Rolle's Theorem

Q.1. Verify Rolle's theorem for the function : $f(x) = e^{2x}$ (sin 2x – cos 2x) , defined in the interval [π/8 , 5π/8] .

Solution : 1

We have , f(x) = e2x($\sin^{2x}$ – $\cos^{2x}$) x ε [π/8, 5π/8]

(1) As sine, cosine and exponential function are always continuous, hence given function

f(x) is continuous in [π/8, 5π/8] .

(2) f'(x) = e^{2x} × 2 ($\sin^{2x}$ – cos 2x) + e^{2x} (2 $\cos^{2x}$ + 2 $\sin^{2x}$)

= 2 e2x ($\sin^{2x}$ – $\cos^{2x}$ + $\cos^{2x}$ + $\sin^{2x}$)

= 2 e^{2x}(2 $\sin^{2x}$) = 4 e^{2x} $\sin^{2x}$.

Thus derivatives exists in the given interval and function is differentiable.

(3) f(π/8) = $e^{\pi/4}$ (sin π/4 – cos π/4) = $e^{\pi/4}$ ×0 = 0 .

f(5π/8) = $e5^{\pi/4}$ (sin $5^{\pi/4}$ – cos $5^{\pi/4}$) = $e5^{\pi/4}$ × 0 = 0 .

Therefore , f(π/8) = f(5π/8)

Now f'(c) = 0

Or, 4 e^{2c} sin2c = 0

Or, sin 2c = 0 [As e^{2c} ≠ 0] Hence, 2c = 0 , π , 2π , 3π ………… .

Or, c = 0 , π/2 , π , 3π/2 ……….. .

Therefore , π/2 ε (π/8 , 5π/8) .

Hence Rolle's theorem is verified.

Q.2. Examine the validity and conclusion of Rolle's theorem for the function :

f(x) = ex . $\sin^{x}$, for all x ε [0, π] .

Solution : 2

We have , f(x) = ex sin ^{x} , for all x ε [0, π]

(1) As exponential function and trigonometric function are continuous , hence their product is also continuous in [0, π] i.e. f(x) is continuous in the given interval.

(2) $f(x) = e^{x} \sin x$

Hence , $f'(x) = e^{x} \sin x + e^{x} \cos x$.

Clearly f(x) exists in the open interval (0, π) .

(3) $f(0) = e^{0}. \sin 0 = 0$; f(π) = eπ.sin π = 0 .

Since f(0) = f(π) = 0 .

Hence all condition of Rolle's theorem is satisfied. Hence there exist 'c' , in 0 < c < π such that f'(c) = 0

Or, $e^{c} (\sin c + \cos c) = 0$

Or, sin c + cos c = 0 [As, $e^{c} \neq 0$]

Or, tan c = −1 = tan 3π/4 .

As , 3π/4 ε [0, π] , **Rolle's theorem is verified.**

Q.3. Verify Rolle's theorem for the function $f(x) = \log [(x^{2} + ab)/\{x (a + b)\}]$, x ε [a, b] and x ≠ 0 .

Solution : 3

(1) $f(x) = \log [(x^{2} + ab)/\{x(a + b)\}] = \log (x^{2} + ab) - \log(a + b) - \log x$

Therefore , f(x) is continuous in a ≤ x ≤ b .

(2) $f'(x) = 2x/(x^{2} + ab) - 1/x$

$= (2x^{2} - x^{2} - ab)/\{x(x^{2} + ab)\}$

$= (x^{2} - ab)/\{x (x^{2} + ab)\}$, which exist in a < x < b .

(3) $f(a) = \log (a^{2} + ab) - \log (a + b) - \log a$

$= \log a + \log (a + b) - \log (a + b) - \log a = 0$

$f(b) = \log (b^2 + ab) - \log (a + b) - \log b$

$= \log b + \log (a + b) - \log (a + b) - \log b = 0$

Hence , f(a) = f(b) .

