MAXIMA & MINIMA

1. Let $f : (0,1) \to \mathbb{R}$ be the function defined as $f(x) = [4x]\left(x - \frac{1}{4}\right)^2\left(x - \frac{1}{2}\right)$, where $[x]$ denotes the greatest integer less than or equal to x. Then which of the following statements is(are) true?

[JEE(Advanced) 2023]

(A) The function f is discontinuous exactly at one point in (0,1)

(B) There is exactly one point in (0,1) at which the function f is continuous but **NOT** differentiable

(C) The function f is **NOT** differentiable at more than three points in (0, 1)

(D) The minimum value of the function f is $-\frac{1}{512}$

2. Let $\alpha = \sum_{k=1}^{\infty} \sin^{2k}\left(\frac{\pi}{6}\right)$.

Let $g : [0, 1] \to \mathbb{R}$ be the function defined by

$g(x) = 2^{\alpha x} + 2^{\alpha(1-x)}$

Then, which of the following statements is/are TRUE? **[JEE(Advanced) 2022]**

(A) The minimum value of $g(x)$ is $2^{\frac{7}{6}}$

(B) The maximum value of $g(x)$ is $1 + 2^{\frac{1}{3}}$

(C) The function $g(x)$ attains its maximum at more than one point

(D) The function $g(x)$ attains its minimum at more than one point

Question Stem for Questions Nos. 3 and 4

Question Stem

Let $f_1 : (0, \infty) \to \mathbb{R}$ and $f_2 : (0, \infty) \to \mathbb{R}$ be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t - j)^j \, dt, \quad x > 0$$

and

$$f_2(x) = 98(x - 1)^{50} - 600(x - 1)^{49} + 2450, \quad x > 0,$$

where, for any positive integer n and real numbers $a_1, a_2, \ldots, a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots, a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0, \infty)$

3. The value of $2m_1 + 3n_1 + m_1n_1$ is _____. **[JEE(Advanced) 2021]**

4. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____. **[JEE(Advanced) 2021]**

5. Consider all rectangles lying in the region

$$\left\{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \le x \le \frac{\pi}{2} \text{ and } 0 \le y \le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is **[JEE(Advanced) 2020]**

(A) $\frac{3\pi}{2}$ (B) π (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

6. Let the function $f: (0, \pi) \to \mathbb{R}$ be defined by

$$f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$$

Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1\pi, \ldots, \lambda_r\pi\}$, where $0 < \lambda_1 < \ldots. < \lambda_r < 1$. Then the value of $\lambda_1 + \ldots + \lambda_r$ is _____.

[JEE(Advanced) 2020]

7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$$

Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) f' has a local maximum at $x = 1$

(B) f is onto

(C) f is increasing on $(-\infty, 0)$

(D) f' is NOT differentiable at $x = 1$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x - 1)(x - 2)(x - 5)$. Define $F(x) = \int_0^x f(t)\,dt$, $x > 0$. Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) F has a local minimum at $x = 1$

(B) F has a local maximum at $x = 2$

(C) $F(x) \neq 0$ for all $x \in (0, 5)$

(D) F has two local maxima and one local minimum in $(0, \infty)$

9. Let $f(x) = \frac{\sin \pi x}{x^2}$, $x > 0$

Let $x_1 < x_2 < x_3 < \ldots < x_n < \ldots$ be all the points of local maximum of f

and $y_1 < y_2 < y_3 < \ldots < y_n < \ldots$ be all the points of local minimum of f.

Then which of the following options is/are correct? **[JEE(Advanced) 2019]**

(A) $|x_n - y_n| > 1$ for every n

(B) $x_1 < y_1$

(C) $x_n \in \left(2n, 2n + \frac{1}{2}\right)$ for every n

(D) $x_{n+1} - x_n > 2$ for every n

10. For every twice differentiable function $f : \mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) There exist $r, s \in \mathbb{R}$, where $r < s$, such that f is one-one on the open interval (r, s)

(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$

(C) $\lim_{x \to \infty} f(x) = 1$

(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

11. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then **[JEE(Advanced) 2017]**

(A) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

(B) $f(x)$ attains its maximum at $x = 0$

(C) $f(x)$ attains its minimum at $x = 0$

(D) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

12. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is - **[JEE(Advanced) 2016]**

