    (d) $R^{-1} = \{(1, 1), (5, 1), (3, 2), (5, 3)\}$
21. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by
$$R = \{(x, y): |x^2 - y^2| < 16\}$$
is given by
    (a) $R_1 = \{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
    (b) $R_2 = \{(2, 2), (3, 2), (4, 2), (2, 4)\}$
    (c) $R_3 = \{(3, 3), (4, 3), (5, 4), (3, 4)$
    (d) none of these
22. Let L be the set of all lines in a plane and R be a relation on L defined by $l_1 R\ l_2$ if and only if $l_1 \perp l_2$ then R is
    (a) reflexive (b) symmetric
    (c) transitive (d) an equivalence relation
23. For non-empty subsets A and B,
    (a) Any subset of $A \times B$ defines a function from A to B.
    (b) Any subset of $A \times B$ defines an equivalence relation
    (c) Any subset of $A \times A$ defines a function on A
    (d) none of these.
24. Let $f(x) = (x + 1)^2 - 1$ $(x \geq -1)$. Then the set $S = \{x : f(x) = f^{-1}(x)\}$ contains
    (a) $\left\{0, -1, \dfrac{-3+i\sqrt{3}}{2}, \dfrac{-3-i\sqrt{3}}{2}\right\}$
    (b) $\{0, 1, -1\}$
    (c) $\{0, -1\}$
    (d) none of these
25. If a set A has n elements then the number of all relations on A is
    (a) 2^{n^2} (b) $2^{n^2} - 1$
    (c) 2^n (d) none of these
26. The function f: $\mathbf{R} \to \mathbf{R}$ given by $f(x) = 3 - 2 \sin x$ is
    (a) one-one (b) onto
    (c) bijective (d) none of these
27. Which of the following are functions?
    (a) $\{(x, y): y^2 = 4ax, x, y \in \mathbf{R}\}$
    (b) $\{(x, y): y = |x|, x, y \in \mathbf{R}\}$
    (c) $\{(x, y): x^2 + y^2 = 1, x, y \in \mathbf{R}\}$
    (d) $\{(x, y): x^2 - y^2 = 1, x, y \in \mathbf{R}\}$
28. If f: $\mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^4 + 2$ then the value of f^{-1} (83) and $f^{-1}(-2)$ respectively are
    (a) ϕ, $\{3, -3\}$ (b) $\{3, -3\}$, ϕ
    (c) $\{4, -4\}$, ϕ (d) $\{4, -4\}$, $\{2, -2\}$.
29. The minimum number of elements that must be added to the relation $R = \{(1, 2), (2, 3)\}$ on the sub set $\{1,2,3\}$ of natural numbers so that it is an equivalence relation is
    (a) 4 (b) 7
    (c) 6 (d) 5
30. Let X be a non-empty set and $P(X)$ be the set of all subsets of X. For $A, B \in P(X)$, ARB if and only if $A \cap B = \phi$ then the relation
    (a) R is reflexive
    (b) R is symmetric
    (c) R is transitive
    (d) R is an equivalence relation
31. Let f be a function satisfying $2f(x) - 3f(1/x) = x^2$ for any $x \neq 0$. Then the value of $f(2)$ is
    (a) – 2 (b) – 7/4
    (c) – 7/8 (d) 4
32. If $f(x) = \begin{vmatrix} 1 & x & (x+1) \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & x(x-1)(x+1) \end{vmatrix}$
then

$f(50) + f(51) + \ldots\ldots + f(99)$ is equal to

(a) 0 (b) 1275
(c) 3725 (d) none of these

33. The domain of definition of the functions $y(x)$ given by the equation $a^x + a^y = a$ $(a > 1)$ is

(a) $0 < x \leq 1$ (b) $0 \leq x \leq 1$
(c) $-\infty < x < 1$ (d) $-\infty < x \leq 0$

34. Let $f(x) = \dfrac{x-[x]}{1+x-[x]}$, where $[x]$ denotes the greatest integer less than or equal to x, then the range of f is

(a) $[0, 1/2]$ (b) $[0, 1)$
(c) $[0, 1/2)$ (d) $[0, 1]$

35. If $2f(x^2) + 3f(1/x^2) = x^2 - 1$, then $f(x^2)$ is

(a) $(1 - x^4)/5x^2$ (b) $(1 - x^2)/5x$
(c) $5x^2/(1 - x^4)$ (d) none of these

36. If $f(x + 3y, x - 3y) = 12xy$, then $f(x, y)$ is

(a) $2xy$ (b) $2(x^2 - y^2)$
(c) $x^2 - y^2$ (d) none of these

37. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$ then $f^{-1}(x)$ is

(a) $(1/2)^{x(x-1)}$ (b) $\dfrac{1}{2}\left(1+\sqrt{1+4\log_2 x}\right)$

(c) $\dfrac{1}{2}\left(1-\sqrt{1+4\log_2 x}\right)$ (d) not defined

38. If $n(A) = 3$ and $n(B) = 5$ then number of one-one functions that can be defined from A to B is

(a) 30 (b) 40
(c) 120 (d) 60

39. Let f: $\{x, y, z\} \to \{1, 2, 3\}$ be a one-one function. If it is given that exactly one of the following statements is true,

Statement-1: $f(x) = 1$, Statement-2: $f(y) \neq 1$, Statement-3: $f(z) \neq 2$.

then $f^{-1}(1)$ is

(a) x (b) y
(c) z (d) none of these

40. The value of $n \in \mathbf{z}$ for which the function

$f(x) = \dfrac{\sin nx}{\sin(x/n)}$ has 4π as its period is

(a) 2 (b) 3
(c) 4 (d) 5

41. Let R be a relation on $\mathbf{N}$ defined by

$$R = \{(m, n): m, n \in N \text{ and } m = n^2\}.$$

Which of the following is true.

(a) $(n, n) \in R \,.\, \forall n \in \mathrm{N}$
(b) $(m, n) \in R \Rightarrow (n, m) \in R$
(c) $(m, n) \in R, (n, p) \in R \Rightarrow (m, p) \in R$
(d) none of these

42. Of the number of three athletic teams in a school, 21 are in the basketball team, 26 in hockey team and 29 in the football team, 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. The total number of members is

(a) 42 (b) 43
(c) 45 (d) none of these

43. E, I, R, O denote respectively the sets of all equilateral, isosceles, right angled and obtuse angled triangles in a plane, then which of the following is not true.

(a) $R \cap E = \phi, R \cap I \neq \phi$
(b) $E \cap O = \phi, O \cap I \neq \phi$
(c) $E \cap I = \phi, E \cap O \neq \phi$
(d) $E \cap I \neq \phi, E \subset I$

44. If $A = \{3^n : n \in \mathbf{N}, n \leq 6\}$, $B = \{9^n : n \in \mathbf{N}, n \leq 4\}$ then which of the following is false

(a) $A \,\Delta\, B = \{6561\}$
(b) $A \sim B = \{3, 27, 243\}$
(c) $A \cap B = \{9, 81, 729\}$
(d) $A \cup B = \{3, 9, 27, 81, 243, 729, 6561\}$

45. If $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = ax + \sin x + a$, then f is one-one and onto for all

(a) $a \in \mathbf{R}$ (b) $a \in \mathbf{R} \sim [-1, 1]$
(c) $a \in \mathbf{R} \sim \{0\}$ (d) $a \in \mathbf{R} \sim \{-1\}$

46. If f: $(0, \pi) \to \mathbf{R}$ is given by $f(x) = \sum_{k=1}^{n}[1+\sin kx]$, $[x]$ denotes the greatest integer function, then the range of $f(x)$ is

(a) $\{n-1, n+1\}$ (b) $\{n\}$
(c) $\{n, n+1\}$ (d) $\{n-1, n\}$

47. If the number of elements in $(A \sim B) \sim C$, $(B \sim C) \sim A$, $(C \sim A) \sim B$ and $A \cap B \cap C$ is 10, 15, 20, and 5 respectively then the number of elements in $(A \Delta B) \Delta C$ is

(a) 35 (b) 50
(c) 40 (d) 45

48. Let $f(x) = \dfrac{x-3}{x+1}, x \neq -1$. Then $f^{2010}(2014)$ (where $f^n(x) = fof\ldots$ of (x) (n times)) is

(a) 2010 (b) 4020
(c) 4028 (d) 2014

Assertion-Reason Type Questions

49. Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ be a relation on set $A = \{1, 2, 3\}$

Statement-1: R is not an equivalence relation on A.
Statement-2: R is function from A to A.

50. **Statement-1:**

If A is a set with 5 elements and B is a set containing 9 elements, then the number of injective mappings from A to B to $9! - 4!$.

Statement-2: The total number of injective mappings from a set with m elements to a set with n elements, $m \leq n$, is $\dfrac{n!}{(n-m)!}$.

51. Let $f(x) = \sin x + \cos x$, $g(x) = \dfrac{\sin x}{1-\cos x}$.

Statement-1: f is neither an odd function nor an even function.

Statement-2: g is an odd function.

52. **Statement-1:** A function $f : R \to R$ satisfies the equation $f(x) - f(y) = x - y \ \forall \ x, y \in R$ and $f(3) = 2$, then $f(xy) = xy - 1$

Statement-2: $f(x) = f(1/x) \ \forall \ x \in R, x \neq 0$, and

$f(2) = 7/3$ if $f(x) = \dfrac{x^2+x+1}{x^2-x+1}$.

