Chapter 9

Binomial Theorem

Solutions (Set-1)

Very Short Answer Type Questions :

1. Expand $\left(x+\frac{1}{x}\right)^5$ by using binomial theorem.

Sol. $\left(x+\frac{1}{x}\right)^5 = {}^5C_0x^5 + {}^5C_1x^4\left(\frac{1}{x}\right) + {}^5C_2(x)^3\left(\frac{1}{x}\right)^2 + {}^5C_3x^2\left(\frac{1}{x}\right)^3 + {}^5C_4x\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$

$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

$\therefore \quad \left(x+\frac{1}{x}\right)^5 = x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

2. Expand $(x^2 + 2a)^5$ by using binomial theorem.

Sol. $(x^2 + 2a)^5 = {}^5C_0(x^2)^5 + {}^5C_1(x^2)^4(2a)^1 + {}^5C_2(x^2)^3(2a)^2 + {}^5C_3(x^2)^2(2a)^3 + {}^5C_4(x^2)(2a)^4 + {}^5C_5(2a)^5$

$= x^{10} + 10x^8a + 40x^6a^2 + 80x^4a^3 + 80x^2a^4 + 32a^5.$

3. Expand $(x + y)^7$ by using the binomial theorem.

Sol. $(x + y)^7 = {}^7C_0x^7 + {}^7C_1x^6y + {}^7C_2x^5y^2 + {}^7C_3x^4y^3 + {}^7C_4x^3y^4 + {}^7C_5x^2y^5 + {}^7C_6xy^6 + {}^7C_7y^7.$

$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$

Hence $(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

4. Expand $(1 + 2x + x^2)^3$ by using the binomial theorem.

Sol. We know that

$(1 + 2x + x^2) = (1 + x)^2$

Now, $(1 + 2x + x^2)^3 = \{(1 + x)^2\}^3 = (1 + x)^6$

Using binomial theorem for positive integral index, we obtain

$(1 + x)^6 = {}^6C_0 + {}^6C_1x + {}^6C_2x^2 + {}^6C_3x^3 + {}^6C_4x^4 + {}^6C_5x^5 + {}^6C_6x^6.$

$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

Hence $(1 + 2x + x^2)^3 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

5. Write the general term in the expansion of $(a^2 - b)^7$.

Sol. We have, $(a^2 - b)^7 = [a^2 + (-b)]^7$

The general term in the above binomial expansion is given by

$T_{r+1} = {}^7C_r(a^2)^{7-r}.(-b)^r$

$= (-1)^r.{}^7C_r(a)^{14-2r}.b^r$

6. If 15th term and 16th term in the expansion of $(1 + x)^{30}$ are equal, then find the value of x.

Sol. In the expansion of $(1 + x)^{30}$, $(r + 1)^{th}$ term is given by

$T_{r+1} = {}^{30}C_r x^r$

$\therefore \quad T_{15} = {}^{30}C_{14}x^{14}$ and $T_{16} = {}^{30}C_{15}x^{15}$

Given, $T_{15} = T_{16}$

$\Rightarrow \quad {}^{30}C_{14}x^{14} = {}^{30}C_{15}x^{15}$

$$\Rightarrow \quad \frac{\lfloor 30}{\lfloor 14 \lfloor 16} = \frac{\lfloor 30}{\lfloor 15 \lfloor 15}x$$

$$\Rightarrow \quad x = \frac{\lfloor 15 \lfloor 15}{\lfloor 14 \lfloor 16}$$

$$= \frac{15\lfloor 14 \lfloor 15}{\lfloor 14 .16 \lfloor 15}$$

$$= \frac{15}{16}$$

$$\therefore \quad x = \frac{15}{16}$$

7. Find the middle term in the expansion of $\left(x - \frac{1}{2x}\right)^{12}$.

Sol. Here $n = 12$ which is even, therefore, $\left(\frac{12}{2} + 1\right)^{th}$ *i.e.*, 7th term will be middle term.

In the expansion of $\left(x - \frac{1}{2x}\right)^{12}$, 7th term is given by

$$T_7 = {}^{12}C_6 x^{12-6}\left(-\frac{1}{2x}\right)^6 = {}^{12}C_6 x^6 \frac{(-1)^6}{(2x)^6}$$

$$= {}^{12}C_6 \times \frac{1}{64} = \frac{\lfloor 12}{\lfloor 6 \lfloor 6} \times \frac{1}{64} = \frac{231}{16}$$

Hence, the middle term $= \frac{231}{16}$.

Short Answer Type Questions :

8. Expand $(1 + x + x^2)^3$ as powers of x.

Sol. $(1 + x + x^2)^3 = (y + x^2)^3$ where $y = 1 + x$.

$= {}^3C_0y^3 + {}^3C_1y^2(x^2)^1 + {}^3C_2y(x^2)^2 + {}^3C_3(x^2)^3$

$= 1(1 + y)^3 + 3.(1 + x)^2.x^2 + 3(1 + x).x^4 + x^6$

$= 1 + 3x^2 + 3x + x^3 + 3x^2 (1 + 2x + x^2) + 3x^4 + 3x^5 + x^6$

$= 1 + 3x^2 + 3x + x^3 + 3x^2 + 6x^3 + 3x^4 + 3x^4 + 3x^5 + x^6$

$= 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$

9. By using binomial theorem, expand $\left(\frac{3x}{2}-\frac{3}{2x}\right)^4$.

Sol. Using binomial theorem for positive integral index, we have

$$\left(\frac{3x}{2}-\frac{3}{2x}\right)^4 = {}^4C_0\left(\frac{3x}{2}\right)^4 + {}^4C_1\left(\frac{3x}{2}\right)^3\left(\frac{-3}{2x}\right)^1 + {}^4C_2\left(\frac{3x}{2}\right)^2\left(\frac{-3}{2x}\right)^2 + {}^4C_3\left(\frac{3x}{2}\right)^1\left(\frac{-3}{2x}\right)^3 + {}^4C_4\left(\frac{-3}{2x}\right)^4$$

$$= \frac{81x^4}{16} + 4\frac{27x^3}{8}\times\left(\frac{-3}{2x}\right) + 6\left(\frac{9x^2}{4}\right)\left(\frac{9}{4x^2}\right) + 4.\left(\frac{3x}{2}\right)\left(\frac{-27}{8x^3}\right) + \frac{81}{4x^4}$$

$$= \frac{81x^4}{16} - \frac{81x^2}{4} + \frac{243}{8} - \frac{81}{4x^2} + \frac{81}{4x^4}$$

Hence, $\left(\frac{3x}{2}-\frac{3}{2x}\right)^4 = \frac{81x^4}{16} - \frac{81x^2}{4} + \frac{243}{8} - \frac{81}{4x^2} + \frac{81}{4x^4}$

10. Which of $(1.01)^{1000000}$ and 10, 000 is larger?

Sol. Writing 1.01 as (1 + 0.01) and using binomial theorem, we have,

$(1 + 0.01)^{1000000}$

$= 1 + {}^{1000000}C_1(0.01)$ + more positive terms

$= 1 + 1000000 \times \frac{1}{100}$ + more positive terms

= 1 + 10000 + Some positive term > 10000

This $(1.01)^{1000000}$ is greater than 10000.

11. Using binomial theorem, prove that $6^n - 5n - 1$ is always divisible by 25, where n is any natural number.

Sol. $6^n - 5n - 1$

$= (1 + 5)^n - 5n - 1$

$= {}^nC_0 + {}^nC_15^1 + {}^nC_25^2 + ... + {}^nC_n5^n - 5n - 1$

$= 1 + 5n + 5^2\,{}^nC_2 + 5^3\,{}^nC_3 + ... + 5n\,{}^nC_n - 5n - 1$

$= 5^2\,\{{}^nC_2 + {}^{5.n}C_3 + 5^2\,{}^nC_4 + ... + 5^{n-2n}C_n\}$.

= 25 {Some positive integer for $n \geq 2$}.

$\Rightarrow$ $6n - 5n - 1$ is a multiple of 25.

When $n = 1$, $6n - 5^n - 1 = 0$, which is divisible by 25

Hence, $6^n - 5n - 1$ is always divisible by 25 for any nature number n.