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

13. Let $f : \mathbb{R} \to (0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable function such that f'' and g'' ar continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim\limits_{x\to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

[JEE(Advanced) 2016]

(A) f has a local minimum at $x = 2$

(B) f has a local maximum at $x = 2$

(C) $f''(2) > f(2)$

(D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

14. A cylindrical container is to be made from certain solid material with the following constraints. It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10mm, then the value of $\frac{V}{250\pi}$ is **[JEE(Advanced) 2015]**

SOLUTIONS

1. Ans. (A, B)

Sol. $$f(x)=\begin{cases}0 & ; \ 0<x<\frac{1}{4}\\ \left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & ; \ \frac{1}{4}\le x<\frac{1}{2}\\ 2\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & ; \ \frac{1}{2}\le x<\frac{3}{4}\\ 3\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & ; \ \frac{3}{4}\le x<1\end{cases}$$

f(x) is discontinuous at $x=\frac{3}{4}$ only

$$f'(x)=\begin{cases}0 & ; \ 0<x<\frac{1}{4}\\ 2\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+\left(x-\frac{1}{4}\right)^2 & ; \ \frac{1}{4}<x<\frac{1}{2}\\ 4\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+2\left(x-\frac{1}{4}\right)^2 & ; \ \frac{1}{2}<x<\frac{3}{4}\\ 6\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+3\left(x-\frac{1}{4}\right)^2 & ; \ \frac{3}{4}<x<1\end{cases}$$

f(x) is non-differentiable at $x=\frac{1}{2}$ and $\frac{3}{4}$

minimum values of f(x) occur at $x=\frac{5}{12}$ whose value is $-\frac{1}{432}$

2. Ans. (A, B, C)

Sol. $\alpha=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^6+...$

$$\alpha=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{3}$$

$\therefore$ $g(x)=2^{x/3}+2^{1/3(1-x)}$

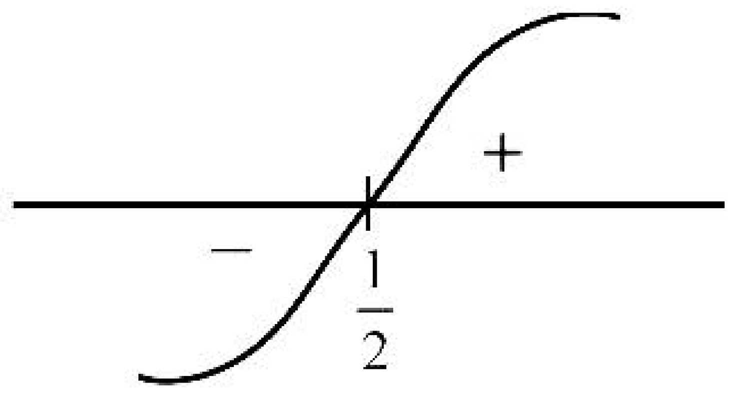
$\therefore$ $g(x)=2^{x/3}+\frac{2^{1/3}}{2^{x/3}}$

where $g(0)=1+2^{1/3}$ & $g(1)=1+2^{1/3}$

$\therefore$ $g'(x)=\frac{1}{3}\left(2^{x/3}-\frac{2^{1/3}}{2^{x/3}}\right)=0$

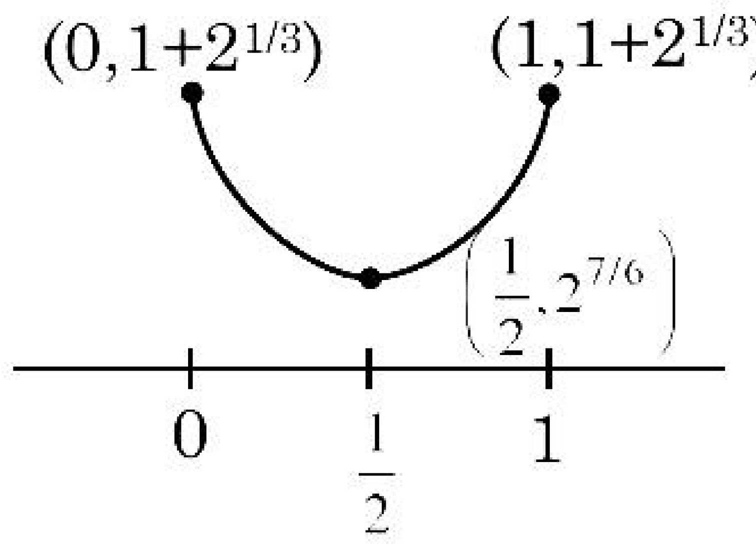
$\Rightarrow 2^{2x/3}=2^{1/3}\Rightarrow x=\frac{1}{2}$ = critical point

