Chapter 12

Progressions

CHAPTER HIGHLIGHTS

- *Progressions*
- *Arithmetic Progression (AP)*
- *Geometric Progression (GP)*
- *Infinite Geometric Progression*
- *Some Important Results*

PROGRESSIONS

In this chapter, we will look at the problems on sequences or progressions of numbers, where the terms of the sequence follow a particular pattern either addition of a constant (arithmetic sequence or arithmetic progression) or multiplication by a constant (geometric sequence or geometric progression). A third type of progression—harmonic progression—has also been defined later.

Arithmetic Progression (AP)

An arithmetic progression is a sequence of numbers in which any number (other than the first) is more (or less) than the immediately preceding number by a constant value. This constant value is called the common difference. In other words, any term of an arithmetic progression can be obtained by adding the common difference to the preceding term.

Let a be the first term of an arithmetic progression, d the common difference, and n the number of terms in the progression.

The n^{th} term is normally represented by T_n, and the sum to n terms of an arithmetic progression is denoted by S_n

$$T_n = n^{\text{th}} \text{ term} = a + (n - 1)\, d$$

S_n = Sum of n terms $= \frac{n}{2} \times [2a + (n-1)\,d]$, then the progression can be represented as a, $a + d$, $a + 2d$, …, $[a + (n - 1)\, d]$. Here, quantity d is to be added to any chosen term to get the next term of the progression.

The sum to n terms of an arithmetic progression can also be written in a different manner.

$$\text{Sum of first } n \text{ terms} = \frac{n}{2} \times [2a + (n-1)d]$$
$$= \frac{n}{2} \times [a + \{a + (n-1)d\}]$$

But, when there are n terms in an arithmetic progression, a is the first term and $\{a + (n - 1)\, d\}$ is the last term. Hence, $S_n = \frac{n}{2} \times$ [first term + last term].

The average of all the terms in an arithmetic progression is called their arithmetic mean (AM). Since average is equal to {sum of all the quantities/number of quantities}, arithmetic progression must be equal to the sum of the terms of the arithmetic progression divided by the number of terms in the arithmetic progression.

Arithmetic mean of n terms in arithmetic progression

$$= \frac{S_n}{n} = \frac{1}{2}\{2a + (n-1)d\}$$
$$= \frac{1}{2} \times (\text{First Term} + \text{Last Term})$$
$$= \frac{(\text{First Term} + \text{Last Term})}{2}$$

i.e. AM is the average of the first and the last terms of the AP.

Arithmetic mean can also be obtained by taking the average of any two terms that are EQUIDISTANT from the two ends of the AP, i.e.

1. The average of the second term from the beginning and the second term from the end will be equal to the AM.
2. The average of the third term from the beginning and the third term from the end will also be equal to the AM and so on.

In general, the average of the k^{th} term from the beginning and the k^{th} term from the end will be equal to the AM.

Conversely, if the AM of an AP is known, the sum to n terms of the series (S_n) can be expressed as

$$S_n = n \times \text{AM}.$$

If three numbers are in arithmetical progression, the middle number is called the arithmetic mean, i.e. if a, b, c are in AP, then b is the AM of the three terms and $b = \frac{a+c}{2}$.

If a and b are in arithmetic progression (AP), then their AM $= \frac{(a+b)}{2}$.

If three numbers are in AP, we can represent the three numbers as $(a - d)$, a, and $(a + d)$.

If four numbers are in AP, we can represent the four numbers as $(a - 3d)$, $(a - d)$, $(a + d)$, and $(a + 3d)$; (in this case, $2d$ is the common difference).

If five numbers are in AP, we can represent the five numbers as $(a - 2d)$, $(a - d)$, a, $(a + d)$, and $(a + 2d)$.

Solved Examples

Example 1

The sixth and the tenth terms of an arithmetic progression are 22 and 38, respectively. Find the first term and the common difference.