53. **Statement-1:** Let $A = \{2, 3, 7, 9\}$ and $B = \{4, 9, 49, 81\}$ $f : A \to B$ is a function defined as $f(x) = x^2$. Then f is a bijection from A to B.

Statement-2: A function f from a set A to a set B is a bijection if $f(A) = B$ and $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$ for all $x_1, x_2 \in A$ and $n(A) = n(B)$.

LEVEL 2

Straight Objective Type Questions

54. The range of the function $f(x) = \cos [x]$ for $-\pi/2 < x < \pi/2$ contains

(a) $\{-1, 1, 0\}$ (b) $\{\cos 1, 1, \cos 2\}$
(c) $\{\cos 1, -\cos 1, 1\}$ (d) $[-1, 1]$.

55. The domain of the function

$f(x) = \log_{10} \dfrac{x-5}{x^2-10x+24} - 3\sqrt{x+5}$ is

(a) $(-5, \infty)$ (b) $(5, \infty)$
(c) $(2, 5) \cup (5, \infty)$ (d) $(4, 5) \cup (6, \infty)$

56. If $g(x) = 1 + \sqrt[3]{x}$ then a function f such that $f(g(x)) = 3 - 3(\sqrt[3]{x}) + x$ is

(a) $f(x) = x^3 - 3x^2 + x + 5$
(b) $f(x) = x^3 + 3x^2 - x - 5$
(c) $f(x) = x^3 - 3x^2 + 5$
(d) $f(x) = x^3 - 3x^2 + 3x + 3$

57. The function $f(x) = \dfrac{x}{e^x-1} + \dfrac{x}{2} + 1$ is

(a) even (b) periodic
(c) odd (d) neither even nor odd

58. Let f and g be two functions defined by $f(x) = \dfrac{x}{x+1}$, $g(x) = \dfrac{x}{1-x}$. Then $(f \, o \, g)^{-1}(x)$ is equal to

(a) x (b) 1
(c) $2x$ (d) none of these

59. Let $f : (-1, 1) \to (0, \pi)$ be defined by

$f(x) = \cot^{-1} \dfrac{2x}{1-x^2}$ Then

(a) f is one-one but not onto
(b) f is onto but not one-one
(c) f is both one-one and onto
(d) f is neither one-one nor onto

60. Let $f : X \to [1, 27]$ be a function by $f(x) = 5 \sin x + 12 \cos x + 14$. The set X so that f is one-one and onto is

(a) $[-\pi/2, \pi/2]$ (b) $[0, \pi]$
(c) $[0, \pi/2]$ (d) none of these

61. Let $f(x) = \dfrac{1-x}{1+x}$. Then $f \, o \, f (\cos x)$ is equal to

(a) $\cos 2x$ (b) $\cos x$
(c) $\tan 2x$ (d) $\tan x$

62. Let $f(x) = \sin \sqrt{x}$, then

(a) $f(x)$ is periodic with period $\sqrt{2\pi}$
(b) $f(x)$ is periodic with period $\sqrt{\pi}$
(c) $f(x)$ is periodic with period $4\pi^2$
(d) none of these

63. Let $X = Y = \mathbf{R} \sim \{1\}$. The function $f : X \to Y$ defined by $f(x) = \dfrac{x+2}{x-1}$ is
   (a) one-one but not onto
   (b) onto but not one-one
   (c) neither one-one nor onto
   (d) one-one and onto

64. If $f(x) = \dfrac{a^x + a^{-x}}{2}$ and $f(x + y) + f(x - y) = K f(x) f(y)$ then K is equal to
   (a) 2 (b) 4
   (c) – 2 (d) none of these

65. If $f(x) = 2 \sin^{-1} \sqrt{x-3}$, then the domain of f is
   (a) [3, 4] (b) [– 1, 1]
   (c) [– π/2, π/2] (d) [6, 8]

66. Let $f(x) = x|x|$ and $g(x) = \sqrt{|x|}$ then the number of elements in the set $\{x \in \mathbf{R} : f(x) = g(x)\}$ is
   (a) 1 (b) 2
   (c) 3 (d) infinitely many

67. Which of the following functions are odd functions
   (a) $f(x) = x\left(\dfrac{a^x + a^{-x}}{a^x - a^{-x}}\right)$
   (b) $f(x) = \dfrac{a^x + x}{a^x - x}$
   (c) $f(x) = \dfrac{a^x - 1}{a^x + 1}$
   (d) $f(x) = x \log_2 \left(x + \sqrt{x^2 + 1}\right)$

68. Let $g(x) = 1 + x - [x]$ and $f(x) = \text{sgn } x$. Then for all x, $f \circ g(x)$ is equal to
   (a) x (b) 1
   (c) $f(x)$ (d) $g(x)$

   Note $f(x) = \text{sgn } x = 1$ if $x > 0$
   $= 0$ if $x = 0$
   $= -1$ if $x < 0$

69. The domain of definition of the function $f(x) = \log_2 (\log_{1/2} (x^2 + 4x + 3)) + \sin^{-1} (2[x]^2 - 3)$, $[x]$ denotes the greatest integer $\le x$ is
   (a) $0 < x \le 1$ (b) $0 \le x \le 1$
   (c) $-\infty < x \le 0$ (d) none of these

70. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin \sqrt{x})^2$, then f and g may be given by
   (a) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$
   (b) $f(x) = \sin x$, $g(x) = |x|$
   (c) $f(x) = x^2$, $g(x) = \sin x$
   (d) f and g cannot be determined

71. If $f(x) = 3x - 5$, then $f^{-1}(x)$
   (a) is given by $\dfrac{1}{3x-5}$
   (b) is given by $\dfrac{x+5}{3}$
   (c) does not exist because f is not one-one
   (d) does not exist because f is not onto

72. Let $f(x) = |x - a|$, $a \ne 0$ then
   (a) $f(x^2) = (f(x))^2$
   (b) $f(|x|) = |f(x)|$
   (c) $f(x + y) = f(x) + f(y)$
   (d) none of these

73. If $n(A) = n(B) = 4$ then number of bijections from A to B is
   (a) 6 (b) 24
   (c) 12 (d) 18

Previous Years' AIEEE/JEE Main Questions

1. Let R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)} be a relation on the set $A = \{1, 2, 3, 4\}$. The relation R is
   (a) not symmetric (b) transitive
   (c) a function (d) reflexive **[2004]**

2. The range of the function is $f(x) = {}^{7-x}P_{x-3}$ is
   (a) {1, 2, 3, 4} (b) {1, 2, 3, 4, 5, 6}
   (c) {1, 2, 3} (d) {1, 2, 3, 4, 5} **[2004]**

3. The domain of the function $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is
   (a) [1, 2] (b) [2, 3]
   (c) [2, 3] (d) [1, 2] **[2004]**

4. If $f : R \to S$ defined by $f(x) = \sin x - \sqrt{3} \cos x + 1$ is onto, then the interval of S is
   (a) [0, 1] (b) [–1, 1]
   (c) [0, 3] (d) [–1, 3] **[2004]**

5. The graph of the function $y = f(x)$ is symmetrical about $x = 2$ then
 (a) $f(x) = f(-x)$ (b) $f(2 + x) = f(2 - x)$
 (c) $f(x + 2) = f(x - 2)$ (d) $f(x) = -f(-x)$ **[2004]**

6. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$
 The relation is
 (a) an equivalence relation
 (b) reflexive and symmetric only
 (c) reflexive and transitive only
 (d) reflexive only **[2005]**

7. A real valued function $f(x)$ satisfies the functional equation
 $$f(x - y) = f(x)f(y) - f(a - x)f(a + y)$$
 where a is a given constant and $f(0) = 1$, $f(2a - x)$ is equal to
 (a) $f(a) + f(a - x)$ (b) $f(-x)$
 (c) $-f(x)$ (d) $f(x)$ **[2005]**

8. Let W denote the words in the English Dictionary. Define the relation R by $R = \{(x, y) \in W \times W$: the words x and y have at least one letter in common$\}$, then R is
 (a) reflexive, not symmetric and transitive.
 (b) not reflexive, symmetric and transitive.
 (c) reflexive, symmetric and not transition.
 (d) reflexive, symmetric and transitive **[2006]**

9. The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log \cos x$ is defined is
 (a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
 (c) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left[0, \frac{\pi}{2}\right)$ **[2007]**

10. Let $\mathbf{R}$ be the real line. Consider the following subsets of the plane $\mathbf{R} \times \mathbf{R}$:
 $S = \{(x, y): y = x + 1 \text{ and } 0 < x < 2\}$
 $T = \{(x, y): x - y \text{ is an integer}\}$
 which of the following is true :
 (a) T is an equivalence relation on R but S is not
 (b) Neither S nor T is an equivalence relation.
 (c) Both S and T are equivalence relation on R.
 (d) S is an equivalence relation but T is not. **[2008]**

11. Let $f : N \to Y$ be a function defined as
 $f(x) = 4x + 3$ where $Y = \{y \in N : y = 4x + 3$ for some $x \in N\}$.
 Show that f is invertible and its inverse is
 (a) $g(y) = \frac{3y + 4}{3}$ (b) $g(y) = 4 + \frac{y + 3}{4}$
 (c) $g(y) = \frac{y + 3}{4}$ (d) $g(y) = \frac{y - 3}{4}$ **[2008]**

12. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then
 (a) $B = C$ (b) $A \cap B = \phi$
 (c) $A = B$ (d) $A = C$ **[2009]**

13. Consider the following relations
 $R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$
 $S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right)\right\}$ $\{m, n, p, q$ are integers such that $n \cdot q \neq 0$ and $qm = pn\}$
 Then
 (a) S is an equivalence relation but R is not an equivalence relation
 (b) R and S both are equivalence relations.
 (c) R is an equivalence relation but S is not an equivalence relation.
 (d) Neither R nor S is an equivalence relation.
 [2010]

14. Let R be the set of real numbers.
 Statement-1:
 $A = \{ (x,y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R.
 Statement-2:
 $B = \{ (x, y) \in R \times R : x = \alpha y$ for some rational numbers $\alpha\}$ in an equivalence relation on R.
 [2011]

15. Let f be a function defined by
 $f(x) = (x - 1)^2 + 1, (x \geq 1)$
 Statement-1:
 The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$
 Statement-2: f is a bijection and
 $f^{-1}(x) = 1 + \sqrt{x - 1}, x \geq 1$. **[2011]**

16. Consider the following relation R on the set of real square matrices of order 3.
 $R = \{(A,B): A = P^{-1} B P$ for some invertible matrix $P\}$
 Statement-1: R is an equivalence relation.
 Statement-2: For any two invertible 3×3 matrices M and N, $(M N)^{-1} = N^{-1}M^{-1}$ **[2011]**

17. Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X$, $Z \subseteq X$ and $Y \cap Z$ is empty is

(a) 2^5 (b) 5^3
(c) 5^2 (d) 3^5. **[2012]**

18. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is
(a) 211 (b) 256
(c) 220 (d) 219 **[2013]**

19. Let P be the relation defined on the set of all real numbers such that
$P = \{(a, b) : \sec^2 a - \tan^2 b = 1\}$. Then P is
(a) reflexive and symmetric but not transitive.
(b) reflexive and transitive but not symmetric.
(c) symmetric and transitive but not reflexive.
(d) an equivalence relation. **[2014]**

20. Let $f(n) = \left[\frac{1}{3}+\frac{3n}{100}\right]n$, where $[n]$ denotes the greatest integer less than or equal to n. Then $\sum_{n=1}^{56} f(n)$ is equal to
(a) 689 (b) 1399
(c) 1287 (d) 56 **[2014]**

21. The function $f(x) = |\sin 4x| + |\cos 2x|$ is a periodic function with period
(a) $\pi/2$ (b) 2π
(c) π (c) $\pi/4$ **[2014]**

22. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \frac{|x|-1}{|x|+1}$ then f is
(a) onto but not one-one
(b) both one-one and onto
(c) one-one but not onto
(d) neither one-one nor onto **[2014]**

23. A relation on the set $A = \{x: |x| < 3,\ x \in \mathbf{Z}\}$, where $\mathbf{Z}$ is the set of integers is defined by $R = \{(x, y) : y = |x|,\ x \neq -1\}$. Then the number of elements in the power set of R is
(a) 32 (b) 16
(c) 8 (d) 64 **[2014]**

24. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having atleast three elements is
(a) 219 (b) 256
(c) 275 (d) 510 **[2015]**

25. In a certain town, 25% of the families own a phone and 15% own a car, 65% families own neither a phone nor a car and 2000 families own both a car and a phone. Consider the following three statements:
(a) 5% families own both a car and a phone
(b) 35% families own either a car or a phone
(c) 40,000 families live in town. Then,
(a) only (a) and (b) are correct
(b) only (a) and (c) are correct
(c) only (b) and (c) are correct
(d) all (a), (b) and (c) are correct **[2015, online]**

26. Let $A = \{x_1, x_2, \ldots x_7\}$ and $B = \{y_1, y_2, y_3\}$ be two sets containing seven and three distinct elements respectively. Then the total number of functions $f : A \to B$ that are onto, if there exist exactly three x in A such $f(x) = y_2$ is equal to
(a) $14.{}^7C_2$ (b) $16.{}^7C_3$
(c) $12.{}^7C_2$ (d) $14.{}^7C_3$ **[2015, online]**

27. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x$, $x \neq 0$ and $S = \{x \in \mathbf{R} : f(x) = f(-x)\}$; then S
(a) is an empty set
(b) contains exactly one element
(c) contains exactly two elements
(d) contains more than two elements **[2016]**

28. If the function $f : [1, \infty[\to [1, \infty[$ is defined by $f(x) = 3^{x(x-1)}$, then $f^{-1}(x)$ is
(a) $\frac{1}{2}\left(1-\sqrt{1+4\log_3 x}\right)$ (b) $\frac{1}{2}\left(1+\sqrt{1+4\log_3 x}\right)$
(c) not defined 2(d) $\left(\frac{1}{3}\right)^{x(x-1)}$ **[2016]**

29. For $x \in \mathbf{R}$, $x \neq 0$, $x \neq 1$, let $f_0(x) = \frac{1}{1-x}$ and $f_{n+1}(x) = f_0(f_n(x))$, $n = 0, 1, 2, \ldots$. Then the $f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$ is equal to
(a) $\frac{8}{3}$ (b) $\frac{4}{3}$
(c) $\frac{5}{3}$ (d) $\frac{1}{3}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The domain of the function
$f(x) = \sqrt{2x-3} + \sin x + \sqrt{x-1}$ is
(a) $(-\infty, 1]$ (b) $[0, 1]$
(c) $\left[\frac{3}{2}, \infty\right)$ (d) $[1, \infty]$ **[2006]**

2. Let $f : (1, \infty) \to (1, \infty)$ be defined by $f(x) = \dfrac{x+2}{x-1}$. Then
   (a) f is $1 - 1$ and onto
   (b) f is $1 - 1$ but not onto
   (c) f is not $1 - 1$ but onto
   (d) f is neither $1 - 1$nor onto **[2008]**
3. Let $A = \{(x, y) : x > 0, y > 0, x^2 + y^2 = 1\}$ and let $B = \{(x, y) : x > 0, y > 0, x^6 + y^6 = 1\}$ Then $A \cap B$
   (a) A (b) B
   (c) ϕ (d) $\{(0, 1), (1, 0)\}$ **[2008]**
4. A school awarded 38 medals in football, 15 in basketball and 20 in cricket. Suppose these medals went to a total of 58 students and only three students got medals in all three sports. If only 5 students got medals in football and basketball, then the number of medals received in exactly two of three sports is
   (a) 7 (b) 9
   (c) 11 (d) 13 **[2008]**
5. Let Q be the set of all rational numbers and R be the relation defined as

   $R = \{(x, y) : 1 + xy > 0, x, y \in Q\}$

   Then relation R is
   (a) symmetric and transitive
   (b) reflexive and transitive
   (c) an equivalence relation
   (d) reflexive and symmetric **[2009]**
6. The domain of the function $f(x) = \dfrac{1}{3 - \log_3(x-3)}$ is
   (a) $(-\infty, 30)$ (b) $(-\infty, 30) \cup (30, \infty)$
   (c) $(3, 30) \cup (30, \infty)$ (d) $(4, \infty)$ **[2009]**
7. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = x^{2009} + 2009x + 2009$
   Then $f(x)$ is
   (a) one-one but not onto
   (b) not one-one but onto
   (c) neither one-one nor onto
   (d) one-one and onto **[2010]**
8. Let f be a function defined on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ by

   $f(x) = 3\cos^4 x - 6\cos^3 x - 6\cos^2 x - 3$

   Then the range of $f(x)$ is
   (a) $[-12, -3]$ (b) $[-6, -3]$
   (c) $[-6, 3)$ (d) $(-12, 3]$ **[2010]**
9. **Statement-1:** The function $f(x) = x^2 e^{-x^2} \sin|x|$ is even

   Statement-2: Product of two odd function is an even function **[2011]**
10. Consider the following relations
    $R_1 = \{(x, y) : x$ and y are integers and $x = ay$ or $y = ax$ for some integer $a\}$
    $R_2 = \{(x, y)$: x and y are integres and $ax + by = 1$ for some integers $a, b\}$
    Then
    (a) R_1, R_2 are not equivalence relations
    (b) R_1, R_2 are equivalence relation
    (c) R_1 is an equivalence relation but R_2 is not
    (d) R_2 is an equivalence relation but R_1 is not **[2012]**
11. Let f and g be functions defined by $f(x) = \dfrac{1}{x+1}$ $x \in \mathbf{R}$, $x \neq -1$ and $g(x) = x^2 + 1$, $x \in \mathbf{R}$. Then go f is
    (a) one-one but not onto
    (b) onto but not one-one
    (c) both one-one and onto
    (d) neither one-one nor onto **[2013]**
12. Let $\mathbf{N}$ be the set of natural number and for $a \in \mathbf{N}$, $a\mathbf{N}$ denotes the set $\{ax : x \in \mathbf{N}\}$.
    If $b\mathbf{N} \cap c\mathbf{N}, = d\mathbf{N}$, where b, c, d are natural numbers greater than 1 and the greatest common divisor of b and c is 1 then d equals
    (a) $\max\{b, c\}$ (b) $\min\{b, c\}$
    (c) bc (d) $b + c$ **[2014]**
13. Let $f(x) = (x + 1)^2 - 1$, $x \geq -1$, then the set $\{x : f(x) = f^{-1}(x)\}$:
    (a) is an empty set
    (b) contains exactly one element
    (c) contains exactly two elements
    (d) contains more than two elements **[2014]**
14. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by
    $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$, then f is
    (a) one-one and onto (b) one-one but not onto
    (c) onto but not one-one (d) neither onto nor one-one **[2015]**
15. If f is a function of real variable x satisfying $f(x + 4) - f(x + 2) + f(x) = 0$, then f is periodic with period
    (a) 8 (b) 10
    (c) 12 (d) 6 **[2016]**