Hence , there exist c ε [a, b] , such that f′(c) = 0 ,

Or, $f'(c) = (c^2 - ab)/\{c(c^2 + ab)\} = 0$

Or, $c^2 - ab = 0 \Rightarrow c^2 = ab \Rightarrow c = \sqrt{(ab)}$.

As, $c = \sqrt{(ab)}$ ε [a, b] , **Rolle's theorem is verified.**

Q.4. Taking the function f(x) = (x – 3) log x, prove that there is at least one value of x in (1, 3) which satisfies x log x = 3 – x.

Solution : 4

(1) As both (x – 3) and log x are continuous functions , hence f(x) is also a continuous in [1, 3] .

(2) f′(x) = (x – 3)/x + log x which exist in (1, 3) .

(3) f(1) = 0 = f(3). Therefore , f(x) satisfies all three conditions. Rolle's theorem is applicable. As such there is at least one value of x in (1, 3) for which f′(x) = 0.

Or, (x – 3)/x + log x = 0 => x log x = 3 – x .

Or, one of the roots of x log x = 3 – x will be in the interval (1, 3) . **[Proved.]**

Q.5. Apply Rolle's theorem to find point (or points) on the curve y = –1 + cos x where the tangent is parallel to the x-axis in [0, 2π] .

Solution : 5

y = f(x) = –1 + cos x ------------------------- (1)

As , cosine function is continuous function for all values of , f(x) is continuous in [0, 2π] and differentiable in (0, 2π) .

Also $f(0) = 0 - 1 + \cos 0 = -1 + 1 = 0$; $f(2\pi) = -1 + 1 = 0$.

Hence , $f(0) = f(2\pi)$.

Thus all the three conditions of Rolle's theorem are satisfied by f(x) in [0, 2π].

Hence , by Rolle's theorem there exist at least one real number x in (0, 2π) such that $f'(x) = 0$.

Now , $f'(x) = -\sin x$, and $f'(x) = 0 \Rightarrow -\sin x = 0 \Rightarrow \sin x = 0 \Rightarrow x = \pi \in [0, 2\pi]$.

When x = π, (i) $y = -1 + \cos \pi = -1 - 1 = -2$. So, there exist a point (π, −2) on the given curve $y = -1 + \cos x$, where the tangent is parallel to the x-axis.

Q.6. Verify Rolle's theorem for the function $f(x) = x^3 - 7x^2 + 16x - 12$ in the interval [2, 3].

Solution : 6

We have $f(x) = x^3 - 7x^2 + 16x - 12$ in [2, 3] .

The function being a polynomial is continuous in [2, 3] and differentiable in (2, 3) . Also f(2) = 23 − 7 × 22 + 16 × 2 − 12 = 0 = f(3) .

Therefore , all the conditions of Rolle's theorem are satisfied , hence there exist at least one value c ε [2, 3] such that f′(c) = 0 .

Now $f'(x) = 3x^2 - 14x + 16$.

And f′(x) = 0 => 3x2 − 14 x + 16 = 0 .

Or, $(x - 2)(3x - 8) = 0$

Or, x = 2, 8/3.

Clearly c = 8/3 ε (2, 3) and f′(c) = 0 .

Hence , **Rolle's theorem is verified.**

Q.7. It is given that Rolle's theorem holds good for the function : $f(x) = x^3 + ax^2 + bx$, x ε [1, 2] at the point x = 4/3. Find the values of a and b.

Solution : 7

We have, $f(x) = x^3 + ax^2 + bx$, $x \varepsilon [1, 2]$

Then $f(1) = 1 + a + b = 0$; $f(2) = 8 + 4a + 2b = 0$

Adding the two we get, $3a + b + 7 = 0$ -------------------------- (i)

Differentiating we get, $f'(x) = 3x^2 + 2ax + b = 0$

And $f'(4/3) = 3(4/3)^2 + 2a(4/3) + b = 0$

Or, $16/3 + 8a/3 + b = 0$

Or, $8a + 3b + 16 = 0$ -------------------------- (ii)

Solving (i) and (ii) we get, $a = -5$, $b = 8$.