$\therefore$ graph of g'(x) =



& $g\left(\frac{1}{2}\right)=2^{\frac{7}{6}}$

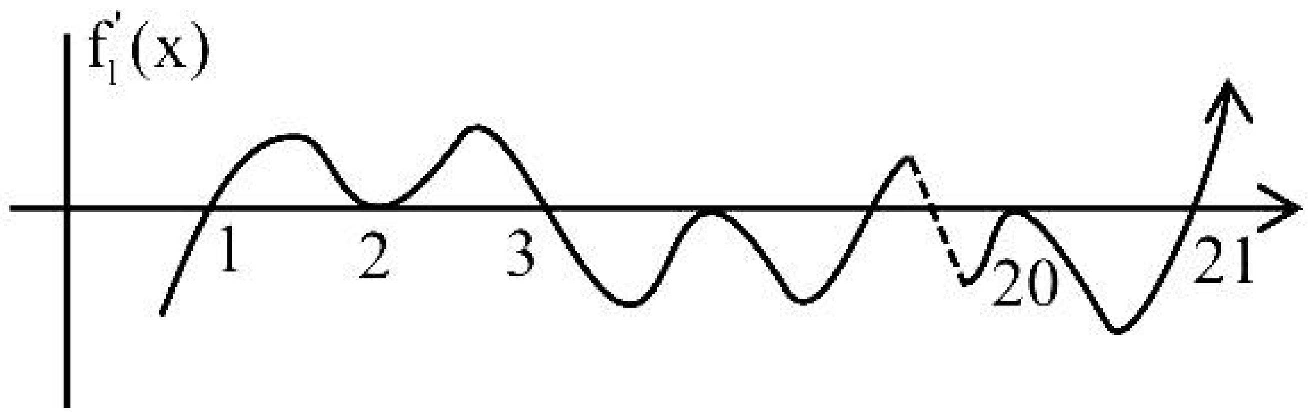
$\therefore$ graph of g(x) in [0, 1]



3. Ans. (57.00)

Sol. $f_1(x)=\int_0^x\prod_{j=1}^{21}(t-j)^j\,dt$

$f_1'(x)=\prod_{j=1}^{21}(x-j)^j=(x-1)(x-2)^2(x-3)^3.....(x-21)^{21}$



So points of minima one 4m + 1 where

m = 0, 1,.....5 $\Rightarrow m_1=6$

Points of maxima are 4m – 1 where

m = 1, 2,.....5 $\Rightarrow n_1=5$

$\Rightarrow 2m_1+3n_1+m_1n_1=57$

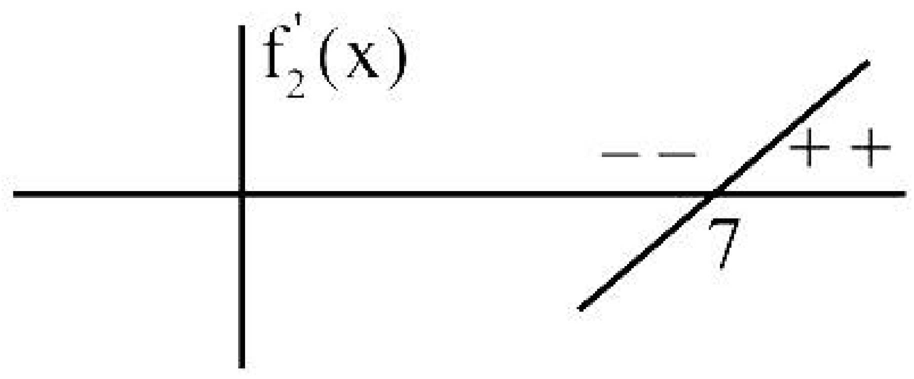
4. Ans. (6.00)