Answers

Concept-based

1. (d)	**2.** (c)	**3.** (b)	**4.** (d)
5. (b)	**6.** (b)	**7.** (d)	**8.** (c)

Level 1

9. (a)	**10.** (c)	**11.** (c)	**12.** (c)

13. (d)	**14.** (c)	**15.** (a)	**16.** (d)
17. (b)	**18.** (b)	**19.** (a)	**20.** (a)
21. (d)	**22.** (b)	**23.** (d)	**24.** (c)
25. (a)	**26.** (d)	**27.** (b)	**28.** (b)
29. (b)	**30.** (b)	**31.** (b)	**32.** (a)
33. (c)	**34.** (c)	**35.** (d)	**36.** (c)
37. (b)	**38.** (d)	**39.** (b)	**40.** (a)
41. (d)	**42.** (b)	**43.** (c)	**44.** (a)
45. (b)	**46.** (c)	**47.** (b)	**48.** (d)
49. (c)	**50.** (d)	**51.** (b)	**52.** (b)
53. (a)			

Level 2

54. (b)	**55.** (d)	**56.** (c)	**57.** (a)
58. (a)	**59.** (c)	**60.** (d)	**61.** (b)
62. (d)	**63.** (d)	**64.** (a)	**65.** (a)
66. (b)	**67.** (c)	**68.** (b)	**69.** (d)
70. (a)	**71.** (b)	**72.** (d)	**73.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (c)	**3.** (b)	**4.** (d)
5. (b)	**6.** (c)	**7.** (b)	**8.** (c)
9. (d)	**10.** (a)	**11.** (d)	**12.** (a)
13. (a)	**14.** (c)	**15.** (a)	**16.** (b)
17. (d)	**18.** (d)	**19.** (c)	**20.** (b)
21. (a)	**22.** (d)	**23.** (b)	**24.** (a)
25. (d)	**26.** (d)	**27.** (c)	**28.** (b)
29. (c)			

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (a)	**3.** (c)	**4.** (b)
5. (d)	**6.** (c)	**7.** (d)	**8.** (a)
9. (b)	**10.** (c)	**11.** (d)	**12.** (c)
13. (c)	**14.** (d)	**15.** (c)	

Hints and Solutions

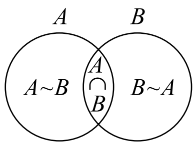
Concept-based

1. $n(A \cup B') = n(A) + n(B') - n(A \cap B')$
   $= n(A) + n(U) - n(B) - (n(A) - n(A \cap B))$
   $= n(U) - n(B) + n(A \cap B)$
   $= 12 - 6 + 2 = 8.$
2. If $a = -1$ than a is not related to a. So R is not reflexive. If $a = -1$, $b = 2$ then $|a| \le b$ but $|b| > a$, so R is not symmetric. If $|a| \le b$ and $|b| \le c$ then $|a| \le b \le |b| \le c$ so $|a| \le c$, hence a is related to c. Thus R is transitive. If $|a| \le b$, and $|b| \le a$ then $a \le |a| \le b \le |b| \le a$. So $a = b$, R is anti symmetric.
3. $f(g(x)) = g(f(x))$ for all $x \in \mathbf{R} \Leftrightarrow f(cx + d) = g(ax + b)$
   $\Leftrightarrow a(cx + d) + b = c(ax + b) + d \Leftrightarrow ad + b = cb + d$
   $\Leftrightarrow f(d) = g(b)$
4. For $f(x)$ to be defined, we have $|x| > x$. Since $|x| \ge x \ \forall\ x \in \mathbf{R}$ and $|x| = x$ for $x \in [0, \infty)$ so $|x| > x$ if $x \in \mathbf{R} = (-\infty, 0)$
5. The function is defined if $-1 \le 1 - 2x \le 1 \Leftrightarrow$
   $-2 \le -2x \le 0$
   $\Leftrightarrow 0 \le x \le 1.$
6. $f_1(x) = \dfrac{f(x)+f(-x)}{2} = \dfrac{1}{2}$ [sin$x$ – cos$x$ – sin$x$ – cos$x$] $= -\cos x$
   $f_2(x) = \dfrac{f(x)-f(-x)}{2} = \dfrac{1}{2}$ [sin$x$ – cos$x$ + sin$x$ + cos$x$] $= \sin x$
7. $-1 \le \sin x \le 1 \Leftrightarrow -1 \le -\sin x \le 1 \Leftrightarrow 0 \le 1 - \sin x \le 2$
8. $f(-x) = (-x)^2 \log\dfrac{1+x}{1-x} = x^2 \log\left(\dfrac{1-x}{1+x}\right)^{-1} =$
   $-x^2 \log\dfrac{1-x}{1+x} = -f(x).$

Level 1

9. $((A \sim B) \cup (B \sim A)) \cap A$
   $= ((A \sim B) \cap A) \cup (B \sim A) \cap \text{A}$
   $= (A \sim B) \cup \phi = (A \sim B) = \{3, 5\}$
10. $(A \cap B) \cap C = \{2 \times 3 \times 5\ x : x \in N\}$
    $= \{30x : x \in \mathbf{N}\}$
11. $X \subset Y \Rightarrow X \cap Y = X, X \cup Y = Y$
    $Y \subset X \Rightarrow X \cap Y = Y, X \cup Y = X.$
    So $X \cap Y = X \cup Y \Rightarrow X = Y$
12. If $X \underset{\neq}{\subset} Y$, let $y \in Y$ and $y \notin X$ then either $y \notin A$. or $y \in A$
    So if $y \in A$, then $y \in A \cap Y$ but $y \notin A \cap X$ and thus $A \cap X \ne A \cap Y$.
    If $y \notin A$, then $y \in A \cup Y$ but $y \notin A \cup X$ and thus $A \cup X \ne A \cup Y$
    So $X \subseteq Y$, similarly $Y \subseteq X$.
    $\Rightarrow \qquad X = Y.$
13. $A = \{7, 8\}$, $B = \{1, 2, 3, ... 9\}$
    $C = \{1, 2, 4, 7, 8, 14, 16, ...\}$
    $B \cap C = \{1, 2, 4, 7, 8\}$
    $A \cup (B \cap C) = \{1, 2, 4, 7, 8\}$
14. $A = \{\text{M, A, T, H, E, I, C, S}\}$
    $B = \{\text{S, T, A, I, C}\}$
    $B \sim A = \phi$, So $A \Delta B = (A \sim B) \cup (B \sim A)$
    $= A \sim B.$
    Verify (a) and (b) are not correct.

15. $n(A \sim B) = n(A) - n(A \cap B)$
 $\Rightarrow \quad 15 = n(A) - 16 \Rightarrow n(A) = 31$
 $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
 $\Rightarrow \quad 36 = 31 + n(B) - 16$
 $\Rightarrow \quad n(B) = 21.$