12. Find the term involving a^2b^5 in the expansion of $(a - 2b)^4 (a + b)^3$.

Sol. $(a - 2b)^4 (a + b)^3 = \{a + (-2b)\}^4 (a + b)^3$

$= \{{}^4C_0a^4 + {}^4C_1a^3 (-2b) + {}^4C_2a^2 (-2b)^2 + {}^4C_3a(-2b)^3 + {}^4C_4 (-2b)^4\}$

$\{{}^3C_0a^3 + {}^3C_1a^2b + {}^3C_2ab^2 + {}^3C_3b^3\}$

$= (a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4) (a^3 + 3a^2b + 3ab^2 + b^3)$

(Multiplying only those terms in the two fraction which produce a term containing a^2b^5)

$= .... + \{(24\ a^2b^2) (b^3) + (-32)ab^3.3ab^2 + (16b^4) (3a^2b)\} + ...$

$= .... + (24 - 96 + 48)a^2b^5 + ...$

Hence the required term is $(-24a^2b^5)$

13. Write the fifth term in the expansion of $\left(2x^2 - \frac{1}{3x^3}\right)^{10}, x \neq 0$.

Sol. General term

$$T_{r+1} = {}^{10}C_r (2x^2)^{10-r} \left(-\frac{1}{3x^3}\right)^r \qquad ...(i)$$

5th term *i.e.*, we have to calculate T_5

$\therefore$ $r = 4$ put $r = 4$ in (i) and get $T_5 - \frac{4480}{27}$

14. Find the two middle terms in the expansion of $\left(2x - \frac{x^2}{4}\right)^9$.

Sol. Here the total number of terms in 9 + 1 = 10 (even)

$\therefore$ There are two middle terms given by

$T_{\frac{9+1}{2}}$ and $T_{\frac{9+3}{2}}$ *i.e.*, T_5 and T_6.

We know that

$$T_{r+1} = {}^nC_r (2x)^{9-r} . \left(-\frac{x^2}{4}\right)^r \qquad ...(i)$$

Putting the value of $r = 4$ and 5, we get T_5 and T_6 respectively.

$$\frac{T_5 = \frac{63}{4}x^{13}}{T_6 = \frac{-63}{32}x^{14}}$$

15. Find the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}, x \neq 0$.

Sol. Here the general term

$$T_{r+1} = {}^{2n}C_r (x)^{2n-r}.\left(\frac{1}{x}\right)^r$$

$$= {}^{2n}C_r\, x^{2n-r-r}$$

$$T_{r+1} = {}^{2n}C_r\, x^{2n-2r} \qquad \text{...(i)}$$

This term will be independent from x if

$2n - 2r = 0$

$\Rightarrow \quad n = r$

Substituting $r = n$ in (i), we get

$$T_{n+1} = {}^{2n}C_n.x^0 = {}^{2n}C_n = \frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}$$

$\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}$ is the required term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$.

16. Find the value of r if the coefficients of $(2r + 4)^{\text{th}}$ term and $(r - 2)^{\text{th}}$ term in the expansion of $(1 + x)^{18}$ are equal.

Sol. $T_{2r+4} = T_{(2r+3)+1} = {}^{18}C_{2r+3}.x^{2r+3}$

$T_{r-2} = T_{(r-3)+1} = {}^{18}C_{r-3}\, x^{r-3}$

Here given that

${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow$ Either $2r + 3 = r - 3$ or $2r + 3 = 18 - (r - 3)$

$\Rightarrow$ Either $r = -6$ or $r = 6$

$\therefore$ $r = 6$ [$\because$ cannot be negative]

17. Find the greatest term in the expansion of $(2 + 3x)^{10}$, when $x = 4$.

Sol. In the expansion of $(2 + 3x)^{10}$,

$$T_{r+1} = {}^{10}C_r\, 2^{10-r}\, (3x)^r \qquad \text{...(i)}$$

$$T_r = {}^{10}C_{r-1}\, 2^{11-r}\, (3x)^{r-1} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{T_{r+1}}{T_r} = \frac{{}^{10}C_r.2^{10-r}(3x)^r}{{}^{10}C_{r-1}2^{11-r}(3x)^{r-1}}$$

$$= \frac{{}^{10}C_r(3x)}{{}^{10}C_{r-1}.2} = \frac{6.{}^{10}C_r}{{}^{10}C_{r-1}} \quad (\because\ x = 4)$$

$$= \frac{6\underline{|10}}{\underline{|10-r}\,\underline{|r}} \times \frac{\underline{|11-r}\,\underline{|r-1}}{\underline{|10}}$$

$$= \frac{6(11-r)}{r}$$

Now, $\frac{T_{r+1}}{T_r} \geq 1$ iff $\frac{6(11-r)}{r} \geq 1$

$\Rightarrow \quad r \leq \frac{66}{7}$

This means that $\frac{T_{r+1}}{T_r} > 1$ if and only if $r \leq 9$. This means that $T_{10} \geq T_9$ but $T_{11} < T_{10}$. Hence T_{10} is the greatest term.

$T_{10} = {}^{10}C_9.2.(3x)^9 = {}^{10}C_1.2.(3 \times 4)^9 = 10 \times 2 \times (12)^9$

$= 2^{20}.3^9.5$

18. Evaluate $\sum_{r=1}^{n} {}^nC_r\, 2^r$.

Sol. $\sum_{r=1}^{n} {}^nC_r 2^r = {}^n C_1 2 + {}^n C_2 2^2 + {}^n C_3 2^3 + ... + {}^n C_n 2^n$

$= \{{}^nC_0 + {}^nC_1.2 + {}^nC_2.2^2 + {}^nC_3 2^3 + ... + {}^nC_n 2^n\} - {}^nC_0$

$= (1 + 2)^n - 1 = 3^n - 1$

$\{\because\ {}^nC_0 + {}^nC_1 n + {}^nC_2 n^2 + ... + {}^nC_n x^n = (1 + x)^n\}$.

19. Use the binomial theorem to find the exact value of $(10.1)^5$.

Sol. $(10.1)^5 = [10 + 0.1]^5$

$= 10^5 + {}^5C_1 10^4 (0.1) + {}^5C_2 10^3 (0.1)^2 + {}^5C_3 10^2 (0.1)^3 + {}^5C_4 10(0.1)^4 + 5C_5 (0.1)^5$

$= 10000 + 5 \times 10^4 \times (0.1) + 10 \times 10^3 (0.01) + 10 \times 10^2 (0.001) + 5 \times 10 \times (0.0001) + 0.00001$

$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001$

$= 105101.00501$.

20. Find the middle term in the expansion of $(1 - 2x + x^2)^n$.

Sol. $(1 - 2x + x^2)^n = [(1 - x)^2]^n = (1 - x)^{2n}$

Here 2^n is an even integer, therefore $\left(\frac{2n}{2}+1\right)^{th}$ i.e., $(n + 1)^{th}$ term will be the middle term.

Now, $(n + 1)^{th}$ term in $(1 - x)^{2n} = {}^{2n}C_n (1)^{2n-n} (-x)^n$

$= {}^{2n}C_n (-x)^n = \frac{\lfloor 2n}{\lfloor n \lfloor n}(-1)^n x^n$

Hence, the middle term = $\frac{\lfloor 2n}{\lfloor n \lfloor n}(-1)^n x^n$

21. Find the coefficient of x^9 in the expansion of $(1 + 3x + 3x^2 + x^3)^{15}$.

Sol. $(1 + 3x + 3x^2 + x^3)^{15} = [(1 + x)^3]^{15} = (1 + x)^{45}$

$\therefore$ Coefficient of x^9 in $(1 + 3x + 3x^2 + x^3)^{15}$

$=$ Coefficient of x^9 in $(1 + x)^{45}$

$= {}^{45}C_5$

$= \frac{\lfloor 45}{\lfloor 9 \lfloor 36}$

22. Given that 4th term in the expansion of $\left(px+\frac{1}{x}\right)^n$ is $\frac{5}{2}$, find n and p.

Sol. Given expression is $\left(Px+\frac{1}{x}\right)^n$

Given $T_4=\frac{5}{2}$

$$\therefore \quad {}^nC_3(Px)^{n-3}\left(\frac{1}{x}\right)^3=\frac{5}{2}$$

$$=\frac{\lfloor n}{\lfloor 3\,\lfloor n-3}.P^{n-3}.x^{n-6}=\frac{5}{2}$$

Since R.H.S. is independent of x

Therefore, $n - 6 = 0$

$\Rightarrow \quad n = 6$

Now $\frac{\lfloor 6}{\lfloor 3\,\lfloor 3}.P^3=\frac{5}{2}$

$\Rightarrow \quad 20P^3=\frac{5}{2}$

$\Rightarrow \quad P^3=\frac{1}{8}=\left(\frac{1}{2}\right)^3$

$\Rightarrow \quad P=\frac{1}{2}$

Hence $n = 6$ and $P=\frac{1}{2}$

Long Answer Type Questions :