Sol. $f_2'(x)=98(x-1)^{50}-600(x-1)^{49}+2450$

$\Rightarrow f_2'(x)=2\times 49\times 50(x-1)^{49}-50\times 12\times 49(x-1)^{48}$

$=50\times 49\times 2(x-1)^{48}(x-1-6)$

$=50\times 49\times 2\,(x-1)^{48}\,(x-7)$



Point of minima = 7

$\Rightarrow m_2=1$

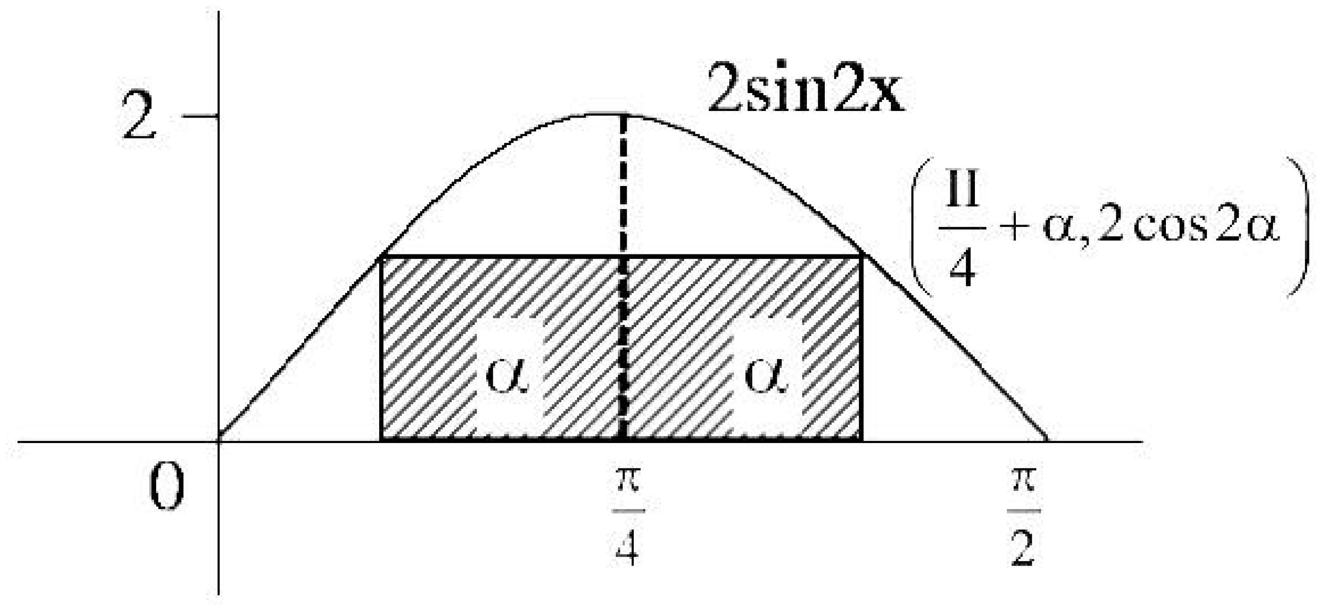
No point of maxima

$\Rightarrow n_2=0$

$6m_2+4n_2+8m_2n_2=6$

5. **Ans. (C)**

Sol.



Perimeter = $2(2\alpha + 2\cos 2\alpha)$

$P = 4(\alpha + \cos 2\alpha)$

$\frac{dP}{d\alpha} = 4(1 - 2\sin 2\alpha) = 0$

$\sin 2\alpha = \frac{1}{2}$

$2\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$

$\frac{d^2P}{d\alpha^2} = -4\cos 2\alpha$

for maximum $\alpha = \frac{\pi}{12}$

Area = $(2\alpha)(2\cos 2\alpha)$

$= \frac{\pi}{6} \times 2 \times \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$