16. $A \cup B$, $A \sim B$, $B \sim A$
 $(A \cup B) \sim (A \sim B) = B$
 $(A \cup B) \sim (B \sim A) = A$
 $A \, \Delta \, B = (A \sim B) \cup (B \sim A)$
 $(A \cup B) \sim (A \, \Delta \, B) = A \cap B$
 and $(A \sim B) \sim A = \phi$
 Thus, the required no is 8.
17. $n(A \sim B) = n(A) - n(A \cap B)$
 $= n(B) - n(A \cap B) = n(B \sim A)$
18. $A \cup B = A \, \Delta \, B \cup (A \cap B)$
 So $A \cup B = A \, \Delta \, B \Rightarrow A \cap B = \phi$.
19. $n(P) = 60$, $n(M) = 58$, $n(P \cup M) = 100$, $N(P \cap M)$
 $= n(P) + M(M) - n(P \cup M)$
20. $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}$
21. $(1, 2) \in R$ but $(1, 2) \notin R_1$ or R_2 or R_3 so $R \neq R_1$ or R_2 or R_3.
22. L_1 is not $\perp L_1$, so R is not reflexive
 $L_1 \perp L_2 \Rightarrow L_2 \perp L_1 \Rightarrow R$ is symmetric.
 $L_1 \perp L_2$ and $L_2 \perp L_3 \Rightarrow L_1 \parallel L_3 \Rightarrow R$ is not transitive.
23. See the definition of function
24. $f(x) = f^{-1}(x) \Rightarrow f(f(x)) = x.$
 $\Rightarrow \quad [(x + 1)^2 - 1 + 1]^2 - 1 = x.$
 $\Rightarrow \quad (x + 1)^2 = x + 1 \Rightarrow x = 0$ or $x = -1$
 So $S = \{0, -1\}$.
25. $A \times A$ has $n \times n = n^2$ elements and any subset of $A \times A$ is a relation on A. The number of such subsets is 2^{n^2}.
26. $-1 \leq \sin x \leq 1 \Rightarrow 1 \leq f(x) \leq 5 \Rightarrow f(x)$ is not onto, $f(x) = f(x + 2\pi) \Rightarrow f$ is not one-one and f is not bijective.
27. In (a) $a \to 2a$ and $a \to -2a$, so it is not a function
 In (b) for each $x \in R$ there is unique $|x| = y \in R$, so it is a function.
28. $f(x) = x^4 + 2 = y \Rightarrow x = (y - 2)^{1/4}$.
 $\Rightarrow \quad f^{-1}(x) = (x - 2)^{1/4}$
 $\Rightarrow \quad f^{-1}(83) = (81)^{1/4} = \pm 3$
 and $f^{-1}(-2) = (-4)^{1/4} = \phi$.
29. R is reflexive if $(x, x). \in R$ for all $x \in \{1, 2, 3\}$
 $\Rightarrow \quad (1, 1), (2, 2), (3, 3) \in R$
 R is symmetric if $(1, 2), (2, 3) \in R$
 $\Rightarrow \quad (2, 1), (3, 2) \in R$.
 R is transitive if $(1, 2), (2, 3) \in R$
 $\Rightarrow \quad (1, 3) \in R$ also $(3, 1) \in R$ as
 R is symmetric.
 So the total numbers of elements is 9.
30. $A \cap A = A \neq \phi$ if $A \neq \phi$, So R is not reflexive.
 $A \cap B = \phi \quad \Rightarrow \quad B \cap A = \phi$
 $ARB \quad \Rightarrow \quad BRA \quad \Rightarrow \quad R$ is symmetric
 Note R is not transitive and not an equivalence relation.
31. $2f(2) - 3f(1/2) = 2^2 = 4$
 $2f(1/2) - 3f(2) = (1/2)^2 = 1/4$
 $\Rightarrow \quad 5(f(2) - f(1/2)) = 4 - 1/4 = 15/4$
 $\Rightarrow \quad f(1/2) = f(2) - 3/4.$
 So $\quad 2f(2) - 3[f(2) - 3/4] = 4$
 $\Rightarrow \quad f(2) = \dfrac{9}{4} - 4 = -\dfrac{7}{4}.$
32. Applying $C_3 \to C_3 - (C_1 + C_2)$ we get $f(x) = 0$ for all x.
33. $a^{x-1} + a^{y-1} = 1$
 $\Rightarrow \quad a^{y-1} = 1 - a^{x-1} \Rightarrow (y - 1)\log a = \log(1 - a^{x-1})$
 R.H.S is defined if $1 - a^{x-1} > 0$
 $\Rightarrow \quad a^{x-1} < 1 = a^{\circ} \Rightarrow x - 1 < 0$ as $a > 1$
 $\Rightarrow \quad x < 1$ and the required domain is $-\infty < x < 1$.
34. If x is an integer, $f(x) = 0$
 if x is not an integer $x - [x] = \dfrac{p}{q}$ where p and q are positive integers and $p < q$, so that $f(x) = \dfrac{p}{p+q} < \dfrac{1}{2}$
 so $0 \leq f(x) < \dfrac{1}{2}$ and the required range is $[0, \dfrac{1}{2}]$
35. $2f(x^2) + 3f(1/x^2) = x^2 - 1$
 $\Rightarrow \quad 2f(1/x^2) + 3f(x^2) = 1/x^2 - 1$
 Subtracting, $f(x^2) - f(1/x^2) = 1/x^2 - x^2$
 $\Rightarrow \quad f(x^2) = 1/x^2.$
36. $f(x + 3y, x - 3y) = 12xy = (x + 3y)^2 - (x - 3y)^2$
 $\Rightarrow \quad f(x, y) = x^2 - y^2.$
37. Let $f(x) = y = 2^{x(x-1)}$
 $\Rightarrow \quad \log y = x(x - 1) \log^2$
 $\Rightarrow \quad x^2 - x - \log_2 y = 0$
 $\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1 + 4 \log_2 y}}{2}$
 $\Rightarrow \quad x = f^{-1}(y) = \dfrac{1 \pm \sqrt{1 + 4 \log_2 y}}{2}$
 $\Rightarrow \quad f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4 \log_2 y}}{2}$ as $f^{-1}(x) \geq 1$.
38. Number of one-one functions $= 5 \times 4 \times 3 = 60$.
39. If we take $f(x) = 1$, then $f(y) \neq 1$ as the function is one-one. So $f(x) \neq 1$
 If $f(z) = 1$, then statements 2 and 3 both are true, so $f(z) \neq 1$ hence $f(y) = 1$ and $f^{-1}(1) = y$.
 [**Note** $f(x) = 2$, $f(y) = 1$, $f(z) = 3$)]
40. $f(x + 4\pi) = f(x)$
 If $\dfrac{\sin n(x + 4\pi)}{\sin \dfrac{x + 4\pi}{n}} = \dfrac{\sin nx}{\sin \dfrac{x}{n}}$
 which holds if $n = 2$.

41. $(n, n) \in R$ only for $n = 1$
$m = n^2 \Rightarrow n = m^2$ unless $m = n = 1$
$m = n^2$ and $n = p^2 \not\Rightarrow m = p^2$ for $m, n, p \neq 1$
hence (a), (b), (c) none is true.

42. Let B, H, F be the sets of the three teams respectively
so $n(B) = 21, n(H) = 26, n(F) = 29,$
$n(H \cap B) = 14$ $n(H \cap F) = 15, n(F \cap B) = 12,$
$n(B \cap H \cap F) = 8$
and $n(B \cup H \cup F) = n(B) + n(H) + n(F) - n(H \cap B)$
$- n(H \cap F) - n(F \cap B) + n(B \cap H \cap F) = 21 + 26$
$+ 29 - 14 - 15 - 12 + 8 = 43$

43. An equilateral triangle cannot be obtuse angled or right angled but is an isosceles triangle.
So $E \subset I$ and $E \cap I = E$.

44. $A = \{3, 9, 27, 81, 243, 729\}, B = \{9, 81, 729, 6561\}$
$A \cup B = \{3, 9, 27, 81, 243, 729, 6561\}$
$A \cap B = \{9, 81, 729\}$
$A \sim B = \{3, 27, 243\}$
$B \sim A = \{6561\}$
$A \Delta B = (A \sim B) \cup (B \sim A) = \{3, 27, 243, 6561\}$

45. For $a \neq 0$, the range of f is $\mathbf{R}$. f is differentiable function so f is one-one and only if f is monotonic.
$$f'(x) = a + \cos x$$
If $a > 1$, $f'(x) > 0$ i.e. f is increasing
If $a < -1$, $f'(x) < 0$ i.e. f is decreasing
Thus f is monotonic if $a \in \mathbf{R} \sim [-1, 1]$.

46. $f(x) = \sum_{k=1}^{n} 1 + [\sin kx] = n + [\sin x] + [\sin 2x] + \cdots$ $+ [\sin nx]$. If $kx \neq \frac{\pi}{2}$ for any $k = 1, 2 \ldots n$ then $0 < kx < \pi$ and $kx \neq \pi/2$ so $0 < \sin kx < 1$. Hence $[\sin kx] = 0$, $k = 1, 2 \ldots n$. i.e. $f(x) = n$. If $kx = \frac{\pi}{2}$ for some k then $x = \frac{\pi}{2k}$, hence $\sin x, \sin 2x, \ldots,$ $\sin(k-1)x$ will lie between 0 and 1 so $[\sin jx] = 0$ $1 \leq j \leq k-1$; $\sin kx = 1$ so $f(x)$ can be $n + 1$ or n.

47. $(A \Delta B) \Delta C$ is disjoint union of $(A \sim B) \sim C$, $(B \sim C) \sim A$, $(C \sim A) \sim B$ and $A \cap B \cap C$.
Threfore, number of elements is $(A \Delta B) \Delta C$ is $10 + 15 + 20 + 5 = 50$.

48. $f^2(x) = fof(x) = f(f(x)) = f\left(\frac{x-3}{x+1}\right)$
$$= \frac{\frac{x-3}{x+1} - 3}{\frac{x-3}{x+1} + 1} = -\frac{x+3}{x-1}$$
$$f^3(x) = f\left(-\frac{x+3}{x-1}\right) = \frac{\frac{-x-3}{x-1} - 3}{\frac{-x-3}{x-1} + 1} = x.$$
Hence $f^{3k}(x) = x$
$$f^{2010}(2014) = f^{3.670}(2014) = 2014.$$

49. $(1, 2) \in R$, But $(2, 1) \notin R \Rightarrow R$ is not symmetric and hence not an equivalence relation.
$\Rightarrow$ Statement-1 is True. Staement-2 is False as $1 \rightarrow 1, 2, 3$.

50. Statement-2 is True. Because if A has m elements, then first elements can be mapped to n elements. For the 2nd element the choice is $n - 1$ and so on. So the total number of injective mappings is $n(n - 1)(n - 2) \ldots (n - m + 1) = n!/(n - m)!$ Which shows that statement-1 is False.

51. $f(-x) = -\sin x + \cos x \neq f(x)$ or $-f(x)$, $\Rightarrow f$ is neither odd nor even. So statement-1 is True
$g(-x) = \frac{-\sin x}{1 - \cos x} = -(g(x)) \Rightarrow g$ is an odd function
$\Rightarrow$ Statement-2 is also True but does not lead to statement-1.