23. Expand $(2 + 3x)^{-5}$ up to four terms in

(i) Ascending power of x

(ii) Descending powers of x

Sol. (i) $(2+3x)^{-5}=\left[2\left(1+\frac{3}{2}x\right)\right]^{-5}$

$$=2^{-5}\left[1+\frac{3}{2}x\right]^{-5}$$

$$=\frac{1}{32}\left[1+(-5)\left(\frac{3}{2}x\right)+\frac{(-5)(-6)}{\lfloor 2}\left(\frac{3}{2}x\right)^2+\frac{(-5)(-6)(-7)}{\lfloor 3}.\left(\frac{3}{2}x\right)^3+.....\right]$$

$$=\frac{1}{32}\left[1-\frac{15}{2}x+\frac{135}{4}x^2-\frac{945}{8}x^3+.....\right]$$

(ii) $(2+3x)^{-5} = \left[3x\left(1+\frac{2}{3x}\right)\right]^{-5}$

$= (3x)^{-5}\left[1+\frac{2}{3x}\right]^{-5}$

$= \frac{1}{243x^5}\left[1+(-5)\left(\frac{2}{3x}\right)+\frac{(-5)\,(-6)}{\underline{|2}}.\left(\frac{2}{3x}\right)^2+\frac{(-5)\,(-6)\,(-7)}{\underline{|3}}\left(\frac{2}{3x}\right)^3+.....\right]$

$= \frac{1}{243\,x^5}\left[1-\frac{10}{3x}+\frac{20}{3x^2}-\frac{280}{27x^3}+.....\right]$

$= \frac{1}{243}\left[\frac{1}{x^5}-\frac{10}{3}.\frac{1}{x^6}+\frac{20}{3}.\frac{1}{x^7}-\frac{280}{27}.\frac{1}{x^8}+.....\right]$

24. By using the binomial expansion, expand $(1 + x + x^2)^3$

Sol. $(1 + x + x^2)^3 = [(1 + x) + x^2]^3$

$= {}^3C_0(1 + x)^3 + {}^3C_1(1 + x)^2.(x^2) + {}^3C_2.(1 + x).\,(x^2)^2 + {}^3C_3.(x^2)^3.$

$= (1 + x)^3 + 3(1 + x)^2.x^2 + 3(1 + x)\,x^4 + x^6$

$= (1 + 3x + 3x^2 + x^3) + (3x^2 + 6x^3 + 3x^4) + (3x^4 + 3x^5) + x^6$

$= x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$

25. (i) Find the 10th term in the expansion of $\left(2x^2+\frac{1}{x}\right)^{12}$

(ii) Find 6th term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$

Sol. We know that in the expansion of $(a + b)^n$, we have $(r + 1)^{th}$ term, $t_{r+1} = {}^nC_r\, a^{n-r}b^r$.

(i) 10th term, $t^{10} = t_{9+1} = {}^{12}C_9\,(2x^2)^{12-9}\left(\frac{1}{x}\right)^9$

$= {}^{12}C_3(2x^2)^3\left(\frac{1}{x^9}\right)$

$= \left(\frac{12\times11\times10}{3\times2\times1}.8x^6.\frac{1}{x^9}\right)$

$= \left(\frac{1760}{x^3}\right)$

(ii) $t_6 = t_{5+1} = (-1)^5\ {}^9C_5\left(\frac{4x}{5}\right)^{9-5}\left(\frac{5}{2x}\right)^5$

$= -{}^9C_4\left(\frac{4x}{5}\right)^4\left(\frac{5}{2x}\right)^5$

$= -\left[\frac{9\times8\times7\times6}{4\times3\times2\times1}.\frac{2^8x^4}{5^4}\times\frac{5^5}{2^5x^5}\right]$

$= -\frac{5040}{x}$

26. Find the middle terms in the expansion of $\left(4x-\frac{x^3}{8}\right)^7$.

Sol. Clearly, the given expression contains 8 terms

$\therefore$ Middle terms are $\left(\frac{8}{2}\right)^{th}$ and $\left(\frac{8}{2}+1\right)^{th}$ terms

i.e., 4^{th} and 5^{th} terms

$$t_4 = t_{3+1} = (-1)^3\ {}^7C_3\ (4x)^{7-3}.\left(\frac{x^3}{8}\right)^3$$

$$= \frac{-7\times6\times5}{1\times2\times3}256x^4.\frac{x^9}{512}$$

$$= -\frac{35}{2}x^{13}$$

$$t_5 = t_{4+1} = (-1)^4\ {}^7C_4(4x)^3\left(\frac{x^3}{8}\right)^4$$

$$= 35\times4\times4\times4.x^3.\frac{x^{12}}{8\times8\times8\times8}$$

$$= \left(+\frac{35}{8}\right)x^{15}$$

Hence, the middle terms are $\left(-\frac{35}{2}x^{13}\right)$ and $\frac{35}{8}x^{15}$.

27. Find the coefficient of $\frac{1}{y^2}$ in the expansion of $\left(y+\frac{c^3}{y^2}\right)^{10}$.

Sol. Suppose that the r^{th} term contains $\frac{1}{y^2}$ or y^{-2}.

In the expansion of $\left(y+\frac{c^3}{y^2}\right)^{10}$, r^{th}

Term is given by

$$T_r = {}^{10}C_{r-1}y^{10-r+1}\left(\frac{c^3}{y^2}\right)^{r-1}$$

$$= {}^{10}C_{r-1}\,y^{11-r}\left(\frac{c^{3r-3}}{y^{2r-2}}\right)$$

$$= {}^{10}C_{r-1}\ y^{11-r-2r+2}.c^{3r-3}$$

Thus, $T_r = {}^{10}C_{r-1}\, y^{13-3r}c^{3r-3}$

Since r^{th} term contains y^2

$\therefore \quad 13 - 3r = -2$

$\Rightarrow \quad r = \frac{15}{3} = 5$

$\therefore \quad 5^{th}$ term contain y^{-2}

$T_5 = {}^{10}C_{5-1}\, c^{15-3}y^{-2} = {}^{10}C_4 c^{12} y^{-2}$

$= \frac{\lfloor 10}{\lfloor 4 \lfloor 10-4}.c^{12} = 210c^{12}.y^{-2}$

Hence, the coefficient of the $\frac{1}{y^2}$ is 210 c^{12}.

28. Find the value of k so that the term independent of x in the expansion of $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$ is 405.

Sol. Let in the expansion of $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$, r^{th} term be independent of x.

In $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$, r^{th} term is given by

$$T_r = {}^{10}C_{r-1}\left(\sqrt{x}\right)^{10-r+1}\left(\frac{k}{x^2}\right)^{r-1}$$

$$= {}^{10}C_{r-1}\left(\sqrt{x}\right)^{11-r}\frac{k^{r-1}}{x^{2r-2}}$$

$$= {}^{10}C_{r-1}x^{\frac{11-r}{2}}.\frac{k^{r-1}}{x^{2r-2}}$$

$$= {}^{10}C_{r-1}x^{\frac{11-r}{2}-2r+2}.k^{r-1}$$

$$= {}^{10}C_{r-1}\, x^{\frac{15-5r}{2}}.k^{r-1}$$

Since r^{th} term is independent of x

$\therefore \quad \frac{15-5r}{2} = 0$

$\Rightarrow \quad r = 3$

Putting $r = 3$

$T_3 = {}^{10}C_2\, k^2$

But according to the question ${}^{10}C_2 k^2 = 405$

$\Rightarrow \quad k = \pm\, 3$

29. If the coefficient of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1 + a)^n$ are in A.P. then prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

Sol. In the expansion of $(1 + a)^n$,

Coefficient of $a^{r-1} = {}^nC_{r-1}$, coefficient of $a^r = {}^nC_r$ and coefficient of $a^{r+1} = {}^nC_{r+1}$

Given ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are in A.P.

$\therefore \quad 2.{}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$

$$\Rightarrow \quad \frac{2\,\lfloor n}{\lfloor (n-r)\,\lfloor r} = \frac{\lfloor n}{\lfloor n-r+1\,\lfloor r-1} + \frac{\lfloor n}{\lfloor r+1\,\lfloor n-r-1}$$

$$= \frac{2}{(n-r)\,\lfloor n-r-1\; r\lfloor r-1} = \frac{1}{(n-r+1)\,(n-r)\,\lfloor n-r-1\;\lfloor r-1} + \frac{1}{\lfloor (n-r-1)\,(r+1)r\lfloor r-1}$$

$$= \frac{2}{r(n-r)} = \frac{1}{(n-r+1)\,(n-r)} + \frac{1}{(r+1)r}$$

$$\Rightarrow \quad \frac{2}{r(n-r)} = \frac{r(r+1)+(n-r)\,(n-r+1)}{(n-r+1)\,(n-r)\,r(r+1)}$$

$\Rightarrow \quad 2[(n - r + 1)(r + 1)] = r(r + 1) + (n - r)\,(n - r + 1)$

$\Rightarrow \quad n^2 - 4nr - n + 4r^2 - 2 = 0$

$\Rightarrow \quad n^2 - n(4r + 1) + 4r^2 - 2 = 0$

30. In the expansion of $(1 - x)^{2n-1}$, a_r is denoted as the coefficient of x^r, then prove that $a_{r-1} + a_{2n-r} = 0$.