6. **Ans. (0.50)**

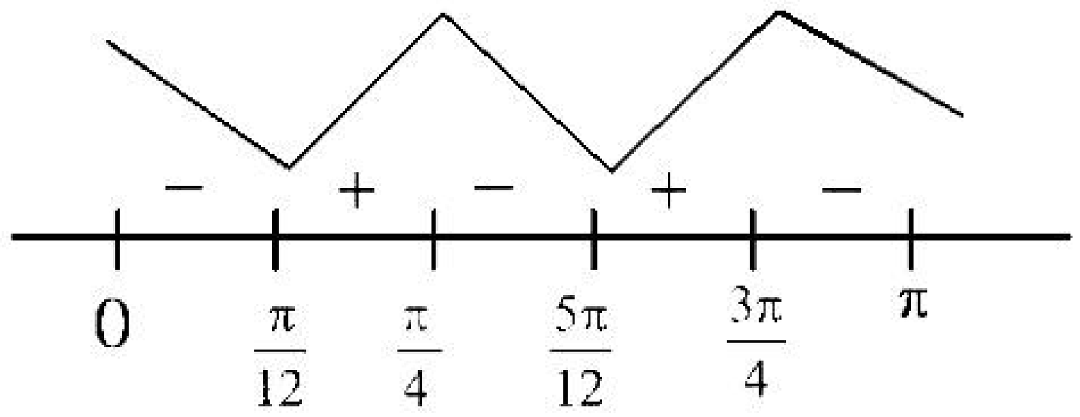
Sol. $f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$

$f(\theta) = \sin^2 2\theta - \sin 2\theta + 2$

$f'(\theta) = 2(\sin 2\theta).(2\cos 2\theta) - 2\cos 2\theta$

$= 2\cos 2\theta(2\sin 2\theta - 1)$

critical points



so, minimum at $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$\lambda_1 + \lambda_2 = \frac{1}{12} + \frac{5}{12} = \frac{6}{12} = \frac{1}{2}$

7. **Ans. (A, B, D)**

Sol.

$$f(x) = \begin{cases} (x+1)^5 - 2x, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$$

for $x < 0$, $f(x)$ is continuous

& $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to 0^-} f(x) = 1$

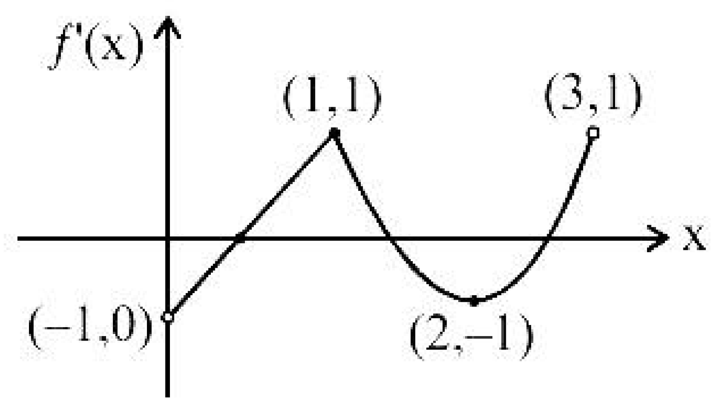
Hence, $(-\infty, 1) \subset$ Range of $f(x)$ in $(-\infty, 0)$

$f'(x) = 5(x+1)^4 - 2$, which changes sign in $(-\infty, 0)$

$\Rightarrow f(x)$ is non-monotonic in $(-\infty, 0)$

For $x \ge 3$, $f(x)$ is again continuous and

$\lim_{x\to\infty} f(x) = \infty$ and $f(3) = \frac{1}{3}$



$\Rightarrow \left[\frac{1}{3}, \infty\right) \subset$ Range of $f(x)$ in $[3, \infty)$

Hence, range of $f(x)$ is $\mathbb{R}$

$$f'(x) = \begin{cases} 2x - 1, & 0 \le x < 1 \\ 2x^2 - 8x + 7, & 1 \le x < 3 \end{cases}$$

Hence f' has a local maximum at $x = 1$ and f' is NOT differentiable at $x = 1$.