52. Taking $y = 3$, $f(x) - f(3) = x - 3$
$\Rightarrow f(x) = x - 3 + f(3) = x - 3 + 2 = x - 1$
$\Rightarrow f(xy) = xy - 1 \Rightarrow$ Statement –1 is True
$$f(x) = \frac{x^2 + x + 1}{x^2 - x + 1} \Rightarrow f(1/x) = \frac{1 + x + x^2}{1 - x + x^2} = f(x)$$
and $f^{(2)} = 7/3 \Rightarrow$ Statement-2 is also True, but does not lead to statement-1.

53 . Statement-2 is true by definition of a bijective mapping, using which statement-1 is also true.

Level 2

54. $-\pi/2 < x < \pi/2 \Rightarrow [x] = -2, -1, 0, 1$
$\Rightarrow f(x) = \cos[x]$
$= \cos(-2), \cos(-1), \cos(0), \cos 1$
$= \cos 2, \cos 1, 1 \cos 1$

55. $\frac{x-5}{x^2 - 10x + 24} > 0$ and $x + 5 > 0$
$\Rightarrow x > 5, x > 6 \Rightarrow x > 6 \Rightarrow x \in (6, \infty)$
or $x < 5, x > 4 \Rightarrow 4 < x < 5 \Rightarrow x \in (4, 5)$

56. Let $g(x) = 1 + \sqrt[3]{x} = y \Rightarrow x^{1/3} = y - 1$
so $f(g(x)) = 3 - 3x^{1/3} + x$
$\Rightarrow f(y) = 3 - 3(y - 1) + (y - 1)^3$
$= y^3 - 3y^2 + 5$

57. $f(-x) = \frac{-x}{e^x - 1} - \frac{x}{2} + 1$
$$= \frac{-xe^x}{1 - e^x} - \frac{x}{2} + 1$$

$$= \frac{x(e^x - 1 + 1)}{e^x - 1} - \frac{x}{2} + 1$$

$$= \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

58. (fog) $(x) = f(g(x)) = f\left(\frac{x}{1-x}\right)$

$$= \frac{\frac{x}{1-x}}{\frac{x}{1-x} + 1} = \frac{x}{x + 1 - x} = x$$

$\Rightarrow$ $(\text{fog})^{-1}$ $(x) = x$

59. $f(x) = \cot^{-1} \dfrac{2x}{1-x^2}$ is clearly one-one as

$f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

Next, let $y = f(x) \Rightarrow \cot y = \dfrac{2x}{1-x^2}$

$$= \frac{2\tan(\theta/2)}{1 - \tan^2(\theta/2)} = \tan\theta$$

$\Rightarrow y = \pi/2 - \theta = \pi/2 - 2\tan^{-1}x$. Taking $x = \tan(\theta/2)$

so $y \in (0, \pi) \Rightarrow 0 < y < \pi$

$\Rightarrow -\pi/4 < \tan^{-1} x < \pi/4$

$\Rightarrow -1 < x < 1 \Rightarrow x \in (-1, 1)$

and thus f is onto.

60. As $-\sqrt{5^2 + 12^2} \le 5 \sin x + 12 \cos x \le \sqrt{5^2 + 12^2}$ for all x

$\Rightarrow 14 - 13 \le 5 \sin x + 2 \cos x \le 14 + 13$

$\Rightarrow 1 \le f(x) \le 27$

f is not one-one as it is periodic.

61. fof $(\cos x) = f\left(\frac{1-\cos x}{1+\cos x}\right) = f(\tan^2(x/2))$

$$= \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \cos x.$$

62. Since $f(x + T) = \sin \sqrt{x+T} \ne \sin\sqrt{x}$ for any value T.

$f(x)$ is not periodic.

63. Let $y = f(x) = \dfrac{x+2}{x-1}$, $y \in Y = \mathbf{R} - \{1\}$

$\Rightarrow x = \dfrac{y+2}{y-1}$, $x \in X = \mathbf{R} \sim \{1\}$

$\Rightarrow f$ is onto

Also $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one.

64. $f(x + y) + f(x - y)$

$$= \frac{a^{x+y} + a^{-(x+y)}}{2} + \frac{a^{x-y} + a^{-(x-y)}}{2}$$

$$= \frac{a^x(a^y + a^{-y}) + a^{-x}(a^{-y} + a^y)}{2}$$

$$= \frac{\left(a^x + a^{-x}\right)\left(a^y + a^{-y}\right)}{2}$$

$= 2.\ f(x)\ f(y) \Rightarrow k = 2.$

65. $x - 3 \ge 0 \Rightarrow x \ge 3$ and $-1 \le \sqrt{x-3} \le 1$

$\Rightarrow x \le 4. \Rightarrow x \in [3, 4].$

66. $x[x] = \sqrt{|x|} \Rightarrow x^2 = x$ if x is an integer

$\Rightarrow \quad x = 0, 1$

67. $f(x) = x\dfrac{a^x + a^{-x}}{a^x - a^{-x}} \Rightarrow f(-x) = f(x) \Rightarrow f$ is not odd.

$f(x) = \dfrac{a^x + x}{a^x - x} \Rightarrow f(-x) = \dfrac{1 - xa^x}{1 + xa^x} \ne -f(x)$

$\Rightarrow f$ is not odd

$f(x) = \dfrac{a^x - 1}{a^x + 1} \Rightarrow f(-x) = \dfrac{1 - a^x}{1 + a^x} = -f(x) \Rightarrow f$ is odd.

$f(x) = x \log_2 \sqrt{\left(x + \sqrt{x^2+1}\right)} \Rightarrow f(-x)$

$= -x \log \left(-x + \sqrt{x^2+1}\right) \ne f(-x).$

68. $g(x) = 1 + x - [x] > 0$ for all x

$\Rightarrow$ (fog) $(x) = f(g(x)) = 1.$

69. The function $f(x) = \log_2 (\log_{1/2}(x^2 + 4x + 3) + \sin^{-1}[x]$ is defined if

$0 < x^2 + 4x + 3 < 1$ and $-1 \le x \le 2$ (I)

$x^2 + 4x + 3 > 0 \Rightarrow x > -1$ or $x < -3$ (II)

$x^2 + 4x + 3 < 1 \Rightarrow -2 - \sqrt{2} < x < -2 + \sqrt{2}$ (III)

So the domain of the function is $-1 < x < -2 + \sqrt{2}$.

70. $g(f(x)) = |\sin x| = \sqrt{\sin^2 x}$

which is satisfied if $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Also it satisfies $f(g(x)) = f\left(\sqrt{x}\right) = (\sin\sqrt{x})^2$

71. $f(x) = 3x - 5 = y \Rightarrow x = \dfrac{y+5}{3} = f^{-1}(y)$

$\Rightarrow f^{-1}(x) = \dfrac{x+5}{3}$

72. $f(x^2) = |x^2 - a| \neq (f(x))^2$

$f(|x|) = ||x| - a| \neq |f(x)|$

$f(x + y) = |x + y - a| \neq |x - a| + |y - a|.$

73. Required number is $4! = 24$.

Previous Years' AIEEE/JEE Main Questions

1. R is not symmetric as $(2, 3) \in R$ but $(3, 2) \notin R$

2. We must have

$7 - x \geq 1,\ x - 3 \geq 0$ and $7 - x \geq x - 3$

$\Rightarrow\ x \leq 6,\ x \geq 3$ and $x \leq 5$

Thus,

$3 \leq x \leq 5$

$\therefore$ Range of $f = \{{}^4P_0,\ {}^3P_1,\ {}^2P_2)$

$= \{1, 3, 2\}$

3. We have $-1 \leq x - 3 \leq 1$ and $9 - x^2 > 0$

$\Rightarrow\ 2 \leq x \leq 4$ and $-3 < x < 3$

$\therefore$ domain of f is $[2, 3,)$

4. We have

$-\sqrt{a^2+b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2+b^2}$

Thus,

$-\sqrt{1+3} \leq \sin x - \sqrt{3} \cos x \leq \sqrt{1+3}$

$\therefore\ -2 + 1 \leq f(x) \leq 2 + 1 \ \Rightarrow\ -1 \leq f(x) \leq 3$

5. A function $f(x)$ is symmetrical about the line $x = a$ if

$$f(a-x) = f(a+x).$$

6. Relation R is clearly reflexive. Since $(6, 12) \in R$ but $(12, 6) \notin R$, R is not symmetric.

Also, as $(3,3) \in R$, $(3, 6) \in R \Rightarrow (3, 6) \in R$;

$(3, 3) \in R$, $(3, 9) \in R \Rightarrow (3, 9) \in R$;

$(3, 3) \in R$, $(3, 12) \in R \Rightarrow (3, 12) \in R$

$(6, 6) \in R$, $(6, 12) \in R \Rightarrow (6, 12) \in R$

Thus, R is transitive.

$\therefore$ R is reflexive and transitive.

7. Putting $x = y = 0$, we get

$$f(0) = f(0)\,f(0) - f(a)f(a)$$

$\Rightarrow\quad f(a)^2 = 1^2 - 1 = 0 \ \Rightarrow f(a) = 0$

Putting $x = a$, $y = a$, we get

$$f(0) = [f(a)]^2 - f(0)\,f(2a)$$

$$1 = 0 - f(2a) \ \Rightarrow\ f(2a) = -1$$

Next, we put $x = 0$, $y = a$ to obtain

$$f(-a) = f(0)\,f(a) - f(a)\,f(2a) = 0$$

Now, putting $x = 2a$, $y = x$, we get

$$f(2a - x) = f(2a)\,f(x) - f(-a)\,f(a + x)$$

$$= (-1)\,f(x) - (0)\,f(a + x) = -f(x)$$

8. Let $\omega \in W$, then $(\omega, \omega) \in R$.

$\therefore$ R is reflexive.