Sol. Given expansion $= (1 - x)^{2n-1}$

rth term in the expansion of $(a + x)^n$

$$= {}^nC_{r-1}\, a^{n-(r-1)}.x^{r-1}$$

Clearly, x^r will occur in $(r + 1)^{\text{th}}$ term in the expansion of $(1 - x)^{2n-1}$.

Now in the expansion of $(1 - x)^{2n-1}$, $(r + 1)^{\text{th}}$ term is given by

$T_{r+1} = {}^{2n-1}C_r(1)2^{n-1-r}(-x)^r$

$\quad = {}^{2n-1}C_r\,(-1)^r.x^r$

Therefore, the coefficient of $x^r = {}^{2n-1}C_r(-1)^r$.

According to the question,

$a_r = (-1)^r\, {}^{2n-1}C_r$

$\therefore \quad a_{r-1} = {}^{2n-1}C_{r-1}(-1)^{r-1}$ and $a_{2n-r} = {}^{2n-1}C_{2n-r}\,(-1)^{2n-r}$

$= (-1)^{-r}.{}^{2n-1}C_{2n-1(2n-r)}$

$$= \frac{(-1)^r}{(-1)^{2r}}.{}^{2n}C_{r-1}$$

$= (-1)^r\, {}^{2n-1}C_{r-1}$

Now, $a_{r-1} + a_{2n-r} = {}^{2n-1}C_{r-1}\,[(-1)^{r-1} + (-1)^r]$

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Chapter 9

Binomial Theorem

Solutions (Set-2)

[Basic of Binomial Theorem for Positive Index]

1. The expansion $(a + x)^n = {}^nC_0\, a^n + {}^nC_1 a^{n-1}x + ... + {}^nC_n x^n$ is valid when n is

(1) An integer (2) A rational number (3) An irrational number (4) A natural number

Sol. Answer (4)

The given expansion is valid for all positive integral values of n (natural number).

2. If the coefficient of rth term and $(r + 1)^{th}$ term in the expansion of $(1 + x)^{20}$ are in the ratio 1 : 2, then r is equal to

(1) 6 (2) 7 (3) 8 (4) 9

Sol. Answer (2)

$$\frac{{}^{20}C_{r-1}}{{}^{20}C_{r-0}} = \frac{1}{2}$$

$\Rightarrow$ $r = 7$

3. When n is any positive integer, the expansion $(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}a + ... + {}^nC_n\, a^n$ is valid only when

(1) $|x| < 1$ (2) $|x| > 1$

(3) $|x| < 1$ and $|a| < 1$ (4) x and a are any two numbers

Sol. Answer (4)

The given expansion is valid whatever x and a may be.

4. The exponent of x occurring in the 7^{th} term of the expansion of $\left(\frac{ax}{2} - \frac{8}{bx}\right)^9$ is

(1) 3 (2) –3 (3) 5 (4) –5

Sol. Answer (2)

The general term

$$T_{r+1} = {}^9C_r \left(\frac{ax}{2}\right)^{9-r} \left(\frac{-8}{bx}\right)^r$$

$$T_7 = {}^9C_6\left(\frac{ax}{2}\right)^{9-6}\left(\frac{-8}{bx}\right)^6$$

$$= {}^9C_6\left(\frac{a}{2}\right)^3\left(\frac{8}{b}\right)^6.\frac{x^3}{x^6}$$

$$= {}^9C_6\left(\frac{a}{2}\right)^3\left(\frac{8}{b}\right)^6.[x^{-3}]$$

5. The term containing a^3b^4 in the expansion of $(a - 2b)^7$ is

(1) 3^{rd} (2) 4^{th} (3) 5^{th} (4) 6^{th}

Sol. Answer (3)

$T_{r+1} = {}^7C_r(a)^{7-r}(-2b)^r$

Since, $a^{7-r} = a^3$

$\Rightarrow$ $7 - r = 3$

$r = 4$

5th term contains a^3

6. The term independent of x in the expansion of $\left(x-\frac{3}{x^2}\right)^{18}$ is

(1) ${}^{18}C_6$ (2) ${}^{18}C_6\,3^6$ (3) ${}^{18}C_{12}$ (4) ${}^{18}C_6\,3^{12}$

Sol. Answer (2)

$$T_{r+1} = {}^{18}C_r(x)^{18-r}.\left(\frac{-3}{x^2}\right)^r$$

$= (-1)^r\,{}^{18}C_r.\,3^r.\,x^{18-r-2r}$

$= (-1)^r\,.\,{}^{18}C_r\,.\,3^r\,x^{18-3r}$ ...(1)

Since the term independent of x.

If $18 - 3r = 0$

$\Rightarrow$ $r = 6$

Put the value of r in (1)., we obtain

$T_7 = (-1)^6\,{}^{18}C_6\,3^6$

$= {}^{18}C_6\,3^6$

7. In the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^{12}$, 5^{th} term from the end is

(1) $-7920\,x^{-4}$ (2) $7920\,x^4$ (3) $7920\,x^{-4}$ (4) $-7920\,x^4$

Sol. Answer (3)

Fifth term from the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^{12}$ is equal to the 5th term from the beginning in the expansion of $\left(\frac{-2}{x^2}+\frac{x^3}{2}\right)^{12}$

$$T_5 = {}^{12}C_4\left(\frac{-2}{x^2}\right)^{12-4}\cdot\left(\frac{x^3}{2}\right)^4$$

$$= \frac{\lfloor 12}{\lfloor 8\,\lfloor 4}\,\frac{2^8}{x^{16}}\times\frac{x^{12}}{2^4}$$

$$= \frac{12\times 11\times 10\times 9}{4\times 3\times 2\times 1}\times\frac{2^8}{2^4}.x^{-4}$$

$$= \frac{12\times 11\times 10\times 9}{4\times 3\times 2\times 1}\times 2^4\times x^{-4}$$

8. The coefficient of a^m and a^n (m, n are positive integer) in the expansion of $(1 + a)^{m+n}$ are

(1) Unequal
(2) Equal
(3) Reciprocal of each other
(4) Additive inverse of each other

Sol. Answer (2)

The coefficient of $a^m = {}^{m+n}C_m = \frac{\lfloor m+n}{\lfloor n\,\lfloor m}$

The coefficient of $a^n = {}^{m+n}C_n = \frac{\lfloor m+n}{\lfloor n\,\lfloor m}$

Hence the coefficient of a^m = The coefficient of a^n

9. The number of terms in the expansion of$\{(a + 4b)^3 (a - 4b)^3\}^2$ is

(1) 7
(2) 6
(3) 8
(4) 32

Sol. Answer (1)

$\{(a + 4b)^3 (a - 4b)^3\}^2$

$\{(a^2 - 16b^2)^3\}^2$

$= (a^2 - 16b^2)^6$

which contains 6 + 1 = 7 terms

10. The number of non-zero terms in the expansion of $(1+3\sqrt{2}a)^9+(1-3\sqrt{2}a)^9$ is

(1) 10
(2) 5
(3) 9
(4) 6

Sol. Answer (2)

$$(1+3\sqrt{2}a)^9+(1-3\sqrt{2}a)^9$$

$$= 2\left\{{}^9C_0 + {}^9C_2\left(3\sqrt{2}a\right)^2 + {}^9C_4\left(3\sqrt{2}a\right)^4 + {}^9C_6\left(3\sqrt{2}a\right)^6 + {}^9C_8\left(3\sqrt{2}a\right)^8\right\}$$

Which contains only 5 terms

11. In the expansion of $\left(2+\frac{1}{3x}\right)^n$, the coefficient of x^{-7} and x^{-8} are equal then n is equal to

(1) 51
(2) 52
(3) 55
(4) 56

Sol. Answer (3)