8. **Ans. (A,B,C)**

Sol. $f(x) = (x-1)(x-2)(x-5)$

$F(x) = \int_0^x f(t)\,dt, x > 0$

$F'(x) = f(x) = (x-1)(x-2)(x-5), x > 0$

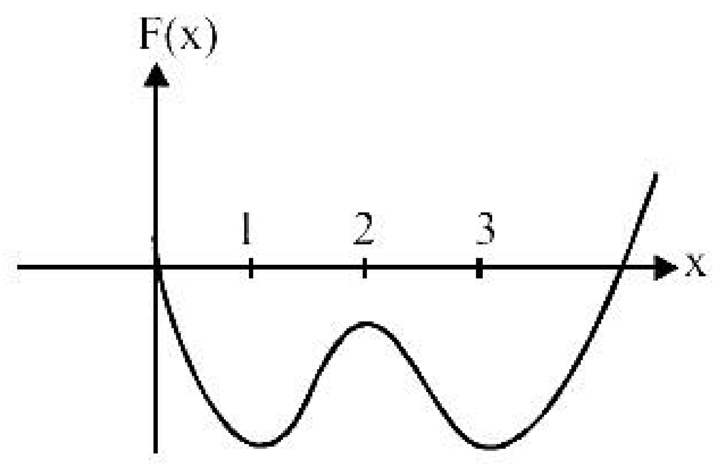
clearly F(x) has local minimum at x = 1,5

F(x) has local maximum at x = 2

$f(x) = x^3 - 8x^2 + 17x - 10$

$$\Rightarrow F(x) = \int_0^x (t^3 - 8t^2 + 17t - 10)\,dt$$

$$F(x) = \frac{x^4}{4} - \frac{8x^3}{3} + \frac{17x^2}{2} - 10x$$

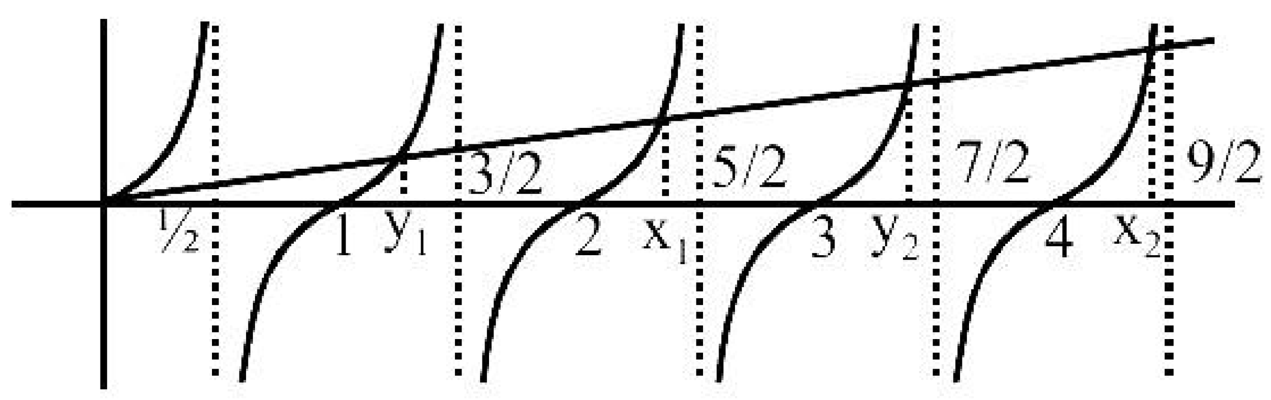


from the graph of y = F(x),

clearly $F(x) \neq 0\ \forall\ x \in (0, 5)$

9. Ans. (A, C, D)

Sol. $f(x) = \frac{\sin \pi x}{x^2}$



$$f'(x) = \frac{2x\cos \pi x\left(\frac{\pi x}{2} - \tan \pi x\right)}{x^4}$$

$\Rightarrow \quad |x_n - y_n| > 1$ for every n

$x_1 > y_1$

$x_n \in (2n, 2n + 1/2)$

$x_{n+1} - x_n > 2.$

10. Ans. (A, B, D)

Sol. $f(x)$ can't be constant throughout the domain. Hence we can find $x \in (r, s)$ such that $f(x)$ is one-one

option (A) is true.

Option (B) :

$$|f'(x_0)| = \left|\frac{f(0) - f(-4)}{4}\right| \leq 1 \text{ (LMVT)}$$

Option (C) :

$f(x) = \sin(\sqrt{85}x)$ satisfies given condition

but $\quad \lim\limits_{x\to\infty} \sin(\sqrt{85})$ D.N.E.