Solution

Let the first term and the common difference be a and d, respectively.

$$a + 5d = 22 \quad (1)$$

$$a + 9d = 38 \quad (2)$$

Subtracting (1) from (2),

$$4d = 16, d = 4$$

Substituting d in (1) or (2),

we get $a = 2$

Example 2

The 12^{th} term, the 14^{th} term, and the last term of an arithmetic progression are 25, 31, and 37, respectively. Find the first term, common difference, and the number of terms.

Solution

Let the first term, the common difference and the number of terms be a, d, and n, respectively.

Given that

$$a + 11d = 25 \quad (3)$$

$$a + 13d = 31 \quad (4)$$

Subtracting (3) from (4),

$$2d = 6$$

$$d = 3$$

Substituting $d = 3$ in (3) or (4),

$$a = -8$$

given, $t_n = -8 + (n - 1)3 = 37$

$$n = 16$$

Example 3

Three terms in arithmetic progression have a sum of 45 and a product of 3240. Find them.

Solution

Let the terms be $a - d$, a and $a + d$.

$$a - d + a + a + d = 45$$

$$a = 15$$

$$(a - d)\, a\, (a + d) = 3240$$

$$15^2 - d^2 = 216$$

$$d = \pm 3$$

If $d = 3$ the terms are 12, 15, and 18. If $d = -3$, the terms are same but in the descending order.

Example 4

The first term and the last term of an arithmetic progression are 9 and 69, respectively. If the sum of all the terms is 468, find the number of terms and the common difference.

Solution

Let the number of terms and the common difference be n and d, respectively,

$$S_n = \frac{n}{2}[9 + 69] = 468$$

$\Rightarrow$ $$39n = 468$$

$$n = 12$$

$$t_n = 9 + 11d$$

$\Rightarrow$ $$11d = 60$$

$\Rightarrow$ $$d = \frac{60}{11}$$

Example 5

The sum of three numbers which are in arithmetic progression is 24. The sum of their square is 200. Find the numbers.

Solution

Let the numbers be $a - d$, a and $a + d$.

Given, $a - d + a + a + d = 24$

$\therefore$ $a = 8$

$$(a - d)^2 + a^2 + (a + d)^2 = 200$$

$$3a^2 + 2d^2 = 200$$

$\Rightarrow$ $d^2 = 4$

$\therefore$ $d = \pm 2$

If $d = 2$, the numbers are 6, 8 and 10. If $d = -2$, the numbers are same, but in the descending order.

Geometric Progression (GP)

Numbers taken in a certain order are said to be in geometrical progression, if the ratio of any (other than the first number) to the preceding one is the same. This ratio is called the common ratio. In other words, any term of a geometric progression can be obtained by multiplying the preceding number by the common ratio.

The common ratio is normally represented by r. The first term of a geometric progression is denoted by a.

A geometric progression can be represented as a, ar, $1ar^2$, …, where a is the first term and r is the common ratio of the geometric progression.

n^{th} term of the geometric progression is ar^{n-1}.

Sum to n terms:

$$\frac{a\left(1-r^n\right)}{1-r} \text{ or } \frac{a\left(r^n-1\right)}{r-1}$$

$$= \frac{xar^{n-1}-a}{r-1} = \frac{r \times \text{Last term} - \text{First term}}{r-1}$$

Thus, the sum to n terms of a geometric progression can also be written as

$$S_n = \frac{r \times \text{Last term} - \text{First term}}{r-1}$$

If n terms $a_1, a_2, a_3, \ldots, a_n$ are in GP, then the geometric mean (GM) of these n terms is given by $= \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n}$

If three terms are in geometric progression then the middle term is a geometric mean of the other two terms, i.e. if a, b, and c are in GP, then b is the geometric mean of the three terms and $b^2 = ac$.

If there are two terms a and b, their geometric mean (GM) is given by $\sqrt{ab}$.

For any two unequal positive numbers a and b, their arithmetic mean is always greater than their geometric mean, i.e.