Coeff. of x^{-7} = coeff. of x^{-8}

$^nC_7.2^{n-7}.(3)^{-7}$

$= {}^nC_8 2^{n-8}\,(3)^{-8}$

$^nC_7 \times 6 = {}^nC_8$

$$\frac{n!}{7!(n-7)!}\times 6=\frac{n!}{8!(n-8)!}$$

$n - 7 = 48$

$n = 55$

12. If in the expansion of $(1 + px)^q$, $q \in N$, the coefficients of x and x^2 are 12 and 60 respectively, then p and q are

(1) 2, 6 (2) 6, 2 (3) –2, 6 (4) –6, 2

Sol. Answer (1)

In the expansion of $(1 + px)^q$, $T_2 = {}^qC_1\,px$ and $T_3 = {}^qC_2\,(px)^2$

Co-efficient of $x = pq = 12$

Co-efficient of $x^2=\dfrac{q(q-1)}{2}p=60$

$\Rightarrow p^2q^2 - p(pq) = 120$

$\Rightarrow 144 - 12p = 120$

$\Rightarrow 12p = 24$

$\Rightarrow p = 2,\ q = 6$

13. The expansion of $\left(x^{\alpha}+\dfrac{1}{x^{\beta}}\right)^n$ has constant term, if

(1) $n\alpha$ is divisible by $n + \beta$
(2) $n\beta$ is divisible by $n + \alpha$
(3) $n\alpha$ is divisible by $\alpha + \beta$
(4) n is divisible by $\alpha + \beta$

Sol. Answer (3)

$T_{r+1} = {}^nC_r\,(x^{\alpha})^{n-r}\,(x^{-\beta})^r$

$^nC_r\,x^{\alpha n - r(\alpha+\beta)}$ is independent of x.

If $\alpha n - r(\alpha + \beta) = 0$

$\Rightarrow \alpha n = r(\alpha + \beta)$

$\Rightarrow \alpha n$ is divisible by $(\alpha + \beta)$

14. The number of rational terms in the expansion of $\left(\sqrt[3]{25}+\dfrac{1}{\sqrt[3]{25}}\right)^{20}$ is

(1) 2 (2) 7 (3) 6 (4) 19

Sol. Answer (2)

$$T_{r+1}={}^{20}C_r(25)^{\frac{20-r}{3}}(25)^{-\frac{r}{3}}$$

$$= {}^{20}C_r(25)^{\frac{20-2r}{3}}$$

Hence T_{r+1} to be rational,

$r = 1, 4, 7, 10, 13, 16, 19$

15. If in the expansion of $(1 + kx)^4$ the coefficient of x^3 is 32, then the value of k is equal to

(1) 2 (2) 4 (3) 8 (4) 1

Sol. Answer (1)

$T_4 = T_{3+1} = {}^4C_3\, k^3x^3$.

Then coefficient of $x^3 = 4k^3 = 32$

$\Rightarrow\ k^3 = 8$

$\Rightarrow\ k = 2$

16. In the expansion of $(x + a)^5$, $T_2 : T_3 = 1 : 3$, then $x : a$ is equal to

(1) 1 : 2 (2) 2 : 1 (3) 2 : 3 (4) 3 : 2

Sol. Answer (3)

$$\frac{1}{3} = \frac{T_2}{T_3} = \frac{{}^5C_1x^4a}{{}^5C_2x^3a^2} = \frac{5x}{10a}$$

$$\Rightarrow \frac{x}{a} = \frac{10}{15}$$

$\Rightarrow\ x : a = 2 : 3$

17. If in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ the coefficient of x^7 is equal to the coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, then ab is equal to

(1) 0 (2) 1 (3) –1 (4) 2

Sol. Answer (2)

$$T_{r+1} = {}^{11}C_r(ax^2)^{11-r}.\frac{1}{(bx)^r}$$

$$= {}^{11}C_r\left(\frac{a^{11-r}}{b^r}\right).x^{22-2r-r}$$

$\Rightarrow\ 22 - 3r = 7$

$\Rightarrow\ r = 5$

The coefficient of x^7 in

$$\left(ax^2 + \frac{1}{bx}\right)^{11} = {}^{11}C_5\frac{a^6}{b^5}$$

In the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$

$$T_{r+1} = {}^{11}C_r(ax)^{11-r}\left[\frac{(-1)^r}{(bx^2)^r}\right]$$

$$= (-1)^r\, {}^{11}C_r\frac{a^{11-r}}{b^r}x^{11-3r}$$

$\therefore$ $11 - 3r = -7$

$\Rightarrow$ $r = 6$

The coefficient of x^{-7} is ${}^{11}C_6 \dfrac{a^5}{b^6}$ and ${}^{11}C_5 \dfrac{a^6}{b^5} = {}^{11}C_6 \dfrac{a^5}{b^6}$

$\Rightarrow$ $ab = 1$

18. Coefficient of x^{12} in the expansion of $(1 + x^2)^{50} \left(x + \dfrac{1}{x}\right)^{-10}$ is

(1) 41 (2) 40 (3) 43 (4) 44

Sol. Answer (2)

$$(1+x^2)^{50}\left(\frac{x^2+1}{x}\right)^{-10}$$

$= x^{10}(1 + x^2)^{40}$

$\therefore$ The coefficient of x^{12} is ${}^{40}C_1 = 40$

19. The term independent of x in the expansion of $\left(\sqrt[6]{x} - \dfrac{2}{\sqrt[3]{x}}\right)^{18}$ is

(1) ${}^{18}C_8\, 2^{12}$ (2) ${}^{18}C_6\, 2^6$ (3) ${}^{18}C_6\, 2^8$ (4) ${}^{18}C_8\, 2^8$

Sol. Answer (2)

$$T_{r+1} = {}^{18}C_r x^{\frac{18-r}{6}}.(-2)^r x^{-\frac{r}{3}}$$

$$= {}^{18}C_r(-2)^r x^{\frac{18-3r}{6}}$$

For independent of r

$18 - 3r = 0$

$r = 6$

20 $(1.003)^4$ is nearly equal to

(1) 1.012 (2) 1.0012 (3) 0.988 (4) 1.003

Sol. Answer (1)

$(1.003)^4 = (1 + 0.003)^4 \approx 1 + 4 \times 0.003 = 1.012$

21. The two consecutive term in the expansion of $(3 + 2x)^{74}$ which have equal coefficients, are

(1) 7^{th} and 8^{th} (2) 11^{th} and 12^{th} (3) 30^{th} and 31^{st} (4) 31^{st} and 32^{nd}

Sol. Answer (3)

Suppose that coefficient of $T_r = {}^{74}C_{r-1}(3)^{75-r}(2x)^{r-1}$ and $T_{r+1} = {}^{74}C_r\, 3^{74-r}(2x)^r$ are equal.

$\Rightarrow$ ${}^{74}C_{r-1}\, 3^{75-r}(2)^{r-1} = {}^{74}C_r\, 3^{74-r}(2)^r$

$\Rightarrow$ $r = 30$

Hence, the coefficients of T_{30} and T_{31} are equal.

22. If the coefficient of r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in the expansion of $(1 + x)^{14}$ are in A.P., then the value of r is equal to

(1) 5 or 9 (2) 4 or 7 (3) 3 or 8 (4) 6 or 10

Sol. Answer (1)

If three consecutive coefficients in the expansion of $(1 + x)^n$ are in A.P., then

$$r = \frac{n \pm \sqrt{n+2}}{2} = \frac{14 \pm \sqrt{14+2}}{2}$$

$= 7 \pm 2$

$\Rightarrow$ 9 or 5

23. Given positive integers $r > 1$, $n > 2$ and the coefficients of $(3r)^{th}$ and $(r + 2)^{th}$ term in the expansion of $(1 + x)^{2n}$ are equal, then, which of the following relation is correct?