Also, if $\omega_1, \omega_2, \in W$ and $(\omega_1, \omega_2) \in R$, then $(\omega_2, \omega_1) \in$ R.

$\therefore$ R is symmetric.

Next, let ω_1 = ink ω_2 = link and ω_3, = let, then $(\omega_1, \omega_2) \in R$,

$$(\omega_2, \omega_3) \in R \text{ but } (\omega_1, \omega_3) \notin R.$$

Thus, R is not transitive.

9. 4^{-x^2} is defined for all $x \in R$.

$\cos^{-1}\left(\dfrac{x}{2} - 1\right)$ is defined if

$-1 \leq \dfrac{x}{2} - 1 \leq 1$ or for $0 \leq x \leq 4$

and log cos x is defined if

$\cos x > 0$ or for $2n\pi - \dfrac{\pi}{2} < x < 2n\pi + \dfrac{\pi}{2}$

where $n \in N$.

Thus, domain of f is $0 \leq x < \dfrac{\pi}{2}$ or $\left[0, \dfrac{\pi}{2}\right)$

10. As $x \neq x + 1$, S is not reflexive

$\Rightarrow S$ is not an equivalence relation.

As $x - x = 0 \in \mathbf{I}$, T is reflexive

$xTy \Rightarrow x - y \in \mathbf{I} \ \Rightarrow y - x \in \mathbf{I} \Rightarrow yTx$

$\therefore$ T is symmetric

Suppose xTy and yTz

$\Rightarrow x - y \in \mathbf{I},\ \ y - z \in \mathbf{I}$

$\Rightarrow (x - y) + (y - z) \in \mathbf{I} \Rightarrow x - z \in \mathbf{I} \Rightarrow xTz$

$\therefore$ T is transitive

11. As f is one-one and onto, f is invertible. So

$y = 4x + 3 \Rightarrow x = \dfrac{1}{4}(y - 3)$

Thus, inverse of f is $g(y) = \dfrac{1}{4}(y - 3)$.

12. $B = ((A \cup B) - A) \cup (A \cap B)$

$= ((A \cup C) - A) \cup (A \cap C) = C.$

13. Note that R is not symmetric as $(0, 1) \in$ R but $(1, 0) \notin$ R.

Next, note that $qm = pm \Leftrightarrow \frac{m}{n} = \frac{p}{q}$

Thus, $\left(\frac{m}{n}, \frac{p}{q}\right) \in S \Rightarrow \frac{m}{n} = \frac{p}{q}$

Clearly, S is reflexive and symmetric

Next, note that

$\left(\frac{m}{n}, \frac{p}{q}\right) \in S$ and $\left(\frac{p}{q}, \frac{r}{s}\right) \in S \Rightarrow \frac{m}{n} = \frac{p}{q}$ and $\frac{p}{q} = \frac{r}{s}$

$\Rightarrow \quad \frac{m}{n} = \frac{r}{s} \quad \Rightarrow \quad \left(\frac{m}{n}, \frac{r}{s}\right) \in \text{S}$

Thus S is transitive so is an equivalence relation.

14. A is *reflexive*

Let $x \in \mathbf{R}$. $(x, x) \in A$ as $x - x = 0$ is an integer.
A is *symmetric*

Suppose $(x, y) \in A \Rightarrow x - y$ is an integer

$\Rightarrow \quad y - x$ is an integer

$\Rightarrow \quad (y, x) \in A$

A is *transitive*

Let $(x, y) \in A$ and $(y, z) \in A$

$\Rightarrow x - y$ and $y - z$ are integers

$\Rightarrow (x - y) + (y - z)$ is an integer

$\Rightarrow x - z$ is an integer.

$\Rightarrow (x, z) \in A$

Thus, A is an equivalence relation.

B is not *symmetric*

Note that $(0, \sqrt{3}) \in B$ as $0 = (0)\sqrt{3}$ and 0 is a rational number, but $(\sqrt{3}, 0) \notin B$ as there is no rational number a such that $\sqrt{3} = (a)(0)$.

$\therefore$ Statement-1 is true but statement-2 is false.
TIP As statement-2 is false, only possible answer is (*d*).

15. Let $y = f(x) = (x - 1)^2 + 1$

$\Rightarrow y - 1 = (x - 1)^2 \quad \Rightarrow x = 1 + \sqrt{y-1}, y \geq 1.$

Thus, $f^{-1}(x) = 1 + \sqrt{x-1}, x \geq 1.$

Now, $f(x) = f^{-1}(x)$

$\Rightarrow (x - 1)^2 + 1 = \sqrt{x-1} + 1$

$\Rightarrow \sqrt{x-1}\,[(x - 1)^{3/2} - 1] = 0$

$\Rightarrow x = 1, 2$

16. As $A = I^{-1} AI$, we get

$(A, A) \in R$ for all A.

$\therefore \quad R$ is reflexive.

Next, assume $(A, B) \in R$

$\Rightarrow$ there exists a non-singular matrix P such that $A = P^{-1}BP$

$\Rightarrow \quad B = PAP^{-1} = (P^{-1})^{-1}AP^{-1}$

Thus, $(B, A) \in R$

$\therefore \quad R$ is symmetric.

Next, assume that $(A, B) \in R$ and $(B, C) \in R$

$\Rightarrow$ there exist non-singular matrices P and Q such that

$A = P^{-1}BP$ and $B = Q^{-1}CQ$

$\therefore \quad A = P^{-1}(Q^{-1}CQ)P$

$= (P^{-1}Q^{-1})\, C(QP)$

$= (QP)^{-1}\, C(QP)$ [using statement-2]

$\Rightarrow (A, C) \in R$

$\therefore$ Statement-1 is true and statement-2 is true and is correct explanation for statement-1.

17. For each $x \in X$, we have the following four choices:

(a) $x \in Y$ and $x \in Z$ (b) $x \notin Y$ and $x \in Z$
(c) $x \in Y$ and $x \notin Z$ (d) $x \notin Y$ and $x \notin Z$

As we do not want $x \in Y \cap Z$, we are left with three choices for each $x \in X$.

Thus, the total number of ways in which Y and Z can be chosen is 3^5.

18. A $\times$ B contain (2) (4) = 8 elements.

The number of subsets of A $\times$ B having 3 or more elements

= Number of subsets of A $\times$ B – [number of subsets with at most 2 elements]

$= 2^8 - ({}^8C_1 + {}^8C_1 + {}^8C_2) = 219$

19. $P = \{(a, b) : \sec^2 a - \tan^2 b = 1\}$

P is not reflexive as $(\pi/2, \pi/2) \notin P$.

P is symmetric.

Suppose $(a, b) \in P$, then

$\sec^2 a - \tan^2 b = 1$

$\Rightarrow \quad (1 + \tan^2 a) - (\sec^2 b - 1) = 1$

$\Rightarrow \quad \sec^2 a - \tan^2 b = 1$

$\Rightarrow \quad (b, a) \in P.$

Suppose $(a, b) \in P, \quad (b, c) \in P$

$\Rightarrow \sec^2 a - \tan^2 b = 1$ and $\sec^2 b - \tan^2 c = 1$

Adding, we get

$\Rightarrow \quad \sec^2 a + \sec^2 b - \tan^2 b - \tan^2 c = 2$

$\Rightarrow \quad \sec^2 a + 1 - \tan^2 c = 2$

$\Rightarrow \quad \sec^2 a - \tan^2 c = 1$

$\therefore\ (a, c) \in P.$

20. $\frac{1}{3}+\frac{3n}{100}<1 \ \Rightarrow n \le 22;$

$1 \le \frac{1}{3}+\frac{3n}{100}<2 \quad \Rightarrow 300 \le 100 + 9n < 600$

$\Rightarrow \quad 23 \le n \le 55;$

Also, $\frac{1}{3}+\frac{3(56)}{100}=2\frac{1}{75}$

Thus, $\sum_{n=1}^{56} f(n) = \sum_{n=1}^{22} f(n)+\sum_{n=23}^{55} f(n)+f(56)$

$= 0 + \sum_{n=23}^{55} f(n)+2(56)$

$= \frac{33}{2}(55+23)+112=1399$

21. Period $|\sin(4x)|$ is $\pi/4$ and period of $|\cos 2x|$ is $\pi/2$.

$\therefore$ period of $f(x) = |\sin 4x| + |\cos (2x)|$ is

$\left(l\text{cm}\left(\frac{1}{2},\frac{1}{4}\right)\right)\pi=\frac{\pi}{2}$

22. $f(-1) = f(1) = 0 \ \therefore f$ is not one-one.

Also $f(x) = \frac{|x|-1}{|x|+1} = 1 - \frac{2}{|x|+1} \le 1$ for all $x \in \mathbf{R}$

Thus f cannot be onto.

23. $A = \{-2, -1, 0, 1, 2\}$

$R = \{(-2, 2), (0, 0), (1, 1), (2, 2)\}$

As R has four elements, the power set of R contains 16 elements.

24. $n(A \times B) = n(A)n(B) = 4.2 = 8$

The number of subsets of $A \times B$ which contains at least three elements.

