Time Domain Response Analysis

Transient and Steady state response

Transient Response:

This part reduces to zero as t → ∞ i.e., $\lim_{t\to\infty} C_{tr}(t) = 0$

Steady state response:

- This is the response of the system as t → ∞
- Mathematically, we can have expressed the time response C(t) as

$$C(t) = C_{tr}(t) + C_{ss}(t)$$

where,

$C_{tr}(t)$ is the Transient Response.

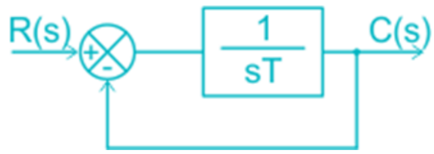
$C_{ss}(t)$ is the Steady State Response.

Time Response of a zero order control

$$\frac{C(s)}{R(s)} = K \quad \text{Gain}$$

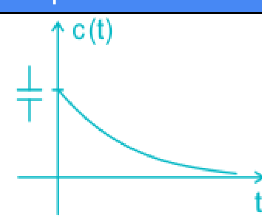
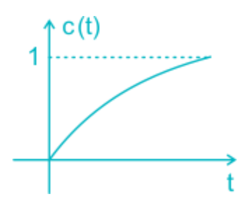
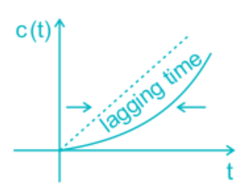
Example: Potentiometer, tachometers etc.

Time Response of a First order control system

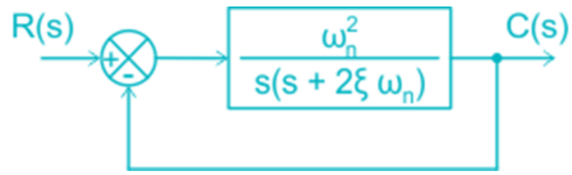


$$\frac{C(s)}{R(s)} = \frac{1}{1 + sT}$$

Time response to different Input signals

Input Signals	Time Response	
Unit impulse, δ(t)	$C(t) = \frac{1}{T} e^{-\frac{t}{T}}$	
Unit step, u(t)	$C(t) = 1 - e^{-\frac{t}{T}}$	
Unit ramp, r(t)	$C(t) = (t - T + Te^{-\frac{t}{T}})$	

Time Response of a Second order control system

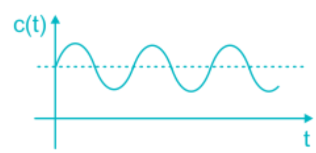
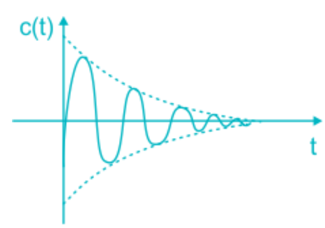


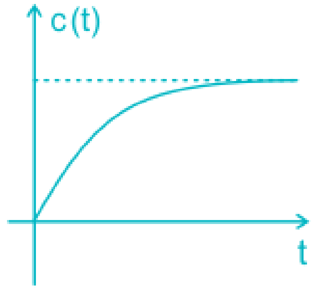
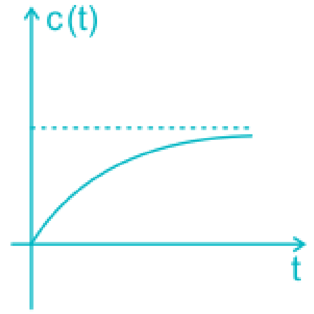
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

With characteristics equation:

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

Roots: $-\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} = -\xi\omega_n \pm j\omega_d$

$\xi = 0$	Undamped oscillations 
$0 < \xi < 1$	Underdamped oscillations 

ξ = 1	Critical damping 
ξ > 1	Over damping 

ξ is called damping ratio and it is given as:

$$\xi = \frac{Actual\ Damping}{Critical\ Damping} = \frac{\xi\omega_n}{\omega_n}$$

Time response specifications of second order (under-damped) control system subjected to unit step input function

Response is given by

$$C(t) = \underbrace{1}_{steady\ state} \underbrace{- \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin[\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right]}_{Transient\ state}$$

where,

$\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$ is damped frequency of oscillations

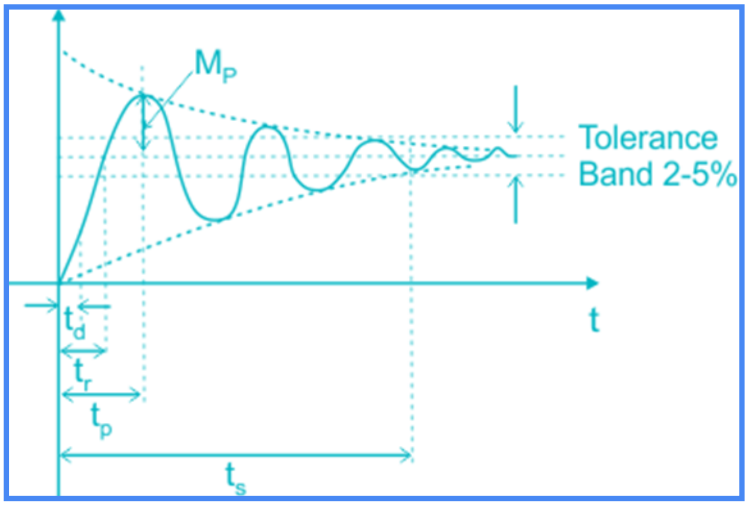
- $\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \cos^{-1}(\xi)$

Step response of second order systems

- When poles are real & lie on the left half of s-plane, step response approaches a steady – state value without oscillations.
- When poles are complex & lie on left half of s-plane, step response approaches a steady state value with damped oscillations.
- When complex conjugate poles lie on imaginary axis, step response will have fixed amplitude oscillations.
- When poles are complex & lie on right half of s-plane, step response approaches infinite with negatively damped oscillations.
- When poles are real & lie on the right half of s-plane, step response approaches an infinite without any oscillations.