> *For any two unequal positive numbers a and b,* $\frac{a+b}{2} > \sqrt{ab}$ *; $(a + b) > 2\sqrt{ab}$ $b) >$*

When there are three terms in geometric progression, we can represent the three terms to be a/r, a, and ar

When there are four terms in geometric progression, we can represent the four terms as $\frac{a}{r^3}$, $\frac{a}{r}$, ar. and ar^3.

(In this case r^2 is the common ratio.)

Infinite Geometric Progression

If $-1 < r < +1$ or $|r| < 1$, then the sum of a geometric progression does not increase infinitely; it 'converges' to a particular value. Such a GP is referred to as an infinite geometric progression. The sum of an infinite geometric progression is represented by S_∞ and is given by the formula $S_\infty = \frac{a}{1-r}$.

Harmonic progression: If the reciprocals of the terms of a sequence are in arithmetic progression, the sequence is said to be a harmonic progression, For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a harmonic progression. In general, the sequence $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$ is a harmonic progression.

If a, b, c are in harmonic progression, b is said to be the harmonic mean of a and c. In general, if $x_1, x_2, \ldots, x_n$ are in harmonic progression, $x_2, x_3, \ldots, x_{n-1}$ are the $n - 2$ harmonic means between x_1 and x_n.

Some Important Results

The results of the sums to n terms of the following series are quite useful and, hence, should be remembered by students.

Sum of the first n natural numbers $= \sum n = \frac{n(n+1)}{2}$

Sum of squares of the first n natural numbers

$$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of cubes of first n natural numbers

$$\sum n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4} = \left[\sum n\right]^2$$

Example 6

Find the 7th term of the geometric progression whose first term is 6 and common ratio is 2.

Solution

n^{th} term of a GP $= ar^{n-1}$

$$7^{th} \text{ term} = 6\,(2^6) = 384$$

Example 7

A geometric progression has its first term as 64 and its common ratio as $\frac{1}{2}$. Find the sum of its first five terms.

Solution

Sum of the first n terms of a GP $= \dfrac{a(1-r^n)}{1-r}$

$$\text{Sum of its first five terms} = \frac{64\left(1-\left(\frac{1}{2}\right)^5\right)}{1-\frac{1}{2}} = 124$$

Example 8

Find the common ratio of the geometric progression whose first and last terms are 5 and $\dfrac{1}{25}$, respectively, and the sum of its terms is $\dfrac{624}{100}$.

Solution

Sum of the terms of a geometric progression whose common ratio is r is given by $\dfrac{r(\text{last term})-(\text{first term})}{r-1}$

$$\frac{r\left(\frac{1}{25}\right)-5}{r-1} = \frac{624}{100}.$$

$$\Rightarrow \quad 4r - 500 = 624r - 624$$

$$\Rightarrow \quad r = \frac{1}{5}.$$

Example 9

Three numbers in geometric progression have a sum of 42 and a product of 512. Find the numbers.

Solution

Let the numbers be $\dfrac{a}{r}$, a and ar.

$$\frac{a}{r} + a + ar = 42$$

$$\left(\frac{a}{r}\right)(a)(ar) = 512$$

$$a = 8$$

$$\frac{8}{r} + 8 + 8r = 42$$

$$8r^2 - 34r + 8 = 0$$

$$8r^2 - 32r - 2r + 8 = 0$$

$$(r-4)(4r-1) = 0$$

$$r = 4 \quad \text{or} \quad \frac{1}{4}$$

If $r = 4$, the numbers are 2, 8 and 32. If $r = \dfrac{1}{4}$, the numbers are same, but in the descending order.

Example 10

The sum of the terms of an infinite geometric progression is 27. The sum of their squares is 364.5. Find the common ratio.