(1) $n = 2r$ (2) $n = 3r$ (3) $n = 2r + 1$ (4) $n = r$

Sol. Answer (1)

$T_{3r} = {}^{2n}C_{3r-1}x^{3r-1}$ and $T_{r+2} = {}^{2n}C_{r+1}x^{r+1}$

Given ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$3r - 1 = r + 1$ or $3r - 1 = 2n - r - 1$

$\Rightarrow$ $2r = 2$ or $4r = 2n$

$\Rightarrow$ $r = 1$ or $n = 2r$

As $r > 1$ $\therefore$ $n = 2r$

24. The coefficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$ is

(1) 61 (2) 59 (3) 0 (4) 60

Sol. Answer (4)

$(1 + x^2)^5 (1 + x)^4$

Coeff. of $x^5 = {}^5C_1.{}^4C_3 + {}^5C_2.{}^4C_1$

$= 20 + 40 = 60$

25. If in the expansion of $(1 + x)^n$, fifth term is 4 times the fourth term and fourth term is 6 times the third term, then the value of n and x is

(1) 11, 2 (2) 2, 11 (3) 3, 12 (4) 12, 3

Sol. Answer (1)

${}^nC_4x^4 = 4 \times {}^nC_3x^3$ ...(i)

${}^nC_3x^3 = 6 \times {}^nC_2\, x^2$ ...(ii)

On solving $n = 11$, $x = 2$

26. The number of terms in the expansion of $(4x^2 + 9y^2 + 12xy)^6$ is

(1) 2 (2) 12 (3) 13 (4) 28

Sol. Answer (3)

$(2x + 3y)^{12}$ so 13 terms

27. In the binomial expansion of $(a - b)^n$, $n \geq 5$, the sum of the 5^{th} and 6^{th} term is zero, then $\frac{a}{b}$ equals

(1) $\frac{n-5}{6}$ (2) $\frac{n-4}{5}$ (3) $\frac{5}{n-4}$ (4) $\frac{6}{n-5}$

Sol. Answer (2)

$$T_5 + T_6 = {}^nC_4 a^{n-4}(-b)^4 + {}^nC_5 a^{n-5}(-b)^5 = 0$$

$$\Rightarrow \frac{a^{n-4}b^4}{a^{n-5}b^5} = \frac{{}^nC_5}{{}^nC_4} \Rightarrow \frac{a}{b} = \frac{n-5+1}{5} = \frac{n-4}{5}$$

28. The sum of the binomial coefficients of $\left(2x+\frac{1}{x}\right)^n$ is equal to 256. Then the constant term in the expansion is

(1) 1120 (2) 2110 (3) 1210 (4) 2210

Sol. Answer (1)

Sum of the binomial coefficients = $2^n = 256$

$\Rightarrow n = 8$

Let T_{r+1} is the constant term then

$$T_{r+1} = {}^nC_r (2x)^{n-r}\left(\frac{1}{x}\right)^r$$

$$T_{r+1} = {}^nC_r 2^{n-r} x^{n-r} x^{-r} = {}^nC_r 2^{n-r} x^{n-2r}$$

For constant term $n - 2r = 0 \Rightarrow r = \frac{n}{2} = \frac{8}{2} = 4$

Hence the constant term = ${}^8C_4 \,.\, 2^{8-4} = {}^8C_4 \times 16 = 1120$

29. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^{2n}$ are in A.P., then

(1) $2n^2 - 9n + 7 = 0$ (2) $2n^2 + 5n + 7 = 0$ (3) $n^2 - 9n + 7 = 0$ (4) $n^2 + 9n - 7 = 0$

Sol. Answer (1)

$(1 + x)^{2n} = 1 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + {}^{2n}C_3 x^3 + ... + {}^{2n}C_{2n} x^{2n}$

Given that

${}^{2n}C_1, {}^{2n}C_2, {}^{2n}C_3$ are in A.P.

$\Rightarrow 2 \,.\, {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$

$$\Rightarrow 2\cdot\frac{2n(2n-1)}{2} = 2n + \frac{2n(2n-1)(2n-2)}{3\times 2}$$

$$\Rightarrow (2n-1) = 1 + \frac{(2n-1)(2n-2)}{3\times 2}$$

$$\Rightarrow 2n - 2 = \frac{(2n-1)(n-1)}{3}$$

$\Rightarrow 6n - 6 = 2n^2 - 3n + 1$

$\Rightarrow 2n^2 - 9n + 7 = 0$

30. $\displaystyle\sum_{r=0}^{n-1} \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}$ equals

(1) $\frac{n}{2}$ (2) $\frac{n+1}{2}$ (3) $\frac{n(n+1)}{2}$ (4) $\frac{n(n-1)}{2(n+1)}$

Sol. Answer (1)

$$\sum_{r=0}^{n-1}\frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}=\sum_{r=0}^{n-1}\frac{1}{1+\frac{{}^nC_{r+1}}{{}^nC_r}}=\sum_{r=0}^{n-1}\frac{1}{1+\left(\frac{\frac{n!}{(r+1)!(n-r)!}}{\frac{n!}{r!(n-r)!}}\right)}$$

$$=\sum_{r=0}^{n-1}\frac{1}{1+\frac{n-r}{r+1}}=\sum_{r=0}^{n-1}\frac{r+1}{n+1}$$

$$=\frac{1}{n+1}[1+2+3+...+n]=\frac{1}{n+1}\cdot\frac{(n+1)(n)}{2}=\frac{n}{2}$$

31. The greatest term in the expansion of $(3 - 5x)^{15}$, when $x=\frac{1}{5}$, is

(1) 6th (2) 5th (3) 5th & 6th (4) 4th & 5th

Sol. Answer (4)

Let T_{r+1} is the greatest term

$\Rightarrow |T_{r+1}| \geq |T_r|$

$$\left|\frac{T_{r+1}}{T_r}\right| \geq 1$$

$$\Rightarrow \left|\frac{{}^{15}C_r\, 3^{15-r}(-5x)^r}{{}^{15}C_{r-1}\, 3^{16-r}(-5x)^{r-1}}\right| \geq 1$$

$$\Rightarrow \left(\frac{15-r+1}{r}\right)\left|\frac{-5x}{3}\right| \geq 1 \qquad \Rightarrow \left(\frac{16-r}{r}\right)\left|\frac{5x}{3}\right| \geq 1$$

$$\Rightarrow \left(\frac{16-r}{r}\right)\left|\frac{5\times\frac{1}{5}}{3}\right| \geq 1 \qquad \Rightarrow \left(\frac{16-r}{r}\right)\left(\frac{1}{3}\right) \geq 1$$

$16 - r \geq 3r$ $\qquad \Rightarrow 16 \geq 4r$

$r \leq 4$

Hence $r = 4$

Greatest term = T_r, T_{r+1} = T_4, T_5

Hence 4th and 5th term are greatest.

32. In the expansion of $(y^{1/5} + x^{1/10})^{55}$, the number of terms free of radical sign are

(1) 5 (2) 6 (3) 50 (4) 56

Sol. Answer (2)

$T_{r+1} = {}^{55}C_r (y^{1/5})^{55-r}(x^{1/10})^r$

$${}^{55}C_r\, y^{\frac{55-r}{5}}\, x^{\frac{r}{10}}$$

The term is free from radical sign if

$\frac{55-r}{5}$ and $\frac{r}{10}$ both are integers

$\Rightarrow r = 0, 15, 25, 35, 45, 55$ makes both integers hence total 6 terms will be free from radical sign.

33. The term independent of x in $(1+x+2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$ is

(1) 25/54 (2) 17/54 (3) 1/6 (4) –17/54

Sol. Answer (2)

The term independent of x in $(1 + x + 2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

$= \text{Coefficient of } x^\circ \text{ in } \left(\frac{3x^2}{2}-\frac{1}{3x}\right) + \text{Coefficient of } x^{-1} \text{ in } \left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9 + 2\times \text{coefficient of } x^{-3} \text{ in } \left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

To find these coefficients let us find general term in $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$, that is given by

$$T_{r+1} = {}^9C_r\left(\frac{3x^2}{2}\right)^{9-r}\left(-\frac{1}{3x}\right)^r = {}^9C_r(-1)^r\left(\frac{3}{2}\right)^{9-r}\left(\frac{1}{3}\right)^r x^{18-2r-r}$$

$$= {}^9C_r(-1)^r\left(\frac{3}{2}\right)^{9-r}\left(\frac{1}{3}\right)^r x^{18-3r}$$

For constant term $18 - 3r = 0 \Rightarrow r = 6$

For coefficient of x^{-1}, $18 - 3r = -1 \Rightarrow 19 = 3r$

$\Rightarrow r = \frac{19}{3}$ which is not an integer, hence coefficient in this case is zero.