Time Constant of under damped response

$$\boxed{\tau = \frac{1}{\xi\omega_n}}$$



Delay time (t_d)

$$\boxed{t_d = \frac{1 + 0.7\xi}{\omega_n}}$$

Rise time (t_r)

$$\boxed{t_r = \frac{\pi - \phi}{\omega_n \sqrt{1 - \xi^2}}}$$

Where,

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \cos^{-1}(\xi)$$

Peak time (t_p)

$\boxed{t_p = \frac{n\pi}{\omega_d}}$;

n = 1, 3, 5, …. For overshoots

n = 2, 4, 6, …. For undershoots

Peak Overshoot (M_p)

Peak percent overshoot, $\%M_p = \frac{c(t_p)-c(\infty)}{c(\infty)} \times 100\%$

Or,

$$\%M_p = e^{-\frac{n\pi\xi}{\sqrt{1-\xi^2}}} \times 100$$

Here, n = 1, 3, 5, …. For overshoots

n = 2, 4, 6, …. For undershoots

Note: Lowest value of damping ratio will provide maximum peak value of overshoot.

Dynamic Error Coefficients

$$K_0 = \lim_{s\to 0} F(s)$$

$$K_1 = \lim_{s\to 0} \frac{d}{ds} F(s)$$

$$K_2 = \lim_{s\to 0} \frac{d^2}{ds^2} F(s)$$

Where, $\boxed{F(s) = \frac{1}{1+G(s)H(s)}}$

Relation between Static and Dynamic Error constants

- $K_1 \simeq \frac{1}{K_v}$
- $K_2 \simeq \frac{1}{K_a}$
- $K_0 \simeq \frac{1}{1+K_p}$

Steady state error & static error coefficients vary with system type

Type '0' system	Step input	Ramp input	Parabolic input
Steady State formula	$\frac{1}{1+K_p}$	$\frac{1}{K_v}$	$\frac{1}{K_a}$
Static Error Constant	K_p = constant	$K_v = 0$	$K_a = 0$
Error	$\frac{1}{(1+K_p)}$	∞	∞

Type '1' system	Step input	Ramp input	Parabolic input
Steady State formula	$\frac{1}{(1+K_p)}$	$\frac{1}{K_v}$	$\frac{1}{K_a}$
Static Error Constant	$K_p = \infty$	K_v = constant	$K_a = 0$
Error	0	$\frac{1}{K_v}$	∞

Type '2' system	Step input	Ramp input	Parabolic input
Steady State formula	$\frac{1}{(1+K_p)}$	$\frac{1}{K_v}$	$\frac{1}{K_a}$
Static Error Constant	$K_p = \infty$	$K_v = \infty$	K_a = constant
Error	0	0	$\frac{1}{K_a}$

Points to Remember

- Finite steady state error varies inversely proportional to the forward path gain.
- As the type of system becomes higher, more steady-state errors are eliminated.
- For type-0 system, steady state error for position input (step input) will be finite.
- For type-1 system, steady state error for velocity input (ramp input) will be finite.
- For type-2 system, steady state error for acceleration input (parabolic input) will be finite.

Settling time (t_s)

$t_s = 3\tau = \frac{3}{\xi \omega_n}$; for 5% tolerance band

$t_s = 4\tau = \frac{4}{\xi \omega_n}$; for 2% tolerance band

Steady State Error

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}}$$

- $|e_{ss}|$ depends on the input and the open-loop transfer function.
- This above formula is only valid for unity feedback system.
- Steady state error is defined only for stable system.
- Steady state error depends on the type of input (i.e., ramp, step etc.) as well as the system type.

Static Error Coefficients

- **Static position error coefficient (K_p)**

$$\boxed{K_p = \lim_{s \to 0} G(s)H(s)}$$

- **Static velocity error coefficient (K_v)**

$$\boxed{K_v = \lim_{s \to 0} s\, G(s)H(s)}$$

- **Static acceleration error coefficient (K_a)**

$$\boxed{K_a = \lim_{s \to 0} s^2 G(s)H(s)}$$

Important Points

1) For unit step input

settling time $T_s = \dfrac{\ln\left[\frac{100}{x}\right]}{\xi w_n}$

Where x = tolerance %
i.e if Tolerance = 2%
then x = 2
This formula is only valid for underdamped system i.e. $0 < \xi < 1$

2) If roots of transfer function are real & unequal

i.e., $T(s) = \frac{1}{(ST_1+1)(ST_2+1)}$

Then it is overdamped system.

$$T_s = \ln\left(\frac{100}{x}\right) \times T$$

Where
T = max (T_1 and T_2)

3) For critically damped system T_s is found by conventional method

But $T_s = \frac{5.8}{\xi w_n}$ for 2% Tolerance

4)

we know $\frac{d}{dt}(step\ response) = Impulse\ Response$

Step response =∫ (impulse Response) dt + C(constant)

To find ‘c’ we use Boundary condition i.e., at t = 0, step response = 0 etc.