$= {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$

$= 2^8 - ({}^8C_0 + {}^8C_1 + {}^8C_2) = 256 - (1 + 8 + 28)$

$= 219$

25. Let total number of families in the town be x. Then $n(P) = 0.25x$, $n(C) = 0.15x$, $n(P' \cap C') = 0.65x$,

$n(P \cup C) = x - n(P' \cap C') = 0.35x$

i.e. 35% of the families own either a car or a phone.

$n(P \cup C) = n(P) + n(C) - n(P \cap C)$

$\Rightarrow 0.35x = 0.25x + 0.15x - n(P \cap C)$

$\Rightarrow n(P \cap C) = 0.05x$

i.e. 5% of the families own a car and a phone

As $0.05x = 2000 \Rightarrow x = 40{,}000$

Thus, (a), (b), (c) are all true.

26. We can choose three elements out of seven in ${}^7C_3$ ways. These three elements are mapped to y_2. The remaining 4 elements are to be mapped to y_1 and y_3. This can be done is 2^4 ways. But these include two ways in which all the four elements are mapped to either y_1 or y_3. Therefore, there are $2^4 - 2 = 14$ ways to map containing four elements to y_1 and y_3. Thus the required number of onto mappings is $14.{}^7C_3$.

27. $f(x) + 2f\left(\frac{1}{x}\right) = 3x \ \forall \ x \ne 0$ (i)

Replacing x by $\frac{1}{x}$, we have

$f\left(\frac{1}{x}\right)+2f(x)=\frac{3}{x}$ (ii)

(i) – 2(ii) gives $f(x) - 4f(x) = 3x - \frac{6}{x}$

$\Rightarrow f(x) = \frac{2}{x}-x$

$f(x) = f(-x) \Rightarrow \frac{2}{x}-x = -\frac{2}{x}+x \Rightarrow \frac{4}{x} = 2x \Rightarrow x^2 = 2$

$x = \pm\sqrt{2}$. Thus S has exactly two elements.

28. Let $y = 3^{x(x-1)}$, so $y \ge 1$

$\Rightarrow \log_3 y = x(x - 1)$

$\Rightarrow x^2 - x - \log_3 y = 0$

$\Rightarrow x = \frac{1\pm\sqrt{1+4\log_3 y}}{2}$. As $x \ge 1$, so

$x = \frac{1}{2}\left[1+\sqrt{1+4\log_3 y}\right].$

Thus $f^{-1}(x) = \frac{1}{2}\left[1+\sqrt{1+4\log_3 y}\right].$

29. $f_1(x) = f_0(f_0(x)) = \frac{1}{1-f_0(x)} = \frac{1}{1-\frac{1}{1-x}}$

$= \frac{1-x}{1-x-1} = 1-\frac{1}{x}$

$f_2(x) = f_0(f_1(x)) = \frac{1}{1-f_1(x)} = \frac{1}{1-\left(1-\frac{1}{x}\right)} = x$

$f_3(x) = f_0(f_2(x)) = f_0(x)$

Thus $f_{3k}(x) = f_0(x)$, $f_{3k+1}(x) = f_1(x)$,

$f_{3k+2}(x) = f_2(x)$ for all $x \in \mathbf{R} \sim \{0, 1\}, k \geq 0$

So, $f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$

$= f_1(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$

$= 1-\frac{1}{3}+1-\frac{3}{2}+\frac{3}{2} = \frac{5}{3}$.

Previous Years' B-Architecture Entrance Examination Questions

1. f is defined if $2x - 3 \geq 0, x \geq 1$

   So the domain of f is $\left[\frac{3}{2}, \infty\right)$.

2. If $\frac{x_1+2}{x_1-1} = \frac{x_2+2}{x_2-1} \Rightarrow x_1x_2 + 2x_2 - x_1 - 2$

   $= x_1x_2 + 2x_1 - x_2 - 2$

   $\Rightarrow 3x_2 = 3x_1 \Rightarrow x_1 = x_2$

   So f is one-one

   If $y = f(x) = \frac{x+2}{x-1} \Rightarrow yx - y = x + 2$

   $\Rightarrow (y - 1)x = y + 2$

   $\Rightarrow x = \frac{y+2}{y-1} \in (1, \infty)$

   Thus f is onto

3. $x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$, so if $(x, y) \in A \cap B$ then $1 = x^4 - x^2y^2 + y^4 = (x^2 + y^2)^2 - 3x^2y^2$

   $= 1 - 3x^2y^2$

   $\Rightarrow x^2y^2 = 0 \Rightarrow x = 0$ or $y = 0$ but $x > 0, y > 0$

   so $A \cap B = \phi$

4. $58 = n(F \cup B \cup C)$

   $= n(f) + n(B) + n(C) - n(F \cap B) - n(B \cap C) - n(F \cap C) + n(F \cap B \cap C)$

   $= 38 + 15 + 20 - n(F \cap B) - n(B \cap C) - n(F \cap C) + 3$

   $\Rightarrow n(F \cap B) + n(B \cap C) + n(F \cap C) = 18$

   Number of medals received in exactly two of three sports is

   $= n(F \cap B \cap C') + n(B \cap C \cap F') + n(F \cap C \cap B')$

   $= n(F \cap B) + n(B \cap C) + n(F \cap C) - 3\, n(F \cap B \cap C)$

   $= 18 - 9 = 9$

5. Since $1 + x^2 > 0$ for all $x \in Q$, so R is reflexive. If $(x, y) \in R \Rightarrow 1 + xy > 0$

   $\Rightarrow 1 + yx > 0 \Rightarrow (y, x) \in R$

   Thus R is symmetric. Take $x = 2$, $y = -\frac{1}{4}$, $z = -1$

   $1 + xy = 1 - \frac{1}{2} = \frac{1}{2} > 0$ so $(x, y) \in R$

   $1 + yz = 1 + \frac{1}{4} = \frac{5}{4} > 0$ so $(y, z) \in R$

   $1 + xz = 1 - 2 = -1 \Rightarrow (x, z) \notin R$. Hence R is not transitive.

6. f is defined if $x > 3$ and $3 \neq \log_3(x - 3)$

   $\Rightarrow x > 3,\ x - 3 \neq 27$ i.e. $x \in (3, \infty) \sim \{30\}$

   $= (3, 30) \cup (30, \infty\}$

7. f is continuous function and $f'(x) = 2009\, x^{2008} + 2009 > 0$. So f is one-one and onto.

8. For $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $0 \leq \cos x \leq 1$. So

   $0 \leq \cos^4 x \leq 1$ and $-5 \leq -2\cos^2 x - 2\cos x - 1 \leq -1$

   $\Rightarrow -4 \leq \cos^4 x - 2\cos^2 x - 2\cos x - 1 \leq -1$

   $\Rightarrow -12 \leq f(x) \leq -3$

9. $f(-x) = (-x)^2\, e^{-(-x)^2} \sin|-x| = x^2 e^{-x^2} \sin|x| = f(x)$. Product of two odd functions is an even function.

10. $x = 1 \cdot x$ so $(x, x) \in R_1$. If $(x, y) \in R_1$ then $x = ay$ or $y = ax$ for some integer a

    $\Rightarrow y = ax$ or $x = ay \Rightarrow (y, x) \in R_1$

    If $x = ay$ or $y = ax$ and $z = by$ or $y = bz$

    for some integer a and b then $x = abz$ or $z = abx$, so $(x, z) \in R_1$. Thus R_1 is an equivalence relation.

    Since $(2, 2) \notin R_2$ so R_2 is not reflexive. Thus R_2 is not an equivalence relation.

11. gof $(x) = g\left(\frac{1}{x+1}\right) = \left(\frac{1}{x+1}\right)^2 + 1 = \frac{1+(x+1)^2}{(x+1)^2} \geq 1$

    so gof is not onto, gof is not one-one as gof $(1) =$ gof (-3).

12. Since g: $cd\,(b, c) = 1$ so $bc =$ l.c.m (b, c). Hence $d = bc$

13. $y = f(x) = (x + 1)^2 - 1 \Rightarrow x = \sqrt{y+1} - 1$

    Hence $f^{-1}(x) = \sqrt{x+1} - 1$. Now $f(x) = f^{-1}(x)$

    $\Leftrightarrow (x + 1)^2 - 1 = \sqrt{x+1} - 1 \Rightarrow (x + 1)^4 = x + 1$

    $\Rightarrow (x + 1)\,[x^3 + 3x^2 + 3x] = 0$

$\Rightarrow (x + 1)\, x(x^2 + 3x + 3) = 0$

$\Rightarrow x = 0, -1$

14. For $x < 0$, $f(x) = \dfrac{e^{-x} - e^{-x}}{e^{x} + e^{-x}} = 0$

$\therefore$ f cannot be one-one

For $x \geq 0$, $f(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \geq 0$

Thus $f(x)$ cannot take negative values so range of f cannot be **R**. Therefore f is not onto.

15. $f(x + 4) - f(x + 2) + f(x) = 0 \ \forall\ x \in \mathbf{R}$

Replace x by $x + 2$,

$f(x + 6) - f(x + 4) + f(x + 2) = 0$

Adding above two equations, we have

$f(x + 6) + f(x) = 0 \ \forall\ x \in \mathbf{R}$

Replace x by $x + 6$, we get

$f(x + 12) + f(x + 6) = 0$

$\Rightarrow f(x + 12) = -f(x + 6) = f(x)$

f is periodic with period 12.