For coefficient of x^{-3}, $18 - 3r = -3$

$3r = 21$

$r = 7$

$\therefore$ The term independent of x

$$= {}^9C_6\left(\frac{3}{2}\right)^3\cdot\left(-\frac{1}{3}\right)^6 + 2\cdot{}^9C_7\left(\frac{3}{2}\right)^2\cdot\left(-\frac{1}{3}\right)^7$$

$$= \frac{17}{54}$$

34. Find the coefficient of x^3 in the expansion of $(1+x+2x^2)\left(2x^2-\frac{1}{3x}\right)^9$.

(1) $-\frac{224}{9}$ (2) $-\frac{112}{27}$ (3) $-\frac{224}{27}$ (4) $-\frac{112}{27}$

Sol. Answer (3)

The expression is

$$(1+x+2x^2)\left(2x^2-\frac{1}{3x}\right)^9$$

$$=\left(2x^2-\frac{1}{3x}\right)^9+x\left(2x^2-\frac{1}{3x}\right)^9+2x^2\left(2x^2-\frac{1}{3x}\right)^9$$

In the expansion of

$$\left(2x^2-\frac{1}{3x}\right)^9$$

General term

$$T_{r+1}={}^9C_r\cdot(2x^2)^{9-r}\cdot\left(-\frac{1}{3x}\right)^r$$

$$={}^9C_r\cdot 2^{18-2r}\cdot x^{9-r}\cdot\left(-\frac{1}{3}\right)^r\cdot x^{-r}$$

$$={}^9C_r\cdot 2^{9-r}\cdot\left(-\frac{1}{3}\right)^r\cdot x^{18-3r}$$

$\because$ we are looking for the coefficient of x^3

$\therefore$ $18-3r=3$

$\Rightarrow$ $3r=15$

$\Rightarrow$ $r=5$

For the coefficient of x^2,

$18-3r=2$

$\Rightarrow r=\frac{16}{3}$ (Not possible)

For the coefficient of x

$18-3r=1$

$\Rightarrow r=\frac{17}{3}$ (Not possible)

Now, the coefficient of x^3 in

$$\left(2x^2-\frac{1}{3x}\right)^9+x\left(2x^2-\frac{1}{3x}\right)^9+2x^2\left(2x^2-\frac{1}{3x}\right)^9$$

= Coefficient of x^3 in $\left(2x^2-\frac{1}{3x}\right)^9$ +

coefficient of x^2 $\left(2x^2-\frac{1}{3x}\right)^9$ in + 2 × Coefficient of x in $\left(2x^2-\frac{1}{3x}\right)^9$

$$={}^9C_5\cdot 2^{9-5}\cdot\left(-\frac{1}{3}\right)^5+0+2(0)$$

$$=\frac{9\times8\times7\times6}{4\times3\times2}\cdot 2^4\cdot-\frac{1}{3^5}=-\frac{224}{27}$$

35. If 6^{th} term of $\left[2^{\log_2\sqrt{9^{x-1}+7}}+\dfrac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right]^7$ is 84, then x =

(1) 4 or 3 (2) 3 or 1 (3) 2 or 1 (4) 1

Sol. Answer (3)

$$T_6 = T_{5+1} = {}^7C_5\left(2^{\log_2\sqrt{9^{x-1}+7}}\right)^2\left(\frac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right)^5 = 84$$

$${}^7C_5(9^{x-1}+7)\cdot\left(\frac{1}{3^{x-1}+1}\right) = 84$$

$$\frac{7\times 6}{2}(9^{x-1}+7)\cdot\left(\frac{1}{3^{x-1}+1}\right) = 84$$

$\dfrac{9^{x-1}+7}{3^{x-1}+1} = 4 \Rightarrow 9^{x-1} + 7 = 4(3^{x-1} + 1)$

$$\Rightarrow \frac{9^x}{9} + 7 = 4\,(3^{x-1}+1)$$

$$9^x + 63 = 36\left(\frac{3^x}{3}+1\right)$$

$\Rightarrow 9^x + 63 = 12.3^x + 36$

Let $3^x = y$

$y^2 = 63 = 12y + 36$

$y^2 - 12y + 27 = 0$

$\Rightarrow y = 3,\ y = 9$

$\Rightarrow 3^x = 3$ or $9 \Rightarrow x = 1, 2$

[Properties of Binomial Coefficients, Differentiation and Integration, Multinomial Theorem]

36. The number of terms which are not similar in the expansion of $(l + m + n)^6$ is

(1) 7 (2) 42 (3) 28 (4) 21

Sol. Answer (3)

$(\ell + m + n)^6 = \{\ell + (m + n)\}^6$

$= {}^6C_0\ell^6 + {}^6C_1\ell^5(m+n) + {}^6C_2\ell^4(m+n)^2 + {}^6C_3\ell^3(m+n)^3 + {}^6C_4\ell^2(m+n)^4 + {}^6C_5\ell(m+n)^5 + {}^6C_6(m+n)^6$ ...(1)

Since the expansion $(m + n)^k$ contains $(k + 1)$ terms therefore

R.H.S. of (1) there are

$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ terms

37. If sum of the coefficients in the expansion of $(2x + 3y - 2z)^n$ is 2187 then the greatest coefficient in the expansion of $(1 + x)^n$

(1) 30 (2) 40 (3) 28 (4) 35

Sol. Answer (4)

Sum is obtained by putting $x = y = z = 1$

Sum = $(2 + 3 - 2)^n$

$\Rightarrow \; 3^n = 2187 \Rightarrow n = 7$

Greatest coefficient = ${}^7C_3 = {}^7C_4 = \dfrac{7\times6\times5}{3\times2\times1} = 35$

38. If $(1 + x)^n = \sum_{r=0}^{n} C_r.x^r$, then $\left(1+\dfrac{C_1}{C_0}\right)\left(1+\dfrac{C_2}{C_1}\right)......\left(1+\dfrac{C_n}{C_{n-1}}\right) =$

(1) $\dfrac{n^{n-1}}{(n+1)!}$ (2) $\dfrac{(n+1)^{n+1}}{(n+1)!}$ (3) $\dfrac{(n+1)^n}{n!}$ (4) $\dfrac{(n+1)^{n+1}}{n!}$

Sol. Answer (3)

The general term = $\left(1+\dfrac{C_r}{C_{r-1}}\right) = T_r$

The given product can be written as $T_1\,T_2\,T_3\,.....\,T_n$

$$T_r = 1+\frac{C_r}{C_{r-1}} = 1+\frac{{}^nC_r}{{}^nC_{r-1}} = 1+\frac{n-r+1}{r} = \frac{n+1}{r}$$

$$\text{Product} = \frac{(n+1)}{1}\cdot\frac{(n+1)}{2}......\frac{(n+1)}{n} = \frac{(n+1)^n}{n!}$$

39. The sum of the last eight coefficients in the expansion of $(1 + x)^{16}$ is equal to

(1) 2^{15} (2) 2^{14} (3) $2^{15} - \dfrac{1}{2}.\dfrac{16!}{(8!)^2}$ (4) 2^{16}

Sol. Answer (3)

$(1+x)^{16} = C_0 + C_1x + C_2x^2 + ... + C_{16}x^{16}$, where $C_r = {}^{16}C_r$

Let $S = C_{16} + C_{15} + \ldots + C_9$

Again $2S = 2C_{16} + 2C_{15} + \ldots + 2C_9$

$= (C_{16} + C_{16}) + (C_{15} + C_{15}) + \ldots. (C_9 + C_9)$

$= (C_0 + C_{16}) + (C_1 + C_{15}) + \ldots. (C_7 + C_9)$

$= (C_0 + C_1 + \ldots\ldots + C_7 + C_8 + \ldots. + C_{16})\;\; C_8$

$= 2^{16} - C_8 = 2^{16} - \dfrac{16!}{8!8!}$

$S = 2^{15} - \dfrac{1}{2}\dfrac{16!}{8!8!}$

40. If n is an integer greater than 1, then $a - {}^nC_1(a-1) + {}^nC_2(a-2) - .... + (-1)^n(a-n) =$

(1) a (2) 0 (3) a^2 (4) 2^n

Sol. Answer (2)

The given expression can be written as

$$a(1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n\,{}^nC_n) + ({}^nC_1 - 2.\,{}^nC_2 + 3.\,{}^nC_3 - 4\,{}^nC_4 + ...)$$

$= a \times 0 + 0 = 0$

41. If $k = \sum_{r=0}^{n} \frac{1}{^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{^nC_r}$ is equal to

(1) nk (2) $\frac{nk}{2}$ (3) $(n-1)k$ (4) $\frac{nk}{3}$

Sol. Answer (2)

$$k = \sum_{r=0}^{n} \frac{1}{^nC_r} \qquad \ldots (i)$$

$$\sum_{r=0}^{n} \frac{r}{^nC_r} = \sum_{r=0}^{n} \frac{r-n+n}{^nC_r} = \sum_{r=0}^{n} \frac{n}{^nC_r} - \sum_{r=0}^{n} \frac{n-r}{^nC_r}$$

$$\Rightarrow \sum_{r=0}^{n} \frac{r}{^nC_r} = nk - \sum_{r=0}^{n} \frac{(n-r)}{^nC_{n-r}} \text{ by (i)}$$

$$= nk - \sum_{r=0}^{n} \frac{r}{^nC_r}$$

$$\Rightarrow 2\sum_{r=0}^{n} \frac{r}{^nC_r} = nk \Rightarrow \sum_{r=0}^{n} \frac{r}{^nC_r} = \frac{nk}{2}$$