Solution

Let the first term and the common ratio be a and r, respectively.

$$\text{Given that} \quad \frac{a}{1-r} = 27 \Rightarrow \left(\frac{a}{1-r}\right)^2 = 729$$

$$\text{And} \quad \frac{a^2}{1-r^2} = 364.5$$

$$\Rightarrow \quad a^2 = 729\,(1-r)^2$$

$$= 364.5\,(1-r^2)$$

$$729\,(1-r)^2 - \frac{729}{2}(1-r)(1+r) = 0$$

$$\frac{729}{2}(1-r)[2(1-r)-(1+r)] = 0$$

$$\Rightarrow \quad (1-r)(1-3r) = 0$$

$$r \neq 1 \quad (\because |r| < 1)$$

$$\therefore \quad r = \frac{1}{3}$$

Example 11

If $|x| < 1$, find the value of $3 + 6x + 9x^2 + 12x^3 + \cdots$

Solution

Let $S = 3 + 6x + 9x^2 + 12x^3 + \cdots$ (7)

$$xS = 3x + 6x^2 + 9x^3 + \quad (8)$$

Subtracting (8) from (7)

$$S(1-x) = 3\,(1 + x + x^2 + \cdots)$$

$$\text{As} \quad |x| < 1, \; S = \frac{3\left(\frac{1}{1-x}\right)}{1-x} = \frac{3}{(1-x)^2}$$

Example 12

Evaluate $\dfrac{1}{1(2)} + \dfrac{1}{2(3)} + \dfrac{1}{3(4)} + \cdots + \dfrac{1}{99(100)}$.

Solution

$$\frac{1}{1(2)} = \frac{1}{1} - \frac{1}{2}$$

$$\frac{1}{2(3)} = \frac{1}{2} - \frac{1}{3}$$

$$\text{Finally} \quad \frac{1}{99(100)} = \frac{1}{99} - \frac{1}{100}$$

The given expression is $1 - \dfrac{1}{100} = \dfrac{99}{100}$

EXERCISES

Direction for questions 1 to 25: Select the correct alternative from the given choices.

1. The sixth term and the eleventh term of an arithmetic progression are 30 and 55, respectively. Find the twenty-first term of the series.

(A) $88\frac{1}{3}$ (B) 105 (C) 110 (D) $92\frac{1}{2}$

2. What is the 15th term of an arithmetic progression whose first term is equal to its common difference and whose 3rd term is 9?

(A) 15 (B) 30 (C) 45 (D) 60

3. If $x + 4$, $6x - 2$, and $9x - 4$ are three consecutive terms of an arithmetic progression, then find x.

(A) 2 (B) 4 (C) 6 (D) 8

4. Find the number of terms and the sum of the terms of the arithmetic progression 32, 28, … 4.

(A) 8; 144 (B) 7; 126
(C) 14; 252 (D) 15; 270

5. Find the sum of the first 31 terms of the arithmetic progression whose first term is 6 and whose common difference is $\frac{8}{3}$.

(A) 1410 (B) 1418 (C) 1426 (D) 1434

6. The sum of five terms of an arithmetic progression is 70. The product of the extreme terms is 132. Find the five terms.