$\Rightarrow \quad$ Incorrect

Option (D) : $g(x) = f^2(x) + (f'(x))^2$

By LMVT $\exists\ x_1 \in (-4, 0)$ such that $|f'(x_1)| \leq 1$

$|f(x_1)| \leq 2 \quad$ (given)

$\Rightarrow \quad g(x_1) \leq 5$

Similarly, we can find some $x_2 \in (0,4)$ such that $g(x_2) \leq 5$

$g(0) = 85 \Rightarrow g(x)$ has maxima in (x_1,x_2) say at α.

$g'(\alpha) = 0$ & $g(\alpha) \geq 85$

$2f'(\alpha)\,(f(\alpha) + f''(\alpha)) = 0$

If $f'(\alpha) = 0 \Rightarrow g(\alpha) = f^2(\alpha) \geq 85$ Not possible

$\Rightarrow \quad f(\alpha) + f''(\alpha) = 0$

$\quad \exists \alpha \in (x_1, x_2) \in (-4, 4)$

option (D) correct.

11. Ans. (B, D)

Sol. Expansion of determinant

$f(x) = \cos 2x + \cos 4x$

$f'(x) = -2\sin 2x - 4\sin 4x = -2\sin x(1 + 4\cos 2x)$

+	−
0	

$\therefore$ maxima at x = 0

$f'(x) = 0 \Rightarrow$

$$x = \frac{n\pi}{2},\ \cos 2x = -\frac{1}{4}$$

$\Rightarrow$ more than two solutions

12. Ans. (C)

Sol. $f(x) = 4\alpha x^2 + \frac{1}{x}; x > 0$

$$f'(x) = 8\alpha x - \frac{1}{x^2}$$

$$= \frac{8\alpha x^3 - 1}{x^2}$$

$f(x)$ attains its minimum at $x = \left(\frac{1}{8\alpha}\right)^{1/3}$

$$f\left(\left(\frac{1}{8\alpha}\right)^{1/3}\right) = 1$$

$$\Rightarrow 4\alpha\left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} = 1$$

$$\Rightarrow 3\alpha^{1/3} = 1 \Rightarrow \alpha = \frac{1}{27}$$

13. Ans. (A, D)

Sol. Using L'Hôpital's Rule

$$\lim_{x\to 2}\frac{f'(x)g(x)+f(x)g'(x)}{f''(x)g'(x)+f'(x)g''(x)}=1$$

$$\Rightarrow \frac{f(2)g'(2)}{f''(2)g'(2)}=1 \Rightarrow \quad f'(2)=f(2)>0$$

option (D) is right and option (C) is wrong

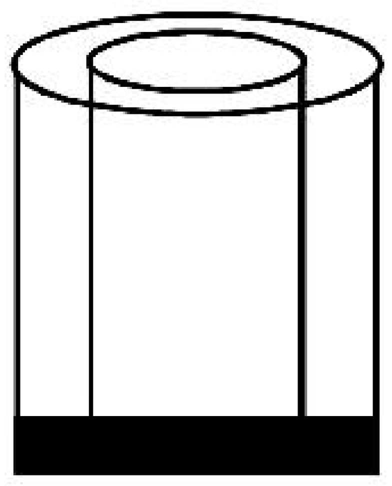
also $f'(2) = 0$ and $f''(2) > 0$

$\therefore$ x = 2 is local minima.

14. Ans. (4)

Sol. Let the inner radius of cylinderical container be r then radius of outer cylinder is (r + 2).

Now, $V = \pi r^2 h$ where h is height of cylinder



Now, Let volume of total material used be T

$$\therefore\ T=\pi\left(\left(r+2\right)^2-r^2\right).\frac{V}{\pi r^2}+\pi\left(r+2\right)^2.2$$

$$\therefore\ T=V\left(\frac{r+2}{r}\right)^2+2\pi\left(r+2\right)^2-V$$

$$\text{Now}\ \frac{dT}{dr}=2V\left(\frac{r+2}{r}\right)\times\left(-\frac{2}{r^2}\right)+4\pi\left(r+2\right)$$

$$\text{Now At } r=10\text{mm}\ \frac{dT}{dr}=0$$

$$\therefore\ 0=\left(r+2\right).4\left(\pi-\frac{V}{r^3}\right)$$

$$\Rightarrow \frac{V}{\pi}=1000\Rightarrow\frac{V}{250\pi}=4$$