42. $\sum_{r=1}^{n} r\,(^nC_r - {^nC_{r-1}})$ is equal to

(1) $2^n + n + 1$ (2) $2^n - n + 1$ (3) $n - 2^n + 1$ (4) $n - 2^n - 1$

Sol. Answer (3)

$$S = \sum_{r=1}^{n} r(^nC_r - {^nC_{r-1}}) = \sum_{r=1}^{n} r\,^nC_r - \sum_{r=1}^{n} r.\,^nC_{r-1} = n.2^{n-1} - \sum_{r=1}^{n} r.\,^nC_{r-1}$$

$$S = n.2^{n-1} - [^nC_0 + 2.\,^nC_1 + 3.\,^nC_2 + 4.\,^nC_3 + ... + n.\,^nC_{n-1}] \qquad \ldots (i)$$

Let $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + ...C_nx^n$ where $C_r = {^nC_r}$

$\Rightarrow x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_nx^{n+1}$

Differentiating

$(1 + x)^n + n(1 + x)^{n-1} = C_0 + 2C_1x + 3C_2x^2 + ...(n+1)C_nx^n$

Putting $x = 1$

$2^n + n\,.\,2^{n-1} = C_0 + 2C_1 + 3C_2 + \ldots.\ (n + 1)C_n$

$\Rightarrow 2^n + n\,.\,2^{n-1} = C_0 + 2C_1 + 3C_2 + \ldots.\ nC_{n-1} + (n + 1)C_n$

$\Rightarrow (C_0 + 2C_1 + 3C_2 + \ldots\ nC_{n-1}) = 2^n + n.2^{n-1} - (n + 1)C_n \ldots\ldots$ …(ii)

If $C_r = {^nC_r}$

Then we may write that

$^nC_0 + 2.^nC_1 + 3.^nC_2 + ...n.^nC_{n-1}$

$= 2^4 + n.2^{n-1} - (n+1)C_n$ ...(iii)

By (i) and (iii)

$S = n.2^{n-1} - [2^n + n.2^{n-1} - n - 1]$

$S = n + 1 - 2^n$

43. The coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$ is

(1) 900 (2) 909 (3) 990 (4) 999

Sol. Answer (3)

$$S = (1+x+x^2+x^3)^{11} = \left(\frac{1-x^4}{1-x}\right)^{11} = (1-x^4)^{11}(1-x)^{-11}$$

$$= (1 + {}^{11}C_1(-x)^4 + ...)\left(1+11x+\frac{11\times 12}{2!}x^2+....\right)$$

Coefficient of $x^4 = \dfrac{11\times 12\times 13\times 14}{4!} - 11 = 990$

[Miscellaneous]

44. The number of zeroes at the end of $(101)^{11} - 1$ is

(1) 8 (2) 4 (3) 6 (4) 2

Sol. Answer (4)

$(101)^{11} - 1 = (1 + 100)^{11} - 1$

$= 1 + {}^{11}C_1(100) + {}^{11}C_2(100)^2 + ... + {}^{11}C_{11}(100)^{11} - 1$

$= 100[{}^{11}C_1 + {}^{11}C_2 100 + {}^{11}C_3(100)^2 + ... + {}^{11}C_{11}(100)^{10}]$

$= 100$ (odd number)

$\therefore$ The number of zeros of the end is 2.

45. If $3^{2n+2} - 8n - 9$ is divided by 64, then remainder is

(1) 1 (2) 0 (3) 7 (4) 2

Sol. Answer (2)

$$(3)^{2(n+1)} - 8n - 9 = (8+1)^{n+1} - 8n - 9$$

$$= {}^{n+1}C_0 8^{n+1} + {}^{n+1}C_1 8^n + ... + {}^{n+1}C_n 8 + {}^{n+1}C_{n+1} - 8n - 9$$

$$= 64({}^{n+1}C_0 8^{n-1} + ...) + 8n + 8 - 8n - 9$$ have remainder = 0

46. 2^{60} when divided by 7, leaves the remainder

(1) 1 (2) 6 (3) 5 (4) 2

Sol. Answer (1)

$$(2^3)^{20} = (7+1)^{20} = {}^{20}C_0 7^{20} + {}^{20}C_1 7^{14} + ... + {}^{20}C_{19} 7 + {}^{20}C_{20} 7^0$$

$$= 7({}^{20}C_0 7^{19} + ... + {}^{20}C_1) + 1$$

Remainder = 1

47. Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$ where $[\cdot]$ denotes the greatest integer function. The value of $R.f$ is

(1) 4^{2n+1} (2) 4^{2n} (3) 4^{2n-1} (4) 4^{-2n}

Sol. Answer (1)

$$R = \left(5\sqrt{5}+11\right)^{2n+1} = I + f$$

Where I is the integral part of R and f is the fractional part of R.

Hence $0 \le f < 1$ … (i)

Let us consider $f' = (5\sqrt{5} - 11)^{2n+1}$

Here also $0 < f' < 1$ …(ii)

$I + f = (5\sqrt{5} + 11)^{2n+1}$

$= {}^{2n+1}C_0(5)^{2n+1} + {}^{2n+1}C_1(5)^{2n}(11) + \ldots + {}^{2n+1}C_{2n+1}(11)^{2n+1}$ …(iii)

$f' = (5 - 11)^{2n+1}$

$= {}^{2n+1}C_0(5)^{2n+1} - {}^{2n+1}C_1(5)^{2n}(11) + \ldots(-1)^{2n+1}\, C_{2n+1}(11)^{2n+1}$ …(iv)

Adding (iii) and (iv)

$I + f - f' = 2[{}^{2n+1}C_1(5\sqrt{5})^{2n}(11) + {}^{2n+1}C_3(5\sqrt{5})^{2n-2}(11)^3 + \ldots.]$

$I + f - f'$ = Even integer

$\Rightarrow$ $I + f - f'$ = Integer

But I is an integer, hence $f - f'$ is an integer.

Also we know

$0 \le f < 1$ …(v)

$0 < f' < 1$

$\Rightarrow$ $-1 < -f' < 0$ …(vi)

Adding (v) and (vi)

$-1 < f - f' < 1$

But $f - f'$ is an integer hence $f - f' = 0 \Rightarrow f = f'$

$\Rightarrow$ $Rf = Rf' = (5 + 11)^{2n+1}(5 - 11)^{2n+1} = (125 - 121)^{2n+1} = 4^{2n+1}$

48. Consider the following statements

S_1: The total number of terms in $(x^2 + 2x + 4)^{10}$ is 21

S_2: The coefficient of x^{10} in $\left(x^2 + \frac{1}{x}\right)^{20}$ is ${}^{20}C_{10}$.

S_3: The middle term in the expansion of $(1 + x)^{12}$ is ${}^{12}C_6x^6$

S_4: If the coefficients of fifth and ninth term in the expansion of $(1 + x)^n$ are same, then $n = 12$

Now identify the correct combination of true statements.

(1) S_1, S_2, S_3, S_4 (2) S_1, S_2 only (3) S_2, S_3 only (4) S_1, S_4 only

Sol. Answer (1)

S_1: $(x^2 + 2x + 1 + 3)^{10} = [(x+1)^2 + 3]^{10}$

$$= {}^{10}C_0(x+1)^{20} + {}^{10}C_1(x+1)^{18}.3 + {}^{10}C_2(x+1)^{16}.3^2 + \ldots\ldots\ldots. + {}^{10}C_{10}3^{10}$$

Total number of different terms = 21

S_{-2}: Let $(r+1)^{th}$ term has x^{10}

Then $T_{r+1} = {}^{20}C_r(x^2)^{20-r}\left(\frac{1}{x}\right)^r = {}^{20}C_r x^{40-2r} \times \frac{1}{x^r}$

$x^{40-3r} = x^{10} \Rightarrow 40-3r = 10 \Rightarrow 30 = 3r \Rightarrow r = 10$

Therefore coefficient of x^{10} is ${}^{20}C_{10}$.

S_3 : Total number of terms in the expansion of $(1 + x)^{12}$ is 13

Therefore middle term is $\left(\frac{n}{2}+1\right)^{th} = (6+1)^{th}$ term

$T_{6+1} = {}^{12}C_6 x^6$

S_4: $T_{4+1} = T_{8+1}$

$\Rightarrow {}^nC_4 = {}^nC_8$

$\Rightarrow {}^nC_{n-4} = {}^nC_8$

$\Rightarrow n - 4 = 8$

$\Rightarrow n = 12$

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