(A) 4, 8, 12, 16, 20 (B) 10, 12, 14, 16, 18
(C) 6, 10, 14, 18, 22 (D) 8, 12, 16, 20, 24

7. The sum to n terms of an arithmetic progression is $5n^2 + 2n$. Find the n^{th} term of the series.

(A) $10n + 5$ (B) $10n - 3$
(C) $5n - 1$ (D) $5n - 2$

8. Which term of the geometric progression 4, $4\sqrt{2}$, 8 … is $64\sqrt{2}$?

(A) 8 (B) 9 (C) 10 (D) 12

9. Find the sixth term of the geometric progression whose first term is 2 and common ratio is 3.

(A) 96 (B) 486 (C) 1458 (D) 162

10. Find the sum of the first 4 terms of a geometric progression whose first term is 6 and whose common ratio is 2.

(A) 90 (B) 84 (C) 96 (D) 102

11. What is the sum of the first 7 terms of a geometric progression whose first term is 1 and 4th term is 8?

(A) 129 (B) 128
(C) 127 (D) None of these

12. If the sum to 37 terms of an arithmetic progression is 703, then find the middle term of the arithmetic progression.

(A) 34 (B) 17 (C) 38 (D) 19

13. Find the sum of the 20 terms of the series 1, (1 + 2), (1 + 2 + 3), (1 + 2 + 3 + 4), (1 + 2 + 3 + 4 + 5), …

(A) 1540 (B) 1435 (C) 1450 (D) 1345

14. If the real numbers a, c and b as well as $a^2 + b^2$, $a^2 + c^2$, and $b^2 + c^2$ are in geometric progression, then which of the following is necessary true?

(A) $a = b$ (B) $b = c$
(C) $a = c$ (D) $a = b = c$

15. How many numbers between 450 and 950 are divisible by both 3 and 7?

(A) 20 (B) 24 (C) 30 (D) 35

16. $S = 2 + 4x + 6x^2 + 8x^3 \ldots$ where $|x| < 1$. Which of the following is the value of S?

(A) $\frac{4}{(1-x)^2}$ (B) $\frac{3}{(1-x)^2}$
(C) $\frac{2}{(1-x)^2}$ (D) $\frac{1}{(1-x)^2}$

17. The sum of the first eight terms of a geometric progression. is 510 and the sum of the first four terms of the geometric progression. is 30. Find the first term of the geometric progression, given that it is positive.

(A) 2 (B) 4 (C) 6 (D) 8

18. Find the integer value of y, if $-x$, $2y$, and $2(y + 3)$ are in arithmetic progression and $(x + 2)$, $2(y + 1)$, and $(5y - 1)$ are in geometric progression.

(A) 2 (B) 3 (C) 4 (D) 5

19. Find the number of terms common to the progressions 2, 8, 14, 20, …, 98 and 6, 10, 14, 18, …, 102.

(A) 7 (B) 6 (C) 8 (D) 9

20. Find the sum of the series $2 + 3x + 4x^2 + 5x^3 + \cdots$ to infinity, if $|x| < 1$.

(A) $\frac{2-x}{(1-x)^2}$ (B) $\frac{2+x}{(1+x)^2}$
(C) $\frac{2-x}{(1+x)^2}$ (D) $\frac{2+x}{(1-x)^2}$

21. The mean of the sequence 3, 8, 17, 30, …, 1227 is ____.

(A) 531 (B) 431 (C) 314 (D) 315

22. Find the value of $-1^2 + 2^2 - 3^2 + 4^2 - 5^2 + 6^2 + \cdots -19^2 + 20^2$

(A) 210 (B) 420 (C) 630 (D) 720

23. Find the sum of the given terms in the following series:

$$\frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{119}+\sqrt{121}}$$

(A) $2\sqrt{3}+1$ (B) 5
(C) $11 - 2\sqrt{3}$ (D) 10

24. If $\log_3 x + \log_{\sqrt[3]{3}} x + \log_{\sqrt[5]{3}} x + \cdots + \log_{\sqrt[23]{3}} x = 432$, then find x.

(A) 9 (B) 27 (C) $3\sqrt{3}$ (D) 81

25. The sum of the first n terms of two arithmetic progressions S_1 and S_2 are in the ratio $11n - 17 : 5n - 21$. Find the ratio of the 16th terms of S_1 and S_2.

(A) 3 : 2 (B) 162 : 67
(C) 9 : 4 (D) 27 : 8

Answer Keys

1. B	**2.** C	**3.** A	**4.** A	**5.** C	**6.** C	**7.** B	**8.** C	**9.** B	**10.** A
11. C	**12.** D	**13.** A	**14.** A	**15.** B	**16.** C	**17.** A	**18.** A	**19.** C	**20.** A
21. B	**22.** A	**23.** B	**24.** B	**25.** B					